Persistence K-theory

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geometry $\rightsquigarrow$ algebra

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$\leftsquigarrow$?

[Geometric Lagrangian Topology ?]

(endow Lagrangian spaces w. metrics)

Motivation

- Compare apples to oranges
  Not all Lagrangians are equivalent (as in Claude's talk) so γ not enough.
- Metric understanding of symplectic rigidity (compare immersed vs embedded; structural results at small and large scale; entropy)
- TDA

I. Algebra (filtered).

a. Setting: triangulated persistence categories.

(TPC) $\mathcal{C}$ is a TPC if:

- it is a persistence category:

$Mor_{\mathcal{C}}(A,B) = \{Mor^{\alpha}_{\mathcal{C}}(A,B)\}_{\alpha\in\mathbb{R}}$ are persistence modules $[\{M_\alpha\}_{\alpha\in\mathbb{R}}$ and maps $i_{\alpha,\beta} : M_\alpha \to M_\beta$, $i_{\beta,\gamma}\circ i_{\alpha,\beta} = i_{\alpha,\gamma}$; $i_{\alpha,\alpha} = \mathbb{1}_{M^\alpha}]$

- $\mathcal{C}_0 = \begin{cases} \text{- same objects as } \mathcal{C} \\ \text{- } \mathrm{Mor}_{\mathcal{C}_0}(A,B) = \mathrm{Mor}^0_{\mathcal{C}}(A,B) \end{cases}$

is triangulated.

- shift functors: $\Sigma^r : \mathcal{C} \longrightarrow \mathcal{C}$ exact (on $\mathcal{C}_0$) + some compatibility relations.

Remark: Certain Fukaya categories naturally admit refinements of this type; Shift functor in this case is: $\Sigma^r : (L, f_L) \longrightarrow (L, f_L + r)$

If $\mathcal{C}$ is a TPC then

$\mathcal{A}\mathcal{C} \subset \mathcal{C}$ sub-category of acyclics

r-acyclic $\overset{\Psi}{A}$: $id_A \in Mor^{0}_{\mathcal{C}}(A,A) \rightsquigarrow 0 \in Mor^{r}_{\mathcal{C}}(A,A)$

Easy to see that $\mathcal{A}\mathcal{C}$ is a TPC.

Moreover: $\boxed{\mathcal{C}_0 / \mathcal{A}\mathcal{C}_0 = \mathcal{C}_\infty}$ (Verdier localization)

$$\mathcal{C}_\infty = \begin{cases} \text{same objects as } \mathcal{C} \\ Mor_{\mathcal{C}_\infty}(A,B) = \lim\limits_{\alpha \to \infty} Mor^{\alpha}_{\mathcal{C}}(A,B) \end{cases}$$

Corollary: $\mathcal{C}_\infty$ is triangulated.

Remark: We say that $\mathcal{C}$ is a TPC refinement of the triangulated category $\mathcal{D}$ if $\mathcal{C}_\infty = \mathcal{D}$.

b. Weighted triangles in a TPC $\mathcal{C}$.

$\phi \in \mathrm{Mor}^{0}_{\mathcal{C}}(A, B)$ is an r-isomorphism if there is an exact triangle in $\mathcal{C}_0$:

$$A \xrightarrow{\phi} B \longrightarrow C \longrightarrow TA, \text{ with } C \text{ } r\text{-acyclic}.$$

A strict exact triangle of weight r in $\mathcal{C}$ is of the form:

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma^{-r} T A$$

has the property that $u, v, w \in \mathrm{Mor}\, \mathcal{C}_0$ and there is an exact triangle in $\mathcal{C}_e$:

$$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[r, "v'"] \arrow[dr, "v"'] & C' \arrow[r, "w'"] \arrow[d, "\phi"] & TA \\ & & C & \end{tikzcd}$$

and an r-isom.

$\phi : C' \longrightarrow C$ + commutativity.

Theorem: $\mathcal{C}_\infty$ carries a triangular weight induced by the weight of the strict exact triangles in $\mathcal{C}$.

Δ = triangulated category. A triangular weight w is a map $w : \mathrm{Exact}\,\Delta_\Delta \to [0,\infty)$ with a weighted form of the octahedral axiom.

$(\mathcal{D}, w)$ (triangulated category
+ w triangular weight

$$\Downarrow$$

fragmentation pseudo-metrics / $\mathrm{Obj}(\mathcal{D})$.

They are defined by infimising the total weight required to decompose an object through iterated exact tr.

[May be viewed as "extensions" of γ to all exact Lag]

$\mathcal{C}$ = TPC. Other associated structures:

$K(\mathcal{C}) := K(\mathcal{C}_0) = \mathrm{Ab}\langle \mathrm{Obj}\, \mathcal{C} \rangle /$ triangle identities

$(B = A + C$ if $A \to B \to C \to TA$ is exact in $\mathcal{C}_0)$

Exact sequence:

$$0 \longrightarrow T\mathcal{C} \longrightarrow K(A\mathcal{C}) \longrightarrow K(\mathcal{C}) \longrightarrow K(\mathcal{C}_\infty) \to 0$$

Pairing (under some finiteness constraints)

$$\bar{\chi} : K(\mathcal{C}) \otimes K(\mathcal{C}) \longrightarrow \Lambda_P =$$ Universal Novikov pol. ring.

Induced by: $\operatorname{Mor}_{\mathcal{C}}(A, B)$ = persistence module =

$$= \bigoplus B_i \quad , \quad B_i = \begin{cases} [x_i, \infty) \\ \text{or} \\ [x_i, y_i) \end{cases}$$

$$\bar{\chi}(A, B) = \sum (-1)^{|B_i|} \left(t^{x_i} - t^{y_i} \right)$$ (Filtered Euler pairing).

II. Main example: Filtered Fukaya category

$(M, \omega = d\lambda)$ Weinstein domain

$\mathcal{D}\mathrm{Fuk}(M)$ = usual derived Fukaya category (as in Seidel's book) generated by

(L, f_L), $L \xrightarrow{i_L} M$ embedded Lagrangians

$i_L^* \lambda = d f_L$.

Developed: FOOO, Seidel, starting from Floer, Hofer, Salamon, Donaldson .....

Recall (L, f_L) are the objects of an A_∞-category $Fuk(M)$. There is a category of A_∞ modules $\boxed{\mathrm{mod}_{Fuk(M)}}$ which is pre-triangulated (may construct cones of morphisms) and

$$\Delta Fuk(M) = H_0\left(\mathrm{Image}(\mathcal{Y})^{\Delta} \right)$$

$\mathcal{Y} : Fuk(M) \longrightarrow \mathrm{Mod}_{Fuk(M)}$ Yoneda functor.

We know that $\mathcal{D}\mathrm{Fuk}(M)$ is triangulated.

Question : Does $\mathcal{D}\mathrm{Fuk}(M)$ admit a natural TPC refinement?

If so the exact triangles in $\mathcal{D}\mathrm{Fuk}(M)$ will carry a (persistence) weight and the objects of $\mathcal{D}\mathrm{Fuk}(M)$ a nice class of pseudo-metrics.

Answer (theorem) : Yes.

To construct $\mathcal{C}$ a TPC such that $\mathcal{C}_\infty = D\mathrm{Fuk}(M)$ we need to track filtrations.

Step 0: $L \pitchfork L' \Rightarrow CF(L, L')$ filtered,

$HF(L, L')$ = persistence module.

Step 1: $L \pitchfork L'$, $L' \pitchfork L''$, $L \pitchfork L'' \Rightarrow$

$$CF(L, L') \otimes CF(L', L'') \longrightarrow CF(L, L'')$$

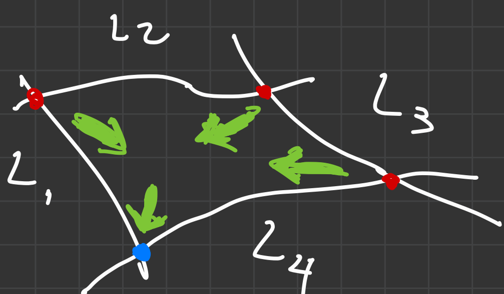
filtered $\Rightarrow HF(L, L') \otimes HF(L', L'') \to HF(L, L'')$ multiplication of persistence modules.

Difficulties (mainly technical)

a) $CF(L,L) = ?$; strict units?

b) energy estimates to make sure that μ_k is filtration preserving.

How to control Ham perturbations so that they do not add up?

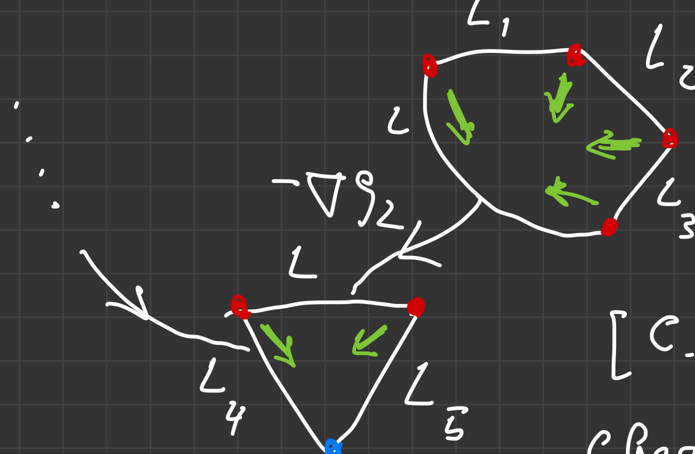


c) Invariance issues when dealing with filtered homological algebra.

Solutions:

a) Use $CF(L, L) = \text{Morse Cplex}\,(g_L : L \to \mathbb{R})$

This leads to using "cluster" moduli spaces:



[C.-Lalonde, FOOO, Charest, Charest-Woodward etc] b). There are two solutions to keep energy "errors" under check:

i) [Biran - C. - Zhang, '23] do the construction for only a finite family of Lagr.

$\mathcal{X} = \{L_1, L_2, \dots, L_m\}$, $L_i \pitchfork L_j$ $\forall i, j$.

$\Rightarrow$ filtered $\overline{Fuk}(\mathcal{X}) \Rightarrow \overline{Mod}_{\overline{Fuk}(\mathcal{X})}$

$\Rightarrow \overline{\Delta Fuk}(\mathcal{X}) = TPC$. It has decent invariance properties.

ii) [Ambrosioni, '23] Use a smart system of perturbations $\rightsquigarrow$ $\overline{\mathcal{D}\mathrm{Fuk}}(M)$ TPC

A little less good w.r to invariance.

In practice both methods can be used equivalently in applications.

<u>For this talk</u>: $\overline{\mathcal{D}\mathrm{Fuk}}(M)$ a TPC that is a refinement of $\mathcal{D}\mathrm{Fuk}(M)$.

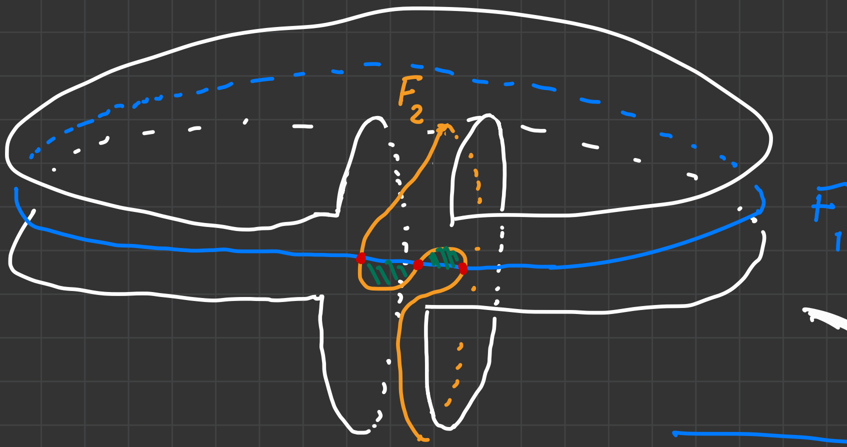
III A problem to look at (w. these methods).

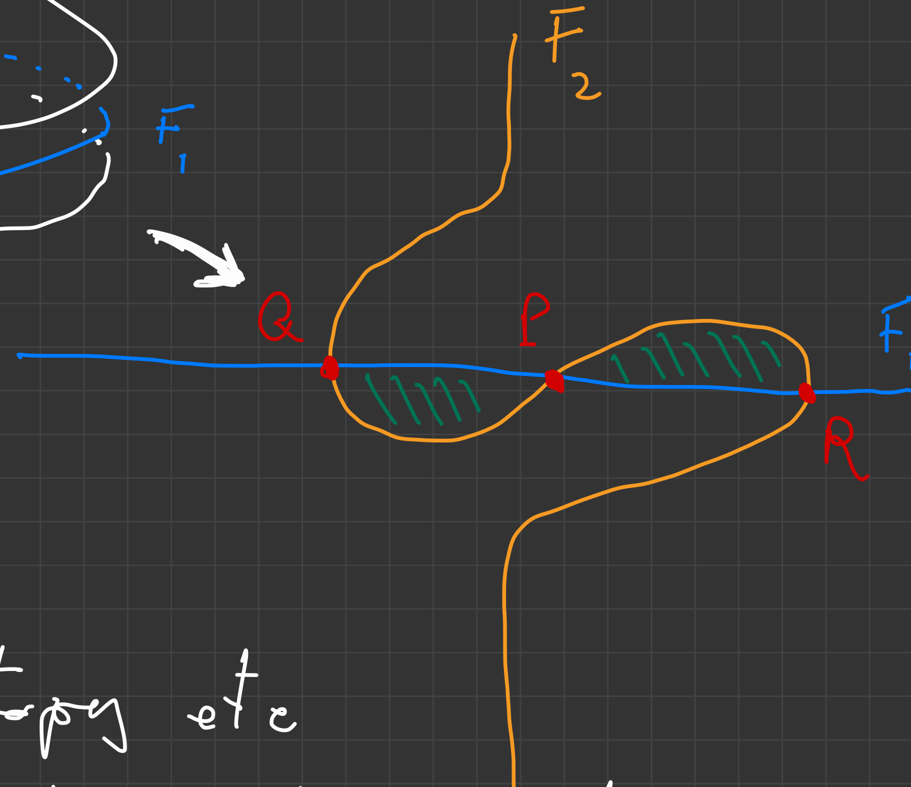
Question:

Given $\mathfrak{F} = \{F_1, F_2, \dots, F_k\}$, F_i embedded exact Lagrangian. Assume $F_i \pitchfork F_j$, $i \neq j$.

Let $F = F_1 \cup F_2 \cdots \cup F_k$ (immersed).

How far is F from an embedded exact Lagrangian (inside $\mathcal{O}S_{\mathfrak{F}}(\mathcal{D}Fuk(M))$) ?





May use:

- surgery
- ham homotopy etc

but preserve exactness at each step.

This means that we want to write some embedded N as the result of n iterated exact triangles in $\Delta\mathcal{F}uk(M)$:

$$D := \begin{cases} \Delta_1 : & X_1 \to 0 \to TF_1 \to TX_1 \\ \Delta_2 : & X_2 \to TF_1 \to Y_2 \to TX_2 \\ & \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \Delta_n : & X_n \to TF_1 \to N \to TX_n \end{cases}$$

With $X_i \in \mathcal{F}$ or $X_i = 0$, use each F_i once ($\approx$ each Δ_i is a surgery or a hom htpy).

Such a $\mathcal{D}$ is a cone decomposition of N

$$\mathcal{D} = \begin{cases} \Delta_1 : & X_1 \to 0 \to TF_1 \to TF_1 \\ \Delta_2 : & X_2 \to TF_1 \to Y_2 \to TF_2 \\ & \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \Delta_n : & X_n \to TF_1 \to N \to TF_R \end{cases}$$

We can associate to $\mathcal{D}$ its total weight.

$$- \; w(\mathcal{D}) = \sum w(\Delta_i).$$

($w(\Delta_i)$ = persitence triangular weight)

We can also associate to the family $\mathcal{F} = \{F_1, \dots, F_k\}$ some numbers:

$gap(\mathcal{F}) := gap(\bar{\chi}([F],[F]))$ is K-theoretic in nature. $[F] = [F_1] + \dots + [F_k] \in K\,\overline{\mathcal{D}Fuk(M)}$

$\bar{\chi} : K \otimes K \longrightarrow \Lambda_P$ is bilinear

given by $\bar{\chi}(L, L) = \chi(L)$ for L embedded, $\bar{\chi}(L, L') = \sum'_{x \in L \cap L'} (-1)^{|x|} t^{\mathcal{A}(x)}$

L, L' embedded and $L \pitchfork L'$, $\mathcal{A}(x) = f_{L'}(x) - f_L(x)$.

$$Q = \overline{\chi}([F],[F]) = k_1 t^{a_1} + k_2 t^{a_2} + \cdots \in \Lambda_P$$

$$\mathrm{gap}(Q) = \min\{(a_i - a_{i-1}),\ a_i \mid a_i > 0,\ i\}$$

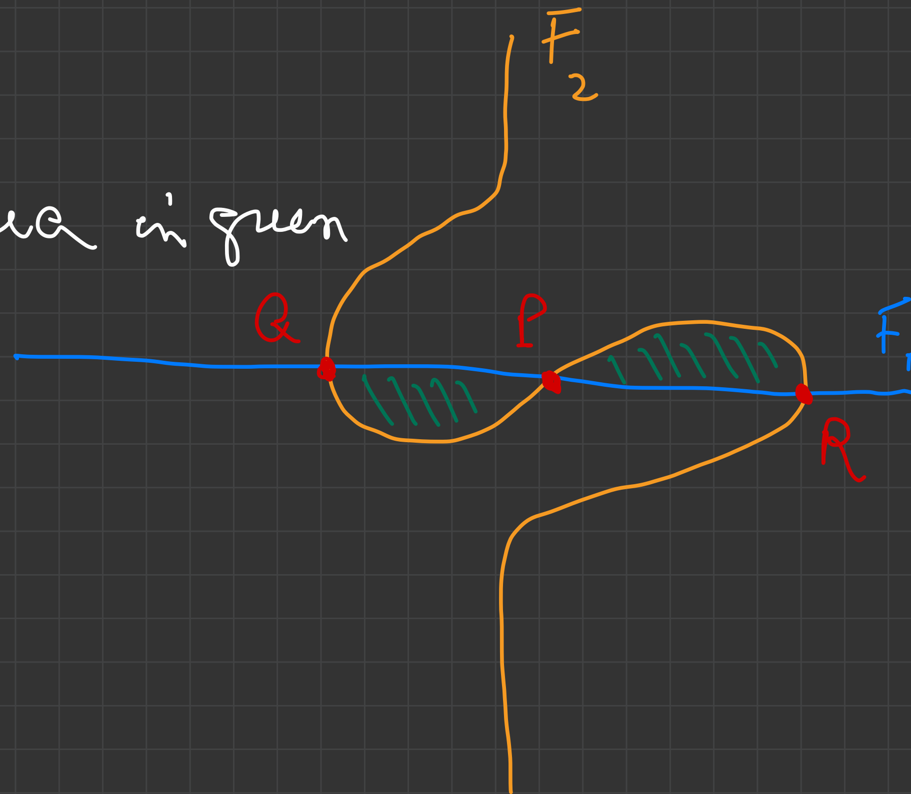
$B(\mathcal{F})$ = max # of bars in $HF(F_i, F_j)$

<u>Theorem</u> If Δ is a decomposition of an embedded N, then:

$$n^2 B(\mathcal{F})\, w(\Delta) \geq \frac{1}{4}\, \mathrm{gap}(\mathcal{F})$$

Remark:

- $\text{gap}(\mathcal{F}) = a =$ area in green
- $B(\tilde{\mathcal{F}}) = 2$



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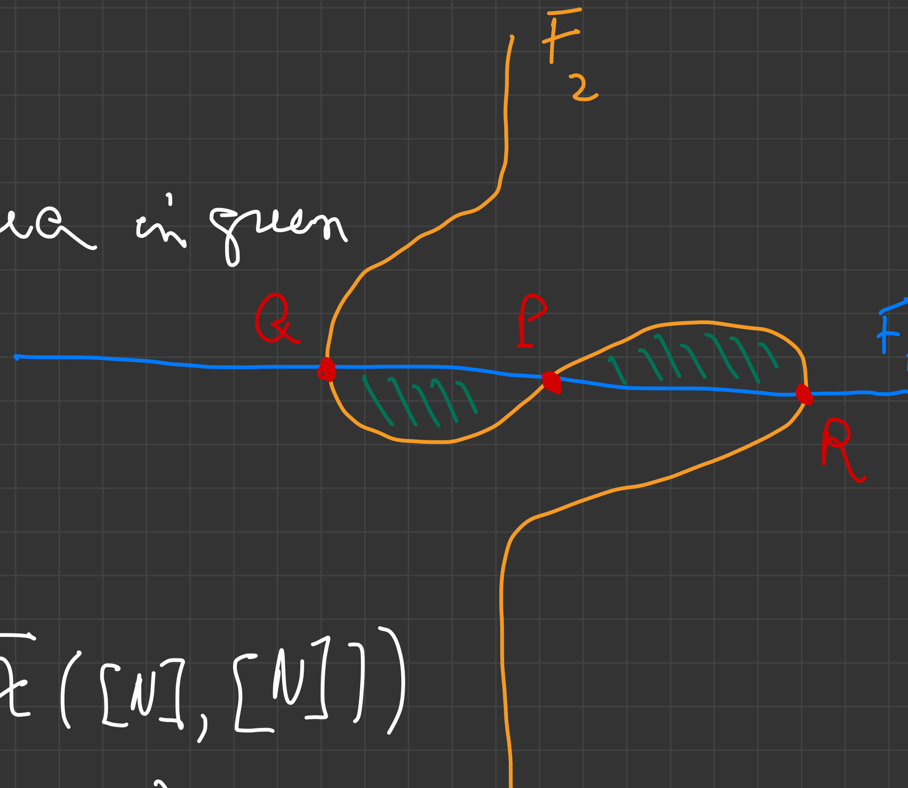

Idea of proof:

$$\operatorname{gap}(N) = \operatorname{gap}\left(\bar{\mathcal{X}}([N],[N])\right)$$
$$= \operatorname{gap}(\mathcal{X}(N)) = 0.$$

$$n^2 B(\mathcal{F})\, w(D) \geq \frac{1}{4} \operatorname{gap}(\mathcal{F})$$

Where D is a decomposition of N

$$n^2 \, B(\mathcal{F}) \, w(\mathcal{D}) \geq \frac{1}{4} \, \mathrm{gap}(\mathcal{F})$$

Where $\mathcal{D}$ is a decomposition of N

$$\mathrm{gap}(N) = 0 \text{ but } \mathrm{gap}(\mathcal{F}) > 0$$

$$n^2 B(\mathcal{F})\, w(\mathcal{D}) \geq \frac{1}{4} \operatorname{gap}(\mathcal{F})$$

where $\mathcal{D}$ is a decomposition of N

$\operatorname{gap}(N) = 0$ but $\operatorname{gap}(\mathcal{F}) > 0$

$$\underbrace{n^2 B(\mathcal{F})}_{\text{\# bars created along the decomp}}$$

$$n^2 B(\mathcal{F})\, w(\mathcal{D}) \geq \frac{1}{4}\, gap(\mathcal{F})$$

Where $\mathcal{D}$ is a decomposition of N

$gap(N) = 0$ but $gap(\hat{\mathcal{F}}) > 0$

$$\underbrace{n^2 B(\tilde{\mathcal{F}})}_{\text{\# bars created along the decomp}} \cdot \underbrace{w(\mathcal{D})}_{\text{estimate of total shift of bars}}$$

$$n^2 B(\mathcal{F}) \, w(\mathcal{D}) \geq \frac{1}{4} \, \mathrm{gap}(\widetilde{\mathcal{F}})$$

where $\mathcal{D}$ is a decomposition of N

$$\mathrm{gap}(N) = 0 \text{ but } \mathrm{gap}(\widetilde{\mathcal{F}}) > 0$$

$$\left. \underbrace{n^2 B(\widetilde{\mathcal{F}})}_{\substack{\text{\# bars created} \\ \text{along the decomp}}} \cdot \underbrace{w(\mathcal{D})}_{\substack{\text{estimate of total} \\ \text{shift of bars}}} \right\} \substack{\text{If big} \\ \text{enough} \\ \text{it can} \\ \text{kill } \mathrm{gap}(\widetilde{\mathcal{F}}).}$$