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Cone-decompositions of Lagrangian cobordisms in Lefschetz fibrations

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Abstract Given a symplectic manifold M we study Lagrangian cobordisms $V \subset E$ where E is the total space of a Lefschetz fibration having M as generic fiber. We prove a generation result for these cobordisms in the appropriate derived Fukaya category. As a corollary, we analyze the relations among the Lagrangian submanifolds $L \subset M$ that are induced by these cobordisms. This leads to a unified treatment—and a generalization—of the two types of relations among Lagrangian submanifolds of M that were previously identified in the literature: those associated to Dehn twists that were discovered by Seidel (Topology 42(5):1003–1063, 2003) and the relations induced by cobordisms in trivial symplectic fibrations described in our previous work (J Am Math Soc 26(2):295–340, 2013; Geom Funct Anal 24(6):1731–1830, 2014).

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1 Introduction

1.1 Main result

The main aim of this paper it to prove a decomposition result for a class of Lagrangian submanifolds with cylindrical ends—called cobordisms—that are embedded in the total space of a Lefschetz fibration $\pi : E \longrightarrow \mathbb{C}$. The results will take place in certain derived Fukaya categories. These are triangulated categories and the decompositions in question are through iterated cone attachments. We refer to [24] for the detailed construction of the derived Fukaya category (we remark that we do not complete with respect to idempotents in this paper).

First, all our Lefschetz fibrations E are assumed to have a positive dimensional fiber (hence $\dim_{\mathbb{R}} E \geq 4$). Moreover, we consider here cobordisms $V \subset E$ with "negative" ends only: outside of a compact subset, the projection of V to $\mathbb{C}$ is a union of rays of the type $\ell_i = (-\infty, a_i] \times \{i\}$, $i \in \mathbb{N}$. Such cobordisms will be called negative-ended. We denote by $\mathcal{L}^*(E)$ the class of these cobordisms (the superscript $-^*$ indicates the monotonicity constraint imposed on the Lagrangians involved). Finally, we work with uniformly monotone Lagrangians and with a class of Lefschetz fibrations that satisfy a strong variant of the monotonicity condition—see Sects. 2.4, and 3.3 for the definitions. The Fukaya categories in this paper will be taken over the universal Novikov ring, $\mathcal{A}$, over the base field $\mathbb{Z}_2$. We work at all times in an ungraded context. We expect the results of this paper to adapt to the graded context (under additional assumptions on the Lagrangians involved). For this adaptation, the results in both [6] and the current paper need to be adjusted to the graded context. As this requires considerable technical additions we prefer not to pursue this aspect in this paper.

We give here the main decomposition result and refer to Sect. 4.1 where the result is restated after making the various ingredients more precise. Our conventions and notation regarding iterated cone decompositions are explained in Sect. 2.5.

Theorem A *Let $E \longrightarrow \mathbb{C}$ be a Lefschetz fibration with m critical values.*

There exists a Fukaya category with objects the cobordisms in $\mathcal{L}^*(E)$. Let $D\mathcal{F}uk^*(E)$ be the associated derived category. Consider one object $V \in \mathcal{L}^*(E)$ with s negative ends. Fix points $z_i \in \ell_i$ along the rays associated to the ends of V and let $L_i = V \cap \pi^{-1}(z_i)$. Let T_j be the thimbles associated to the curves t_j as in Fig. 1, and let $\gamma_i L_i \subset E$, $2 \leq i \leq s$, be the (union of) parallel transports of L_i along the curve γ_i, in the same figure.

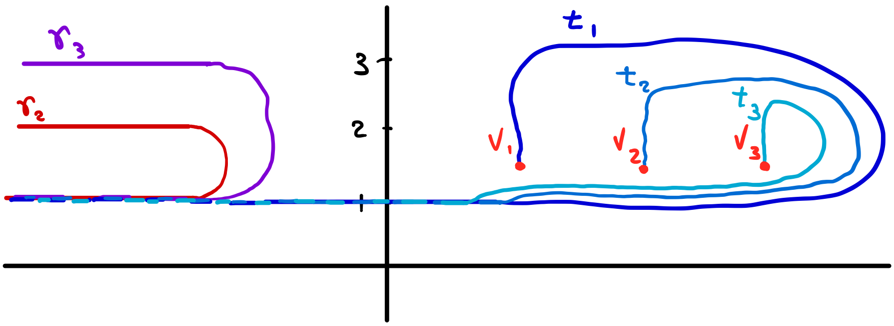

Fig. 1 The curves γ_i, and the curves t_j emanating from the critical values v_j of the Lefschetz fibration

There exist finite rank $\mathcal{A}$*-modules* E_k*,* $1 \leq k \leq m$*, and an iterated cone decomposition taking place in* $D\mathcal{F}uk^*(E)$*:*

$$V \cong (T_1 \otimes E_1 \to T_2 \otimes E_2 \to \cdots \to T_m \otimes E_m \to \gamma_s L_s \to \gamma_{s-1} L_{s-1} \to \cdots \to \gamma_2 L_2).$$

The $\mathcal{A}$-modules E_i are made explicit in the proof—see (28). They are obtained as Floer homologies between V and certain Lagrangian spheres in an auxiliary Lefschetz fibration associated to E.

Lagrangian cobordisms in the total space of a Lefschetz fibration $\pi : E \to \mathbb{C}$ with compact fiber are, in some sense, the simplest non-compact Lagrangians in E. The Theorem says that any such cobordism decomposes in terms of the simplest cobordisms available: those that project onto a curve in $\mathbb{C}$. Moreover, the only curves that are required for such decompositions are the γ_i's and the t_j's.

1.2 Some consequences

Cobordisms are of interest not only for their own sake but also because they can be used to define relations among their ends, in the sense of the usual cobordism relation. In this direction, one of the main consequences of Theorem A is that each such cobordism V produces an iterated cone decomposition inside $D\mathcal{F}uk^*(M)$, where $M = \pi^{-1}(z_1)$ is the general fiber of E. This cone decomposition expresses the end L_1 of V as an iterated cone involving the ends $L_i, i \geq 2$ and the vanishing cycles of the singularities of π. Thus, cobordisms in E and the triangular decompositions in the (derived) Fukaya category of the fiber are intimately related—see Corollary 5.1.1.

To discuss a further consequence, recall that to any triangulated category $\mathcal{C}$ one can associate a Grothendieck group $K_0\mathcal{C}$ defined as the quotient of the free abelian group generated by the objects of $\mathcal{C}$ modulo the relations $B = A + C$ associated to each exact triangle $A \to B \to C$. We remark that in this paper we work with ungraded categories, hence our Grothendieck groups will always be 2-torsion. Another application of Theorem A—see Sect. 5.3—is to give a description of the Grothendieck group $K_0 D\mathcal{F}uk^*(M)$ as an "algebraic" cobordism group. We focus here on the case of the trivial fibration $E = \mathbb{C} \times M$ even if we establish the relevant results in more generality in the paper. Recall from [6] the definition of the cobordism group $\Omega^*_{Lag}(M)$. It is the quotient of the free abelian group generated by the objects in $\mathcal{L}^*(M)$ (the precise constraints $*$ appear in Sect. 2.4) modulo the relations $L_1 + L_2 + \cdots + L_s = 0$ for each negative-ended cobordism $V \subset \mathbb{C} \times M$ whose ends are $L_1, \ldots, L_s$. For every $i \in \mathbb{N}$ there is a natural restriction operation that associates to a cobordism V its ith end. We will see that these operations admit extensions to all objects of $D\mathcal{F}uk^*(\mathbb{C} \times M)$ (i.e. also to the "non-geometric" objects). The ith "end" of an object $\mathcal{M}$ in $D\mathcal{F}uk^*(\mathbb{C} \times M)$ is denoted by $[\mathcal{M}]_i \in \mathcal{O}b(D\mathcal{F}uk^*(M))$. It is natural to define an algebraic cobordism group $\Omega^*_{Alg}(M)$ as the free abelian group generated by the (isomorphism classes of) objects of $D\mathcal{F}uk^*(M)$ modulo the relations $\sum_i [\mathcal{M}]_i = 0$ for each object $\mathcal{M}$ of $D\mathcal{F}uk^*(\mathbb{C} \times M)$. Equivalently, $\Omega^*_{Alg}(M)$ is defined in a similar way to $\Omega^*_{Lag}(M)$ only that the generators and relations now come also from the non-geometric objects in $D\mathcal{F}uk^*(M)$ and $D\mathcal{F}uk^*(\mathbb{C} \times M)$. There is an obvious map $q : \Omega^*_{Lag}(M) \to \Omega^*_{Alg}(M)$. A consequence of Theorem A, Corollary 5.3.2, is that there exists a group isomorphism

$$\Theta_{Alg} : \Omega^*_{Alg}(M) \to K_0 D\mathcal{F}uk^*(M)$$

so that the composition $\Theta_{Alg} \circ q$ coincides with the Lagrangian Thom morphism

$$\Theta : \Omega^*_{Lag}(M) \to K_0 D\mathcal{F}uk^*(M) \tag{1}$$

previously introduced in [6]. One of the reasons why this is of interest is that this result should shed some light on the kernel of Θ which is at present somewhat mysterious. Another implication of the fact that Θ_{Alg} is an isomorphism appears in Corollary 5.4.1 which asserts that the obvious map $\Omega^*_{Lag}(M) \to QH(M)$, that associates to the cobordism class of a Lagrangian L its homology class $[L] \in QH_n(M)$, admits an extension to $\Omega^*_{Alg}(M)$. Here $QH(M)$ stands for the quantum homology of the ambient manifold M.

We also obtain a periodicity result for K_0—Corollary 5.5.1:

$$K_0(D\mathcal{F}uk^*(\mathbb{C} \times M)) \cong \mathbb{Z}_2[t] \otimes K_0(D\mathcal{F}uk^*(M)). \tag{2}$$

Here t is a formal variable whose role will become clear in the proof (roughly speaking, different powers of t are used to label the K_0-classes associated to different ends of a cobordism, or more generally, the "ends" of an object of $D\mathcal{F}uk^*(\mathbb{C} \times M)$).

1.3 Relation to previous work

Theorem A can be viewed as a simultaneous generalization of the two previously known methods to produce exact triangles in the derived Fukaya category.

The first such method is due to Seidel [23], [24, Chapter III, Section 17] and, in its basic form, it associates an exact triangle of the form:

$$\tau_S L \to L \to S \otimes HF(S, L) \tag{3}$$

to the Dehn twist $\tau_S : M \to M$ corresponding to a Lagrangian sphere S and any $L \in \mathcal{L}^*(M)$. Seidel works in an exact setting, but as we will see below, this triangle remains valid in the monotone context too. Other cases have been treated in the literature too, e.g. see [18] for the case of Lagrangians with vanishing Maslov class in Calabi–Yau manifolds as well as [28] for a generalization to fibered Dehn twists. Seidel also considers a Fukaya category $\mathcal{F}uk(\pi)$ associated to a Lefschetz fibration $\pi : E \to \mathbb{C}$, [24,25]. In our setting, this category corresponds to the full and faithful subcategory of $\mathcal{F}uk^*(E)$ generated by the thimbles T_i. He also proves a decomposition result for this category that, in our context, essentially implies the statement of Theorem A in the special case when V has a single end. This category is related to mirror symmetry questions (cobordisms with a single end appear e.g. in [12]). Cobordisms with multiple ends as well as a category somewhat similar to $\mathcal{F}uk^*(E)$ appear in the recent paper [2].

The second method to construct exact triangles appears in our previous paper [6]. It is shown there that if $V \subset \mathbb{C} \times M$ is a cobordism, then the ends of V are related by a cone-decomposition in $D\mathcal{F}uk^*(M)$. This decomposition coincides with the one in Corollary 5.1.1 below when E is the trivial fibration $\mathbb{C} \times M$. Nevertheless, we remark that the statement of Theorem A—which concerns decompositions of cobordisms—is new even for the trivial fibration.

The exact triangle associated to a Dehn twist and the exact triangle obtained through the cobordism machinery coincide when there is a single, transverse intersection between S and L. This can be shown by methods already in the literature. For example, this follows from a combination of the results from [22] and [6] (see also [10,18] for an earlier approach). In this case, Seidel's exact triangle coincides with the surgery exact sequence which is associated to a specific cobordism (in $\mathbb{C} \times M$) whose ends are $\tau_S L$, L, S. This cobordism is constructed as the trace of the Lagrangian surgery at the intersection point $S \cap L$. Theorem A and its proof go beyond this case and further clarify the interplay between these two constructions.

1.4 Outline of the paper

The necessary background on Lefschetz fibrations, monotonicity conditions and other basic conventions and notation are given in Sect. 2. We pursue in Sect. 3 with the definition of the Fukaya category of cobordisms $\mathcal{F}uk^*(E)$. This construction is very similar to the construction of the Fukaya category of cobordisms in the trivial fibration, as described in [6], but there are some modifications required and they are discussed here. The Proof of Theorem A is contained in Sect. 4 and it consists of three main ingredients. The first one deals with decompositions of cobordisms V'—called remote with respect to E—that are included in the total space E' of a Lefschetz fibration that coincides with E over the upper half-plane. The defining property of such a V' is that it can be moved inside E' away from the critical points of $E \longrightarrow \mathbb{C}$, so that its only intersection with an object X of $\mathcal{F}uk^*(E)$ occurs in the region where both V' and X are cylindrical. We show in Sect. 4.3 that such a remote cobordism viewed as a module over $\mathcal{F}uk^*(E)$ admits a decomposition just as the one in the statement of Theorem A but without any of terms $T_i \otimes E_i$. The second step, in Sect. 4.4, shows how to transform a general cobordism V into a remote one. It is done by placing V inside a new Lefschetz fibration E' obtained from E by adding singularities over the lower half-plane and showing that the cobordism $V' \subset E'$ obtained as an iterated Dehn twist of V, $V' = (\tau_{S_m} \circ \cdots \circ \tau_{S_i} \circ \cdots \circ \tau_{S_1})(V)$, where S_i are certain matching cycles in E', is remote with respect to E. The third ingredient—in Sect. 4.5—is Seidel's exact triangle for which we provide a new proof reflecting our cobordism perspective. These ingredients are put together in Sect. 4.6. In short, the cobordism $V' = (\tau_{S_m} \circ \cdots \circ \tau_{S_1})(V)$ is remote with respect to E and thus, by the first step, it admits a certain decomposition involving the ends of V, but as it is obtained by an iterated Dehn twist from V, it can be related to V by another decomposition, involving the matching cycles S_i, by using the relevant Seidel exact triangles. The two decompositions combine as in the statement of Theorem A. The Corollaries of Theorem A described above are proven in Sect. 5. The paper ends with Sect. 6 that is focused on a class of Lagrangian cobordisms in real Lefschetz fibrations.

From a technical standpoint, we rely heavily on Seidel's work, in particular [24]. We also make heavy use of the constructions in our previous papers [5,6].

For the sake of brevity a number of technical arguments are omitted from this paper. These points are clearly indicated in the text and the arguments themselves all appear in an expanded, earlier version [3] of the current paper.

2 Background

2.1 Leschetz fibrations

Lefschetz fibrations will play a central role in this paper. From the symplectic viewpoint there are several versions of this notion in the literature. Our setup is similar to [23,24] but with some modifications.

We begin with Lefschetz fibrations having a compact fiber.

Definition 2.1.1 A Lefschetz fibration with compact fiber consists of the following data:

i. A symplectic manifold (E, Ω_E) without boundary, endowed with a compatible almost complex structure J_E.
ii. A Riemann surface (S, j) (which is generally not assumed to be compact; typically we will have $S = \mathbb{C}$).
iii. A proper (J_E, j)-holomorphic map $\pi : E \longrightarrow S$. (In particular all regular fibers of π are closed manifolds.)
iv. We assume that π has a finite number of critical points. Moreover, we assume that every critical value of π corresponds to precisely one critical point of π. We denote the set of critical points of π by $\text{Crit}(\pi)$ and by $\text{Critv}(\pi) \subset S$ the set of critical values of π.
v. All the critical point of π are ordinary double points in the following sense. For every $p \in \text{Crit}(\pi)$ there exist a J_E-holomorphic chart around p (hence J_E is integrable on this chart) and a j-holomorphic chart around $\pi(p)$ with respect to which π is a holomorphic Morse function.

For $z \in S$ we denote by $E_z = \pi^{-1}(z)$ the fiber over z. We will sometimes fix a base-point $z_0 \in S \setminus \mathrm{Critv}(\pi)$ and refer to the symplectic manifold $(M := \pi^{-1}(z_0), \omega_M := \Omega_E|_M)$ as "the" fiber of the Lefschetz fibration. We will also use the following notation: for a subset $\mathcal{S} \subset S$ we denote $V|_{\mathcal{S}} = \pi^{-1}(\mathcal{S}) \cap V$.

Our constructions work for the most part also when the fiber is not compact.

Let (M, ω_M) be a (non-compact) symplectic manifold which is convex at infinity. We define *a Lefschetz fibration* $\pi : E \longrightarrow S$ *with fiber* (M, ω_M) to be as in Definition 2.1.1 with the following modifications. Firstly, properness in condition iii is removed (thus allowing for the fibers to be non-compact). Secondly, the map $\pi : E \setminus \pi^{-1}(\mathrm{Critv}(\pi)) \longrightarrow S \setminus \mathrm{Critv}(\pi)$ is now explicitly assumed to be a smooth locally trivial fibration. Finally, E is assumed to satisfy the following additional condition which is a variant of the notion of boundary horizontality that appears in [23] and [24].

Assumption T_∞ (Triviality at infinity) *Let $\pi : E \longrightarrow S$ be as above. We say that E is trivial at infinity if there exists a subset $E^0 \subset E$ with the following properties:*

(1) For every compact subset $K \subset S$, $E^0 \cap \pi^{-1}(K)$ is also compact. (In other words, $\pi|_{E^0} : E^0 \longrightarrow S$ is a proper map.)
(2) Set $E^\infty = E \setminus E^0$ and $E^\infty_{z_0} = E^\infty \cap \pi^{-1}(z_0)$, where $z_0 \in S \setminus \mathit{Critv}(\pi)$ is a fixed base-point. Then there exists a trivialization $\phi : S \times E^\infty_{z_0} \longrightarrow E^\infty$ of $\pi|_{E^\infty} : E^\infty \longrightarrow S$ such that

$$\phi^* \Omega_E = \omega_S \oplus \omega_M|_{E^\infty_{z_0}}, \text{ and } \phi^* J_E = j \oplus J_0$$

where ω_S is a positive (with respect to j) symplectic form on S and J_0 is a fixed almost complex structure on $M = \pi^{-1}(z_0)$, compatible with ω_M.

This extended definition generalizes the preceding one: if M is compact we take $E^0 = E$ and $E^\infty = \emptyset$. From now on, unless otherwise stated, by a Lefschetz fibration we mean one with compact fiber that satisfies Definition 2.1.1 or, more generally, with a non-compact fiber that is convex at infinity and satisfies the conditions above, including T_∞. To avoid distinguishing in various arguments the case of a 0-dimensional fiber from the rest, we will assume from now on that all Lefschetz fibrations have positive dimensional fibers. (See also the setting and definitions in Sect. 3.3.)

To a Lefschetz fibration as above we can associate a connection $\Gamma = \Gamma(\Omega_E)$ on $E \setminus \mathrm{Crit}(\pi)$ with horizontal distribution $\mathcal{H} \subset T(E)$ given for every $x \in E \setminus \mathrm{Crit}(\pi)$ by

$$\mathcal{H}_x = \big\{ u \in T_x(E) \mid \Omega_E(\xi, u) = 0 \ \ \forall\, \xi \in T^v_x(E) \big\},$$

where $T^v_x(E)$ is the vertical tangent space at x. The connection Γ induces parallel transport maps. Let $\lambda : [a, b] \longrightarrow \mathbb{C} \setminus \mathrm{Critv}(\pi)$ be a smooth path. We denote by $\Pi_\lambda : E_{\lambda(a)} \longrightarrow E_{\lambda(b)}$ the parallel transport along λ with respect to the connection Γ. Notice that even when the fiber of E is not compact, parallel transport is still well defined. Indeed, thanks to assumption T_∞, the connection Γ is trivial at infinity with respect to the trivialization ϕ. In particular, relative to the trivialization ϕ, parallel transport becomes the identity at infinity in the sense that $\phi^{-1} \circ \Pi_\lambda \circ \phi(\lambda(a), x) = (\lambda(b), x)$ for every $x \in E^{\infty}_{z_0}$.

It is well known that Π_λ is a symplectomorphism, where we endow the fibers of π with the symplectic structure induced by Ω_E (see e.g. [14, Chapter 8], [13, Chapter 6]). If λ is a loop starting and ending at $z \in \mathbb{C} \setminus \mathrm{Critv}(\pi)$ then the symplectomorphism $\Pi_\lambda : E_z \longrightarrow E_z$ is also called the holonomy of Γ along λ. If the loop λ is contractible (within $\mathbb{C} \setminus \mathrm{Critv}(\pi)$) then the holonomy Π_λ is in fact a Hamiltonian diffeomorphism of E_z (see [13, Section 6.4]).

Let $\lambda : [a, b] \longrightarrow \mathbb{C} \setminus \mathrm{Critv}(\pi)$ be a smooth embedding and $L \subset E_{\lambda(a)}$ a Lagrangian submanifold. Consider the images of L under the parallel transport along λ, namely $L_t := \Pi_{\lambda|_{[a,t]}}(L) \subset E_{\lambda(t)}$, $t \in [a, b]$ and set

$$\lambda L := \cup_{t \in [a,b]} L_t.$$

Then λL is a Lagrangian submanifold of (E, Ω_E). We call λL the *trail* of L along λ.

We refer the reader to [24] for the foundations of the symplectic theory of Lefschetz fibrations and to [13, Chapter 6] and [14, Chapter 8] for symplectic fibrations.

2.2 Lagrangians with cylindrical ends

Let $\pi : E \longrightarrow \mathbb{C}$ be a Lefschetz fibration and $\mathcal{U} \subset \mathbb{C}$ an open subset containing $\mathrm{Critv}(\pi)$. The following terminology is useful. A horizontal ray $\ell \subset \mathbb{C}$ is a half-line of the type $(-\infty, -a_\ell] \times \{b_\ell\}$ or $[a_\ell, \infty) \times \{b_\ell\}$ with $a_\ell > 0$, $b_\ell \in \mathbb{R}$. The imaginary coordinate b_ℓ is also referred to as the "height" of ℓ.

Definition 2.2.1 A Lagrangian submanifold (without boundary) $V \subset (E, \Omega_E)$ is said to have *cylindrical ends outside of* $\mathcal{U}$ if the following conditions are satisfied:

i. For every $R > 0$, the subset $V \cap \pi^{-1}([-R, R] \times \mathbb{R})$ is compact.
ii. $\pi(V) \cap \mathcal{U}$ is bounded.
iii. $\pi(V) \setminus \mathcal{U}$ consists of a finite union of horizontal rays, $\ell_i \subset \mathbb{C}$, $i = 1, \dots, r$. Moreover, for every i we have $V|_{\ell_i} = \ell_i L_i$ for some Lagrangian $L_i \subset E_{\sigma_i}$, where $\sigma_i \in \mathbb{C}$ stands for the starting point of the ray ℓ_i, and $\ell_i L_i$ is the trail of L_i along ℓ_i as defined above. (Note that we do allow $r = 0$, i.e. that V has no ends at all.)

In case all the heights of the rays ℓ_i are positive integers $b_{l_i} \in \mathbb{N}^*$ the Lagrangian V is called a *cobordism* in E.

In short, over each of the rays appearing in $\pi(V) \setminus \mathcal{U}$ the Lagrangian submanifold V is the trail under parallel transport of L_i along ℓ_i—see Fig. 2. The rays of the form $(-\infty, a_l] \times \{b_l\}$ give rise to the *negative ends* of V and the rays of the form $[a_l, \infty) \times \{b_l\}$ correspond to the *positive ends* of V. We will mostly work in the paper with V's that have only negative ends.

The role of the condition ii above is to exclude the possibility that $\pi^{-1}(\mathcal{U})$ entirely covers some of the ends of V. For most of the time we will work with subsets $\mathcal{U}$ that are U-shaped (see Fig. 4), and then condition ii is automatically satisfied (in view of

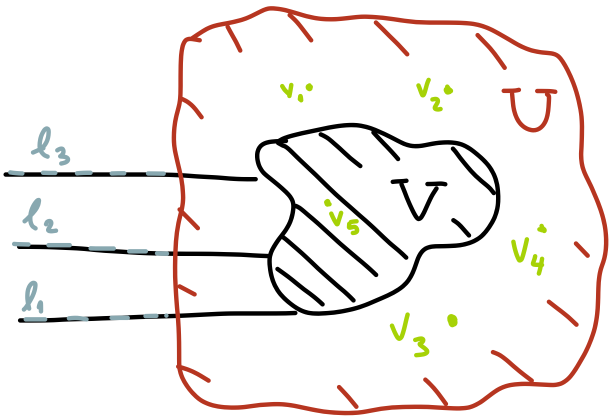

Fig. 2 A Lagrangian V with cylindrical ends outside $\mathcal{U}$ in a Lefschetz fibration $\pi : E \to \mathbb{C}$ with critical values v_i

condition i). However, occasionally we will have to consider $\mathcal{U}$'s that are not bounded in the horizontal direction (see e.g. Sect. 4.4 and Fig. 16), and then condition ii is necessary.

The above notion of cobordism extends the definition of Lagrangian cobordism as given for the trivial fibration $E \approx \mathbb{C} \times M$ in [5]. The terminology "trivial fibration" is slightly imprecise because we have not specified a (topological) trivialization of the fibration $E \longrightarrow \mathbb{C}$ at infinity (and in general there is no canonical one). Moreover, even when one fixes such a trivialization the parallel transport along a ray ℓ_i might not be trivial (even not at infinity), hence the actual ends of V at infinity are not well defined. In view of that, we will often work with a restricted type of Lefschetz fibrations, called tame, where this imprecision is not present and that have a number of additional technical advantages. We will see later on that this does not restrict the generality of our theory.

Definition 2.2.2 Let $\pi : E \longrightarrow \mathbb{C}$ be a Lefschetz fibration. Let $U \subset \mathbb{C}$ be a closed subset, let $z_0 \in \mathbb{C} \setminus U$ be a base point and (M, ω_M) be the fiber over z_0. We say that this Lefschetz fibration is tame outside of U if there exists a trivialization

$$\psi_{E,\mathbb{C}\setminus U} : (\mathbb{C}\setminus U) \times M \longrightarrow E|_{\mathbb{C}\setminus U}$$

such that $\psi^*_{E,\mathbb{C}\setminus U}(\Omega_E) = c\omega_{\mathbb{C}} \oplus \omega_M$, where $\omega_{\mathbb{C}}$ is the standard symplectic structure on $\mathbb{C} \cong \mathbb{R}^2$ and $c > 0$ is a constant. The manifold (M, ω_M) is called the generic fiber of π.

It follows from the definition that all the critical values of π must be contained inside U. Sometimes it will be more natural to fix the complement of U, say $\mathcal{W} = \mathbb{C} \setminus U$, and say that the fibration is tame over $\mathcal{W}$. Given a tame Lefschetz fibration, the set U (sometimes denoted by U_E for clarity), the point z_0 and the symplectic trivialization $\psi_{E,\mathbb{C}\setminus U}$, are all viewed as part of the fixed data associated to the fibration. Moreover, we will assume that the set U is so that there exists $a_U > 0$ sufficiently large with the property that U is disjoint from both quadrants:

$$Q_U^- = (-\infty, -a_U] \times [0, +\infty), \;\; Q_U^+ = [a_U, \infty) \times [0, +\infty) \tag{4}$$

The constant a_U will be considered as part of the data associated to a tame Lefschetz fibration. The cobordism relation, as defined in [5], admits an obvious extension in a tame Lefschetz fibration.

Definition 2.2.3 Fix a Lefschetz fibration that is tame outside $U \subset \mathbb{C}$ with fiber (M, ω) over $z_0 \in \mathbb{C}\backslash U$. Let $(L_i)_{1 \leq i \leq k_-}$ and $(L'_j)_{1 \leq j \leq k_+}$ be two families of closed Lagrangian submanifolds of M. We say that these two families are Lagrangian cobordant in E, if there exists a Lagrangian submanifold $V \subset E$ with the following properties:

i. There is a compact set $K \subset E$ so that $V \cap \pi^{-1}(U) \subset V \cap K$ and $V \backslash K \subset \pi^{-1}(Q_U^+ \cup Q_U^-)$.
ii. $V \cap \pi^{-1}(Q_U^+) = \coprod_j ([a_U, +\infty) \times \{j\}) \times L'_j$
iii. $V \cap \pi^{-1}(Q_U^-) = \coprod_i ((-\infty, -a_U] \times \{i\}) \times L_i$

The formulas at ii, iii are written with respect to the trivialization of the fibration over $\mathbb{C}\backslash U$.

The manifold V is obviously a Lagrangian cobordism in the sense of Definition 2.2.1 and—because of tameness—its ends at ∞ are well defined so that we can say that V is a cobordism from the Lagrangian family (L'_j) to the family (L_i). We write $V : (L'_j) \rightsquigarrow (L_i)$ or $(V; (L_i), (L'_j))$.

2.3 From general Lefschetz fibrations to tame ones

We will now see that it is always possible to pass from a general Lefschetz fibration $\pi : E \longrightarrow \mathbb{C}$, as in Sect. 2.1, to a tame one.

Proposition 2.3.1 *Let $\pi : E \longrightarrow \mathbb{C}$ be a Lefschetz fibration and let $\mathcal{N} \subset \mathbb{C}$ be an open subset that contains all the critical values of π and has the shape depicted in Fig. 3. Let $\mathcal{W} \subset \mathbb{C}$ be another open subset of the shape depicted in Fig. 3 with $\overline{\mathcal{W}} \cap \overline{\mathcal{N}} = \emptyset$ and dist$(\overline{\mathcal{W}}, \overline{\mathcal{N}}) > 0$. Then there exists a symplectic structure Ω' on E and a trivialization $\varphi : \mathcal{W} \times M \longrightarrow E|_{\mathcal{W}}$ with the following properties:*

(1) On $\mathcal{W} \times M$ we have $\varphi^*\Omega' = c\omega_{\mathbb{C}} \oplus \omega_M$ for some $c > 0$.
(2) Ω' coincides with Ω_E on all the fibers of π.
(3) $\Omega' = \Omega_E$ on $\pi^{-1}(\mathcal{N})$.
(4) There exists an Ω'-compatible almost complex structure J'_E on E which coincides with J_E on $\pi^{-1}(\mathcal{N})$ and such that the projection $\pi : E \longrightarrow \mathbb{C}$ is (J'_E, i)-holomorphic.

In particular, when endowed with the symplectic structure Ω', the Lefschetz fibration $\pi : E \longrightarrow \mathbb{C}$ is tame over $\mathcal{W}$.

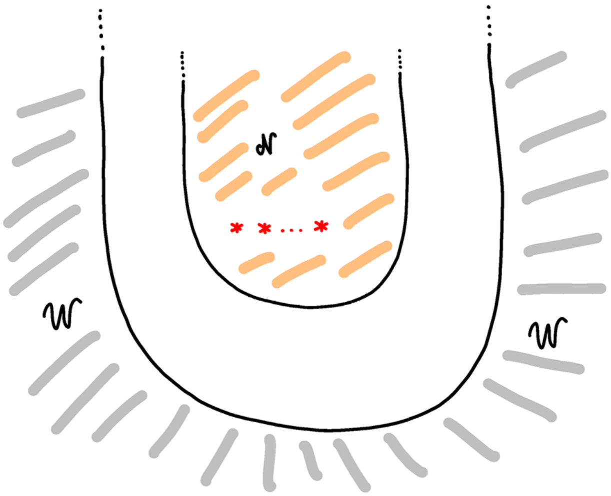

Fig. 3 A Lefschetz fibration $\pi : E \to \mathbb{C}$; the domains $\mathcal{N}$ and $\mathcal{W}$ and, in *red*, the critical values of π (color figure online)

The proof of this Proposition is elementary (full details are included in [3]).

Remark 2.3.2 It is easy to pass from a cobordism in a general Lefschetz fibration to a cobordism in a tame fibration. Indeed, let $\pi : E \longrightarrow \mathbb{C}$ be a Lefschetz fibration and $V \subset E$ a Lagrangian submanifold with cylindrical ends. Let $\mathcal{N} \subset \mathbb{C}$ be a subset as in Proposition 2.3.1 and assume that V has cylindrical ends outside of $\mathcal{N}'$, where $\mathcal{N}' \subset \mathcal{N}$ is a slightly smaller subset than $\mathcal{N}$ which contains $\text{Critv}(\pi)$ and is of the same shape as $\mathcal{N}$. Denote the horizontal rays corresponding to the ends of V by $\ell_i \subset \mathbb{C}, i = 1, \ldots, r$ and by $L_i \subset E_{\sigma_i}$ the corresponding Lagrangians over the starting points of these rays. Let $\mathcal{W} \subset \mathbb{C}$ be a subset as in Proposition 2.3.1 and consider the new symplectic structure Ω' on E provided by that proposition. By performing parallel transport of the L_i's along the horizontal rays ℓ_i, but this time with respect to the connection corresponding to (E, Ω') we obtain a new Lagrangian submanifold $V' \subset (E, \Omega')$ with the following properties:

i. V' coincides with V over $\mathcal{N}$.
ii. V' has cylindrical ends outside of $\mathcal{N}$.
iii. Over $\mathcal{W}$, V' looks like

$$V'|_{\mathcal{W}} = \cup_{i=1}^{r} \ell_i' \times L_i',$$

where $\ell_i' = \ell_i \cap \mathcal{W}$ and L_i' is the image of the parallel transport of L_i (with respect to the connection $\Gamma(\Omega')$) along the portion of ℓ_i that connects $\mathcal{N}'$ with $\mathcal{W}$.

2.4 Conventions relative to the definition of Fukaya categories

The purpose of this section is to fix some relevant notation. We assume familiarity with the construction of the Fukaya category $\mathcal{F}uk^*(M)$ (and its derived category) of uniformly monotone, closed Lagrangian submanifolds of a symplectic manifold M which is assumed to be either closed or convex at infinity. This is described in Seidel's book [24, Sections 8–12] in the exact case (the minor adjustments required in the monotone case are described, for instance, in [6]). We also assume familiarity with the variant of this construction that applies to monotone cobordisms in $\mathbb{C} \times M$, as defined in [6]. Here $*$ encodes a uniform monotonicity constraint imposed on the objects of $\mathcal{F}uk^*(M)$ (see below). This constraint is necessary to define the A_∞-operations.

Fix a symplectic manifold (M, ω), compact or convex at infinity. Given a closed Lagrangian submanifold $L \subset M$ there are two morphisms

$$\mu : \pi_2(M, L) \to \mathbb{Z}, \ \omega : \pi_2(M, L) \to \mathbb{R}$$

given, the first, by the Maslov index and, the second, by integration of ω. We say that L is monotone if $\omega(\alpha) = \rho\mu(\alpha)$ for some constant $\rho \geq 0$ and if the number

$$N_L = \min\{\mu(\alpha) : \alpha \in \pi_2(M, L), \ \omega(\alpha) > 0\}$$

is at least 2. We allow $\rho = 0$ in the definition of monotonicity. In this case, ω vanishes on $\pi_2(M, L)$ (such Lagrangians are sometime called weakly exact) and we set $N_L = \infty$.

For a connected monotone Lagrangian L and for a generic almost complex structure J compatible with ω, the number (mod 2) of J-holomorphic disks of Maslov number 2 with boundary on L that pass through a generic point of L is an invariant denoted by d_L (and is defined in detail, for instance, in [4]). Note that in case $\rho = 0$ we set $d_L = 0$ by definition.

In order to define the Fukaya category of M we first need to specify its underlying class of Lagrangian submanifolds. In what follows we will mainly consider two classes of Lagrangians $\mathcal{L}^{(0)}(M)$ and $\mathcal{L}^{(\rho,1)}$, which are defined as follows:

a. The class $\mathcal{L}^{(0)}(M)$: consists of all closed monotone Lagrangians $L \subset M$ with $d_L = 0$. This includes in particular all Lagrangians with $N_L \geq 3$ as well as the case $\rho = 0$.
b. The class $\mathcal{L}^{(\rho,1)}(M)$: consists of all the closed monotone Lagrangians $L \subset M$ with $d_L = 1$ and with monotonicity constant ρ, where $\rho > 0$ is a prescribed positive real number.

Of course one could restrict also to some subclasses of the above. For example, when M is exact it makes sense to restrict to the subclass $\mathcal{L}^{(\mathrm{ex})}(M) \subset \mathcal{L}^{(0)}(M)$ of exact Lagrangian submanifolds.

To simplify the notation we denote any of these two choices by $\mathcal{L}^*(M)$, where the symbol $*$ stands for either (0) in the first case, or for $(\rho, 1)$ in the second case. Lagrangians in the class $\mathcal{L}^*(M)$ will be called *uniformly monotone* of class $*$. In what follows we will work also with uniformly monotone *negative-ended* Lagrangian cobordisms in the total space of a Lefschetz fibration $E \longrightarrow \mathbb{C}$. Similarly to the Lagrangians in M we will denote the various classes of uniformly monotone Lagrangian cobordisms in E by $\mathcal{L}^*(E)$, where the definition of these classes is the same as above except that the Lagrangians in E are assumed to be cobordisms rather than compact.

Our conventions regarding Floer homology are the following. The two coefficients rings of interest will be $\mathbb{Z}_2$ and the universal Novikov ring $\mathcal{A}$ over $\mathbb{Z}_2$:

$$\mathcal{A} = \left\{ \sum_{k=0}^{\infty} a_k T^{\lambda_k} \ : \ a_k \in \mathbb{Z}_2, \ \lambda_k \in \mathbb{R}, \ \lim_{k \to \infty} \lambda_k \to \infty \right\}.$$

Unless otherwise stated, Floer homology will be taken in this paper with coefficients in $\mathcal{A}$. We work at all times in an ungraded setting so that Floer homology itself is ungraded. For background on Floer homology we refer to Floer's original paper [9] and in the monotone setting to Oh [16,17].

The Fukaya A_∞-category $\mathcal{F}uk^*(M)$ has as objects the Lagrangians in $\mathcal{L}^*(M)$,

$$\mathcal{O}b(\mathcal{F}uk^*(M)) = \mathcal{L}^*(M).$$

Let $L, L' \in \mathcal{L}^*(M)$. We denote by $(CF(L, L'; J), d)$ the Floer complex associated to L and L' and by $HF(L, L')$ the Floer homology. The morphisms in $\mathcal{F}uk^*(M)$ are $\mathrm{Mor}_{\mathcal{F}uk^*(M)}(L, L') = CF(L, L')$. The A_∞ structural maps are multilinear maps

$$\mu_k : CF(L_1, L_2) \otimes CF(L_2, L_3) \otimes \cdots \otimes CF(L_k, L_{k+1}) \to CF(L_1, L_{k+1})$$

that satisfy the relation $\mu \circ \mu = \sum \mu(-, -, \ldots, \mu, \ldots, -, -) = 0$. These maps are such that μ_1 is differential of the Floer complex and, for $k > 1$, μ_k is defined by:

$$\mu_k(x_1, \ldots, x_k) = \sum_y \sum_{u \in \mathcal{M}_0(x_1, \ldots, x_k; y)} T^{\omega(u)} y, \tag{5}$$

at least when the L_i's are in general position, $x_i \in L_i \cap L_{i+1}$, $y \in L_1 \cap L_{k+1}$. Here $\mathcal{M}_0(x_1, \ldots, x_k; y)$ is the 0-dimensional moduli space of J-holomorphic polygons with $k + 1$ sides that have k "inputs" asymptotic—in order—to the intersection points x_i and one "exit" asymptotic to y.

Consider next the category of A_∞-modules over the Fukaya category

$$mod(\mathcal{F}uk^*(M)) := fun(\mathcal{F}uk^*(M), Ch^{opp})$$

where Ch^{opp} is the opposite of the dg-category of chain complexes over $\mathcal{A}$. This category is triangulated in an A_∞ sense. There is a Yoneda embedding $\mathcal{Y} : \mathcal{F}uk^*(M) \to mod(\mathcal{F}uk^*(M))$, the functor associated to an object $L \in \mathcal{L}^*(M)$ being $CF(-, L)$. The derived Fukaya category $D\mathcal{F}uk^*(M)$ is the homology category associated to the triangulated completion of the image of the Yoneda embedding inside $mod(\mathcal{F}uk^*(M))$ and is a usual triangulated category. We emphasize that in the construction of $D\mathcal{F}uk^*(-)$ we do not complete with respect to idempotents.

Remark 2.4.1 Our notation is homological and it coincides with the one in [6] except that we use here the universal Novikov ring $\mathcal{A}$ in the place of $\mathbb{Z}_2$ (this is needed for a certain part of our results to hold). Using $\mathcal{A}$ has some advantages as the compactness of the relevant moduli spaces is easier to achieve and there is no need to require the vanishing of the morphisms $\pi_1(L) \to \pi_1(M)$ (which is assumed in Equation (8) from [6]). Moreover, when $d_L = 0$ we can work with Lagrangians of varying monotonicity constants ρ.

2.5 Conventions on cone decompositions and K_0

We now briefly fix the notation for writing iterated cone-decompositions in a triangulated category $\mathcal{C}$. Suppose that there are exact triangles:

$$C_{i+1} \to Z_i \to Z_{i+1}$$

with $1 \leq i \leq n$ and with $X = Z_{n+1}$, $Z_0 = C_0$. We write such an iterated cone-decomposition as

$$X = (C_{n+1} \to (C_n \to (C_{n-1} \to \cdots \to C_0))\cdots).$$

In fact we can omit the parentheses in this notation without ambiguity since there is an isomorphism between the following two iterated cones:

$$((A \to B) \to C) \cong (A \to (B \to C)).$$

This follows immediately from the axioms of a triangulated category together with the fact that we work here in an ungraded setting. In short, we write:

$$X = (C_{n+1} \to C_n \to C_{n-1} \to \cdots \to C_0).$$

There is a slight abuse of notation in the last formula in that, in the absence of the relevant parentheses, the arrows in the formula do not independently correspond to morphisms in the category $\mathcal{C}$. The formula should be interpreted as saying that X can be expressed as an iterated cone attachment with the objects $C_0, \ldots, C_{n+1}$ as described above.

The Grothendieck group of a triangulated category $\mathcal{C}$ is the abelian group generated by the objects of $\mathcal{C}$ modulo the relations generated by $B = A+C$ for each exact triangle

$$A \to B \to C.$$

We denote the Grothendieck group of $\mathcal{C}$ by $K_0(\mathcal{C})$. Notice that, with our terminology, if

$$L_1 = (L_n \to L_{n-1} \to L_{n-2} \to \cdots \to L_2),$$

then, because we work in an ungraded setting, in $K_0(\mathcal{C})$ we have the relation $L_n + L_{n-1} + \cdots + L_1 = 0$. Moreover, our version of $K_0(\mathcal{C})$ is always 2-torsion.

3 The Fukaya category of negative-ended cobordisms in Lefschetz fibrations

We discuss here the construction of Fukaya categories of cobordisms in Lefschetz fibrations.

3.1 Cobordisms in tame fibrations

We start with the case of tame fibrations. We consider a Lefschetz fibration $\pi : E \to \mathbb{C}$ that is tame outside $U \subset \mathbb{C}$ (see Definition 2.2.2) and has as generic fiber the symplectic manifold (M, ω). We will also assume that U is U-shaped, as in Fig. 4 and

$$U \subset Int([-a_U, +a_U] \times [0, \infty)). \tag{6}$$

The main object of study in this paper is the Fukaya category $\mathcal{F}uk^*(E)$, where $*$ is the monotonicity class of E as defined in Definition 3.3.1. It has as objects the cobordisms V as in Definition 2.2.3 such that the following additional conditions are satisfied:

i. V is monotone in the class $*$.
ii. $V \subset \pi^{-1}(\mathbb{R} \times [\frac{1}{2}, +\infty))$
iii. V has only negative ends that all belong to $\mathcal{L}^*(M)$. In particular, with the notation from Definition 2.2.3, $k_+ = 1$ and $L'_1 = \emptyset$.

This collection of Lagrangians of E with the properties above will be denoted by $\mathcal{L}^*(E)$. In other words, $\mathcal{O}b(\mathcal{F}uk^*(E)) = \mathcal{L}^*(E)$. Such an object is represented schematically in Fig. 4. In view of Definition 2.2.3 and of our conventions, all such V's are cylindrical outside the strip $[-a_U, a_U] \times \mathbb{R}$.

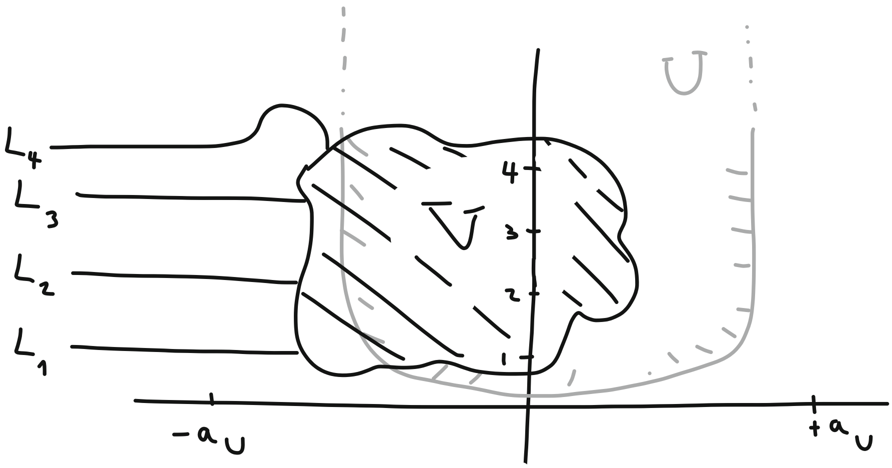

Fig. 4 The projection on $\mathbb{C}$ of an object $V \in \mathcal{O}b(\mathcal{F}uk^*(E))$ together with the set U outside which E is tame

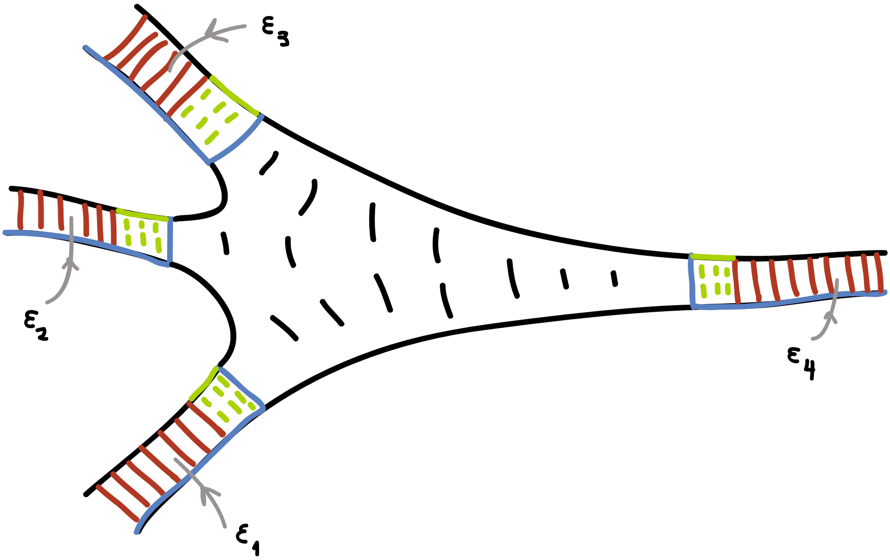

Fig. 5 The constraints imposed on a transition function for a domain with three entries and one exit: in the *red region* the function α_r equals $(s, t) \to t$; along the *blue arcs* the function α_r vanishes; the *green region* is a transition region. There are no additional constraints in the *black region* (color figure online)

We call the objects $V \in \mathcal{L}^*(E)$ *negative-ended cobordisms*: they are cobordisms from the void set to a family $(L_1, \ldots, L_s)$.

The operations μ_k of the Fukaya category $\mathcal{F}uk^*(E)$ are defined following closely the construction in [6]. We briefly recall below some of the technical details that will be needed later in the paper.

3.1.1 The case of a compact fiber

We consider first the case when M is compact. The modifications required when M is convex at infinity will be discussed after that. For convenience, we assume that $a_U < \frac{1}{2}$. A construction similar to the one below can be performed for any other value of a_U.

As always, the operations μ_k of our Fukaya category are defined in terms of counting (with coefficients in $\mathcal{A}$) perturbed J-holomorphic polygons. To define the appropriate moduli spaces, we need two additional structures in comparison to the construction of the category $\mathcal{F}uk^*(-)$ in [24].

The first is a choice of *transition functions* associated to a system of strip-like ends whose role is to allow perturbed J-holomorphic polygons to be transformed by a change of variables into curves that project holomorphically onto certain regions of $\mathbb{C}$. Transition functions are a family of functions $\alpha_r : S_r \to [0, 1]$ defined for each $(k+1)$-pointed disk S_r compatible with a fixed choice of strip like ends (the parameter r belongs to the appropriate moduli space of $(k+1)$-pointed disks $\mathcal{R}^{k+1}$). The definition and the properties of this family are exactly as in §3.1 [6] (to avoid confusion with other notation, we denote here transition functions by α_r while in [6]

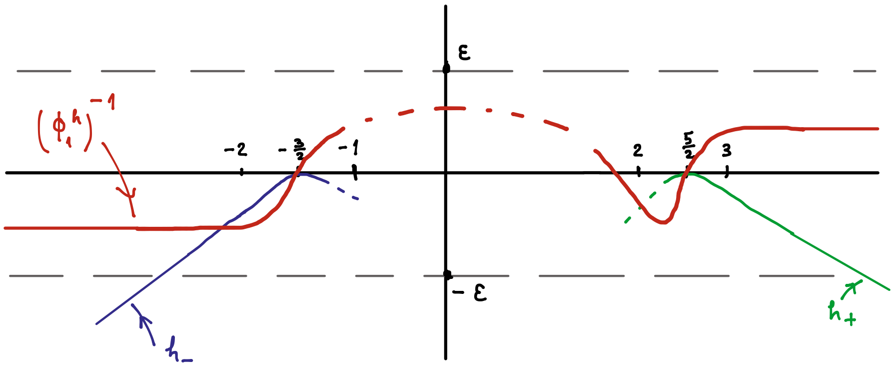

Fig. 6 The graphs of h_- and h_+ and the image of $\mathbb{R}$ by the Hamiltonian diffeomorphism $(\phi_1^h)^{-1}$. The profile of the functions h_- at $-3/2$ and h_+ at $5/2$ are the "bottlenecks"

they are denoted by a_r). We recall below the main properties of these functions for a fixed r.

i. For each parametrization of an entry strip-like end $\epsilon_i : Z^- = (-\infty, 0] \times [0, 1] \to S_r$, $1 \leq i \leq k$, we have:
   a. $\alpha_r \circ \epsilon_i(s, t) = t, \forall (s, t) \in (-\infty, -1] \times [0, 1]$.
   b. $\frac{\partial}{\partial s}(\alpha_r \circ \epsilon_i)(s, 1) \leq 0$ for $s \in [-1, 0]$.
   c. $\alpha_r \circ \epsilon_i(s, t) = 0$ for $(s, t) \in ((-\infty, 0] \times \{0\}) \cup (\{0\} \times [0, 1])$.

ii. For the exit strip-like end $\epsilon_{k+1} : Z^+ = [0, \infty) \times [0, 1] \to S_r$ we have:
   a'. $\alpha_r \circ \epsilon_{k+1}(s, t) = t, \forall (s, t) \in [1, \infty) \times [0, 1]$.
   b'. $\frac{\partial}{\partial s}(\alpha_r \circ \epsilon_{k+1})(s, 1) \geq 0$ for $s \in [0, 1]$.
   c'. $\alpha_r \circ \epsilon_{k+1}(s, t) = 0$ for $(s, t) \in ([0, +\infty) \times \{0\}) \cup (\{0\} \times [0, 1])$.

The second structure is a choice of *profile functions*. The role of the profile functions is to prescribe how cobordisms intersect in the region where they are cylindrical. This is needed both for ensuring compactness of the relevant moduli spaces and also to be able to draw certain algebraic consequences from the way the curves v above project onto $\mathbb{C}$. The definition of a profile function is as in §3.2 [6]: this is a function $h : \mathbb{R}^2 \to \mathbb{R}$ with support in the union of the sets

$$W_i^+ = [2, \infty) \times [i - \epsilon, i + \epsilon] \quad \text{and} \quad W_i^- = (-\infty, -1] \times [i - \epsilon, i + \epsilon], \; i \in \mathbb{Z},$$

where $0 < \epsilon < 1/4$ and whose restriction to each of the sets $F_i^+ = [2, \infty) \times [i - \epsilon/2, i + \epsilon/2]$ and $F_i^- = (-\infty, -1] \times [i - \epsilon/2, i + \epsilon/2]$ is respectively of the form $h(x, y) = h_\pm(x)$. The properties of the smooth functions $h_\pm$ are given in §3.2 [6] and they are pictured in Fig. 6. The critical points $(-3/2, i)$ of h_- and $(5/2, i)$ of h_+ are called *bottlenecks*.

To define the μ_k's we pick for each pair of cobordisms $V, V' \subset E$ the Floer datum $\mathscr{D}_{V,V'} = (\bar{H}_{V,V'}, J_{V,V'})$. This consists of a Hamiltonian $\bar{H}_{V,V'} : [0, 1] \times E \to \mathbb{R}$ and an almost complex structure $J_{V,V'}$ on E which is compatible with Ω_E. The conditions on this Floer datum are exactly as in [6], page 1762. In particular, we require that there exists a compact set $K_{V,V'} \subset (-\frac{5}{4}, \frac{9}{4}) \times \mathbb{R} \subset \mathbb{C}$ such that

$\bar{H}_{V,V'}(t,(x,y,p)) = h(x,y) + H_{V,V'}(t,p)$ for $(x+iy,p)$ outside of $\pi^{-1}(K_{V,V'})$, for some $H_{V,V'} : [0,1]\times M \to \mathbb{R}$ and the projection $\pi : E \to \mathbb{C}$ is $(J_{V,V'}(t), (\phi_t^h)_* i)$-holomorphic outside of $\pi^{-1}(K_{V,V'})$ for every $t \in [0,1]$.

For a $(k+1)$-pointed disk S_r, let $C_i \subset \partial S_r$ be the connected components of ∂S_r indexed so that C_1 goes from the exit to the first entry, C_i goes from the $(i-1)$th entry to the ith one, $2 \le i \le k$, and C_{k+1} goes from the kth entry to the exit.

Following Seidel's scheme from [24, Section 9], we now need to choose additional perturbation data. For every collection of cobordisms V_i, $1 \le i \le k+1$ we choose a perturbation datum $\mathscr{D}_{V_1,\ldots,V_{k+1}} = (\Theta, \mathbf{J})$ consisting of a family of forms $\Theta = \{\Theta^r\}_{r\in\mathcal{R}^{k+1}}$, where $\Theta^r \in \Omega^1(S_r, C^\infty(E))$ is a 1-form on S_r with values in smooth functions on E and $\mathbf{J} = \{J_z\}_{z\in\mathcal{S}^{k+1}}$ is a family of Ω_E-compatible almost complex structure on E that are parametrized by $z \in S_r$. The forms Θ^r induce forms $Y^r = Y^{\Theta^r} \in \Omega^1(S_r, C^\infty(TE))$ with values in (Hamiltonian) vector fields on E via the relation $Y(\xi) = X^{\Theta(\xi)}$ for each $\xi \in TS_r$. The relevant Cauchy–Riemann equation associated to $\mathscr{D}_{V_1,\ldots,V_{k+1}}$ is:

$$u : S_r \to E, \quad Du + J(z,u)\circ Du \circ j = Y + J(z,u)\circ Y \circ j, \quad u(C_i) \subset V_i. \quad (7)$$

Here j stands for the complex structure on S_r. The map u satisfies $u(C_i) \subset V_i$ and u is required to be asymptotic—in the usual Floer sense—to appropriate Hamiltonian chords γ_i on each respective strip-like end. The perturbation data $\mathscr{D}_{V_1,\ldots,V_{k+1}}$ are constrained by a number of additional conditions that are exactly as in [6], pp. 1763–1764. In summary, there are the usual asymptotic conditions ensuring the compatibility of the form Θ and the Floer data, there is a special expression of Θ on S_r, $\Theta|_{S_r} = d\alpha_r \otimes \bar{h} + \Theta_0$ for some $\Theta_0 \in \Omega^1(S_r, C^\infty(E))$ (subject to constraints as in [6], page 1764) where $\alpha_r : S_r \to \mathbb{R}$ are the transition functions and $\bar{h} = h \circ \pi$. Finally, for a certain compact set $K_{V_1,\ldots,V_{k+1}} \subset (-\frac{3}{2}, \frac{5}{2}) \times \mathbb{R}$ we have that the projection π is $(J_z, (\phi^h_{a_r(z)})_*(i))$-holomorphic on $\pi^{-1}(K_{V_1,\ldots,V_{k+1}})$ for every $r, z \in S_r$. (All the relevant conditions are identical to the corresponding ones in [6] except that the writing $K \times M \subset \mathbb{C}\times M$ there, which makes sense in a trivial fibration, needs to be replaced here by $\pi^{-1}(K) \subset E$). Using these choices of data the morphisms $CF(V, V'; \mathscr{D}_{V,V'})$, between two objects V and V' are the elements of the $\mathcal{A}$-vector space generated by the Hamiltonian chords $\gamma : [0,1] \to E$ of $\bar{H}_{V,V'}$ with $\gamma(0) \in V$ and $\gamma(1) \in V'$. The A_∞ structural maps μ_k are defined by summing—as in (5), with coefficients in $\mathcal{A}$—pairs (r,u) with $r \in \mathcal{R}^{k+1}$ and u a finite energy solution of (7) that belongs to a 0-dimensional moduli space. The Gromov compactness and regularity arguments work just as in [6]. In particular, the proof of compactness uses the naturality transformation that will be recalled below in Sect. 3.1.2. The choice of strip-like ends, transition functions and profile function (in particular, the placement of the bottlenecks) changes the resulting A_∞-category only up to quasi-equivalence.

Once the category $\mathcal{F}uk^*(E)$ is constructed, the derived category $D\mathcal{F}uk^*(E)$ is defined as usual, by considering the A_∞-modules $mod(\mathcal{F}uk^*(M)) := fun(\mathcal{F}uk^*(E), Ch^{opp})$ and by letting $D\mathcal{F}uk^*(E)$ be the homological category associated to the triangulated closure of the image of the Yoneda functor $\mathcal{Y} : \mathcal{F}uk^*(E) \to mod(\mathcal{F}uk^*(E))$.

3.1.2 The naturality transformation

Assume that $u : S_r \to E$, $r \in \mathcal{R}^{k+1}$, is a solution of (7) as in Sect. 3.1. Define $v : S_r \to E$ by the formula:

$$u(z) = \phi^{\bar{h}}_{\alpha_r(z)}(v(z)), \tag{8}$$

where $\alpha_r : S_r \to [0, 1]$ is the transition function.

The Floer equation (7) for u transforms into the following equation for v:

$$Dv + J'(z, v) \circ Dv \circ j = Y' + J'(z, v) \circ Y' \circ j. \tag{9}$$

Here $Y' \in \Omega^1(S_r, C^\infty(TM))$ and J' are defined by:

$$Y = D\phi^{\bar{h}}_{\alpha_r(z)}(Y') + d\alpha_r \otimes X^{\bar{h}}, \quad J_z = (\phi^{\bar{h}}_{\alpha_r(z)})_* J'_z. \tag{10}$$

The map v satisfies the following moving boundary conditions:

$$\forall\, z \in C_i, \quad v(z) \in (\phi^{\bar{h}}_{\alpha_r(z)})^{-1}(V_i). \tag{11}$$

The asymptotic conditions for v at the punctures of S_r are as follows. For $i = 1, \ldots, k$, $v(\epsilon_i(s, t))$ tends as $s \to -\infty$ to a time-1 chord of the flow $(\phi^{\bar{h}}_t)^{-1} \circ \phi_t^{\bar{H}_{V_i, V_{i+1}}}$ starting on V_i and ending on $(\phi^{\bar{h}}_1)^{-1}(V_{i+1})$. (Here $\epsilon_i(s, t)$ is the parametrization of the strip-like end at the i'th puncture.) Similarly, $v(\epsilon_{k+1}(s, t))$ tends as $s \to \infty$ to a chord of $(\phi^{\bar{h}}_t)^{-1} \circ \phi_t^{\bar{H}_{V_1, V_{k+1}}}$ starting on V_1 and ending on $(\phi^{\bar{h}}_1)^{-1}(V_{k+1})$.

Let $v' = \pi \circ v : S_r \to \mathbb{C}$. It is easy to see—as in [6]—that v' is holomorphic over $\mathbb{C} \setminus ([-\frac{3}{2} + \delta', \frac{5}{2} - \delta'] \times \mathbb{R})$ for small enough $\delta' > 0$. As discussed in [6], the holomorphicity of v' is used to prove the compactness of the moduli spaces defining the μ_k's as well as to deduce certain algebraic properties of these operations out of the geometric properties of v'.

3.1.3 The case of a non-compact fiber

We now assume that (M, ω) is non-compact and convex at infinity and that the Lefschetz fibration E satisfies the conditions in Sect. 2.1 as well as the Assumption T_∞ from page 6. Additionally, we continue to assume that E is tame outside a U-shaped subset $U \subset \mathbb{C}$ as in Sect. 3.1.

From Assumption T_∞ we deduce that there is a trivialization $\phi : \mathbb{C} \times M^\infty \to E^\infty$ with respect to which both the symplectic form and the almost complex structure split so that, in particular, $\phi^* J_E = j \oplus J_0$ where J_0 is a fixed almost complex structure on M compatible with ω and with the symplectic convexity of M. Recall also that $E^0 = E \setminus E^\infty$.

The objects of the category $\mathcal{F}uk^*(E)$ are the same as before. By Definition 2.2.3, any cobordism V has the property that $V \cap \pi^{-1}(z)$ is compact for any $z \in \mathbb{C}$. The construction of the category $\mathcal{F}uk^*(E)$ proceeds exactly in the same fashion as in the compact case with an additional requirement: all the almost complex structures involved are required to coincide with J_E outside a large enough neighborhood of E^0. More precisely, for any two objects $V, V' \in \mathcal{O}b(\mathcal{F}uk^*(E))$ we require that $J_{V,V'}$ coincide with J_E outside a neighborhood of E^0 that contains both V and V'. Similarly, each almost complex structure J_z in the family $\mathbf{J}$ that is part of the perturbation data associated to the collection of cobordisms $V_1, \ldots, V_{k+1}$ has to coincide with J_E outside of a neighborhood of E^0 that contains all of the V_i's. Finally, as explained in Sect. 3.1.2 the actual curves u that appear in the μ_k's are transformed into curves v which satisfy equations that are holomorphic with respect to almost complex structures of the form $J'_z = (\phi^{\overline{h}}_{\alpha_r(z)})^{-1}_* J_z$. Due to the splitting provided by the trivialization ϕ and because $\overline{h} = h \circ \pi$ these structures are also split at ∞ (along the fiber) and, by using the trivialization ϕ, it follows that J'_z restricted to the fiber direction coincides with J_0 (away from a compact subset). Therefore, over E^∞ one can again use ϕ to project such a curve v on M^∞ thus getting a new curve v' that away from a compact is J_0-holomorphic. The usual compactness arguments for manifolds that are symplectically convex at infinity apply to this v' and thus compactness can be achieved.

3.2 Fukaya categories of negative-ended cobordisms in general Lefschetz fibrations

In this section we use the construction in Sect. 3.1 to associate a Fukaya A_∞-category to a general Lefschetz fibration. Let $\pi : E \to \mathbb{C}$ be a Lefschetz fibration as in Sect. 2.1. The category we intend to construct will depend on a tame Lefschetz fibration $\pi : E_\tau \to \mathbb{C}$ associated to E and will be denoted by $\mathcal{F}uk^*(E; \tau)$. The parameter τ indicates the particular choice of a tame symplectic structure on E.

We first fix an additional notation. For two constants $r < 0 < s$, put $S_{r,s} = [r, s] \times \mathbb{R} \subset \mathbb{C}$. Fix constants $x < 0 < y$ such that all the singularities of the fibration E are contained in the interior of $\pi^{-1}(S_{x,y})$. We also assume that the critical values of π are included in the upper half plane.

The objects of the category $\mathcal{F}uk^*(E; \tau)$ are cobordisms V in E—in the sense of Definition 2.2.1—that are cylindrical outside $S_{x-3,y+3}$ and satisfy the following additional constraints:

i. V is monotone of class $*$.
ii. $V \subset \pi^{-1}(\mathbb{R} \times [\frac{1}{2}, +\infty))$
iii. V has only negative ends belonging to $\mathcal{L}^*(M)$.

Condition *iii* means that for some point z along one of the rays ℓ_i associated to the ends of V the Lagrangian $V \cap \pi^{-1}(z)$ belongs to $\mathcal{L}^*(M)$ (this does not depend on the choice of the point z). To define the morphisms and the operations μ_k we proceed as follows. We fix a Lefschetz fibration $\pi : E_\tau \to \mathbb{C}$ that is tame outside a set U whose interior contains $[x-4, y+4] \times (-1, \infty)$ and coincides with E over $[x-4, y+4] \times [-\frac{1}{2}, \infty)$. Such a fibration exists due to the results from Sect. 2.3. Each object $V \in \mathcal{O}b(\mathcal{F}uk^*(E; \tau))$ corresponds to an object $\overline{V} \in \mathcal{O}b(\mathcal{F}uk^*(E_\tau))$ that is obtained, as in Remark 2.3.2, by cutting off the ends of V along the line $\{x - \frac{7}{2}\} \times \mathbb{R}$ and extending them horizontally by parallel transport in the fibration E_τ. It is easy to see that the subcategory of $\mathcal{F}uk^*(E_\tau)$ that consists of all the objects $\overline{V}$ obtained in this way is quasi-equivalent to $\mathcal{F}uk^*(E_\tau)$ itself because each object of this larger category is quasi-isomorphic to one of the $\overline{V}$'s. Notice however that the category $\mathcal{F}uk^*(E_\tau)$ contains more objects than those of the form $\overline{V}$, an example is provided in Fig. 20. We now put $\mathrm{Mor}_{\mathcal{F}uk^*(E;\tau)}(V, V') = \mathrm{Mor}_{\mathcal{F}uk^*(E_\tau)}(\overline{V}, \overline{V}')$ and similarly we define all operations in $\mathcal{F}uk^*(E;\tau)$ associated to $V_1, \ldots, V_{k+1}$ by means of the corresponding operations associated to $\overline{V}_1, \ldots, \overline{V}_{k+1}$ in $\mathcal{F}uk^*(E_\tau)$.

It is clear, by construction, that there is an inclusion:

$$\mathcal{F}uk^*(E;\tau) \to \mathcal{F}uk^*(E_\tau)$$

which is a quasi-equivalence.

Remark 3.2.1 There is a derived Fukaya category of cylindrical Lagrangians in $\mathbb{C} \times M$ with ends of arbitrary heights in $[1, \infty)$ and not only with integral heights, as described above. The construction is the following. First, given any infinite sequence of strictly increasing reals in $[1, \infty)$ that tends to ∞, $S = \{a_1, \ldots, a_n, \ldots\}$, there is a Fukaya category $\mathcal{F}_S$ of cobordisms with ends in S that is defined just as in the case of $S = \mathbb{N}^*$. The sets S are ordered by inclusion in an obvious way and this order implies the existence of inclusion functors among the $\mathcal{F}_S$ as well as induced functors among the respective derived categories $D\mathcal{F}_S$. At the derived level the inclusion functors are easily seen to be compatible and the derived category in question is given as $\mathrm{colim}_S(D\mathcal{F}_S)$. The decompositions results in the paper are expected to extend directly to this setting, however we do not pursue here this construction.

3.3 Strongly monotone Lefschetz fibrations

In order to prove our decomposition results that involve thimbles and vanishing spheres we need to ensure that all these are monotone Lagrangian submanifolds (with the right monotonicity parameters) so that they are objects in the same Fukaya category. We discuss here the required constraints on the Lefschetz fibrations and related notation.

Let $\pi : E \longrightarrow \mathbb{C}$ be a Lefschetz fibration as in Definition 2.1.1. Fix a base point $z_0 \in \mathbb{C}$ and let $M = \pi^{-1}(z_0)$ be the fiber over z_0, endowed with the symplectic structure $\omega = \Omega_E|_M$ induced from E. Denote by $x_1, \ldots, x_m \in E$ the critical points of π and by $v_1, \ldots, v_m \in \mathbb{C}$ the corresponding critical values of π. Fix m smooth paths $\lambda_1, \ldots, \lambda_m \subset \mathbb{C}$ such that for every k, λ_k starts at v_k and ends at z_0 and such that except of their end points none of the paths λ_k passes through the critical values of π. Denote by $S_1, \ldots, S_m \subset M$ the Lagrangian vanishing spheres associated to the paths $\lambda_1, \ldots, \lambda_m$.

Definition 3.3.1 (*Strongly monotone Lefschetz fibrations*) We say that $\pi : E \longrightarrow \mathbb{C}$ is a *strongly monotone* Lefschetz fibration if the following conditions hold:

(1) In case $\dim_{\mathbb{R}} M \geq 4$ we require that M is a monotone symplectic manifold, that is $\omega = 2\rho c_1$ on $\pi_2(M)$ for some $\rho \geq 0$.
(2) In case $\dim_{\mathbb{R}} M = 2$ we require that (E, Ω_E) is a monotone symplectic manifold. Note that this implies that M is monotone too and we define ρ as in point (1) above.

In addition to the above we make the following assumptions. Denote by $c_1^{\min} \in \mathbb{Z}_{>0} \cup \{\infty\}$ the minimal Chern number of M. Then:

(i) If $\rho > 0$ and $c_1^{\min} = 1$ then we require that $d_{S_1} = \cdots = d_{S_m}$ (see Sect. 2.4 for the definition of d_{S_k}). Denote the latter number by $d_E \in \mathbb{Z}_2$. In case $d_E = 0$ we put, by convention, $* = (0)$ and if $d_E = 1$ we set $* = (\rho, 1)$.
(ii) If $\rho = 0$ we put $* = (0)$ and we also set $d_E = 0$.
(iii) If $c_1^{\min} > 1$ we take $* = (0)$ and, again, $d_E = 0$.

We will refer to $*$ from Definition 3.3.1 as the monotonicity class of the Lefschetz fibration E. This depends only on the fibration E. The purpose of the assumptions and notation at (i),(ii) and (iii) is to ensure that all vanishing spheres belong to the same uniformly monotone Fukaya category of montonicity class $*$. For each such sphere S the invariant d_S equals d_E. In short, the vanishing spheres need to be uniformly monotone and their monotonicity class determines the monotonicity class $*$ of the Fukaya category $\mathcal{F}uk^*(M)$ that is under study. There is one exception to the conventions above, namely when E has no critical values at all, i.e. $E \approx \mathbb{C} \times M$ is the trivial fibration. In this case we only assume that M is monotone and allow to choose the monotonicity class $*$ to be arbitrary, subject to the restrictions made in Sect. 2.4.

Remark 3.3.2 It is easy to see that when $\dim_{\mathbb{R}} M \geq 4$, (M, ω) is monotone iff (E, Ω_E) is monotone and in that case $c_1^{\min}(E) = c_1^{\min}(M)$. This is so because under this dimension assumption, the morphism, $\pi_2(M) \to \pi_2(E)$, induced by inclusion, is surjective. We also have $c_1(E)|_{H^2(M)} = c_1(M)$. Moreover, the monotonicity of the symplectic manifold (E, Ω_E) implies that the spheres $S_1, \ldots, S_k \subset M$ are all monotone (even when $\dim_{\mathbb{R}} M = 2$).

Standard arguments show that Definition 3.3.1 is independent of the choice of paths $\lambda_1, \ldots, \lambda_m$ and, moreover, that the procedure that modifies the symplectic structure of a Lefschetz fibration to render it tame, does not affect the property of being strongly monotone (for details see [3]).

Given a Lefschetz fibration $\pi : E \to \mathbb{C}$ that is strongly monotone the notation $\mathcal{F}uk^*(E, \tau)$ will be used for the Fukaya category constructed as in 3.2 with the value of $*$ as above. Thus, all the vanishing spheres and thimbles associated to the singularities of π are $*$-monotone and are objects, respectively, in $\mathcal{F}uk^*(M)$ and $\mathcal{F}uk^*(E)$ (where M is the fiber of E).

4 Decomposing cobordisms

Fix a strongly monotone Lefschetz fibration $\pi : E \to \mathbb{C}$ and a Fukaya category $\mathcal{F}uk^*(E; \tau)$ as defined in Sect. 3.2. This section contains the main result of the paper. It claims that each object V of $D\mathcal{F}uk^*(E; \tau)$ admits an iterated cone decomposition in terms of simpler objects. We will also see later in the paper that $D\mathcal{F}uk^*(E; \tau)$ is independent of τ.

4.1 Statement of the main result

We will restate here Theorem A after providing the precise definitions of the objects involved.

To fix ideas, we assume that π has m critical points $x_k \in E, k = 1, \ldots, m$ of corresponding critical values $v_k = (k, \frac{3}{2}) \in \mathbb{C}$. Consider a Fukaya category $\mathcal{F}uk^*(E; \tau)$ of uniformly monotone negative-ended cobordisms $V \subset E$ that are cylindrical outside $\pi^{-1}(S_{x-3,y+3})$ with the two constants $x < 0 < y$ fixed, and so that all the singularities of π are contained in $\pi^{-1}(S_{x,y})$. See Sect. 3.2 for the definition. In particular, τ indicates that the morphisms and operations in $\mathcal{F}uk^*(E; \tau)$ are defined by means of the Fukaya A_∞-category $\mathcal{F}uk^*(E_\tau)$ associated to a tame Lefschetz fibration $\pi : E_\tau \to \mathbb{C}$ that agrees with E over $[x - 4, y + 4] \times [-\frac{1}{2}, \infty)$. The objects of $\mathcal{F}uk^*(E; \tau)$ are collected in the set $\mathcal{L}^*(E)$.

4.1.1 *The "atoms" of the decomposition*

Our first task is to describe the simpler objects that form the basic pieces of our decomposition. We make use of two types of smooth curves in the plane.

(I) These curves are denoted by γ_i, $i \geq 2$ and are so that $\gamma_i : \mathbb{R} \to \mathbb{C}$ is a smooth embedding with

$$\gamma_i(\mathbb{R}) \subset (-\infty, x) \times [\frac{1}{2}, +\infty), \;\; \gamma_i(-1, 1) \subset [x-2, x-1] \times [1, i]$$

and:

$$\gamma_i((-\infty, -1]) = (-\infty, x-2] \times \{1\}, \;\; \gamma_i([+1, +\infty)) = (-\infty, x-2] \times \{i\}.$$

(II) The second type of curve is denoted by t_k. For $1 \leq k \leq m$ the curve t_k is given by a smooth embedding $t_k : (-\infty, 0] \to \mathbb{C}$ so that we have

$$\begin{aligned} t_k(0) = v_k, \;\; & t_k((-\infty, -2]) \\ &= (-\infty, x-2] \times \{1\}, \;\; t_k((-\infty, 0)) \subset (-\infty, m+1) \times [1, 3] \end{aligned}$$

and t_k turns once around all the points $v_{k+1}, v_{k+2}, \ldots, v_m$.

Both types of curves are pictured in Fig. 7.

Let $x - 3 < a < x - 2$ and fix the points $z_k = (a, k) \in \mathbb{R}^2 \approx \mathbb{C}$, $k \in \mathbb{N}$. Set also $z_* = (a, 1) \in \mathbb{R}^2$ (of course, $z_1 = z_*$, we use this double notation because we want to view z_* as a base-point). Let (M_{z_k}, ω_{z_k}) be the fiber of π over the point z_k. There are two families of Lagrangian cobordisms in $\mathcal{L}^*(E)$ that are associated to the geometric data given above.

(I') For each Lagrangian in $L \in \mathcal{L}^*(M_{z_k})$ we consider the trail $\gamma_k L$ of L along the curve γ_k. This is a well-defined Lagrangian in E and, further, $\gamma_k L \in \mathcal{L}^*(E)$.

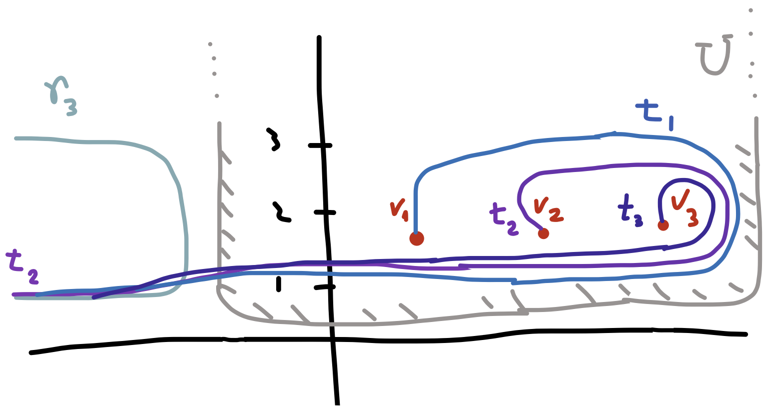

Fig. 7 The curves γ_3 and t_1, t_2, t_3 for a fibration E with three critical points

(II') Denote by T_i the thimble associated to the singularity x_i and the curve t_i. Denote by $S_i \subset M_{z_*}$ the vanishing sphere associated to the singularity x_i such that T_i is the trail of S_i along t_i. Since E is strongly monotone it follows from Sect. 3.3 that $T_i \in \mathcal{L}^*(E)$.

4.1.2 The decomposition

We now reformulate Theorem A in the setting and notation above. Recall that $\mathcal{L}^*(E)$ stands for the collection of *negative-ended* Lagrangian cobordisms in E of monotonicity class $*$ that satisfy some additional properties as defined in Sect. 3.1. Recall also that we use the Novikov ring $\mathcal{A}$ as coefficients at all times.

Theorem 4.1.1 (Theorem A reformulated) *Let $V \in \mathcal{L}^*(E)$, and assume V has s cylindrical negative ends $L_i = V|_{z_i}$, $1 \leq i \leq s$. There exist finite rank $\mathcal{A}$-modules E_k, $1 \leq k \leq m$, and an iterated cone decomposition taking place in $D\mathcal{F}uk^*(E; \tau)$:*

$$V \cong (T_1 \otimes E_1 \to T_2 \otimes E_2 \to \cdots \to T_m \otimes E_m \to \gamma_s L_s \to \gamma_{s-1} L_{s-1} \to \cdots \to \gamma_2 L_2).$$

Moreover, the category $D\mathcal{F}uk^*(E; \tau)$ is independent of τ (up to equivalence).

The Proof of Theorem 4.1.1 follows from an analogue result—Theorem 4.2.1, stated in the first subsection below—which applies to tame Lefschetz fibrations. The three subsequent Sects. 4.3–4.5 form the technical heart of the paper. They provide the arguments that are put together in Sect. 4.6 to show Theorem 4.2.1. The decomposition in the statement of Theorem 4.1.1 follows directly from that provided by Theorem 4.2.1. The modules E_i are explicitly identified along the proof—see Eq. (28). The independence of $D\mathcal{F}uk^*(E; \tau)$ from the choice of τ is postponed to Sect. 5 as it is an immediate consequence of Corollary 5.2.1 which is itself deduced from Theorem 4.2.1.

4.2 Decomposition of cobordisms in strongly monotone tame fibrations

Assume now that the Lefschetz fibration $\pi : E \to \mathbb{C}$ is tame outside the set U—as in Definition 2.2.2—and is so that:

i. the set U contains $[0, m+1] \times [\frac{1}{2}, K]$ and $U \subset \mathbb{R} \times [0, +\infty)$.
ii. as before, π has m critical points $x_k \in E$ of corresponding critical values $v_k = (k, \frac{3}{2})$.
iii. we fix $a_U > 0$ sufficiently large so that the set $\{z \pm d \mid z \in U, \ 0 \leq d \leq 4\}$ is disjoint from both quadrants

$$Q_U^- = (-\infty, -a_U] \times [0, +\infty), \quad Q_U^+ = [a_U, \infty) \times [0, +\infty).$$

In this setting we again first define the "simple" pieces that appear in the relevant decomposition. They again involve two types of curves, again denoted by γ_i and t_j, and are defined as at the points (I) and (II) in Sect. 4.1.1 but by using instead of the constant x the value $-a_U + 3$. As a consequence, the position of these curves relative to the set U is as in Fig. 7. We then define the two families of associated Lagrangians as at the points (I') and (II'). Notice that the Lagrangian $\gamma_k L$ is a product $\gamma_k L = \gamma_k \times L$. This is because the fibration is trivial over the complement of U and γ_k is entirely contained in this complement. At the same time, because of condition iii above, $\gamma_k L$ as well as T_j are cobordisms in the sense of Definition 2.2.3 (relative to the constant a_U). Finally, assume that $L \in \mathcal{L}^*(M)$. Thus the $\gamma_k L$'s are objects of $\mathcal{L}^*(E)$, and by Sect. 3.3 the same holds for the T_j's.

We reformulate again Theorem A in this context and $\mathcal{F}uk^*(E)$ denotes the Fukaya category as constructed in Sect. 3.1, and subject to the strong monotonicity conditions in Sect. 3.3.

Theorem 4.2.1 *Let $V \in \mathcal{L}^*(E)$, $V : \emptyset \to (L_1, \ldots, L_s)$. There exist finite rank $\mathcal{A}$-modules E_k, $1 \leq k \leq m$, and an iterated cone decomposition taking place in $D\mathcal{F}uk^*(E)$:*

$$V \cong (T_1 \otimes E_1 \to T_2 \otimes E_2 \to \cdots \to T_m \otimes E_m \to \gamma_s \times L_s \to \gamma_{s-1} \times L_{s-1} \to \cdots \to \gamma_2 \times L_2).$$

4.3 Decomposition of remote Yoneda modules

We consider here two Lefschetz fibrations $E, \hat{E}$, where $\hat{E}$ is an "extension" of E (the precise meaning is given below). The main result of this subsection is that if a cobordism W in $\hat{E}$ can be separated from the singularities of E, then W can be decomposed, when viewed as a module over $\mathcal{F}uk^*(E)$, as an iterated cone as claimed in Theorem 4.2.1 but with all the modules $E_i = 0$. We assume the "tame" setting of Sect. 4.2. The strong monotonicity assumption on E is not necessary for the results of this subsection, and we fix a monotonicity class $*$.

Fix a large constant $K > 0$ and consider a Lefschetz fibration $\hat{\pi} : \hat{E} \to \mathbb{C}$ so that:

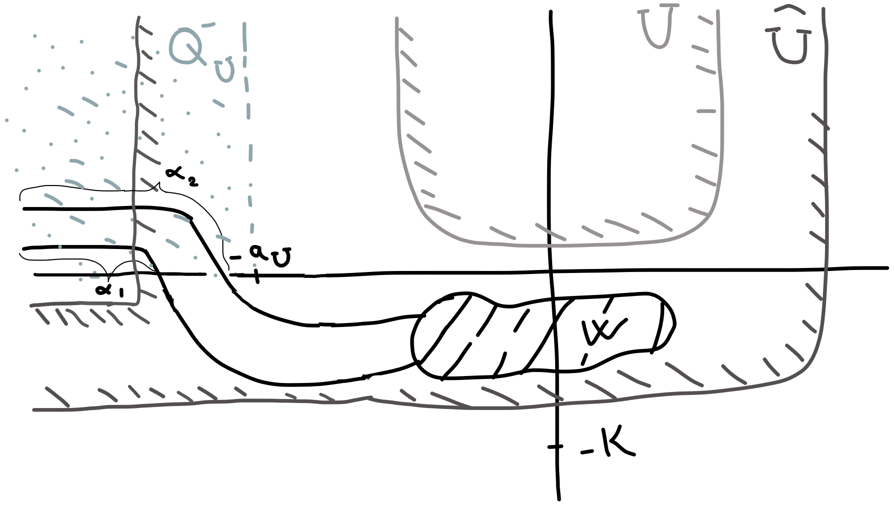

Fig. 8 The domains $\hat{U}$, U, the quadrant Q_U^- and the cobordism W that is remote relative to E

i. $\hat{\pi}$ is tame outside $\hat{U}$, with $U \subset \hat{U}$ and is so that for some constant $a_{\hat{U}} > a_U$ the quadrants $Q_{\hat{U}}^- = (-\infty, -a_{\hat{U}}] \times [0, +\infty)$ and $Q_{\hat{U}}^+ = [a_{\hat{U}}, \infty) \times [0, +\infty)$ are disjoint from $\hat{U}$.
ii. $\hat{U} \subset \mathbb{R} \times [-K, +\infty)$.
iii. $\hat{E}|_{\mathbb{R}\times[-\frac{1}{2},+\infty)} = E|_{\mathbb{R}\times[-\frac{1}{2},+\infty)}$ including their symplectic structures.

Similarly to the definition of the category $\mathcal{F}uk^*(E)$ in Sect. 3.1 we consider a Fukaya category $\mathcal{F}uk^*(\hat{E})$ whose objects are cobordisms $W \subset \hat{E}$ as in Definition 2.2.3 so that W is monotone of class $*$, W has only negative ends $L_1, \ldots, L_s$ (all in $\mathcal{L}^*(M)$) and

$$W \subset \hat{\pi}^{-1}\left(\mathbb{R} \times [-K + \frac{1}{2}, \infty)\right).$$

Following Definition 2.2.3, the cobordism W is cylindrical and the ends of W project to rays of the form $(-\infty, -a_{\hat{U}}] \times \{k\}$ with $k \in \mathbb{N}^*$.

A cobordism W as before is called *remote* relative to E if, in addition,

$$W \subset \hat{\pi}^{-1}(\mathbb{R} \times (-\infty, 0] \cup Q_U^-). \tag{12}$$

In this case, we deduce that $W \cap \pi^{-1}(U) = \emptyset$ (this explains the terminology, in the sense that W is remote from all the singularities of π). See Fig. 8. Because $\hat{U}$ might contain an unbounded region not included in the upper half plane (in the figure this region goes through the third quadrant, it could as well also intersect the fourth quadrant but that is irrelevant for the argument), the conditions i,ii,iii allow for $\hat{E}$ to have more singularities than E.

Given property ii from Sect. 3.1, it is clear that such remote cobordisms W are not objects of $\mathcal{F}uk^*(E)$. On the other hand, each object of $\mathcal{F}uk^*(E)$ is an object of

$\mathcal{F}uk^*(\hat{E})$. Moreover, by a simple application of the open mapping theorem, we see that there is an inclusion of A_∞-categories

$$\mathrm{Incl}^{E,\hat{E}} : \mathcal{F}uk^*(E) \to \mathcal{F}uk^*(\hat{E}). \tag{13}$$

The relevant argument is as follows. All objects of $\mathcal{F}uk^*(E)$ project to the upper half plane so that the J-polygons that compute the operations μ^k of $\mathcal{F}uk^*(\hat{E})$ (for objects that are in $\mathcal{F}uk^*(E)$) project to maps $v : S_r \to \mathbb{C}$ with boundary inside the upper half plane. Our choice of almost complex structures implies that such a curve v can be assumed—after applying the change of coordinates as in Sect. 3.1.2—to be holomorphic outside (possibly a slightly bigger set containing) U and, by the open mapping theorem, we deduce that v can not extend outside of the region where E and $\hat{E}$ coincide. Thus, for objects picked in $\mathcal{F}uk^*(E)$, the operations μ_k are the same in $\mathcal{F}uk^*(\hat{E})$ and in $\mathcal{F}uk^*(E)$.

Let $\mathcal{Y}(W)$ be the Yoneda module associated to an object $W \in \mathcal{O}b(\mathcal{F}uk^*(\hat{E}))$. We denote by W_E the pull-back module:

$$W_E = (\mathrm{Incl}^{E,\hat{E}})^*(\mathcal{Y}(W)) \tag{14}$$

In case W is remote with respect to E we say that the module W_E is a remote $\mathcal{F}uk^*(E)$-module.

Proposition 4.3.1 *With the terminology above, assume that $W \in \mathcal{O}b(\mathcal{F}uk^*(\hat{E}))$ is remote relative to E, $W : \emptyset \rightsquigarrow (L_1, \dots, L_s)$, then the $\mathcal{F}uk^*(E)$-module W_E is quasi-isomorphic to an iterated cone of $\mathcal{F}uk^*(E)$-modules of the form*

$$W_E \simeq (\mathcal{Y}(\gamma_s \times L_s) \to \mathcal{Y}(\gamma_{s-1} \times L_{s-1}) \to \cdots \to \mathcal{Y}(\gamma_2 \times L_2)). \tag{15}$$

In particular, W_E corresponds to an object of $D\mathcal{F}uk^*(E)$.

Before proceeding to the proof, we will first fix some algebraic facts that will be useful along the way. We then sketch the main line of proof as well as the main ideas required.

Suppose that $\mathcal{A}$ is an A_∞-category. Let $\mathcal{M}$ and $\mathcal{N}$ be two A_∞ modules over $\mathcal{A}$ and let $i : \mathcal{N} \to \mathcal{M}$ be a module morphism. We say that i is an inclusion if all the higher components i^k, $k \geq 2$, vanish and $i^1 : \mathcal{N}(X) \to \mathcal{M}(X)$ is an injective map for all objects X of $\mathcal{A}$. We refer to $\mathcal{N}$ as a submodule of $\mathcal{M}$. Obviously, the structural maps of $\mathcal{N}$ are given by restriction from the structural maps of $\mathcal{M}$. If $\mathcal{N}$ is a submodule of $\mathcal{M}$, then the quotient module $\mathcal{M}/\mathcal{N}$ (over $\mathcal{A}$) is defined as $(\mathcal{M}/\mathcal{N})(X) = \mathcal{M}(X)/\mathcal{N}(X)$ and its structural maps are induced from those of $\mathcal{M}$. In the same context, it is useful to note that by the definition of a cone in the category of A_∞ modules (and because we work without grading) the following is true. If $i : \mathcal{N} \to \mathcal{M}$ is an inclusion, then there exists a morphism of modules $j : \mathcal{M}/\mathcal{N} \to \mathcal{N}$ so that $\mathcal{M} = \mathrm{cone}(j)$. Moreover, there is a natural morphism $\mathrm{cone}(i) \to \mathcal{M}/\mathcal{N}$ which is a quasi-isomorphism. When i is an inclusion as before, we will sometimes refer to the sequence $0 \to \mathcal{N} \xrightarrow{i} \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{N} \to 0$ as an exact sequence of A_∞ modules (notice however that the category of A_∞ modules over $\mathcal{A}$ is not abelian).

With these algebraic preliminaries the sketch of the proof is as follow. We start with the remark that if W is remote relative to E, then it can be repositioned in such a way that the resulting cobordism W' only intersects the objects X of $\mathcal{F}uk^*(E)$ along the cylindrical parts of W' and X, in a pattern similar to a rectangular lattice, as in Fig. 10. The modules W_E and W'_E are quasi isomorphic but the structural maps of the module W'_E are much easier to understand then those of W_E. Their properties lead to the decomposition in the statement. Namely, we want to show that W'_E contains a sequence of submodules $0 = W'_{E,1} \subset W'_{E,2} \subset \cdots \subset W'_{E,s} = W'_E$ so that, for each i, the quotient $W'_{E,i}/W'_{E,i-1}$ is quasi isomorphic to $\mathcal{Y}(\gamma_i \times L_i)$. In view of the algebraic preliminaries above, this is sufficient to prove the proposition. For each $X \in \mathcal{O}b(\mathcal{F}uk^*(E))$ the definition of $W'_{E,i}$ is simple: $W'_{E,i}(X)$ is generated by the intersection points between X and the first i cylindrical branches of W'. As a consequence $W'_{E,i}/W'_{E,i-1}(X)$ is generated by the intersection points between the ith branch of W' and X. If the curve γ_i is pushed a bit lower in the plane (below the line $\mathbb{R} \times \{i\}$), as needed to achieve transversality when computing $\mathcal{Y}(\gamma_i \times L_i)(X) = CF(X, \gamma_i \times L_i)$, and if it is also repositioned so that it covers the planar projection of the ith cylindrical branch of W', we see that $\mathcal{Y}(\gamma_i \times L_i)(X)$ and $W'_{E,i}/W'_{E,i-1}(X)$ may be assumed to have the same generators. Furthermore, it is not difficult to also show that the structural maps of the two modules are identified. This argument basically reduces the proof of the proposition to being able to show that the structural maps of W'_E restrict to $W'_{E,i}$ (for all i) as A_∞ module maps. The idea to show this is simple: assuming that the curves u that contribute to the structural maps project holomorphically in the plane, use the open mapping theorem together with orientation constraints reflecting the position of the output and the input of $\pi \circ u$ to show that if the input of u is an intersection of X with the ith branch of W', then the output can not be an intersection of X and any jth branch of W' for $j > i$. While this is easy to see if $\pi \circ u$ would be holomorphic, there are some serious technical difficulties to implement this argument because the Eq. (7) satisfied by u involves Hamiltonian perturbations so that $\pi \circ u$ is not holomorphic. To address this problem we use specific profile functions as in Sect. 3.1 together with the naturality change of coordinates in Sect. 3.1.2: instead of basing the argument above on the curves u we use the curves v where u and v are related through (8) and, as described in Sect. 3.1.2, the curve $v' = \pi \circ v$ is now holomorphic in the region of the plane that we are interested in. At this point, we would like to apply the open mapping theorem and orientation constraints mentioned above to the curve v' but there is yet one more difficulty we encounter. To apply these arguments we need for the boundary constraints of v' to go through the intersection points of W' and X in the pattern of the initial lattice. The difficulty is that, as can be seen from Sect. 3.1.2, the curves v' verify moving boundary conditions that might not leave fixed the corners of the lattice. This last technicality is dealt with by picking special Hamiltonian perturbations in the form of "snaky" perturbation maps.

The proof below makes essential use of constructions that appear in [6] and it is organized in three steps: the first is the geometric repositioning of W; the second sets up the "snaky" perturbations; the third puts these elements together and analyzes the properties of the resulting curves v'.

Proof of Proposition 4.3.1 We start by repositioning W by using a horizontal Hamiltonian isotopy in $\hat{E}$. By definition, this is an isotopy possibly not with compact support, whose support contains a neighborhood of the singularities of $\hat{E}$, and which slides the ends of W along themselves just as in Definition 2.2.3 in [6]. It is immediate to see that such isotopies do not change the isomorphism type of objects in $\mathcal{F}uk^*(\hat{E})$.

By applying such an isotopy to W we may assume that not only $W \subset \hat{\pi}^{-1}(\mathbb{R} \times (-\infty, 0] \cup Q_U^-)$ as in the definition of remote cobordisms but that, moreover, the intersection

$$W^- = W \cap \pi^{-1}(Q_U^-)$$

coincides with a disjoint union of cylindrical ends of W. In other terms

$$W^- = \cup_{i=1}^{s} \alpha_i \times L_i$$

where α_i are curves in $\mathbb{C}$ as in Fig. 8. In particular, for any object $X \in \mathcal{O}b(\mathcal{F}uk^*(E))$, the intersection $W \cap X$ consists of a union of intersections of the ends of W with the ends of X and is included in the quadrant Q_U^-.

Step 1: *Repositioning* W. Here we replace the module W_E with a quasi-isomorphic module W_E' corresponding to a cylindrical Lagrangian W' that not only has the property that its intersection with any cobordism in $X \in \mathcal{O}b(\mathcal{F}uk^*(E))$ reduces to the intersection of the cylindrical parts of X and W' but, moreover, the ends of W' are at half integer heights and thus are distinct from those of X. The advantage of using this W' is that the module multiplication maps of W_E' are easier to describe compared to those of W_E.

To make this precise, we include the two A_∞-categories $\mathcal{F}uk^*(E)$ and $\mathcal{F}uk^*(\hat{E})$ in two other A_∞-categories, respectively, $\mathcal{F}uk^*_{\frac{1}{2}}(E)$ and $\mathcal{F}uk^*_{\frac{1}{2}}(\hat{E})$. These two categories have objects that are again cobordisms as before with the difference that their ends have heights $\in \frac{1}{2}\mathbb{Z} \subset \mathbb{Q}$. In other words, compared with Definition 2.2.3, the difference is that $V \cap \pi^{-1}(Q_U^-) = \cup_{i \in \mathbb{N}^*}((-\infty, -a_U] \times \{\frac{i}{2}\}) \times L_i$. The inclusion $\mathcal{F}uk^*(E) \to \mathcal{F}uk^*_{\frac{1}{2}}(E)$ is obvious and is clearly full and faithful and similarly for the two categories associated to $\hat{E}$. We now perturb W by a (non-horizontal) Hamiltonian isotopy so as to obtain an object W' of $\mathcal{F}uk^*_{\frac{1}{2}}(\hat{E})$ that differs from W only inside $(-\infty, -a_U - 2] \times [\frac{1}{2}, +\infty)$ and is so that the ends of W' restricted to $(-\infty, -a_U - 4 - s] \times [\frac{1}{2}, +\infty)$ are of the form $(-\infty, -a_U - 4 - s] \times \{i - \frac{1}{2}\} \times L_i$ (for all the definitions involved to be coherent we might need to enlarge here the set $\hat{U}$). In other words, the ends of W' are shifted down by $\frac{1}{2}$ compared to the ends of W. Let W_E' be the $\mathcal{F}uk^*(E)$-module obtained as pull-back over the inclusions

$$\mathcal{F}uk^*(E) \to \mathcal{F}uk^*(\hat{E}) \to \mathcal{F}uk^*_{\frac{1}{2}}(\hat{E})$$

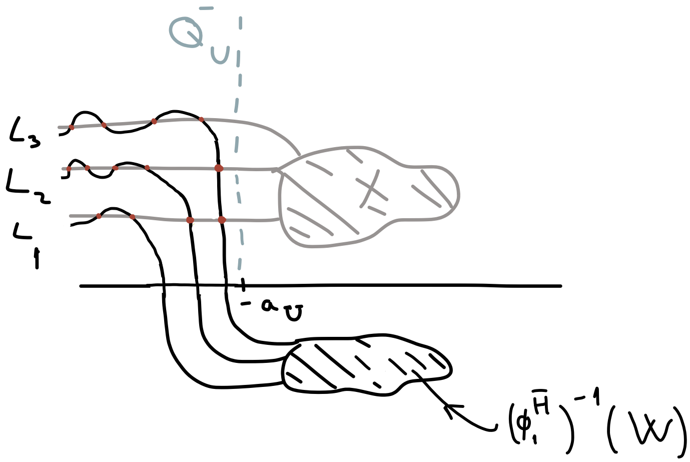

Fig. 9 The projections on $\mathbb{C}$ of $(\phi_1^{\bar{H}_{X,W}})^{-1}(W)$ and of X. The ends of $(\phi_1^{\bar{H}_{X,W}})^{-1}(W)$ are below those of X at infinity

from the $\mathcal{F}uk^*_{\frac{1}{2}}(\hat{E})$-module $\mathcal{Y}(W')$.

The two modules W_E and W'_E are quasi-isomorphic. This is a direct consequence of the definition of $\mathrm{Mor}_{\mathcal{F}uk^*(\hat{E})}(X, W) = CF(X, W)$. This uses a perturbation of W in which its negative ends are "moved" down compared to those of X. More precisely, recall from §3 in [6] (see also Figure 8 there) that $CF(X, W)$ is defined by using a specific profile function h and an associated Hamiltonian $\bar{H}_{X,W}$. With these choices $CF(X, W)$ is identified with $CF(X, (\phi_1^{\bar{H}_{X,W}})^{-1}(W))$ (under the assumption that X and $(\phi_1^{\bar{H}_{X,W}})^{-1}(W)$ intersect transversely). The projection of $(\phi_1^{\bar{H}_{X,W}})^{-1}(W)$ to $\mathbb{C}$ is as in Fig. 9. On the other hand the ith end of W' is, by construction, below the horizontal line $\mathbb{R} \times \{i\}$ and therefore the complexes $CF(X, W)$ and $CF(X, W')$ are quasi-isomorphic. Further, this quasi-isomorphism extends to a quasi-isomorphism of the modules W_E and W'_E.

To summarize this first step, we have replaced in our argument the cobordism W by the cobordism W'. Moreover, by a further horizontal Hamiltonian isotopy, we may assume that W' has a projection as in Fig. 10. More precisely, we assume that $(W')^- = W' \cap \pi^{-1}(Q_U^-)$ is a disjoint union of components $\alpha_i \times L_i$ so that α_i is obtained by rounding the corner of the union of two intervals $(-\infty, -a_U - 4 - s + i] \times \{i - \frac{1}{2}\} \cup \{-a_U - 4 - s + i\} \times [0, i - \frac{1}{2}]$. In particular, the intersections of X and W' project onto $\mathbb{C}$ to the points $b_{ij} = \{-a_U - 4 - s + i\} \times \{j\}$ with $i > j$, $i, j \in \mathbb{N}^*$, $i = 1, 2, \ldots, s$; b_{ij} is precisely the projection of the intersection of the ith end of W' with the jth end of X.

We may also assume, by a slight additional horizontal isotopy, that $W' \cap \pi^{-1}(\mathbb{R} \times [-\frac{1}{2}, \infty))$ is a union of cylindrical ends.

Step 2 : *"Snaky" perturbation data.* This step of the proof consists in choosing the perturbation data used in the definition of $\mathcal{F}uk^*(E)$ and $\mathcal{F}uk^*(\hat{E})$ in a convenient way. Recall that W' is already fixed as discussed at step 1. The perturbation data in question

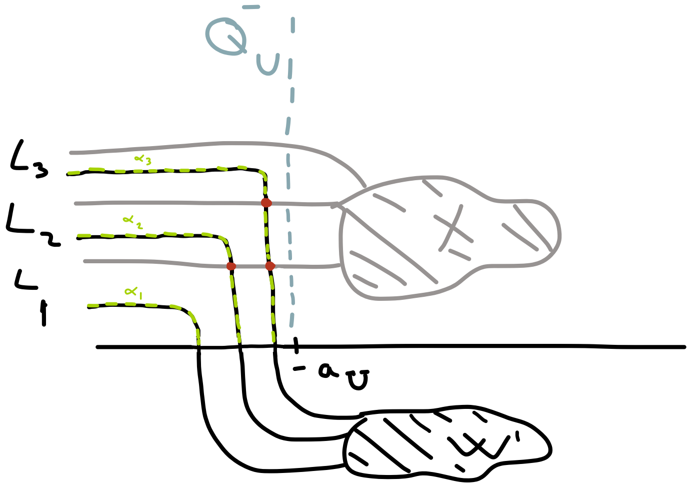

Fig. 10 The remote cobordism $W' \subset \hat{E}$, $X \in \mathcal{O}b(\mathcal{F}uk^*(E))$ and the curves α_i. The height of the ith end of W' is $i - \frac{1}{2}$ while the ith end of X has height i

are chosen as described in Sect. 3.1 except that the profile function h as well as the almost complex structure $\mathbf{J}$ will be picked with some additional properties described below.

We start with the choice of the profile function h. As can be seen from Sect. 3.1 the fundamental ingredients in the definition of h are the functions $h_\pm$. We start with h_+: the only requirement in this case is that $h_+ : [a_U + \frac{3}{2}, \infty) \to \mathbb{R}$ has its single critical point (the bottleneck) at $a_U + 2$. In other words the difference with respect to the construction in Sect. 3.1 (which uses the conventions in §3.2 [6]) is that the value $\frac{1}{2}$ is replaced with a_U. In fact, as we only consider cobordisms without positive ends the choice of h_+ is not particularly important as long as the bottlenecks are away from U. We now discuss the function h_-. This is a smooth function $h_- : (-\infty, -a_U - 1] \to \mathbb{R}$ with the following additional properties—see Fig. 11:

- a$'$. The function h_- has critical points $o_i = -a_U - 3 - i$, $i = 0, 1, \ldots, s$ that are non-degenerate local maxima.
- a$''$. The function h_- has critical points $o'_i = -a_U - \frac{7}{2} - i$, $i = 0, 1, \ldots, s-1$ that are non-degenerate local minima.
- a$'''$. h_- has no other critical points than those at a$'$, a$''$ above and for all $x \in (-\infty, a_U - 4 - s]$ we have $h_-(x) = \alpha^- x + \beta^-$ for some constants $\alpha^-, \beta^-, \alpha^- > 0$.

Beyond this, the properties of the function h are obtained by direct analogy with those given in Sect. 3.1.1 but by changing the critical points conditions by the three conditions a$'$, a$''$, a$'''$ above. In particular, the set W_i^- now becomes $W_i^- = (-\infty, -a_U - 1] \times [i - \epsilon, i + \epsilon]$ and $F_i^- = (-\infty, -a_U - 1] \times [i - \epsilon/2, i + \epsilon/2]$. From this point on, the construction continues along the same approach as in Sect. 3.1.1. In particular, the properties of the family Θ and those of $\mathbf{J}$ are just the same as before but they are relative to sets $K_{V_1,\ldots,V_{k+1}}$ that satisfy different requirements compared to those in Sect. 3.1.1.

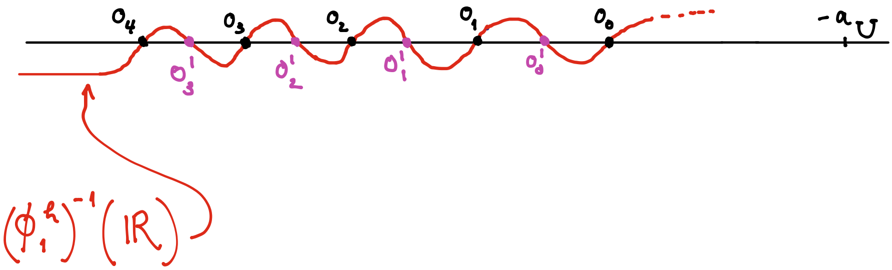

Fig. 11 The graph of $(\phi_1^h)^{-1}(\mathbb{R})$ for $s = 4$

We now discuss the two properties required of $K_{V_1,\ldots,V_{k+1}}$. We start by underlining that, because we care here about a module structure, while $V_1, \ldots, V_k$ are elements of $\mathcal{L}^*(E)$, V_{k+1} is either an element of $\mathcal{L}^*(E)$ or $V_{k+1} = W'$. Further, we fix small disks $D_{ij} \subset \mathbb{C}$ of radius smaller than $\frac{1}{8}$ that are respectively centered at the points (o'_i, j), $i = 0, \ldots, s-1$, $j \in \{1, \ldots, s_{V_1,\ldots,V_{k+1}}\}$. We denote by $D'_{ij} \subset D_{ij}$ the disk with the same center but with radius half of that of D_{ij}. Recall, that $s_{V_1,\ldots,V_{k+1}}$ is the smallest $l \in \mathbb{N}$ so that $\pi(V_1 \cup V_2 \cup \cdots \cup V_{k+1}) \subset [\frac{1}{2}, l)$. We also pick a compact set $Z \subset \mathbb{R} \times (-\infty, -\frac{1}{4}]$ which contains in its interior $\pi(W') \cap \mathbb{R} \times (-\infty, -\frac{1}{2}]$ (recall that W' is cylindrical outside $\pi^{-1}(\mathbb{R} \times (-\infty, -\frac{1}{2}])$ as well as a slightly bigger set $Z' \subset \mathbb{R} \times (-\infty, -\frac{1}{4}]$. We require:

$$K_{V_1,\ldots,V_{k+1}} \supset \cup_{i,j} D'_{ij} \cup \left[-a_U - \frac{11}{4}, a_U + \frac{7}{4}\right] \times \left[\frac{1}{4}, s_{V_1,\ldots,V_{k+1}} + 1\right] \cup Z. \quad (16)$$

and

$$K_{V_1,\ldots,V_{k+1}} \subset \cup_{i,j} D_{ij} \cup [-a_U - \frac{13}{4}, a_U + 2) \times [\frac{1}{8}, +\infty) \cup Z'. \quad (17)$$

It is useful to keep in mind that $K_{V_1,\ldots,V_{k+1}}$ contains the non-cylindrical part of the cobordisms V_i as well as the points (o'_i, j).

We now will see that this class of perturbation data is sufficient to ensure the regularity and the compactness of the moduli spaces appearing in the definition of the category $\mathcal{F}uk^*(E)$ and of the $\mathcal{F}uk^*(E)$-module W'_E. In the next section we will use these specific perturbations to extract the exact triangles claimed in the statement.

Let $u : S_r \to E$ be a solution of (7) that satisfies the boundary and asymptotic conditions required to define the multiplications μ_k for $\mathcal{F}uk^*(E)$ or for the definition of the module W_E. In the first case the boundary conditions are along cobordisms $V_1, \ldots, V_{k+1}$ ($V_i \in \mathcal{L}^*(E)$, in particular, V_i projects on the upper half plane). In the second case, the curve is defined on a punctured polygon so that the component C_i of the boundary of the polygon is mapped to V_i for $1 \leq i \leq k$ and the $(k+1)$th component C_{k+1} is mapped to W'.

By the change of variables in Sect. 3.1.2, (and by taking h sufficiently small) we deduce that there exists some small $\delta > 0$ so that if $u : S_r \to E$ satisfies (7) with the choice of perturbation data as just above and if $v : S_r \to E$ is defined by $u(z) = \phi^{\bar{h}}_{a_r(z)}(v(z))$, then $v' = \pi \circ v$ is holomorphic outside of the set

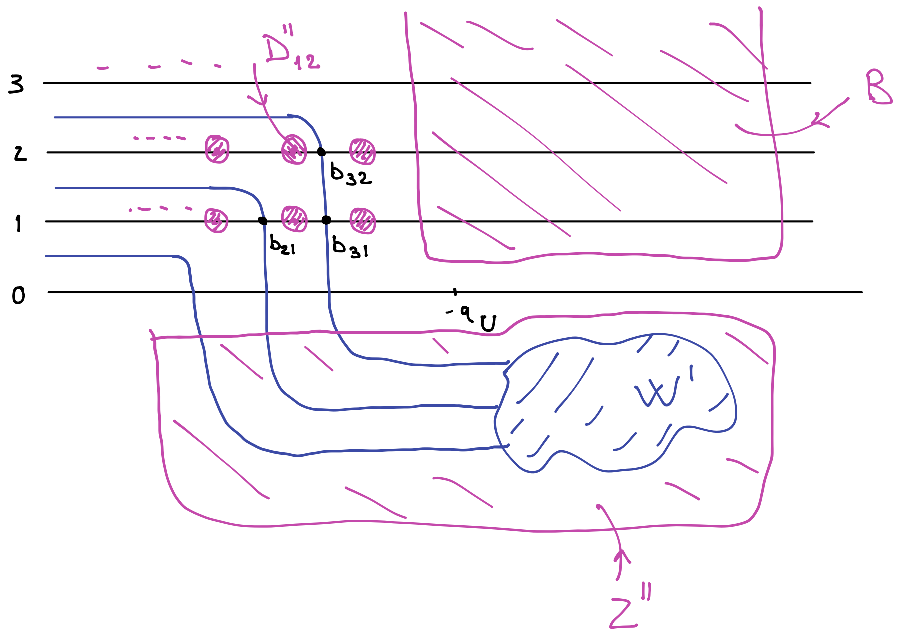

Fig. 12 The set $\widehat{K}$ outside which v' is holomorphic is the union of all the regions in *pink*: the disks D''_{ij}, the box $B = [-a_U - \frac{13}{4} - \delta, a_U + 2 + \delta] \times [\frac{1}{8} - \delta, +\infty)$ and the neighborhood Z'' of the non-cylindrical part of $\pi(W')$. Are also pictured the points b_{ij} (for $s = 3$). The non cylindrical part of the cobordisms $X \subset E$ projects inside B (color figure online)

$$\widehat{K} = \cup_{i,j} D''_{ij} \cup \left[-a_U - \frac{13}{4} - \delta, a_U + 2 + \delta\right] \times [\frac{1}{8} - \delta, +\infty) \cup Z'', \qquad (18)$$

where D''_{ij} is a disk with the same center as D_{ij} but slightly bigger and, similarly, Z'' is a set slightly bigger than Z'—see Fig. 12. In view of this transformation, compactness for the relevant moduli spaces follows without difficulty by the "bottleneck" argument in §3.3 [6]. Thus, the only issue that requires some attention is regularity. Denote

$$K' = \cup_{i,j} D'_{ij} \cup \left[-a_U - \frac{11}{4}, a_U + \frac{7}{4}\right] \times \left[\frac{1}{4}, s_{V_1,\dots,V_{k+1}} + 1\right] \cup Z.$$

Given that $K' \subset K_{V_1,\dots,V_k}$, the perturbation data can be chosen freely over K' and thus, for all moduli spaces consisting of curves whose image intersects $\pi^{-1}(K')$ regularity can be handled in the standard fashion as in [24]. Therefore, we are left to analyze the curves $u : S_r \to E$ so that $\pi(u)$ has an image disjoint from K'. Assume first that u appears in the definition of the higher structures of $\mathcal{F}uk^*(E)$. In this case, the condition $\pi^{-1}(K') \cap Image(u) = \emptyset$ implies that all the boundary of u projects onto $\mathbb{C}$ along a single line $(-\infty, -a_U - 2] \times \{j\}$. Given that $(o'_i, j) \in K'$, it follows that the image of $\pi(u)$ can not cross any of the points (o'_i, j), nor can it have one of these points as asymptotic limit. As a consequence, the asymptotic limits of u have to project to just one of the points (o_i, j). But by now taking a look to v' which is holomorphic around (o_i, j) one sees immediately that v' and thus $\pi(u)$ has to be constant (indeed, (o_i, j) can not be the exit point of v' by an application of the open mapping theorem). The second possibility to consider is if u appears in the definition of the module structure of W'_E. It is immediate, in this case too that $\pi^{-1}(K') \cap Image(u) = \emptyset$ implies that all asymptotic limits of u coincide with a single point b_{ij} (which is, of course, also of the from (o_i, j)). It is easy to see by an application of the open mapping theorem that in this case $\pi(u)$ has again to be constant. To conclude this argument, the only moduli spaces for which regularity is in question consist of curves u so that $\pi(u)$ is constant equal to one of the point (o_i, j). That means that these curves take values in the fiber over (o_i, j) and, because o_i is a local maximum of h_-, one can see, as in §4.2 [6] that by picking regular data in the fiber these moduli spaces are regular too.

Thus the regularity of all the moduli spaces involved is ensured by generic choices of data. We work from now on with such data associated to the "snaky" perturbations constructed at this step.

Step 3: *The proof of* (15). We will show now that there is a sequence of $\mathcal{F}uk^*(E)$-modules $\widetilde{L}_i$, $W'_{E,i}$, $i = 1, \dots, s$, with $W'_{E,i}$ being submodules of W'_E, so that:

i. $W'_{E,1} = 0$, $W'_{E,s} = W'_E$ and for $i \geq 2$ there exist exact sequences of $\mathcal{F}uk^*(E)$-modules

$$0 \to W'_{E,i-1} \to W'_{E,i} \to \widetilde{L}_i \to 0$$

ii. there exists a quasi-isomorphism of $\mathcal{F}uk^*(E)$-modules

$$\widetilde{L}_i \simeq \mathcal{Y}(\gamma_i \times L_i),$$

where $\mathcal{Y}$ is the Yoneda embedding for $\mathcal{F}uk^*(E)$.

These points immediately imply the statement of Proposition 4.3.1. We now proceed to the construction of $W'_{E,i}$ and to prove the points i, ii above.

Let $X \in \mathcal{L}^*(E)$ and let W' be the remote cobordism as discussed at the first step. We now assume "snaky" perturbations picked as described at the second step. In particular, the complex $CF(X, W')$ is well defined. The generators of this complex are identified with the intersection $X \cap (\phi_1^{\bar{h}})^{-1}(W')$. Notice that due to the choice of snaky perturbations $\pi(X \cap (\phi_1^{\bar{h}})^{-1}(W')) = \{b_{rs}\}_{r,s}$ see Fig. 13. We now put

$$P_{rs}(X) = X \cap \left(\phi_1^{\bar{h}}\right)^{-1} (W') \cap \pi^{-1}(b_{rs})$$

and we define

$$W'_{E,i}(X) = \mathcal{A}\left\langle \cup_{1 \leq r \leq i; s < r} P_{rs} \right\rangle \subset CF(X, W').$$

In other words, the generators of $W'_{E,i}(X)$ are the intersection points of X with the first i branches of W'. It is clear from the construction that $W'_{E,1} = 0$ and that $W'_{E,s} = W'_E$.

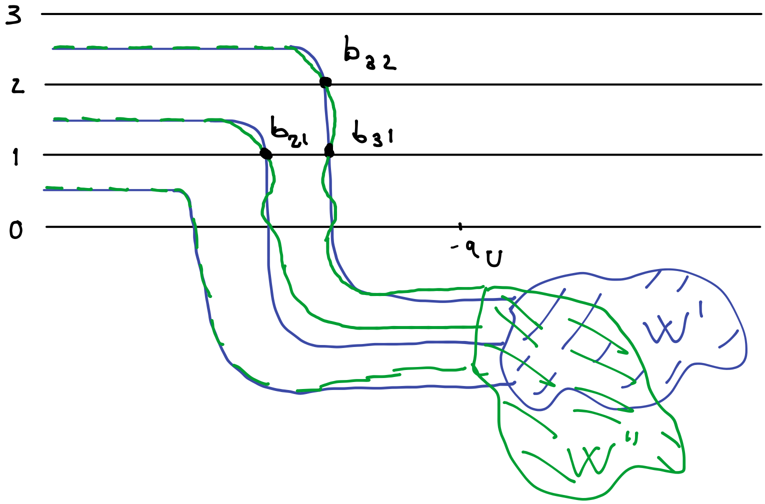

Fig. 13 The cobordism W' and its perturbation $W'' = (\phi_1^{\bar{h}})^{-1}(W')$

We will show now that, for each $1 \leq i \leq s$, the structural maps μ_k of W'_E when restricted to $W'_{E,i}$ have values into $W'_{E,i}$. In other words

$$\mu_k|_{W'_{E,i}} : CF(V_1, V_2) \otimes \cdots \otimes CF(V_{k-1}, V_k) \otimes W'_{E,i}(V_k) \to W'_{E,i}(V_1). \tag{19}$$

This property immediately implies that the $W'_{E,i}$ are indeed A_∞-modules and moreover that the inclusions of vector spaces $W'_{E,i-1}(-) \subset W'_{E,i}(-)$ are actually inclusions of $\mathcal{F}uk^*(E)$-modules. The modules $\widetilde{L}_i$ defined as the respective quotients. With these definition for $W'_{E,i}$ and assuming (19), point ii follows because the quotient $\widetilde{L}_i$ is naturally identified (up to quasi-isomorphism) with $\mathcal{Y}(\gamma_i \times L_i)$. Indeed, $\widetilde{L}_i$ is basically the module obtained from $W'_{E,i}$ by "forgetting" the first $i-1$ branches of W'. As such, it coincides with the module $\mathcal{Y}(\bar{\gamma}_i \times L_i)$ where $\bar{\gamma}_i$ is the curve γ_i repositioned so that it covers the intersection of the ith branch of $\pi(W')$ with $\mathbb{R} \times [\frac{1}{2}, \infty)$. By a horizontal Hamiltonian isotopy we see that this module is quasi-isomorphic to $\mathcal{Y}(\hat{\gamma}_i \times L_i)$ where $\hat{\gamma}_i$ is the curve γ_i translated by $(0, -\frac{1}{2})$ in $\mathbb{R}^2$. Finally, the modules $\mathcal{Y}(\hat{\gamma}_i \times L_i)$ and $\mathcal{Y}(\gamma_i \times L_i)$ are quasi-isomorphic because, by the definition of the profile function h_- (from Sect. 3.1), when computing $CF(V, V'; H) \cong CF(V, (\phi_1^{\bar{H}_{V,V'}})^{-1}(V'))$ (with V, V' cobordisms) the ends of V' are translated below the lines $\mathbb{R} \times \{i\}$—see also Fig. 6—following the same pattern as the translation by $(0, -\frac{1}{2})$.

In summary, to conclude the proof of the proposition it remains to show (19).

Our argument is based on properties of the curve $v' = \pi(v)$ where v is related to a curve $u : S_r \to E$ by Eq. (8) and u is a solution of (7) contributing to the module structural map μ_k. Here S_r is the disk with $k+1$ boundary punctures, of which k are the entries and the last one is an exit puncture. The last entry, denoted m, is the "module" entry and is asymptotic to a generator of $CF(V_k, W'_{E,i})$. The exit, denoted e, is asymptotic to a generator of $CF(V_1, W'_{E,i})$.

To proceed, we will make the following simplifying assumption: we assume that the transition functions α_r used in the definition of moduli spaces associated to the module operations are so that:

$$\alpha_r(z) = 1 \quad \forall z \in C_{k+1}, \tag{20}$$

where C_{k+1} is the component of the boundary of the punctured disk S_r that joins m to e. (See Fig. 5 for an illustration of the case $k = 3$, where C_4 bounds both ϵ_3 and ϵ_4.)

In other words, we use transition functions as in Sect. 3.1.1, except that we add (20) and we modify accordingly conditions *i. c* and *ii. c'* in Sect. 3.1.1 as follows. Assuming that ϵ is the parametrization of the strip like end associated to the last entry, we replace *i. c* by the constraint $\alpha_r \circ \epsilon(s,t) = 0$ for $(s,t) \in (-\infty, 0] \times \{0\}$. Notice that this is compatible with conditions *i. a* and *i. b* when applied to ϵ (with the derivative at *i. b* being 0). Similarly, assuming now that ϵ is the parametrization of the strip like end associated to the exit, we replace *ii. c*' with $\alpha_r \circ \epsilon(s,t) = 0$ for $(s,t) \in [0, \infty) \times \{0\}$. Again this condition is compatible with *ii. a'* and *ii. b'*. By imposing (20) *just to the moduli spaces appearing in the definition of modules* over $\mathcal{F}uk^*(E)$ (and not to those defining the A_∞-operations in $\mathcal{F}uk^*(E)$ itself) we easily see that, on one hand, condition (20) is compatible with gluing and splitting and, moreover, it does not contradict the definition of the operations in $\mathcal{F}uk^*(E)$ itself. At the same time, this means that we get two possibly different definitions for the Yoneda modules of objects in $\mathcal{F}uk^*(E)$: one using the original conditions in Sect. 3.1.1 and the other making use of (20). However, it is easy to see that the two resulting modules are quasi-isomorphic and thus our simplifying condition does not affect any further arguments.

The geometric advantage of this simplifying assumption on α_r is that v no longer satisfies a moving boundary condition along C_{k+1}, rather v maps all of C_{k+1} to $W'' = (\phi_1^{\bar{h}})^{-1}(W')$. Note also that, by the definition of h, and the position of $\pi(W')$ relative to the ends of cobordisms $\in \mathcal{L}^*(E)$—as in Fig. 13—we have that W'' is just a close perturbation of W' and $\pi(W'')$ intersects the horizontal lines of positive, integral imaginary coordinates transversely and in the same points as $\pi(W')$.

Our claim (19) reduces to showing that if $v'(m) = b_{\alpha\beta}$ and $v'(e) = b_{rs}$, then $r \leq \alpha$ (we recall $v' = \pi \circ v$).

We first fix some notation relative to certain regions in Q_U^-. Denote by F the region given as

$$F = \bigcup_{0 \leq t \leq 1,\ j \in \mathbb{Z}} \phi_{-t}^{h}((-\infty, -a_U] \times \{j\}) \cup W''.$$

In short, F is the set swiped by all the potential boundary conditions of the curves v'. Further, we denote $\widehat{F} = F \cup \widehat{K}$ [see (18)] and we put $G = \mathbb{C} \backslash \widehat{F}$—see Fig. 14.

From step 2 we know that v' is holomorphic over G and clearly, the boundary of S_r is so that $v'(\partial S_r) \cap G = \emptyset$. It is an elementary fact (see e.g. [6, Proposition 3.3.1]) that as soon as $Image(v')$ intersects a connected component of G, the full component has to be contained in $Image(v')$. In particular this implies that $Image(v')$ can not intersect an unbounded component of G.

Each point b_{ij} is in the closure of four components of G that meet, basically, as four quadrants at b_{ij}. Our argument will make use of the following:

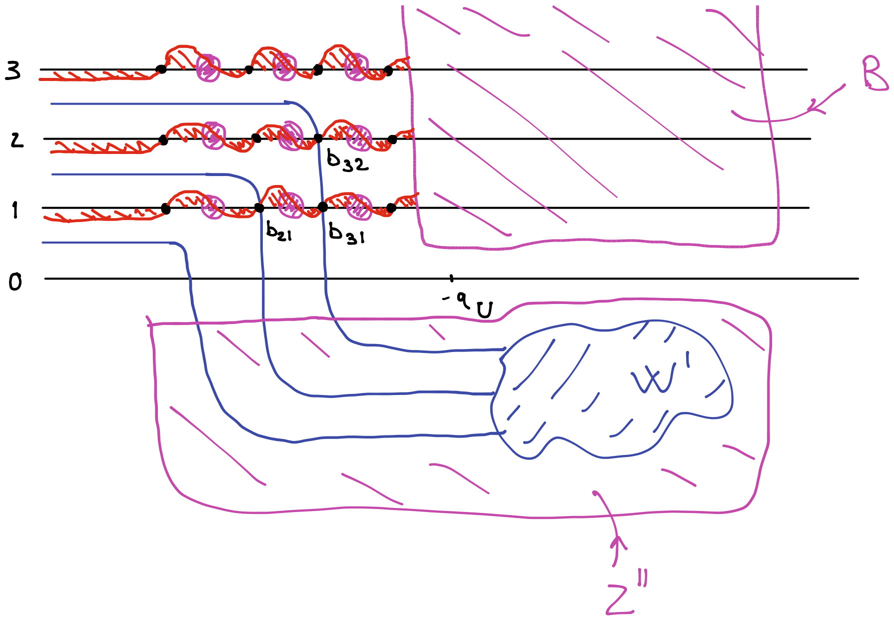

Fig. 14 The region $\widehat{F}$ is the union of $\widehat{K}$ (the union of all the *pink regions*) and F (the region in *red*) (color figure online)

Lemma 4.3.2 *Suppose that b_{ij} is different from both $v'(e)$ and $v'(m)$ and that the component corresponding to the fourth quadrant at b_{ij} is in the image of v', then at least one among the first or third quadrants are also in the image of v'.*

An illustration of the statement of the Lemma is given in Fig. 15. The claim of the Lemma is that if the green region (South-East of b_{42}) having b_{42} in its boundary is included in $Image(v')$, then one of the yellow regions next to b_{42} (North-East and South-West to b_{42}) is also contained in this image.

Proof of Lemma 4.3.2 Consider a small segment $I \subset \pi(W'')$ that ends up at b_{ij} and is included in the closure of the fourth quadrant (the quadrants here are defined by the vertical and horizontal lines in Fig. 15). We have $I \subset Image(v')$. Let $x \in I$. If x is the image of a point $z \in Int(S_r)$, then, by the open mapping theorem, the image of v' also intersects the third quadrant which implies our claim. Thus it is sufficient to consider the case when all the points of I are in the image of boundary points of S_r. The only boundary component that is mapped to W'' is C_{k+1}, hence $I \subset v'(C_{k+1})$. Moreover, as b_{ij} is not the asymptotic image of the ends of C_{k+1}, it follows that $b_{ij} \in v'(C_{k+1})$. Let $z \in C_{k+1}$ be such that $v'(z) = b_{ij}$. As shown at step 2, v' is holomorphic outside of $\widehat{K}$ and thus, in particular, around b_{ij}. Given that (around b_{ij}) $v'(C_{k+1})$ is contained in the vertical line through b_{ij} and, due to the bottleneck structure around b_{ij}, the open mapping theorem implies that $Image(v')$ intersects the region of G corresponding to the first quadrant. This ends the proof of the lemma. □

We return to the Proof of Proposition 4.3.1. (Figure 15 continues to be a relevant illustration for the proof.) Recall $v'(m) = b_{\alpha\beta}$, $v'(e) = b_{rs}$. Assume that $r > \alpha$. As

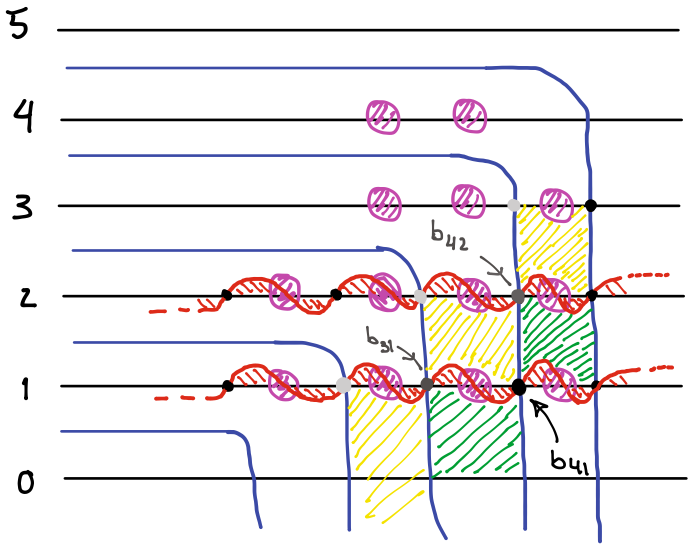

Fig. 15 We take here $s \geq 5$ and in *blue* are the projections of the ends of W''. Assume $v'(m) = b_{41}$ and suppose $v'(e) = b_{rs}$ with $r \geq 4$; v' exits b_{41} through one of the *green regions* which is therefore included in $Image(v')$; Lemma 4.3.2 applied to b_{42} and b_{41} shows that one of the *yellow regions* $\subset Image(v')$; by applying again Lemma 4.3.2 to one of the *upper left corners* of the *yellow regions*—in *light gray*—we get that an unbounded region of G is contained in $Image(v')$. Thus, we reach a contradiction in three steps (color figure online)

m is an entry point, for orientation reasons, $Image(v')$ has to contain at least one of the first or third quadrants at $b_{\alpha\beta}$. In both cases, the upper left corner of the respective quadrant, that we denote by $b_{i_1 j_1}$, is such that $i_1 \leq \alpha$. Thus Lemma 4.3.2 can be applied to $b_{i_1 j_1}$ and it implies that the first or third quadrant at $b_{i_1 j_1}$ is contained in $Image(v')$. Let $b_{i_2 j_2}$ be the upper left corner of the respective quadrant. We have $i_2 \leq i_1$. This process can be pursued recursively, thus getting a sequence of points $b_{i_1 j_1}, b_{i_2 j_2}, \ldots$ and associated quadrants $\subset Image(v')$ by picking at each step the upper left corner of a quadrant obtained from Lemma 4.3.2 applied to the previous point in the sequence. This process continues till one of the quadrants in question is an unbounded region. But this contradicts the fact that the image of v' can not intersect such a region. □

4.4 Disjunction via Dehn twists

The purpose of this subsection is to show that certain Dehn twists of a cobordism are Hamiltonian isotopic to remote cobordisms and therefore can be decomposed by means of Proposition 4.3.1. Monotonicity assumptions are not required in this part but we still work with tame Lefschetz fibrations.

The idea is the following. Given a cobordism $V \subset E$, we first add specific singularities to E (with critical values in the lower half plane) so that we can join each initial singularity x_i of E to one of the "new" ones, x_i', by a matching cycle S_i. We then notice that, with appropriate choices for the matching cycles and the other elements of the construction, the iterated Dehn twist $\tau_{S_m} \circ \cdots \circ \tau_{S_i} \circ \cdots \circ \tau_{S_1}$ transforms V into a remote cobordism V'. The proofs will only be sketched here and we refer to [3] for full details.

4.4.1 The case of a single singularity

We start with the core of the geometric argument. This appears in the case of a fibration with a single singularity.

Fix $S \subset M$, a framed (or parametrized) Lagrangian sphere. We use Seidel's terminology [23,24] so that this means S is Lagrangian together with a parametrization $e : S^n \to S$. Consider a Lefschetz fibration $\pi : E \to \mathbb{C}$ which is tame outside $U \subset \mathbb{R} \times [\frac{1}{4}, +\infty) \subset \mathbb{C}$ and with a single singularity x_1 so that the vanishing cycle corresponding to x_1 coincides with S. We assume that the singularity has critical value $v_1 = (1, \frac{3}{2})$. Fix also a negative-ended cobordism $V \subset E$ with ends $L_1, L_2, \ldots, L_s$.

For the construction described below it is useful to refer to Fig. 16 (which contains also details that will be relevant only later on). We will make use of an auxiliary Lefschetz fibration $\hat{\pi} : \hat{E} \to \mathbb{C}$ that coincides with E over the upper half plane and that has an additional critical point x_1' with corresponding critical value $v_1' = (-1, -\frac{3}{2})$ and a matching cycle $\hat{S}_\gamma \subset \hat{E}$ that projects onto $\mathbb{C}$ to a path joining v_1' to v_1. More precisely, $\hat{E}$ has the following properties. The fibration $\hat{E}$ is tame outside a set $\hat{U}$ (as pictured in Fig. 16), $\hat{U} \subset (-\infty, a_{\hat{U}}] \times [-K, +\infty)$. Moreover, let D be a disk around v_1' that is included in the lower half plane but is not completely included in $\hat{U}$. Let $v_0 \in \partial D \backslash \hat{U}$. Fix also a path γ that joins v_1 to v_0. Denote by T_γ the thimble originating at x_1 and whose planar projection is γ. The boundary of T_γ is identified to the vanishing cycle S and, as subset in $\pi^{-1}(v_0)$, we denote it by S_0. The fibration $\hat{\pi} : \hat{E} \to \mathbb{C}$ is such that it admits the sphere S_0 as vanishing cycle also relative to the singularity x_1'. If we extend the curve γ to a curve (that we will continue to denote by γ) that joins v_1 to v_1' this is covered by a matching cycle $\hat{S}_\gamma \subset \hat{E}$. Given that E is trivial over the lower half-plane, the construction of $\hat{E}$ follows directly from the constructions in §16 [24].

For further use, we now fix another thimble T originating at x_1 and whose projection is the vertical half-line $\{1\} \times [\frac{3}{2}, \infty)$.

Proposition 4.4.1 *There exists a curve γ, depending on V, and a framed Lagrangian sphere S' in $\hat{E}$, hamiltonian isotopic to the matching sphere $\hat{S}_\gamma$ so that the image of V under the Dehn-twist along S', $V' = \tau_{S'} V$, is disjoint from T and the intersection $V' \cap S'$ is contained in D.*

Proof of Proposition 4.4.1. Fix $K^n \longrightarrow K \subset N$, a parametrized Lagrangian sphere and let L be another Lagrangian submanifold of the symplectic manifold (N, ω). Assume that L is transverse to K and $L \cap K = \{p_1, \ldots, p_r\}$. Fix an additional point $p_0 \in K$ and a small neighborhood of it $\mathcal{V} \subset K$.

Since the Dehn twist τ_K sends K to itself, $\tau_K L$ is transverse to K and it intersects K in r points $q_1, \ldots, q_r$. Moreover, we can arrange the choices made in the definition of the Dehn twist so that all the points $q_1, \ldots, q_r$ belong to $\mathcal{V}$, $1 \leq j \leq r$.

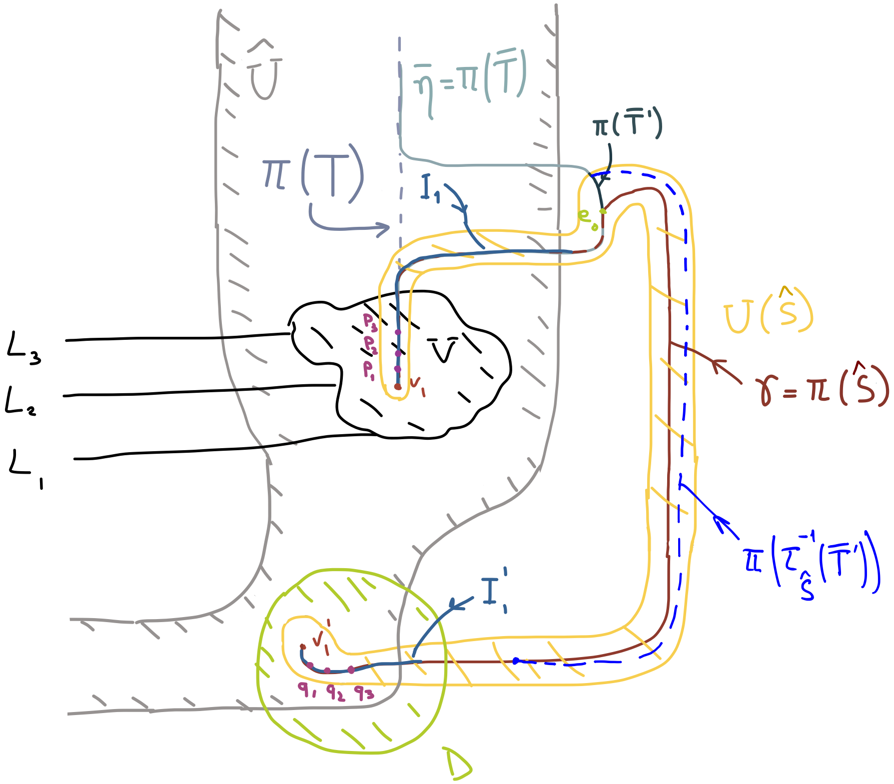

Fig. 16 The Lefschetz fibration $\hat{\pi} : \hat{E} \to \mathbb{C}$ coincides with E over the upper half-plane; $\hat{\pi}$ has two singularities of critical values v_1 and v_1' and is symplectically trivial outside of $\hat{U}$. Are pictured (in projection on $\mathbb{C}$): the "straight" vertical thimble T and its deformation $\bar{T}$; the matching cycle $\hat{S}$ that coincides with $\bar{T}$ from v_1 to e_0; the disk D; $\hat{S} \cap V = \{p_1, p_2, p_3\}$; the neighborhood $U(\hat{S})$ where $\tau_{\hat{S}}$ is supported; the portion $\bar{T}'$ of $\bar{T}$ that differs from $\hat{S}$ and is included in $U(\hat{S})$; the projections I_1, I_1' of two disks K_1, K_1' in $\hat{S}$ around the two singularities of $\hat{\pi}$ so that $\hat{S} \backslash (K_1 \cup K_1')$ lies inside a trivial symplectic fibration. Notice that the domain $\hat{U}$ is generally unbounded along some additional directions compared to the domain outside which E is tame. This is required so that the fibration $\hat{E}$, that agrees with E over the upper half plane, has additional singularities compared to E. Our choice is for this unbounded direction to be in the lower left corner, as in the picture

We now return to our setting and to Fig. 16. We apply the observation above to the case when the ambient manifold N is replaced with $\hat{E}$, the sphere K is replaced with the matching cycle $\hat{S}_\gamma$ and the Lagrangian V takes the place of L. We denote in this proof $\hat{S} = \hat{S}_\gamma$ to shorten notation. We deduce that (with appropriate choices) $\tau_{\hat{S}} V$ only intersects $\hat{S}$ in a set of points $q_1, q_2, \ldots$ equal in number to the number of intersections between V and $\hat{S}$ and all these points can be assumed to belong to D. We now consider a thimble $\bar{T}$ which is Hamiltonian isotopic to T and which, in the region that interests us, projects onto the plane along the curve $\bar{\eta}$ in Fig. 16. This thimble $\bar{T}$ is composed of three regions: it coincides with $\hat{S}$ along a curve included in γ that starts at v_1 and ends at the point e_0; a second region $\bar{T}'$ is equal to the intersection of

$\bar{T}$ with $U(\hat{S})\backslash\hat{S}$ where $U(\hat{S})$ is a small Weinstein neighbourhood of $\hat{S}$ (inside which $\tau_{\hat{S}}$ is supported); finally, the rest of $\bar{T}$, which is outside of $U(\hat{S})$.

A delicate observation is next: one can perform this construction so that $\bar{T}'$ is also disjoint from $\tau_{\hat{S}}V$. This is equivalent to $\tau_{\hat{S}}^{-1}(\bar{T}') \cap V = \emptyset$. The details needed to justify this fact are tedious but this statement essentially follows from the direction of the half twist (in the plane) of the matching sphere. More explicitly, the main idea is as follows. We use a parametrization of the sphere $\hat{S}$ with the following properties: the region of $\hat{S}$ that projects to the interval I_1 in Fig. 16 is identified to a spherical cap K_1 on the sphere $\hat{S}$ so that $(U(V) \cap \hat{S}) \subset K_1$ (where $U(V)$ is a small neighbourhood of V), there is also a spherical cap K_1' that projects to the interval I_1' and so that the points q_i belong to K_1'. The parametrization of $\hat{S}$ is such that these two caps are antipodal and very small. We let $U(K_1)$ be the restriction of the Weinstein neighbourhood $U(\hat{S})$ to the cap K_1. With this notation, the claim reduces to show $\tau_{\hat{S}}^{-1}(\bar{T}') \cap U(K_1) = \emptyset$. Therefore, we need to understand the effect of $\tau_{\hat{S}}^{-1}$ on $\bar{T}'$. Recall that the inverse Dehn twist reduces to the antipodal map on the zero section of $U(\hat{S})$, and, away from the zero section, it coincides with an adequate reparametrization of the inverse normalized geodesic flow, so that it is the identity on the boundary of $U(\hat{S})$, [24]. In particular, $\tau_{\hat{S}}^{-1}$ permutes the caps K_1 and K_1'. Moreover, because $\bar{T}'$ projects to the "left" of $\hat{S}$ an easy calculation shows that for each $(x, v) \in \bar{T}' \subset T^*(\hat{S})$ with $v \neq 0$, the negative normalized geodesic flow keeps (x, v) away from K_1 for all times $t \in [0, \pi]$. In Fig. 16 we draw a possible image of $\tau_{\hat{S}}^{-1}(\bar{T}')$ (this is not quite realistic as $\pi(\tau_{\hat{S}}^{-1}(\bar{T}'))$ does not need to be a curve in general). In summary, we deduce $\tau_{\hat{S}}^{-1}(\bar{T}') \cap V = \emptyset$. We refer to [3] for more details.

The last step in the proof is simple. Let ψ be the Hamiltonian isotopy that carries $\bar{T}$ to T. It is easy to see that we may assume that $\psi(V) = V$ and in that case if we put $S' = \psi(\hat{S})$ we then have $\tau_{S'}V \cap T = \emptyset$ (where $\tau_{S'}$ is defined by using the parametrization of S' induced from that of $\hat{S}$). □

Corollary 4.4.2 *With the notation in Proposition 4.4.1 the cobordism $\tau_{S'}V$ is hamiltonian isotopic—via an isotopy with compact support—to a cobordism that is remote relative to E.*

Proof We already know from Proposition 4.4.1 that $V' = \tau_{S'}V$ is disjoint from T. Consider an Ω-compatible almost complex structure J on E with the additional property that $\pi : E \longrightarrow \mathbb{C}$ is J-holomorphic. The function $Im(\pi) : E \to \mathbb{R}$ defines a Morse function on E whose negative gradient flow ξ (with respect to the metric induced by (Ω, J)) is also Hamiltonian (this follows from the Cauchy–Riemann equations that are satisfied because we assume π to be J-holomorphic). Moreover, ξ has the thimble T as a stable manifold. Write $\xi = X^H$ with $H : E \to \mathbb{R}$. Now consider a smooth function $\eta : \mathbb{C} \to \mathbb{R}$ so that $\eta(z) = 1$ if $z \in [-a_U - 1, a_U + 1] \times [-\frac{1}{4}, +\infty)$ and $\eta(z) = 0$ if $z \in ((-\infty, -a_U-2]\times\mathbb{R})\cup([-a_U-2, a_U+2]\times(-\infty, -\frac{1}{2}])\cup([a_U+2, \infty)\times\mathbb{R})$. Let ξ' be the Hamiltonian flow of the function $(\eta\circ\pi)H$ defined on $\hat{E}$. After sufficient time, the flow ξ' isotopes V' to a new cobordism V'' that is included in $\hat{\pi}^{-1}(\mathbb{R}\times(-\infty, 0]\cup Q_U^-)$. Therefore, V'' is remote relative to E. Moreover, as the ends of V' are not moved by

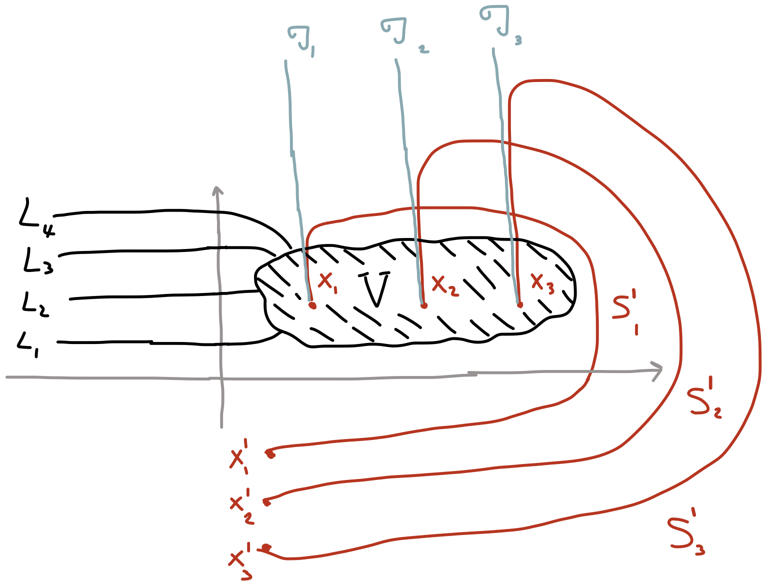

Fig. 17 The cobordism $V : \emptyset \rightsquigarrow (L_1, L_2, L_3, L_4)$, the Lagrangian spheres S'_1, S'_2, S'_3 together with the vertical thimbles $\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3$ so that $V'' = \tau_{S'_m} \circ \tau_{S'_{m-1}} \circ \cdots \circ \tau_{S'_1}(V)$ is disjoint from the $\mathscr{T}_i$'s

this isotopy, it is easy to see that, by a further truncation of ξ', V'' is hamiltonian isotopic to V' through a compactly supported isotopy. □

4.4.2 Multiple singularities

Consider a Lefschetz fibration $\pi : E \to \mathbb{C}$ as in Sect. 4.2, thus tame and possibly with more than one singularity.

We fix $V \in \mathcal{O}b(\mathcal{F}uk^*(E))$, $V : \emptyset \rightsquigarrow (L_1, \ldots, L_s)$. The purpose of this subsection is to describe an extension of Proposition 4.4.1 and Corollary 4.4.2 to the case of multiple singularities.

We will consider a fibration $\hat{\pi} : \hat{E} \to \mathbb{C}$ that extends E and has one more singularity x'_i for each singular point x_i, $1 \leq i \leq m$, of π so that the vanishing cycles of x_i and x'_i can be related by matching cycles $\hat{S}_i$ that are the analogues of the matching cycle $\hat{S}_\gamma$ from Proposition 4.4.1. The specific positioning of the corresponding critical values v'_i in the plane $\mathbb{C}$ is important and is as in Fig. 17. We then obtain Lagrangian spheres, S'_i that are hamiltonian isotopic to $\hat{S}_i$ (as in Fig. 17) and we then consider the image of V under the iterated Dehn twist

$$V' = \tau_{\hat{S}_m} \circ \tau_{\hat{S}_{m-1}} \circ \cdots \circ \tau_{\hat{S}_1}(V)$$

inside $\hat{E}$ as well as the following Hamiltonian isotopic copy of it $V'' = \tau_{S'_m} \circ \tau_{S'_{m-1}} \circ \cdots \circ \tau_{S'_1}(V)$ obtained by applying an iterated Dehn twist along the Lagrangian spheres S'_j which are Hamiltonian isotopic to the $\hat{S}_j$'s.

Let $\mathscr{T}_i$ be the vertical thimble with origin the critical point x_i and projecting to the vertical half-line $\{i\} \times [\frac{3}{2}, \infty)$. The thimbles $\mathscr{T}_i$ generalize the thimble T considered earlier (just before Proposition 4.4.1) in the context of one singularity to the case of multiple singularities. We denote them by $\mathscr{T}_i$ (this avoids confusion with the thimbles T_i that are horizontal at infinity and are associated to the curves t_i, see Fig. 7).

Corollary 4.4.3 *It is possible to construct $\hat{E}$ and the Lagrangian spheres S'_i so that the cobordism V'' is disjoint from all the thimbles $\mathscr{T}_i$. As a consequence, there exists a compactly supported Hamiltonian isotopy ϕ so that the cobordism $\phi(V'') \subset \hat{E}$ is remote relative to E. In particular, in $D\mathcal{F}uk^*(E)$, there exists a cone decomposition:*

$$V'_E \cong (\gamma_s \times L_s \to \gamma_{s-1} \times L_{s-1} \to \cdots \to \gamma_2 \times L_2).$$

Proof The first part of the proof is to construct iteratively fibrations $\hat{\pi}_i : \hat{E}_i \to \mathbb{C}$ with $\hat{E}_0 = E$ and with the final fibration $\hat{E} = \hat{E}_m$ so that $\hat{E}_{i+1}$ extends $\hat{E}_i$ and has one more singularity, x'_{i+1}, compared to $\hat{E}_i$. At each step we also construct the matching cycles $\hat{S}_i$ joining x_i to x'_i and their Hamiltonian isotopic images S'_i so that the relevant properties are satisfied. Here are more details on the induction step. Assume that $\hat{E}_k$ has already been constructed together with the matching cycles $\hat{S}_i$ and their hamiltonian isotopic copies S'_i, $1 \leq i \leq k$ so that $V''_k = \tau_{S'_k} \circ \tau_{S'_{k-1}} \circ \cdots \circ \tau_{S'_1}(V)$ is disjoint from $\mathscr{T}_i$, $1 \leq i \leq k$. We now consider the cobordism V''_k and the vertical thimble $\mathscr{T}_{k+1}$ and we apply to them the construction described in the Proof of Proposition 4.4.1. This produces first a new fibration $\hat{E}_{k+1}$ that has an additional singularity denoted now by x'_{k+1}. Here, the only difference with respect to the construction of $\hat{E}$ in Proposition 4.4.1 is that the coordinates of the critical value v'_{k+1} associated to x'_{k+1} is now $(-1, -k - \frac{3}{2})$ and the set $\hat{U}$, outside which $\hat{E}_{k+1}$ is tame, is extended appropriately inside the third-quadrant. Further, just as in the Proof of Proposition 4.4.1 we can construct the deformed thimble $\bar{\mathscr{T}}_{k+1}$ as well as the matching cycle $\hat{S}_\gamma$ so that $\hat{S}_\gamma$ coincides with $\bar{\mathscr{T}}_{k+1}$ over a certain sub-segment of γ. Two important points should be made here: first, the place of V in the Proof of Proposition 4.4.1 is taken here by V''_{k+1}; second $\mathscr{T}_{k+1}$ as well as $\bar{\mathscr{T}}_{k+1}$ and $\hat{S}_\gamma$ are all disjoint from $\mathscr{T}_i$ for $i \leq k$. Now, again as in the Proof of Proposition 4.4.1, we obtain that there exists a hamiltonian isotopy ψ_{k+1} supported outside a neighborhood of V''_{k+1} so that $S'_{k+1} = \psi_{k+1}(\hat{S}_\gamma)$ has the property that $V''_{k+1} = \tau_{S'_{k+1}} V''_k$ is disjoint from $\mathscr{T}_{k+1}$. One additional point appears here: it is easy to see that the isotopy ψ_{k+1} can be assumed to leave fixed $\mathscr{T}_i$ for $i \leq k$. By defining V''_{k+1} by using a sufficiently small neighborhood $U(S'_{k+1})$ of S'_{k+1} so that $U(S'_{k+1}) \cap \mathscr{T}_i = \emptyset$ for all $i \leq k$, we also deduce $V''_{k+1} \cap \mathscr{T}_i = \emptyset$ $1 \leq i \leq k$ and the induction step is completed.

We now put $V'' = V''_m$ and we know that V'' is disjoint from all the thimbles $\mathscr{T}_i$. Constructing the horizontal isotopy that transforms V'' into a cobordism V''' remote relative to E is a simple exercise by, possibly, iterating the construction in Corollary 4.4.2.

Finally, the cone-decomposition in the statement follows by applying to V''' Proposition 4.3.1. □

The following Lemma establishes monotonicity properties for $\hat{E}$ that will be used later on in Sect. 4.6 when proving Theorems 4.2.1 and A.

Lemma 4.4.4 *If the Lefschetz fibration $E \longrightarrow \mathbb{C}$ is strongly monotone (see Definition 3.3.1) then the extended fibration $\hat{E} \longrightarrow \mathbb{C}$ is strongly monotone too and has the same monotonicity class $*$. The matching spheres $\hat{S}_j \subset \hat{E}$ are monotone of class $*$ and if the cobordism $V \subset E$ is monotone of class $*$ then it continues to be monotone of the same class when viewed as a cobordism in $\hat{E}$.*

The proof is based on standard arguments and we omit it here. We refer to [3] for the proof.

Remark 4.4.5 The "doubling" of singularities used in Proposition 4.4.1 first appeared in a somewhat different form and with a different purpose in the work of Seidel [24]. It is likely that Proposition 4.4.1 can be proved also along an approach closer to Seidel's constructions involving bifibrations. The basic idea along this line would be to construct the fibration $\hat{E}$ by symmetrizing the restriction of the fibration E to the upper half-plane by a rotation σ by 180° around the origin in $\mathbb{C}$. This gives rise to a specific matching cycle that projects to a segment joining the singular value v_1 to its "mirror" v_1'. By restricting to a suitable disk D containing this segment, we see that the Dehn twist around this vanishing cycle is identified to the rotation σ (Lemma 18.2 in [24]). At the same time if V is assumed to be a Lagrangian without ends and included in D, then $\sigma(V)$ is remote. However, as V is in general more complicated this argument does not work without further adjustments and thus we preferred to give a direct geometric argument.

4.5 A cobordism viewpoint on Seidel's exact triangle

The last essential ingredient for the Proof of Theorem 4.2.1 is Seidel's exact triangle [23,24]. This exact triangle fits extremely well with the cobordism perspective and, in part because of this, we present a new proof for it that is based on the cobordism machinery. Another reason for presenting this argument is to explain why the Novikov ring $\mathcal{A}$ is required in the Proof of Theorem 4.2.1: this is precisely in establishing Seidel's exact triangle. We give all the main ideas and constructions involved in this proof, but, in the interest of brevity, we skip a number of details. These particular points are clearly indicated in the text and the arguments in question can all be found in [3].

Apart from Seidel's original proof [23] and the one presented here there is yet another proof of the same result, due to Mak–Wu [15]. The latter is very different than our proof, but interestingly it is also based on the theory of Lagrangian cobordism.

4.5.1 The exact triangle

Recall that we work with monotone Lagrangians of class $*$ and that Floer complexes and Fukaya categories are ungraded and with coefficients in the universal Novikov ring $\mathcal{A}$.

Let (X^{2n+2}, ω) be a symplectic manifold which is either closed or symplectically convex at infinity. Throughout this section we add the assumption that $\dim_{\mathbb{R}} X \geq 4$. (The reason for this restriction will be explained in Remark 4.5.2 below.) Let S a parametrized Lagrangian sphere in X, i.e. a Lagrangian submanifold $S \subset X$ together with a diffeomorphism $i_S : S^{n+1} \longrightarrow S$. Recall that we denote by $\tau_S : X \longrightarrow X$ the Dehn twist associated to S. Assume further that $S \subset X$ is monotone and denote by $*$ its monotonicity class. Following the conventions of the paper, we write $\mathcal{F}uk^*(X)$ for the Fukaya category of monotone closed Lagrangian submanifolds of X of monotonicity class $*$.

The following important result was proved by Seidel [23] in the exact case.

Proposition 4.5.1 *Let X, S be as above and let $Q \subset X$ be another monotone closed Lagrangian submanifold of monotonicity class $*$. Then in $D\mathcal{F}uk^*(X)$ there is an exact triangle of the form:*

$$\begin{tikzcd} \tau_S(Q) \arrow[r] & Q \arrow[d] \\ & S \otimes HF(S, Q) \arrow[ul] \end{tikzcd} \tag{21}$$

Remark 4.5.2 We restrict ourselves to $\dim_{\mathbb{R}} X \geq 4$ for the following reason. The proof uses an auxiliary Lefschetz fibration $\mathcal{E}$ with a single singularity and with general fiber X. Moreover, we will use a version of the Fukaya category of cobordisms in $\mathcal{E}$. For this to work we need $\mathcal{E}$ to be strongly monotone (see Definition 3.3.1). This easily follows from the monotonicity of X when $\dim_{\mathbb{R}} X \geq 4$. However, when $\dim_{\mathbb{R}} X = 2$ this might not be the case anymore. It seems plausible that this difficulty can be overcome (since in dimension 4, i.e. the dimension of $\mathcal{E}$, for a generic almost complex structure there are no holomorphic disks with non-positive Maslov numbers).

Proof of Proposition 4.5.1. The first step in the proof is to use general cobordism machinery to reduce the statement to Lemma 4.5.3 below. This is achieved as follows.

By the geometric interpretation of the monodromy around an isolated Lefschetz singularity—[1,23]—there exists a Lefschetz fibration $\pi : \mathcal{E} \to \mathbb{C}$ with a single singularity (chosen at the origin) and with general fiber X which is tame outside the region $\mathcal{U}$ as in Fig. 18 whose monodromy around the origin (when turning in trigonometric sense) is τ_S. Note that $\mathcal{E}$ is strongly monotone of class $*$. This follows immediately from the Definition 3.3.1 (recall we have assumed that $\dim_{\mathbb{R}} X \geq 4$).

Let $\gamma' \subset \mathbb{C}$ be the curve from Fig. 18. Similarly to [6] γ' gives rise to an inclusion functor

$$\mathcal{I}_{\gamma'} : \mathcal{F}uk^*(X) \longrightarrow \mathcal{F}uk^*(\mathcal{E})$$

whose action on objects is $\mathcal{I}_{\gamma'}(N) = \gamma' N$, where $\gamma' N \subset \mathcal{E}$ stands for the trail of N along the curve γ'. Here, by $\mathcal{F}uk^*(\mathcal{E})$ we mean the Fukaya category of cobordisms in $\mathcal{E}$ of monotonicity class $*$ but we do not require the cobordisms to be only negative-ended. This category is defined, following the recipe in [6] as described in Sect. 3.1, but by also using perturbations and bottlenecks associated to the positive ends. For the purpose of the proof below, it is actually enough to restrict to a subcategory whose objects are cobordisms in $\mathcal{E}$ that project to curves in $\mathbb{C}$.

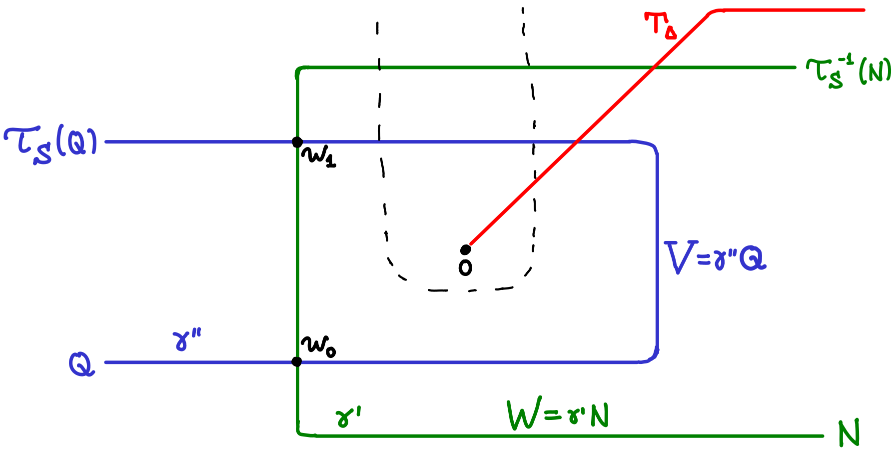

Fig. 18 The cobordisms V, W and T_Δ

View $W = \mathcal{I}_{\gamma'}N = \gamma' N$ as a cobordism in $\mathcal{E}$. Next, consider the curve $\gamma'' \subset \mathbb{C}$ as depicted in Fig. 18 and fix a base point $w_0 \in \gamma'' \cap (\mathbb{C}\setminus\mathcal{U})$. Define $V \subset (\mathcal{E}, \Omega)$ to be the Lagrangian submanifold obtained as the trail of $Q \subset \mathcal{E}_{w_0} = X$ along γ''. Clearly both V and W are monotone and by standard arguments (see [7] and also [5, Remark 2.2.4]) we have $d_V = d_Q$ and $d_W = d_N$. It follows that both V and W are monotone of class $*$ hence are legitimate objects of the Fukaya category $\mathcal{F}uk^*(\mathcal{E})$ as considered in this section.

Since the fibration $(\mathcal{E}, \Omega)$ is symplectically trivial over $\mathcal{W} = \mathbb{C}\setminus\mathcal{U}$, the lower end of V is identified with Q and due to the homotopy class of γ'' (in $(\mathbb{C}\setminus\{0\},\ \text{rel}\ \infty)$) the upper end of V is a Lagrangian submanifold of X which is Hamiltonian isotopic to $\tau_S(Q)$. Similarly, the lower end of W is cylindrical over N and the upper end is cylindrical over $\tau_S^{-1}(N)$.

Denote by $\mathcal{Y}_X : \mathcal{F}uk^*(X) \longrightarrow mod(\mathcal{F}uk^*(X))$ and $\mathcal{Y} : \mathcal{F}uk^*(\mathcal{E}) \longrightarrow mod(\mathcal{F}uk^*(\mathcal{E}))$ the Yoneda embeddings associated to the Fukaya categories of X and $\mathcal{E}$ respectively. When no confusion may arise we will simplify the notation and denote the module $\mathcal{Y}_X(L)$ associated to a Lagrangian $L \subset X$ simply by L and similarly for Lagrangians in $\mathcal{E}$.

We now analyze the pullback module $\mathcal{I}_{\gamma'}^* V \in mod(\mathcal{F}uk^*(X))$. Similar arguments to §4.4 [6] (see also Sect. 4.3 in this paper, in particular the exact sequence at Step 3i on page 30) show that we have a quasi-isomorphism:

$$\mathcal{I}_{\gamma'}^* V \simeq \text{cone}\big(\tau_S(Q) \xrightarrow{\varphi} Q\big), \tag{22}$$

for some homomorphism of A_∞-modules φ that is induced by counting holomorphic strips (and polygons) going from the intersection of V with W at the $\tau_S(Q)$ end to the intersection of V and W at the Q end—see Fig. 18.

Let $T_\Delta \subset \mathcal{E}$ be the thimble corresponding to the “diagonal” curve Δ depicted in Fig. 18. This thimble is monontone of class $(*)$ and we view it as an object of $\mathcal{F}uk^*(\mathcal{E})$

and consider the $\mathcal{F}uk^*(\mathcal{E})$-module

$$\mathcal{M} = T_\Delta \otimes CF(T_\Delta, V), \tag{23}$$

where the second factor in the tensor product is regarded as a chain complex (see Chapter 3c in [24]).

The A_∞-operations μ_k, $k \geq 2$, induce a homomorphism of modules $\mathcal{M} \longrightarrow V$. Pulling back by $\mathcal{I}_{\gamma'}$, this homomorphism induces a homomorphism of $\mathcal{F}uk^*(X)$-modules:

$$\nu : \mathcal{I}^*_{\gamma'}\mathcal{M} \longrightarrow \mathcal{I}^*_{\gamma'}V. \tag{24}$$

We claim that Proposition 4.5.1 reduces to the next statement:

Lemma 4.5.3 *The homomorphism ν is a quasi-isomorphism.*

This is due to the following quasi-isomorphisms:

$$\mathcal{I}^*_{\gamma'}\mathcal{M} = \mathcal{I}^*_{\gamma'}T_\Delta \otimes CF(T_\Delta, V) \simeq S \otimes CF(S, Q). \tag{25}$$

Here we identify S and its image under the Yoneda embedding.

Outline of the Proof of Lemma 4.5.3 We will discuss the main ideas in this proof and omit some technicalities. We refer to [3] for full details.

By the general theory of A_∞-categories, in order to prove Lemma 4.5.3 it is enough to show that for every Lagrangian $N \in \mathcal{O}b(\mathcal{F}uk^*(X))$ the map

$$\mu_2 : CF(\gamma' N, T_\Delta) \otimes CF(T_\Delta, V) \longrightarrow CF(\gamma' N, V) \tag{26}$$

is a quasi-isomorphism. Recall that $\gamma' N$ stands for the trail of N along γ'. We fix such an N and we again denote $W = \gamma' N$.

The first step in the proof is geometric and is based on the well-known fact that the function $\mathrm{Re}(\pi) : \mathcal{E} \to \mathbb{R}$ is Morse with a single singularity at the origin and that its gradient with respect to the standard metric is Hamiltonian. Denote by ϕ_t the negative gradient flow of this function. The positive horizontal thimble originating at 0 is the stable manifold of $\mathrm{Re}(\pi)$ and the negative horizontal thimble is the unstable manifold of $\mathrm{Re}(\pi)$. To start this stage in the proof, we use the flow ϕ_t^{-1} to push W to the right in picture Fig. 18 thus getting $\widetilde{W}$; similarly, we use the flow ϕ_t to push V to the left in the same picture thus getting $\widetilde{V}$—see Fig. 19. It is easy to see that these isotopies can be assumed to be horizontal.

Consider now the thimble T_Δ as in Fig. 19.

Due to the invariance of the respective Floer homologies with respect to horizontal Hamiltonian isotopies, we have the quasi-isomorphisms:

$$CF(\widetilde{W}, T_\Delta) \simeq CF(W, T_\Delta) = CF(N, S),$$
$$CF(T_\Delta, \widetilde{V}) \simeq CF(T_\Delta, V) = CF(S, Q),$$

and

$$CF(\widetilde{W}, \widetilde{V}) \simeq CF(W, V).$$

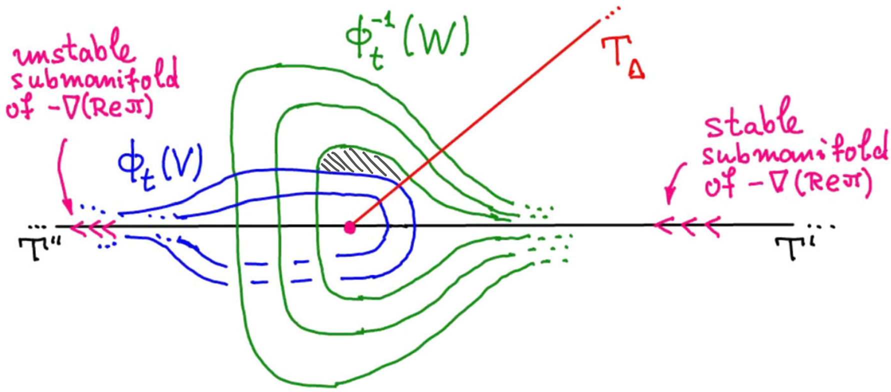

Fig. 19 The cobordisms V, W after the flows ϕ_t and ϕ_t^{-1} are applied to them for large time t together with one "short" triangle in *gray*

Therefore, to show the statement it is sufficient to prove that:

$$\mu_2 : CF(\widetilde{W}, T_\Delta) \otimes CF(T_\Delta, \widetilde{V}) \to CF(\widetilde{W}, \widetilde{V}) \quad \text{is an isomorphism.} \tag{27}$$

To show this we start by analyzing the complex $CF(\widetilde{W}, \widetilde{V})$. Assuming all relevant intersections are generic, by standard Morse theory, if W is pushed enough to the right, $\widetilde{W}$ intersects a neighborhood $\mathcal{N}$ around the singularity in a number n_1 of copies $\widetilde{W}_i$ of the stable manifold of $\mathrm{Re}(\pi)$. Here n_1 is equal to the number of intersections of W with the unstable manifold of $\mathrm{Re}(\pi)$. Similarly, $\widetilde{V}$ intersects $\mathcal{N}$ in n_2 copies $\widetilde{V}_j$ of the unstable manifold of $\mathrm{Re}(\pi)$ and n_2 is equal to the number of intersections of V with the stable manifold of $\mathrm{Re}(\pi)$. The interpretation of the stable and unstable manifolds as thimbles (and our transversality assumptions) immediately imply that n_1 equals the number of intersection points $N \cap S$ and n_2 is the number of intersections $S \cap Q$. Moreover, each $\widetilde{W}_i$ intersects precisely once each $\widetilde{V}_j$ in a point denoted z_{ij}. Similarly, we analyze the intersections $\widetilde{W} \cap T_\Delta$ and $T_\Delta \cap \widetilde{V}$. Each point $x_i \in \widetilde{W} \cap T_\Delta$ is given as the intersection of one W_i with T_Δ, $x_i = \widetilde{W}_i \cap T_\Delta$. Similarly, each point $y_j \in \widetilde{V} \cap T_\Delta$ is the intersection of one V_j with T_Δ. Thus, we have an isomorphism of vector spaces

$$\chi : CF(\widetilde{W}, T_\Delta) \otimes CF(T_\Delta, \widetilde{V}) \to CF(\widetilde{W}, \widetilde{V})$$

induced by $(x_i, y_j) \to z_{ij}$.

The proof ends by analyzing the multiplication μ_2 in (27). When W is pushed enough to the right and V is pushed enough to the left and $\mathcal{N}$ is small enough, then μ_2 decomposes as a sum of two parts. The first part is given by "short" triangles that are completely contained inside $\mathcal{N}$ and that relate the intersection points of the Lagrangians $\widetilde{W}_i$, T_Δ, $\widetilde{V}_j$ precisely through the map χ. The projection to $\mathbb{C}$ of the "short" triangles can be seen in Fig. 19. The second part consists of "long" triangles, that go out of $\mathcal{N}$. It is then shown that there is a uniform bound $\delta > 0$ so that all short triangles are of energy lower than δ and all long triangles are of energy at least 2δ (both steps, while intuitively clear, require work—see [3]). As a consequence, because we work over $\mathcal{A}$ and the dominant term in μ_2 is χ which is an isomorphism, we deduce that the product μ_2 is an isomorphism too. □

Remark 4.5.4 The maps that appear in the exact triangle are explicitly identified in the proof. The first map is φ from (22) ; the second map is the composition $Q \to \text{cone}(\varphi) \cong S \otimes HF(S, Q)$ with $Q \to \text{cone}(\varphi)$ given by the inclusion in the cone and the isomorphism $\text{cone}(\varphi) \cong S \otimes HF(S, Q)$ given by Lemma 4.5.3; the third map in the exact triangle is given by the composition $S \otimes HF(S, Q) \cong \text{cone}(\varphi) \to \tau_S Q$. It is not difficult to see that these morphisms coincide with the corresponding morphisms in Seidel's exact triangle.

4.5.2 Second version of the exact triangle: the case when X is a Lefschetz fibration

Here we assume that X is the total space of a tame Lefschetz fibration $\pi_X^{2n+2} : X \longrightarrow \mathbb{C}$, $n \geq 1$, as defined in Sect. 2. We denote by $\mathcal{F}uk^*(X)$ the Fukaya category of X whose objects are negative-ended *Lagrangian cobordisms* in X of monotonicity class $*$ as defined in Sect. 3.1.

Proposition 4.5.5 *For X as above, let $S \subset X$ be a monotone Lagrangian sphere of class $*$ and let $Q \subset X$ be a monotone Lagrangian cobordism (possibly without ends) of the same monotonicity class. Then in $D\mathcal{F}uk^*(X)$ there is an exact triangle as in* (21).

The proof is very similar to the Proof of Proposition 4.5.1, the only difference being that now Q is allowed to be a cobordism rather than just a closed Lagrangian (and similarly for the objects of $\mathcal{F}uk^*(X)$). There are a variety of technicalities to deal with, mainly related to the interaction of the perturbation data and the condition T_∞. We refer to [3] for the proof of this Proposition.

4.6 The decomposition in Theorem A

To construct this decomposition we start with the Proof of Theorem 4.2.1.

4.6.1 Proof of Theorem 4.2.1

Assume for the moment that we are in the setting of Sect. 4.2. Thus, $\pi : E \to \mathbb{C}$ is a tame, strongly monotone Lefschetz fibration with the properties listed there.

Let $V : \emptyset \rightsquigarrow (L_1, \ldots, L_s)$ and consider the Lefschetz fibration $\hat{\pi} : \hat{E} \to \mathbb{C}$ obtained from E by adding singularities as described in Sect. 4.4.2. By Lemma 4.4.4, $\hat{E}$ is strongly monotone. The cobordism V continues to be monotone in $\hat{E}$ and the matching spheres $\hat{S}_j$ are monotone too. Moreover, all these Lagrangians are of monotonicity class $*$. Recall also that by assumption $\dim_{\mathbb{R}} E \geq 4$. Consider now the cobordism

$$V' = \tau_{\hat{S}_m} \circ \tau_{\hat{S}_{m-1}} \circ \cdots \circ \tau_{\hat{S}_1}(V) \subset \hat{E}.$$

Given $W \in \mathcal{L}^*(\hat{E})$ we rewrite the exact triangle in Proposition 4.5.5 as

$$W = (S \otimes HF(S, W) \to \tau_S W)$$

and deduce that in $D\mathcal{F}uk^*(\hat{E})$ we have the following decomposition of V:

$$V \cong \left(\hat{S}_1 \otimes E_1 \to \hat{S}_2 \otimes E_2 \to \cdots \to \hat{S}_m \otimes E_m \to V'\right),$$

where

$$E_i = HF\left(\hat{S}_i, \tau_{\hat{S}_{i-1}} \circ \cdots \circ \tau_{\hat{S}_1}(V)\right). \tag{28}$$

Notice that in $D\mathcal{F}uk^*(E)$ we have $T_i \cong (\mathrm{Incl}^{E,\hat{E}})^*(\hat{S}_i)$ where $\mathrm{Incl}^{E,\hat{E}}$ is the inclusion (13) and T_i are the thimbles in the statement of Theorem 4.2.1. Thus, in $D\mathcal{F}uk^*(E)$ we have the decomposition:

$$V \cong (T_1 \otimes E_1 \to T_2 \otimes E_2 \to \cdots \to T_m \otimes E_m \to V'). \tag{29}$$

By Corollary 4.4.3 we know that inside $D\mathcal{F}uk^*(E)$ we have:

$$V' \cong (\gamma_s \times L_s \to \gamma_{s-1} \times L_{s-1} \to \cdots \to \gamma_2 \times L_2) \tag{30}$$

Splicing together (29) and (30) we obtain:

$$V \cong (T_1 \otimes E_1 \to \cdots \to T_m \otimes E_m \to \gamma_s \times L_s \to \cdots \to \gamma_2 \times L_2)$$

which concludes the Proof of Theorem 4.2.1. □

4.6.2 The decomposition in Theorem A

We assume the setting from Theorem 4.1.1 (which we recall is just a more precise reformulation of Theorem A) and recall a bit of the necessary background. The fibration $\pi : E \to \mathbb{C}$ is no longer assumed to be tame but continues to be strongly monotone. All the singularities of π are included in $\pi^{-1}(S_{x,y})$, $x < 0 < y$ and there is a tame fibration $\pi : E_\tau \to \mathbb{C}$ that coincides with E over $[x-4, y+4] \times [-\frac{1}{2}, \infty)$ and is tame outside of a set U that contains $[x-4, y+4] \times (-1, \infty)$. Recall also the category $\mathcal{F}uk^*(E_\tau)$ whose objects are cobordisms (with only negative ends) as in Definition 2.2.3. In particular, these cobordisms have ends that project to the axes $(-\infty, -a_U] \times \{i\} \subset \mathbb{C}$. The constant a_U satisfies $-a_U < x - 4$. Recall from Sect. 3.2 that the objects of the category $\mathcal{F}uk^*(E; \tau)$ are uniformly monotone cobordisms $V \subset E$ that are cylindrical outside $S_{x-3,y-3}$ and the operations μ_k of $\mathcal{F}uk^*(E; \tau)$ are defined by means of the corresponding operations in the category $\mathcal{F}uk^*(E_\tau)$ associated to the tame fibration E_τ.

The decomposition in Theorem 4.1.1 (and thus that in Theorem A) follows rapidly from that in Theorem 4.2.1. Indeed, recall from Sect. 3.2 that we have an inclusion:

$$\mathcal{F}uk^*(E; \tau) \to \mathcal{F}uk^*(E_\tau) \tag{31}$$

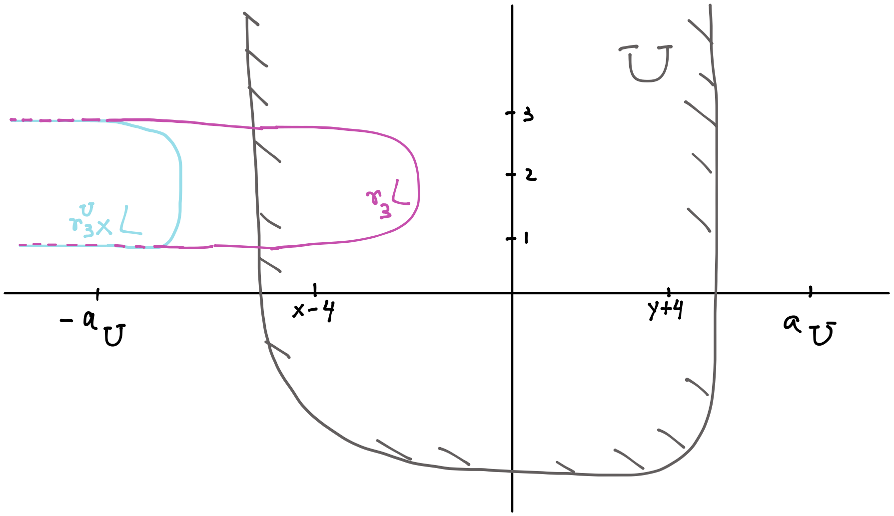

Fig. 20 The Lagrangian $\gamma_3^U \times L$ is an object in $\mathcal{F}uk^*(E_\tau)$ but is not cylindrical outside of $[x-3, y+3] \times \mathbb{R}$ and thus it not an object in $\mathcal{F}uk^*(E, \tau)$

that is a quasi-equivalence and which, on objects, is defined by $V \to \overline{V}$ where $\overline{V}$ is obtained by cutting off the ends of V along the line $\{x - \frac{7}{2}\} \times \mathbb{R}$ and extending them horizontally by parallel transport in the fibration E_τ. As E_τ is a tame fibration, Theorem 4.2.1 can be applied to it. We deduce decompositions involving two types of curves in the plane, the t_k's and γ_i's as in Fig. 7. The curves γ_i appearing here are included in $(-\infty, -a_U + 3] \times [0, \infty)$ and they are away from U. For reasons that will become clear in a moment, it is convenient to refine the notation for these curves such as to explicitly indicate their dependence on U. Thus we will further denote them by γ_i^U.

The decomposition result that we want to show here—for the statement of Theorem 4.1.1—applies to $\mathcal{F}uk^*(E; \tau)$. It again involves the same thimbles T_k associated to the curves t_k as before as well as certain "trails" denoted in Theorem 4.1.1 by $\gamma_i L_i$. It is important to notice at this point that the curves γ_i appearing in the statement of Theorem 4.1.1 do not coincide with the γ_i^U's above—see also Fig. 20. Indeed, following the definition in Sect. 4.1.1, these curves have image inside $(-\infty, x) \times [\frac{1}{2}, \infty)$ and they "bend" inside $[x-2, x-1] \times [1, \infty]$, while γ_i^U is away from U and thus away from $(x-4, y+4) \times \mathbb{R}$.

Nonetheless, for $L \in \mathcal{L}^*(M)$ and any curve γ_i consider the cobordism $\overline{\gamma_i L}$ as an object of $\mathcal{F}uk^*(E_\tau)$. This object is quasi-isomorphic to $\gamma_i^U \times L$ (this can be proved directly, but also follows easily from Theorem 4.2.1 itself). As a consequence, we may replace in the decomposition given by Theorem 4.2.1 the objects $\gamma_i^U \times L_i$ by the objects $\overline{\gamma_i L_i}$ and by pulling back the resulting decomposition from $\mathcal{F}uk^*(E_\tau)$ to $\mathcal{F}uk^*(E; \tau)$ via the inclusion (31) we obtain the decomposition claimed in Theorem 4.1.1. □

5 Some consequences

We assume in this section that $\pi : E \to \mathbb{C}$ is a Lefschetz fibration which is tame outside of $U \subset \mathbb{C}$ and is strongly monotone. Let (M, ω) be the generic fiber. The fibration E

has singularities $x_1, \dots, x_m$ of respective critical values $v_1, \dots, v_m$ (assumed to be, for simplicity, $v_k = (k, \frac{3}{2})$). Denote by $O \in \mathbb{C}$ the origin. We will also assume that $O \notin U$. Connect each critical value v_k to O by a straight segment, and denote by $S_k \in \pi^{-1}(O) = M$ the vanishing cycle associated to that path.

5.1 Descent: from decompositions in $D\mathcal{F}uk^*(E)$ to decompositions in $D\mathcal{F}uk^*(M)$

Corollary 5.1.1 *As in Theorem 4.2.1, let $V \in \mathcal{L}^*(E)$, $V : \emptyset \to (L_1, \dots, L_s)$. Then there exists an iterated cone decomposition that depends on V and takes place in $D\mathcal{F}uk^*(M)$:*

$$\begin{aligned} L_1 \cong \big(&\widetilde{\tau}_{2,\dots,m}^{-1} S_1 \otimes E_1 \to \widetilde{\tau}_{3,\dots,m}^{-1} S_2 \otimes E_2 \to \cdots \\ &\to \widetilde{\tau}_{i+1,\dots,m}^{-1} S_i \otimes E_i \to \cdots \to S_m \otimes E_m \to L_s \to L_{s-1} \to \cdots \to L_2\big), \end{aligned} \tag{32}$$

where $\widetilde{\tau}_{i,\dots,m}$ stands for the composition:

$$\widetilde{\tau}_{i,\dots,m} = \tau_{S_i} \circ \tau_{S_{i+1}} \circ \cdots \circ \tau_{S_m}.$$

Proof In this proof it is convenient to consider again the category $D\mathcal{F}uk^*_{\frac{1}{2}}(E)$ from Sect. 4.3. Recall that the difference between this category and $D\mathcal{F}uk^*(E)$ is that the objects V of the underlying category $\mathcal{F}uk^*_{\frac{1}{2}}(E)$ are more general cobordisms than those given in Definition 2.2.3 in that the y-coordinates of the ends of V are allowed to be in $\frac{1}{2}\mathbb{Z}$. In other words, V has only negative ends and

$$V \cap \pi^{-1}(Q_U^-) = \coprod_i \left((-\infty, -a_U] \times \frac{i}{2} \right) \times L_i.$$

We now consider curves η_i as in Fig. 21.

These curves satisfy

$$\begin{aligned} \eta_i((-\infty, -1]) &= (-\infty, -a_U - 2] \times \frac{2i-1}{2}, \\ \eta_i([1, +\infty)) &= (-\infty, -a_U - 2] \times \frac{2i+1}{2} \end{aligned}$$

and $\eta_i(\mathbb{R}) \subset Q_U^-$.

As shown in [6] §4 there exists an A_∞-functor:

$$i^{\eta_j} : \mathcal{F}uk^*(M) \to \mathcal{F}uk^*_{\frac{1}{2}}(E)$$

which acts on objects by $L \longmapsto \eta_j \times L$. Consider now the pull-back functor:

$$(i^{\eta_j})^* : mod(\mathcal{F}uk^*_{\frac{1}{2}}(E)) \to mod(\mathcal{F}uk^*(M)).$$

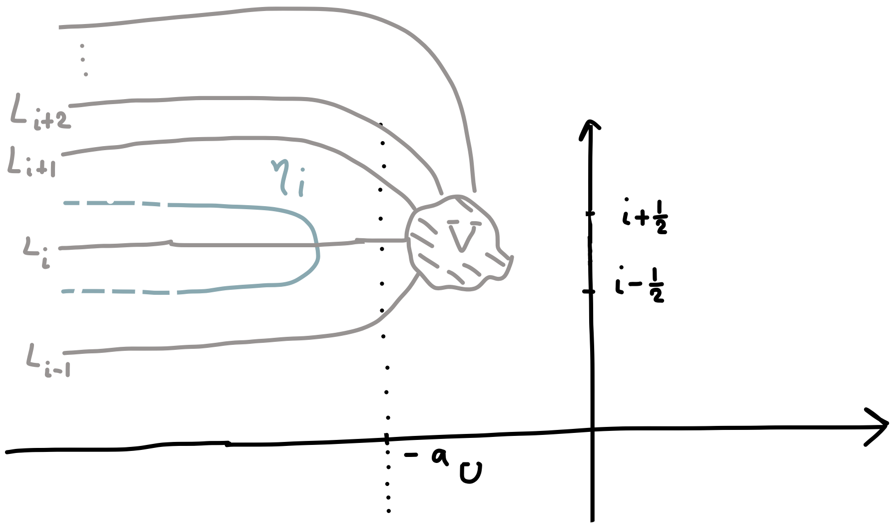

Fig. 21 The auxiliary curves η_i together with the cobordism $V \in \mathcal{L}^*(E)$

Notice that there is a full and faithful embedding $e : \mathcal{F}uk^*(E) \to \mathcal{F}uk^*_{\frac{1}{2}}(E)$. Consider the Yoneda embeddings $\mathcal{Y} : \mathcal{F}uk^*(E) \to mod(\mathcal{F}uk^*(E))$ and $\mathcal{Y}_{\frac{1}{2}} : \mathcal{F}uk^*_{\frac{1}{2}}(E) \to mod(\mathcal{F}uk^*_{\frac{1}{2}}(E))$. Let $\mathcal{Y}' : \mathcal{F}uk^*(E) \to mod(\mathcal{F}uk^*_{\frac{1}{2}}(E))$ be $\mathcal{Y}' = \mathcal{Y}_{\frac{1}{2}} \circ e$. The homology category associated to the triangular completion $(Image(\mathcal{Y}'))^\wedge$ of the image of $\mathcal{Y}'$ inside $mod(\mathcal{F}uk^*_{\frac{1}{2}}(E))$ is easily seen to be quasi-equivalent to $D\mathcal{F}uk^*(E)$.

For an object $V \in \mathcal{F}uk^*(E)$ let $\mathcal{M}'_V = \mathcal{Y}'(V)$. Notice that $(i^{\eta_j})^*(\mathcal{M}'_V)$ is precisely the Yoneda module associated to the j-end of V. Thus i^{η_j} takes Yoneda modules to Yoneda modules and given that $H(Image(\mathcal{Y}')^\wedge) = D\mathcal{F}uk^*(E)$ we deduce that the functor $(i^{\eta_j})^*$ induces a triangulated functor

$$\mathcal{R}_j : D\mathcal{F}uk^*(E) \to D\mathcal{F}uk^*(M) \tag{33}$$

that we will refer to as the restriction to the jth end.

The decomposition in the statement is obtained by applying $\mathcal{R}_1$ to the decomposition in Theorem 4.2.1. Symplectic Picard–Lefschetz theory shows that the end of the thimble T_k is Hamiltonian isotopic to $(\tau^{-1}_{S_m} \circ \tau^{-1}_{S_{m-1}} \circ \tau^{-1}_{S_{k+1}})(S_k) = \widetilde{\tau}^{-1}_{k+1,\ldots,m} S_k$ and its projection to $\mathbb{C}$ has y-coordinate 1. Clearly, the end of $\gamma_k \times L_k$ over $y = 1$ is L_k for $k \geq 2$ and, similarly, the end of V over $y = 1$ is L_1. $\square$

Remark 5.1.2 The functor $\mathcal{R}_j$ from (33) can also be interpreted in a different fashion. We can view it as the triangulated functor induced by an A_∞-functor $\widetilde{\mathcal{R}}_j : \mathcal{F}uk^*(E) \to \mathcal{F}uk^*(M)$ that, on objects, associates to each cobordism $V : \emptyset \rightsquigarrow (L_1, \ldots, L_s)$ its jth end, L_j. It is not difficult to see that, with appropriate choices of auxiliary structures, such a functor is indeed defined and that it induces at the derived level precisely $\mathcal{R}_j$.

At the derived level we also have $\mathcal{R}_j \circ i^{\eta_j} = \mathrm{id}$. Notice also that the pull-back functor

$$\widetilde{\mathcal{R}}_j^* : mod(\mathcal{F}uk^*(M)) \to mod(\mathcal{F}uk^*(E))$$

takes the Yoneda module $\mathcal{Y}(L)$ to the Yoneda module $\mathcal{Y}(\eta_j \times L) = i^{\eta_j}(L)$.

Remark 5.1.3 Let $V' \subset (\mathbb{C} \times M, \omega_{\mathbb{C}} \oplus \omega)$ be a negative-ended cobordism in a trivial fibration with ends $L_1, \ldots, L_k$. Theorem A and its Corollary 5.1.1 associate to V' a cone decomposition

$$L_1 \cong (L_k \to L_{k-1} \to \cdots \to L_2). \tag{34}$$

At the same time, the results of [6] imply another cone decomposition of the zero module (associated to the empty right-hand end) : $0 \cong (L_k \to L_{k-1} \to \cdots \to L_2 \to L_1)$. This isomorphism is equivalent to an isomorphism $L_1 \cong (L_k \to L_{k-1} \to \cdots \to L_2)$. We remark that the latter cone decompositions of L_1 in fact coincides with the one in (34). This can be easily proved by following the definitions of the modules $W'_{E,i}$ that are introduced at the Step 3 of the Proof of Proposition 4.3.1 and comparing them with the modules $\overline{\mathcal{M}}_i$ from [6, §4.4.2]. See the expanded version of this paper [3, §6.2] for more details.

5.2 Ascent: from $D\mathcal{F}uk^*(M)$ to the category $D\mathcal{F}uk^*(E)$

We start with some algebraic notation. Let $\mathcal{B}$ be an A_∞-category (over a given ring $\mathcal{A}$, e.g. the Novikov ring) and $R_1, \ldots R_m$ a collection of m objects of $\mathcal{B}$. The following construction is a straightforward extension of the notion of directed A_∞-category as it appears in [24] (see, in particular, (5m) there).

Consider the ordered set $I_m = (\alpha_1, \ldots, \alpha_m)$ (where the α_i's stand for distinct formal characters) and let $\mathbb{N}_{+m}$ be the disjoint union $\mathbb{N} \cup I_m$ ordered strictly in a way that respects the orders of $\mathbb{N}$ and of I_m and so that each element in I_m is strictly bigger than any element of $\mathbb{N}$. We still denote the resulting order relation by $\geq$. For any two $i, j \in \mathbb{N}_{+m}$ we put $\xi^{i,j} = 1$ if $i \geq j$ and $\xi^{i,j} = 0$ if $i < j$ and we let $\xi^{i_1,i_2,\ldots,i_{k+1}} = \xi^{i_1,i_2}\xi^{i_2,i_3}\ldots\xi^{i_k,i_{k+1}}$.

We denote by $\mathbb{N}_{+m} \otimes \mathcal{B}$ the unique A_∞-category with the properties:

i. The objects of $\mathbb{N}_{+m} \otimes \mathcal{B}$ are couples (i, L) with $i \in \mathbb{N}_{+m}$ and L an object of $\mathcal{B}$ with the constraint that if $i \in I_m$, then $L = R_i$. We will write the couples (i, L) as $i \times L$.
ii. The morphisms of $\mathbb{N}_{+m} \otimes \mathcal{B}$ are defined by:

$$\mathrm{Mor}(i \times L, j \times L') = \xi^{i,j}\mathrm{Mor}_{\mathcal{B}}(L, L')$$

except if $i = j \in I_m$. In this case $\mathrm{Mor}(i \times R_i, i \times R_i) = \mathcal{A}e_{R_i}$. Here e_{R_i} is, by definition, a strict unit in the category $\mathbb{N}_{+m} \otimes B$.
iii. We denote by

$$\mu_k : \mathrm{Mor}(L_1, L_2) \otimes \mathrm{Mor}(L_2, L_3) \otimes \cdots \otimes \mathrm{Mor}(L_k, L_{k+1}) \to \mathrm{Mor}(L_1, L_{k+1})$$

the multiplications in $\mathcal{B}$. Consider successive indices $(i_1, i_2, \ldots, i_{k+1})$ so that no two successive indexes i_r, i_{r+1} satisfy $i_r = i_{r+1} \in I_m$. Then the multiplications in $\mathbb{N}_{+m} \otimes \mathcal{B}$ are given by:

$$\begin{aligned} \mu'_k : &\mathrm{Mor}(i_1 \times L_1, i_2 \times L_2) \otimes \mathrm{Mor}(i_2 \times L_2, i_3 \times L_3) \\ &\otimes \cdots \otimes \mathrm{Mor}(i_k \times L_k, i_{k+1} \times L_{k+1}) \\ &\to \ \mathrm{Mor}(i_1 \times L_1, i_{k+1} \times L_{k+1}) \\ \mu'_k &= \xi^{i_1,\ldots,i_{k+1}} \mu_k. \end{aligned} \tag{35}$$

In case for some index r we have $i_r = i_{r+1} \in I_m$, then μ'_k is completely described by the requirement that e_{R_i} is a strict unit: μ'_k vanishes if $k \neq 2$ and $\mu'_2(a, e_{R_i}) = a$, $\mu'_2(e_{R_i}, b) = b$.

The notation $\mathbb{N}_{+m} \otimes \mathcal{B}$ is slightly imprecise as this category actually depends on the choice of objects $R_1, \ldots, R_m$. Moreover, there is obviously an abuse of notation here as $\mathbb{N}_{+m} \otimes \mathcal{B}$ is not a tensor product (there is no addition among the objects etc).

In case the A_∞-category $\mathcal{B}$ is such that the objects R_i have strict units $e'_{R_i} \in \mathrm{Mor}_{\mathcal{B}}(R_i, R_i)$, then by taking $e_{R_i} = e'_{R_i}$, Eq. (35) applies without treating separately the case $i_r = i_{r+1} \in I_m$. In general, when the R_i's do not have strict units, we view the e_{R_i}'s as formal elements which are part of the construction of $\mathbb{N}_{+m} \otimes \mathcal{B}$.

Corollary 5.2.1 *Let $E \longrightarrow \mathbb{C}$ be a Lefschetz fibration with m critical values and generic fiber M. Then there exist Lagrangians spheres $R_1, \ldots, R_m \in \mathcal{L}^*(M)$ and an equivalence of categories:*

$$\mathcal{I} : D(\mathbb{N}_{+m} \otimes \mathcal{F}uk^*(M)) \to D\mathcal{F}uk^*(E).$$

Proof Consider the full and faithful subcategory $\mathcal{F}(E)$ of $\mathcal{F}uk^*(E)$ whose objects consist of the following two collections:

i. $\gamma_{i+2} \times L$ with $i \in \mathbb{N}$ and $L \in \mathcal{L}^*(M)$. Here γ_k, $k \geq 2$, are the plane curves defined in Sect. 4.1.1 (see also Fig. 7).
ii. the thimbles T_j, $j \in I_m$.

The generation Theorem 4.2.1 combined with the algebraic Lemma 3.34 in [24] implies that there is an equivalence of categories

$$D\mathcal{F}(E) \to D\mathcal{F}uk^*(E)$$

induced by the inclusion

$$\mathcal{F}(E) \to \mathcal{F}uk^*(E).$$

We will now show the existence of a quasi-equivalence of A_∞-categories:

$$\Xi : \mathbb{N}_{+m} \otimes \mathcal{F}uk^*(M) \to \mathcal{F}(E).$$

To this end we first pick a specific family of objects $R_1, \ldots, R_m$ in $\mathcal{F}uk^*(M)$. By definition, these objects are the following Lagrangian spheres:

$$R_{m+1-i} := \widetilde{\tau}^{-1}_{i+1,\ldots,m}(S_i), \quad i = 1, \ldots, m$$

—see Corollary 5.1.1 for the notation. For $i \in \mathbb{N}$, and $L \in \mathcal{L}^*(M)$, we define $\Xi'(i \times L) = \gamma_{i+2} \times L$. For $i \in I_m$ we define $\Xi'(i \times R_i) = T_{m+1-i}$.

It is not difficult to see—as in the construction of the inclusion functor $\mathcal{I}_{\gamma,h}$ in [6], in particular Proposition 4.2.3 there—that by using appropriate choices for the curves γ_i as well as almost complex structures and perturbation data, we can describe the morphisms and higher products in $\mathcal{F}(E)$ by the formulas corresponding to $\mathbb{N}_{+m} \otimes \mathcal{F}uk^*(M)$. There is however one exception concerning this correspondence and due to it the map Ξ' can not be assumed directly to be a morphism of A_∞ categories: the difficulty comes from the fact that the objects T_j of $\mathcal{F}(E)$ do not, in general, have strict units. However, there is an algebraic argument—Lemma 5.20 in §(5n) in [24]—that applies also to our case with minor modifications and implies that we can replace Ξ' by a true A_∞ functor: $\Xi : \mathbb{N}_{+m} \otimes \mathcal{F}uk^*(M) \to \mathcal{F}(E)$ that acts on objects in the same way as Ξ' and so that Ξ is a quasi-equivalence. Clearly, this implies the equivalence of the associated derived categories and the existence of $\mathcal{I}$. □

Remark 5.2.2 a. Corollary 5.2.1 extends a result of Seidel [24, §18] (see also [25]) which provides a similar description for the subcategory of $D\mathcal{F}uk^*(E)$ which is generated by the thimbles T_i.

b. It is easy to see by direct calculation that there are inclusions $\mathcal{J}_s : D\mathcal{F}uk^*(M) \to D(\mathbb{N}_{+m} \otimes \mathcal{F}uk^*(M))$ induced by $L \to (s, L)$ for all $s \in \mathbb{N}$. The compositions $\mathcal{J}'_s = \mathcal{I} \circ \mathcal{J}_s$ have a simple geometric interpretation. Consider the inclusion $i^{\gamma_{s+2}} : \mathcal{F}uk^*(M) \to \mathcal{F}uk^*(E)$ which acts on objects as $L \to \gamma_{s+2} \times L$. This induces a functor $i^{\gamma_{s+2}} : D\mathcal{F}uk^*(M) \to D\mathcal{F}uk^*(E)$ that coincides with $\mathcal{J}'_s$.

c. An obvious by-product of Corollary 5.2.1 is that the derived categories $D\mathcal{F}uk^*(E; \tau)$ from the statement of Theorem 4.1.1 are independent of the choice of tame fibration E_τ up to equivalence. Together with Sect. 4.6.2 this concludes the Proof of Theorem 4.1.1.

5.3 The Grothendieck group

The purpose of this section is to discuss a variety of consequences of Theorem 4.2.1 in what concerns the morphism Θ from (1) as well as the Grothendieck group itself.

5.3.1 Cobordism groups and the Grothendieck group.

We start by defining the appropriate cobordism groups that will be of interest to us here. Let $E \longrightarrow \mathbb{C}$ be a tame Lefschetz fibration. Let $\Omega^*_{Lag}(M; E)$ be the abelian group defined as the quotient of the free abelian group generated by the Lagrangians $L \in \mathcal{L}^*(M)$-modulo the relations $\mathcal{R}^E_{cob}$ generated by the cobordisms $V : \emptyset \rightsquigarrow (L_1, \ldots, L_s)$, $V \in \mathcal{L}^*(E)$ in the sense that to each such V we associate the relation $L_1 + \cdots + L_s \in \mathcal{R}^E_{cob}$. Notice that all vanishing spheres $S \subset M$ (associated to any path between a critical value of π and O) belong to $\mathcal{R}^E_{cob}$, hence their cobordism class is $0 \in \Omega^*_{Lag}(M; E)$. In case $\pi : E \longrightarrow \mathbb{C}$ is the trivial fibration (i.e. E splits symplectically as $E = \mathbb{C} \times M$ and $\pi = \mathrm{pr}_{\mathbb{C}}$) we will abbreviate $\Omega^*_{Lag}(M; E)$ by $\Omega^*_{Lag}(M)$.

We now consider the Grothendieck group, $K_0(D\mathcal{F}uk^*(M))$, that is associated to the triangulated category $D\mathcal{F}uk^*(M)$. We are interested in a quotient of this Grothendieck group that is associated to E and is defined as:

$$K_0(D\mathcal{F}uk^*(M); E) = K_0(D\mathcal{F}uk^*(M))/\mathcal{S}_E$$

where $\mathcal{S}_E$ is the subgroup generated by a collection of vanishing spheres S_k, respectively associated to the singularities x_k (by choosing a path from each x_k to a fixed base point).

Corollary 5.3.1 *The groups $\mathcal{S}_E$ and $K_0(D\mathcal{F}uk^*(M); E)$ do not depend on the choices of paths used to define the collection of vanishing spheres. Moreover, there exists a morphism of groups:*

$$\Theta^E : \Omega^*_{Lag}(M; E) \to K_0(D\mathcal{F}uk^*(M); E)$$

that is induced by $L \to L$.

This morphism extends the Lagrangian Thom morphism initially constructed in [6] and already mentioned at (1)

$$\Theta : \Omega^*_{Lag}(M) \to K_0(D\mathcal{F}uk^*(M))$$

Proof We first discuss the independence of $K_0(D\mathcal{F}uk^*(M); E)$ of the choices of the vanishing spheres S_i. Assume for instance that one of these spheres, say Σ_1—that is the end of a thimble S_1 that projects to a segment k_1 from v_1 to O—is replaced with a sphere Σ_1 which is the end of a thimble K_1, associated to a different path, k_1. By the results of Seidel [24], the difference between S_1 and Σ_1 (up to hamiltonian isotopy) can be described as follows: one sphere is obtained from the other by applying a symplectic diffeomorphism ϕ which can be written as a word in the elements $\tau_{S_2}, \ldots, \tau_{S_m}$ (i.e. ϕ is a composition of Dehn twists and their inverses along spheres from the collection $S_2, \ldots, S_m$). From Seidel's exact triangle as given in Proposition 4.5.1 we see that the subgroups generated, respectively, by $S_1, S_2, \ldots, S_m$ and $\Sigma_1, S_2, \ldots, S_m$ are the same.

The existence of the morphism Θ^E is now an immediate consequence of the decomposition in Corollary 5.1.1. $\square$

5.3.2 The Grothendieck group as an algebraic cobordism group.

We now focus our attention on the category $\mathcal{F}uk^*(E)$.

For each module $\mathcal{M} \in \mathcal{O}b(D\mathcal{F}uk^*(E))$, define $[\mathcal{M}]_j \in \mathcal{O}b(D\mathcal{F}uk^*(M))$ by

$$[\mathcal{M}]_j = \mathcal{R}_j(\mathcal{M})$$

where $\mathcal{R}_j$ are the restriction functors defined in the Proof of Corollary 5.1.1 (see also Remark 5.1.2). This extends to all objects in $D\mathcal{F}uk^*(E)$ the operation that associates to a cobordism V its jth end. For each object $\mathcal{M}$ of $D\mathcal{F}uk^*(E)$ there are only finitely many non-vanishing $[\mathcal{M}]_j$'s.

We now define another group $\Omega^*_{Alg}(M; E)$, which we call the *algebraic cobordism group*, as the free abelian group generated by all the *isomorphisms types of* objects $\in \mathcal{O}b(D\mathcal{F}uk^*(M))$ modulo the relations

$$[\mathcal{M}]_1 + [\mathcal{M}]_2 + [\mathcal{M}]_3 + \cdots = 0$$

for each $\mathcal{M} \in \mathcal{O}b(D\mathcal{F}uk^*(E))$. The group $\Omega^*_{Alg}(M; E)$ can be viewed as an algebraic cobordism group in the following sense. The generators of this group are the (isomorphism type of) objects of $D\mathcal{F}uk^*(M)$, thus they are obtained by completing algebraically the objects of $\mathcal{F}uk^*(M)$ as in the construction of the derived Fukaya category. Similarly, the relations defining the group are again an algebraic completion—in a similar sense but now involving the category $\mathcal{F}uk^*(E)$—of the relations providing $\Omega^*_{Lag}(M; E)$. There is an obvious group morphism:

$$q : \Omega^*_{Lag}(M; E) \to \Omega^*_{Alg}(M; E).$$

Corollary 5.3.2 *There is a group isomorphism*

$$\Theta^E_{Alg} : \Omega^*_{Alg}(M; E) \to K_0(D\mathcal{F}uk^*(M); E)$$

such that $\Theta^E = \Theta^E_{Alg} \circ q$.

Proof Throughout the proof we abbreviate $K_0 = K_0(D\mathcal{F}uk^*(M); E)$.

At the level of generators we define Θ^E_{Alg} to be the identity. The surjectivity of Θ^E_{Alg} is clear as well as the relation $\Theta^E = \Theta^E_{Alg} \circ q$. The only two things to check are that this map is well-defined and injective.

To show that Θ^E_{Alg} is well-defined we need to prove that if $\mathcal{M}$ is an object of $D\mathcal{F}uk^*(E)$, then $\sum_i [\mathcal{M}]_i = 0$ in $K_0(D\mathcal{F}uk^*(M); E)$. To see this recall that, by the definition of $D\mathcal{F}uk^*(E)$, there are $V_j \in \mathcal{L}^*(E)$ so that:

$$\mathcal{M} \cong (V_m \to V_{m-1} \to \cdots \to V_2 \to V_1).$$

By Theorem 4.2.1, in K_0 we have:

$$\sum_i [V_j]_i = 0, \quad \forall j.$$

Moreover, for all i, we have the following cone decomposition of $[\mathcal{M}]_i$ in $D\mathcal{F}uk^*(M)$:

$$[\mathcal{M}]_i \cong ([V_m]_i \to [V_{m-1}]_i \to \cdots \to [V_2]_i \to [V_1]_i)$$

because the functor $\mathcal{R}_i$ is triangulated. This means that in K_0:

$$\sum_i [\mathcal{M}]_i = \sum_{i,j} [V_j]_i = 0.$$

This concludes the proof of the well-definedness of the map Θ^E_{Alg}.

It remains to show that Θ^E_{Alg} is injective. We start by proving the injectivity in the case when $E = \mathbb{C} \times M$ is the trivial fibration. We omit E from the notation of Θ_{Alg} in this case and, similarly, we put $\Omega_{Alg}(M) = \Omega_{Alg}(M; \mathbb{C} \times M)$. Assume that

$$\mathcal{M} \to \mathcal{M}' \to \mathcal{M}''$$

is an exact triangle of $\mathcal{F}uk^*(M)$-modules. The injectivity of Θ_{Alg} follows by constructing for each such triangle an object T in $D\mathcal{F}uk^*(\mathbb{C} \times M)$ so that $[T]_1 = \mathcal{M}''$, $[T]_2 = \mathcal{M}'$ and $[T]_3 = \mathcal{M}$. Indeed, this implies that all the relations that are used in the definition of K_0 also appear among the relations that define $\Omega^*_{Alg}(M)$ which means that Θ_{Alg} is invertible.

To construct this object T we proceed as follows. We first recall that, by definition, $\mathcal{M}''$ is—up to isomorphism—the cone over a module map $f : \mathcal{M} \to \mathcal{M}'$.

Now recall the A_∞-category $\mathbb{N} \otimes \mathcal{F}uk^*(M)$ as in Sect. 5.2 (notice that now $m = 0$). We first construct an object $\tilde{T}$ of $\mathbb{N} \otimes \mathcal{F}uk^*(M)$. This consists of two steps. First, for each $\mathcal{F}uk^*(M)$-module $\mathcal{N}$ and each curve γ_i we define a $\mathbb{N} \otimes \mathcal{F}uk^*(M)$-module denoted by $\gamma_i \times \mathcal{N}$. On objects $\gamma_j \times L$ we put $(\gamma_i \times \mathcal{N})(\gamma_j \times L) = \xi^{j,i}\mathcal{N}(L)$. The A_∞-module operations are defined by a direct adaptation of the formulas giving the operations in $\mathbb{N} \otimes \mathcal{F}uk^*(M)$. The second step is to define a morphism

$$\bar{f} : \gamma_3 \times \mathcal{M} \to \gamma_2 \times \mathcal{M}'.$$

We then define $\tilde{T}$ by $\tilde{T} = cone(\bar{f})$. The morphism $\bar{f}$ is induced by f and is given by a formula again perfectly similar to the formula of the multiplication in $\mathbb{N} \otimes \mathcal{F}uk^*(M)$, but using f instead of μ_k and replacing $\mathrm{Mor}(i_k \times L_k, i_{k+1} \times L_{k+1})$ by $(\gamma_3 \times \mathcal{M})(\gamma_{i_k - 2} \times L_{k+1})$ and $\mathrm{Mor}(i_1 \times L_1, i_{k+1} \times L_{k+1})$ by $(\gamma_2 \times \mathcal{M}')(\gamma_{i_1 - 2} \times L_1)$. We now consider the sequence of functors, the first two being equivalences and the last a full and faithful embedding:

$$D(\mathbb{N} \otimes \mathcal{F}uk^*(M)) \to D\mathcal{F}(\mathbb{C} \times M) \to D\mathcal{F}uk^*(\mathbb{C} \times M) \to D\mathcal{F}uk^*_{\frac{1}{2}}(\mathbb{C} \times M). \tag{36}$$

Here, the A_∞-category $D\mathcal{F}(\mathbb{C} \times M)$ is defined as in the Proof of Corollary 5.2.1. We now use the composition of the functors in (36) to define $[\mathcal{H}]_j = (i^{\eta_j})^*(\mathcal{H})$ for each module $\mathcal{H}$ in $D(\mathbb{N} \otimes \mathcal{F}uk^*(M))$—see the Proof of Corollary 5.1.1 for the definition of i^{η_j}. We take T to be the image of $\tilde{T}$ by the first two equivalences in (36) and we claim that:

a. for each object $\mathcal{N}$ in $D\mathcal{F}uk^*(M)$ we have that $[(\gamma_i \times \mathcal{N})]_j \cong \mathcal{N}$ if $i = j$ or $j = 1$ and is 0 otherwise. Moreover, $(i^{\eta_1})^*(\bar{f}) \cong f$.
b. $[T]_1 = \mathcal{M}''$, $[T]_2 = \mathcal{M}'$, $[T]_3 = \mathcal{M}$ and $[T]_i = 0$ whenever $i \geq 4$.

Notice that point b concludes the proof for $E = \mathbb{C} \times M$. Given that the equivalences in (36) are triangulated, point b follows directly from a. Thus, it remains to check a. For this we notice that pull-back respects triangles and as each object $\mathcal{N}$ is isomorphic to an iterated cone of objects $L \in \mathcal{F}uk^*(M)$ it is enough to verify the statement for the Yoneda modules $\gamma_i \times L$, $L \in \mathcal{L}^*(M)$. But for these modules the statement is obvious. The statement for $\bar{f}$ follows in a similar fashion.

We are left to show the more general statement for a Lefschetz fibration $\pi : E \to \mathbb{C}$ that is not trivial. For this we recall that for each thimble T_i we have $(i^{\eta_1})^*(T_i) = \tilde{\tau}^{-1}_{i+1,\ldots,m} S_i$. Thus, by the definition of the groups involved, we have a quotient map

$$\Omega^*_{Alg}(M)/\mathcal{S}'_E \to \Omega^*_{Alg}(M; E) \xrightarrow{\Theta^E_{Alg}} K_0(D\mathcal{F}uk^*(M); E), \tag{37}$$

where $\mathcal{S}'_E$ is the subgroup generated by the vanishing spheres of π. To conclude the proof of the corollary it is enough to show that the composition of maps in (37) is an isomorphism. Recall that

$$K_0(D\mathcal{F}uk^*(M); E) = K_0(D\mathcal{F}uk^*(M))/\mathcal{S}_E$$

and notice that the isomorphism Θ_{Alg}—associated to the trivial fibration $\mathbb{C} \times M$—has the property that $\Theta_{Alg}(\mathcal{S}'_E) = \mathcal{S}_E$. Therefore the composition of maps in (37) is an isomorphism and this concludes the proof. □

5.4 Comparison with ambient quantum homology

There is an obvious morphism:

$$i : \Omega^*_{Lag}(M) \to QH(M)$$

that associates to each Lagrangian L its homology class $[L] \in H_n(M; \mathbb{Z}_2) \subset QH(M)$. From the point of view of Corollary 5.3.2 it is natural to expect that i factors through a morphism:

$$i' : \Omega^*_{Alg}(M) \to QH(M).$$

This is indeed true as we will see below.

Corollary 5.4.1 *Consider a module $\mathcal{M} \in \mathcal{O}b(D\mathcal{F}uk^*(M))$. Such a module admits a cone-decomposition (up to quasi-isomorphism)*

$$\mathcal{M} \cong (L_s \to L_{s-1} \to \cdots \to L_1).$$

With this notation, the equation

$$i'(\mathcal{M}) = \sum_j [L_j] \in QH(M) \tag{38}$$

provides a well-defined group morphism

$$i' : \Omega^*_{Alg}(M) \to QH(M)$$

so that $i = i' \circ q$.

Proof While this definition of i' seems very simple the fact that i' is a well-defined morphism of groups is somewhat surprising. We only know a proof of this fact which follows from the indirect construction that we give below.

We will write i' as a composition of two morphisms $i' = \tilde{i}' \circ \Theta_{Alg}$ where Θ_{Alg} : $\Omega^*_{alg}(M) \to K_0(D\mathcal{F}uk^*(M))$ is the isomorphism in Corollary 5.3.2 and

$$\tilde{i}' : K_0(D\mathcal{F}uk^*(M)) \to QH(M)$$

is a morphism that is known to experts, see for instance § 5 in [26]. The definition of $\tilde{i}'$ is somewhat subtle so we review it here.

The morphism $\tilde{i}'$ is a composition of morphisms:

$$K_0(D\mathcal{F}uk^*(M)) \xrightarrow{f_1} K_0(\mathcal{Y}(\mathcal{F}uk^*(M))^\wedge)$$
$$\xrightarrow{f_2} HH_*(\mathcal{Y}(\mathcal{F}uk^*(M))^\wedge) \xrightarrow{f_3} HH_*(\mathcal{F}uk^*(M)) \xrightarrow{f_4} QH(M).$$

Here, the category $\mathcal{Y}(\mathcal{F}uk^*(M))$ is the Yoneda image of $\mathcal{F}uk^*(M)$; $(\mathcal{Y}(\mathcal{F}uk^*(M))^\wedge$ is its triangular completion (as A_∞-category); $HH_*(\mathcal{B})$ is the Hochschild homology of the A_∞-category $\mathcal{B}$ with values in itself (generally denoted by $HH_*(\mathcal{B}, \mathcal{B})$). The morphisms involved are as follows: f_1 is an obvious isomorphism that reflects the definition of the triangular structure of $D\mathcal{F}uk^*(M)$, the morphism f_2 sends each module in $\mathcal{M} \in \mathcal{Y}(\mathcal{F}uk^*)^\wedge$ to the Hochschild homology class of its unit endomorphism $e_\mathcal{M} \in \hom(\mathcal{M}, \mathcal{M})$. The latter descends to K_0 because, as it follows from Proposition 3.8 in [24], if $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}''$ is an exact triangle in a triangulated A_∞-category $\mathcal{A}$, then $e_\mathcal{M} = e_{\mathcal{M}'} + e_{\mathcal{M}''}$ in $HH_*(\mathcal{A})$. The morphism f_3 comes from the fact that the natural inclusion

$$\mathcal{F}uk^*(M) \to \mathcal{Y}(\mathcal{F}uk^*(M))^\wedge$$

induces an isomorphism in Hochschild homology (this is sometimes referred to as a form of Morita invariance. See [27] for the analogous though different context of dg-categories); f_3 is the inverse of this isomorphism. Finally, f_4 is the open-closed map (see for instance [26] where it is defined for in the exact case, the adaptation to the monotone setting is immediate). □

Remark 5.4.2 Assume that $\mathcal{M}'$ is another module in $D\mathcal{F}uk^*(M)$ as in the statement of the corollary such that $\mathcal{M}' \cong \mathcal{M}$ and

$$\mathcal{M}' = (L'_r \to L'_{r-1} \to \cdots \to L'_1).$$

The existence of i' then implies that $\sum_j [L'_j] = \sum_k [L_k]$. It is interesting to note that the only way we know to show this fact is through the indirect method contained in the proof of the Corollary.

5.5 The periodicity isomorphism (2)

In view of Corollary 5.2.1 it is natural to expect that $K_0(D\mathcal{F}uk^*(E))$ can be calculated in terms of $K_0(D\mathcal{F}uk^*(M))$. We will give here such a calculation but only in the case when E is the trivial fibration $E = \mathbb{C} \times M$. An analogous statement for non-trivial fibrations is expected to also hold, but would require further algebraic elaboration.

Corollary 5.5.1 *There exists a canonical isomorphism*

$$K_0(D\mathcal{F}uk^*(\mathbb{C} \times M)) \cong \mathbb{Z}_2[t] \otimes K_0(D\mathcal{F}uk^*(M))$$

induced by the map that sends $\mathcal{M} \in \mathcal{O}b(D\mathcal{F}uk^*(\mathbb{C} \times M))$ to $\sum_{i\geq 2} t^{i-2} \otimes \mathcal{R}_i(\mathcal{M})$, where $\mathcal{R}_i$ is the restriction functor from (33).

Proof From Corollary 5.2.1 it is enough to show that

$$K_0(D(\mathbb{N} \otimes \mathcal{F}uk^*(M))) \cong \mathbb{Z}_2[t] \otimes K_0(D\mathcal{F}uk^*(M)).$$

To simplify notation we denote $G_1 = K_0(D(\mathbb{N} \otimes \mathcal{F}uk^*(M)))$ and $G_2 = \mathbb{Z}_2[t] \otimes K_0(D\mathcal{F}uk^*(M))$. Given a module $\mathcal{M}$ which is an object of $D(\mathbb{N} \otimes \mathcal{F}uk^*(M))$ we use the composition in (36) to define the restriction modules $[\mathcal{M}]_i$ that are objects of $D\mathcal{F}uk^*(M)$ and define the sum $\phi(\mathcal{M}) = \sum_{i\geq 2} t^{i-2} \otimes [\mathcal{M}]_i \in G_2$. Because the restriction functors $\mathcal{R}_j$ are triangulated it is easy to see that this map descends to a morphism $\phi : G_1 \to G_2$. The construction of the modules $\gamma_i \times \mathcal{N}$ in the Proof of Corollary 5.3.2, in particular point (a) in the course of that proof, shows that ϕ is surjective. To show that ϕ is injective we construct an inverse $\psi : G_2 \to G_1$. We define $\psi(t^i \otimes \mathcal{N}) = \gamma_{i+2} \times \mathcal{N}$ for each object in $\mathcal{N} \in D\mathcal{F}uk^*(M)$, where we have used here the notation from the Proof of Corollary 5.2.1. Once we show that ψ is well defined (in other words, that it respects the relations giving K_0) it immediately follows that it is an inverse of ϕ by the point (a) in the Proof of Corollary 5.3.2. But again as in the Proof of Corollary 5.3.2, namely the construction of $\tilde{T}$, it is easy to see that the map $\mathcal{N} \mapsto \gamma_i \times \mathcal{N}$ respects triangles. As a consequence, ψ is well defined and this concludes the proof. ☐

6 Real Lefschetz fibrations

Real Lefschetz fibrations have recently been studied from the topological and real algebraic geometry viewpoints (see e.g. [8,19–21]). Lagrangian cobordism is naturally related to this notion and we describe this relationship in the first subsection below. We then pursue with a construction of such fibrations and, in the last subsection, with a concrete example.

6.1 Lagrangian cobordism and real Lefschetz fibrations

Let $\pi : E \longrightarrow \mathbb{C}$ be a Lefschetz fibration endowed with a symplectic structure Ω, as in Definition 2.1.1. Denote by (M, ω) the general fiber of (E, Ω). Let $c_E : E \longrightarrow E$ be an anti-symplectic involution, i.e. $c_E^*\Omega = -\Omega$ and $c_E \circ c_E = \mathrm{id}$. Assume further that c_E covers the standard complex conjugation $c_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$, namely $\pi \circ c_E = c_{\mathbb{C}} \circ \pi$. Denote by $V = \mathrm{Fix}(c_E)$ the fixed point locus of c_E. Note that the projection $\pi(V)$ of V to $\mathbb{C}$ is a subset of $\mathbb{R}$. The following proposition shows that V is a Lagrangian cobordism and also gives a criterion for its monotonicity.

Proposition 6.1.1 *Under the above assumptions, V is a Lagrangian cobordism with at most one positive end and at most one negative one (but possibly without any ends at all). Its projection $\pi(V) \subset \mathbb{R}$ is of the form $\cup_{j \in \mathcal{S}} \overline{I}_j$, where $\mathcal{S}$ is a subset of the set of connected components of $\mathbb{R} \setminus Critv(\pi)$, I_j stands for the path connected component corresponding to j and $\overline{I}_j$ is the closure of I_j. Thus $\partial \pi(V)$ is a subset of $Critv(\pi) \cap \mathbb{R}$.*

Moreover, for every $z \in \mathbb{R} \setminus Critv(\pi)$ the part of V lying over z, $V_z := E_z \cap V$, coincides with the fixed point locus of the anti-symplectic involution $c_E|_{E_z}$ hence is either empty or a smooth Lagrangian submanifold of E_z (possibly disconnected). In particular, the Lagrangians corresponding to the ends of V (if they exist) are real with respect to restriction of c_E to the regular fibers over $\mathbb{R}$ at $\pm\infty$.

If (E, Ω) is monotone, then $V \subset E$ is a monotone Lagrangian (possibly with $N_V = 1$). Denoting by $c_1^{\min}(E)$ the minimal Chern number on spherical classes in E and by N_V the minimal Maslov number of V we have: if $c_1^{\min}(E)$ is odd then $c_1^{\min}(E) | N_V$, and if $c_1^{\min}(E)$ is even then $\frac{1}{2} c_1^{\min}(E) | N_V$.

If $\dim_{\mathbb{C}} M \geq 2$ and (M, ω) is monotone then (E, Ω) is monotone too and $c_1^{\min}(E) = c_1^{\min}(M)$, hence V is a monotone Lagrangian cobordism.

The proof is straightforward and can be found in the expanded version of this paper [3, §6.5].

6.2 Constructing real Lefschetz fibrations

Here we show how linear systems associated to real projective embeddings give rise to real Lefschetz fibrations.

Let X be a smooth complex projective variety endowed with a real structure, namely an anti-holomorphic involution $c_X : X \longrightarrow X$. Let $\mathscr{L}$ be a very ample line bundle on X. Assume $\mathscr{L}$ is endowed with a real structure $c_{\mathscr{L}}$ compatible with c_X. By this we mean a bundle map (fiberwise linear over $\mathbb{R}$) $c_{\mathscr{L}} : \mathscr{L} \longrightarrow \mathscr{L}$ which is an anti-holomorphic involution and covers c_X, i.e. $\mathrm{pr} \circ c_{\mathscr{L}} = c_X \circ \mathrm{pr}$, where $\mathrm{pr} : \mathscr{L} \longrightarrow X$ is the bundle projection.

Denote by $H^0(\mathscr{L})$ the space of holomorphic sections of $\mathscr{L}$ and by $\mathbf{P} := \mathbb{P}\big(H^0(\mathscr{L})\big)^*$ the projectivization of its dual (i.e. the space of hyperplanes in $H^0(\mathscr{L})$). Denote by $\mathbf{P}^* := \mathbb{P}H^0(\mathscr{L})$ the projectivization of the space of sections itself. Note that $\mathbf{P}^*$ is the dual projective space of $\mathbf{P}$.

The real structure of $\mathscr{L}$ induces a real structure c_H on $H^0(\mathscr{L})$ defined by $c_H(s) = c_{\mathscr{L}} \circ s \circ c_X$. Denote by $H^0_{\mathbb{R}}(\mathscr{L}) \subset H^0(\mathscr{L})$ the space of real sections of $\mathscr{L}$ (i.e. sections s with $c_H(s) = s$). The real structure c_H descends to real structures on $\mathbf{P}^*$ and $\mathbf{P}$ which, by abuse of notation, will both be denoted by c_H. The fixed point locus of c_H on $\mathbf{P}$ will be denoted by $\mathbf{P}_{\mathbb{R}}$ and that on $\mathbf{P}^*$ by $\mathbf{P}^*_{\mathbb{R}}$.

Consider now the projective embedding $\iota : X \hookrightarrow \mathbf{P}$ defined using the sections of $\mathscr{L}$. This embedding is real in the sense that it commutes with (c_X, c_H). Furthermore, there is an isomorphism between $\mathbf{P}$ and $\mathbb{C}P^N$ which sends c_H to the standard real structure $c_{\mathbb{C}P^N}$ of $\mathbb{C}P^N$ (hence $\mathbf{P}_{\mathbb{R}}$ is sent under this isomorphism to $\mathbb{R}P^N$). We fix once and for all such an isomorphism. Denote by $\omega_{\mathbb{C}P^N}$ the standard symplectic structure of $\mathbb{C}P^N$ normalized such that $\int_{\mathbb{C}P^1} \omega_{\mathbb{C}P^n} = 1$. Since $c_{\mathbb{C}P^N}$ is anti-symplectic with respect to $\omega_{\mathbb{C}P^N}$ the previously mentioned isomorphism yields a Kähler form $\omega_{\mathbf{P}}$ on $\mathbf{P}$ and therefore also a Kähler form ω_X on X so that c_X is anti-symplectic with respect to ω_X.

Let $\Delta(\mathscr{L}) \subset \mathbf{P}^*$ be the discriminant locus (a.k.a. the dual variety of X), which by definition is the variety consisting of all section $[s] \in \mathbf{P}^*$ (up to a constant factor) which are somewhere *non-transverse* to the zero-section. Denote by $\Delta_{\mathbb{R}}(\mathscr{L}) = \Delta(\mathscr{L}) \cap \mathbf{P}^*_{\mathbb{R}}$ its real part.

Let $\ell \subset \mathbf{P}^*$ be a line which is invariant under c_H and intersects $\Delta(\mathscr{L})$ only along its smooth strata and transversely. Fix an isomorphism $\ell \approx \mathbb{C}P^1$ and endow ℓ with a standard Kähler structure ω_ℓ normalized such that $\int_\ell \omega_\ell = 1$. Consider the symplectic manifold $\ell \times X$ endowed with the symplectic structure $\omega_\ell \oplus \omega_X$. For every $\lambda \in \mathbf{P}^*$ denote by $\Sigma^{(\lambda)} = s^{-1}(0) \subset X$ the zero locus corresponding to a section s representing λ. Note that for all $\lambda \notin \Delta(\mathscr{L})$, the variety $\Sigma^{(\lambda)}$ is smooth. We endow these varieties with the symplectic structure induced from ω_X. The complement of the discriminant, $\mathbf{P}^* \backslash \Delta(\mathscr{L})$, is path connected (since $\mathrm{codim}_{\mathbb{C}}(\Delta(\mathscr{L}) \subset \mathbf{P}^*) \geq 1$). Therefore all the symplectic manifolds $\Sigma^{(\lambda)}$, $\lambda \in \mathbf{P}^* \backslash \Delta(\mathscr{L})$, are mutually symplectomorphic.

For every $\lambda \in \mathbf{P}^*_{\mathbb{R}} \backslash \Delta_{\mathbb{R}}(\mathscr{L})$ the manifold $\Sigma^{(\lambda)}$ has a real structure induced by c_X. Denote its real part by $\Sigma^{(\lambda)}_{\mathbb{R}}$. We stress that *in contrast to $\mathbf{P}^* \backslash \Delta(\mathscr{L})$, its real part $\mathbf{P}^*_{\mathbb{R}} \backslash \Delta_{\mathbb{R}}(\mathscr{L})$ is in general disconnected and the topology of $\Sigma^{(\lambda)}_{\mathbb{R}}$ depends on the connected component λ belongs to.* Define now

$$\widehat{E} = \{(\lambda, x) \mid \lambda \in \ell,\ x \in \Sigma^{(\lambda)}\} \subset \ell \times X.$$

Since $\ell \pitchfork \Delta(\mathscr{L})$ it follows that $\widehat{E}$ is a smooth complex variety and we endow it with the symplectic structure $\widehat{\Omega}$ induced by $\omega_\ell \oplus \omega_X$. The projection $\pi : \widehat{E} \longrightarrow \ell$ (induced from $\ell \times X \to \ell$) is a Lefschetz fibration (with base $\ell \approx \mathbb{C}P^1$). The fact that the critical points of π are non-degenerate follows from our assumptions on the intersection of ℓ and $\Delta(\mathscr{L})$. The involutions c_H and c_X induce an anti-holomorphic involution on $\widehat{E}$ which is also anti-symplectic with respect to $\widehat{\Omega}$.

Let $D \subset \ell$ be a closed disk which is invariant under c_H. Identify $\ell \backslash D$ with $\mathbb{C}$ via an orientation preserving diffeomorphism which commutes with $(c_H, c_{\mathbb{C}})$, where $c_{\mathbb{C}}$ is the standard conjugation on $\mathbb{C}$. The real part $\ell_{\mathbb{R}} \backslash D$ of $\ell \backslash D$ is sent by this diffeomorphism to $\mathbb{R}$.

By restricting π to the complement of D we obtain a Lefschetz fibration $E = \pi^{-1}(\ell \backslash D)$ over $\ell \backslash D \cong \mathbb{C}$. We endow E with the symplectic structure Ω coming from $\widehat{\Omega}$ and by a slight abuse of notation denote its projection by $\pi : E \longrightarrow \mathbb{C}$. Restricting the preceding anti-symplectic involution of $\widehat{E}$ to E we obtain an anti-symplectic involution c_E on E which covers the standard conjugation $c_{\mathbb{C}}$ as in Sect. 6.1. The critical values of π are precisely $(\ell \backslash D) \cap \Delta(\mathscr{L})$. Some of them lie on $\ell_{\mathbb{R}}$ (i.e. the real axis) and the others come in pairs of conjugate points.

Note that $\ell_{\mathbb{R}} \backslash \Delta(\mathscr{L})$ might have several connected components. If $\lambda', \lambda'' \in \ell_{\mathbb{R}} \backslash \Delta(\mathscr{L})$ are in the same component then $\Sigma_{\mathbb{R}}^{(\lambda')}$ and $\Sigma_{\mathbb{R}}^{(\lambda'')}$ are diffeomorphic, but otherwise not necessarily.

Consider now the fixed point locus $V = \mathrm{Fix}(c_E) \subset E$. By Proposition 6.1.1, V is a Lagrangian cobordism. Its ends correspond to $\Sigma_{\mathbb{R}}^{(\lambda_-)}$ and $\Sigma_{\mathbb{R}}^{(\lambda_+)}$, where $\lambda_-, \lambda_+ \in \ell_{\mathbb{R}} \backslash D$ are close enough to the two boundary points of $\ell_{\mathbb{R}} \cap D$. As hinted above, any of $\Sigma_{\mathbb{R}}^{(\lambda_\pm)}$ might be disconnected or even void.

We now address monotonicity. Assume that $\dim_{\mathbb{C}} X \geq 3$ and that the symplectic manifold $(\Sigma^{(\lambda)}, \omega_X|_{\Sigma^{(\lambda)}})$, $\lambda \notin \Delta(\mathscr{L})$, is monotone. By Proposition 6.1.1, $V \subset E$ is a monotone Lagrangian. Here is an algebraic–geometric criterion that assures monotonicity of the $\Sigma^{(\lambda)}$'s. For an algebraic variety X we denote by K_X its canonical class. The following follows easily from adjunction.

Proposition 6.2.1 *Let X be a Fano manifold with $\dim_{\mathbb{C}} X \geq 3$ and write $-K_X = rD$, with $r \in \mathbb{N}$ and D a divisor class. Further, suppose that the very ample line bundle $\mathscr{L}$ is related to D by $\mathscr{L} = qD$ with $0 < q \in \mathbb{Q}$ and $q < r$. Then the symplectic manifolds $(\Sigma^{(\lambda)}, \omega_X|_{\Sigma^{(\lambda)}})$, $\lambda \notin \Delta(\mathscr{L})$, are monotone. In particular V is a monotone Lagrangian cobordism (possibly with $N_V = 1$).*

6.3 A concrete example: real quadric surfaces

We present here a concrete example of a real Lefschetz fibration associated to a pencil of complex quadric surfaces in $\mathbb{C}P^3$. The example can be easily generalized to higher dimensions.

Let $X = \mathbb{C}P^3$ and $\mathscr{L} = \mathcal{O}_{\mathbb{C}P^3}(2)$, both endowed with their standard real structures (induced by complex conjugation). Note that $\mathscr{L}$ is very ample and give rise to the so called degree-2 Veronese embedding $\iota : X \hookrightarrow \mathbb{C}P^9$. More precisely, using homogeneous coordinates $[X_0 : X_1 : X_2 : X_3]$ on $\mathbb{C}P^3$ identify the space $H^0(\mathscr{L})$ of sections of $\mathscr{L}$ with the space of quadratic homogeneous polynomials $\lambda(\underline{X})$ in the variables $\underline{X} = (X_0, X_1, X_2, X_3)$:

$$\lambda(\underline{X}) = \sum_{0 \le i \le j \le 3} a_{i,j} X_i X_j. \tag{39}$$

Taking $X_i X_j$, $0 \le i \le j \le 3$, as a basis for this space we obtain an identifications $\mathbf{P} \cong \mathbb{C}P^9$ under which the projective embedding $\iota : X \hookrightarrow \mathbb{C}P^9$ is given by:

$$[z_0 : z_1 : z_2] \longmapsto [z_0^2 : z_0 z_1 : \cdots : z_i z_j : \cdots : z_2 z_3 : z_3^2],$$

where the coordinates on the right-hand side go over all (i, j) with $0 \le i \le j \le 3$.

The hyperplane section corresponding to the polynomial λ is the quadric surface

$$\Sigma^{(\lambda)} = \left\{[z_0 : z_1 : z_2 : z_3] \mid \lambda(z_0, z_1, z_3, z_3) = 0\right\} \subset \mathbb{C}P^3.$$

A straightforward calculation shows that $\lambda \in \Delta(\mathscr{L})$ if and only if

$$\det \begin{pmatrix} 2a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & 2a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & 2a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & 2a_{33} \end{pmatrix} = 0. \tag{40}$$

This shows that the discriminant $\Delta(\mathscr{L})$ is a variety of degree 4 in $\mathbf{P}^* \cong \mathbb{C}P^9$. The smooth stratum of $\Delta(\mathscr{L})$ consists of those λ's where the matrix in (40) has rank 3. The real part $\Delta_{\mathbb{R}}(\mathscr{L})$ of $\Delta(\mathscr{L})$ consists of those polynomials λ which in addition to (40) have real coefficients (i.e. $a_{i,j} \in \mathbb{R}\ \forall i, j$).

It is well known that for $\lambda \notin \Delta(\mathscr{L})$ the variety $\Sigma^{(\lambda)}$ is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. As a symplectic manifold $(\Sigma^{(\lambda)}, \iota^* \omega_{\mathbb{C}P^9})$ is symplectomorphic to $(\mathbb{C}P^1 \times \mathbb{C}P^1, 2\omega_{\mathbb{C}P^1} \oplus 2\omega_{\mathbb{C}P^1})$.

Consider now the following two sections $\lambda_0, \lambda_1 \in \mathbf{P}^* \setminus \Delta(\mathscr{L})$:

$$\lambda_0(\underline{X}) = X_0^2 + X_1^2 + X_2^2 - X_3^2, \qquad \lambda_1(\underline{X}) = X_0 X_3 - X_1 X_2.$$

Denote the real part of $\Sigma^{(\lambda_i)}$ by $L^{(\lambda_i)}$, $i = 0, 1$. Then $L^{(\lambda_1)}$ is a Lagrangian torus and moreover there is a symplectomorphism $\phi^{(\lambda_1)} : \Sigma^{(\lambda_1)} \longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ such that $\phi^{(\lambda_1)}(L^{(\lambda_1)})$ is the torus $T = \mathbb{R}P^1 \times \mathbb{R}P^1$. For example, one can take $\phi^{(\lambda_1)}$ to be the inverse of the map:

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \ni ([z_0 : z_z], [w_0 : w_1]) \longmapsto [z_0 w_0 : z_0 w_1 : z_1 w_0 : z_1 w_1] \in \Sigma^{(\lambda_1)}.$$

A straightforward calculation shows that this is a symplectomorphism and that it sends the torus T to $L^{(\lambda_1)}$. By similar considerations, there is a symplectomorphism $\phi^{(\lambda_0)} : \Sigma^{(\lambda_0)} \longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ that sends $L^{(\lambda_0)}$ to the "anti-diagonal" Lagrangian sphere $S = \{(z, \bar{z}) \mid z \in \mathbb{C}P^1\} \subset \mathbb{C}P^1 \times \mathbb{C}P^1$. For example one can take $\phi^{(\lambda_0)}$ to be the inverse of the map:

$$\begin{aligned} &\mathbb{C}P^1 \times \mathbb{C}P^1 \ni ([z_0 : z_z], [w_0 : w_1]) \\ &\quad \longmapsto [z_0 w_1 + z_1 w_0 : i(z_0 w_1 - z_1 w_0) : z_1 w_1 - z_0 w_0 : z_1 w_1 + z_0 w_0] \in \Sigma^{(\lambda_0)}. \end{aligned}$$

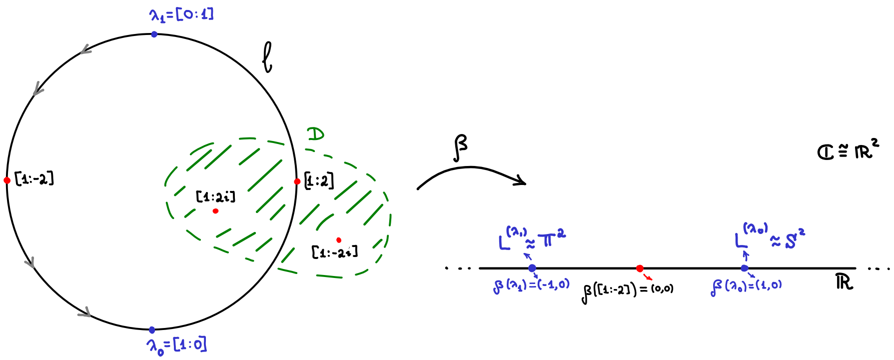

Fig. 22 The real pencil ℓ on the *left*, and the image of $\ell \setminus D$ under β in $\mathbb{C}$

We now consider the pencil $\ell \subset \mathbf{P}^*$ that passes through the two points λ_0 and λ_1. Clearly ℓ is invariant under the anti-holomorphic involution c_H. We can parametrize ℓ by

$$\mathbb{C}P^1 \ni [t_0 : t_1] \longmapsto \lambda_{[t_0:t_1]} := t_0\lambda_0 + t_1\lambda_1.$$

The intersection $\ell \cap \Delta(\mathscr{L})$ occur for the following values of $[t_0 : t_1]$ (see left side of Fig. 22):

$$[t_0 : t_1] \in \big\{[1 : 2], [1 : -2], [1 : 2i], [1 : -2i]\big\}. \tag{41}$$

Moreover ℓ intersects $\Delta(\mathscr{L})$ only along its smooth stratum and $\ell \pitchfork \Delta(\mathscr{L})$.

We now appeal to the construction in Sect. 6.2. We will identify $\mathbb{C} \cong \mathbb{R}^2$ in the obvious way. Choose a disk $D \subset \ell$ which is invariant under c_H and contains the points $[1 : 2], [1 : 2i], [1 : -2i]$ but not $[1 : -2]$. Fix an orientation preserving diffeomorphism $\beta : \ell \setminus D \longrightarrow \mathbb{R}^2$ such that:

$$\beta(\lambda_1) = (-1, 0), \quad \beta(\lambda_0) = (1, 0), \quad \beta([1 : -2]) = (0, 0).$$

See Fig. 22. From now on we use the identification β implicitly and write $\lambda_1 = (-1, 0)$, $\lambda_0 = (1, 0)$.

Restricting $\widehat{E}$ to $\ell \setminus D$ and applying a base change via β we obtain a Lefschetz fibration $\pi : E \longrightarrow \mathbb{C}$ with general fiber $\mathbb{C}P^1 \times \mathbb{C}P^1$ and with a real structure. Since the minimal Chern number of the general fiber is $c_1^{\min} = 2$, E is a strongly monotone Lefschetz fibration in the sense of Definition 3.3.1. Its monotonicity class is $* = (0)$.

The projection π has exactly one critical value at $0 \in \mathbb{C}$ (corresponding to $[1 : -2] \in \ell$). The real part V of E is a cobordism with one negative end associated to $L^- = L^{(\lambda_1)}$ which is a Lagrangian torus, and one positive end associated to $L^+ = L^{(\lambda_0)}$ which is a Lagrangian sphere. By Proposition 6.2.1, V is monotone and a simple calculation shows that it has minimal Maslov number $N_V = 2$. Interestingly we have $N_{L^-} = 2$ while $N_{L^+} = 4$. Note also that $d_{L^-} = d_{L^+} = 0$, hence V is of the right monotonicity class $* = (0)$.

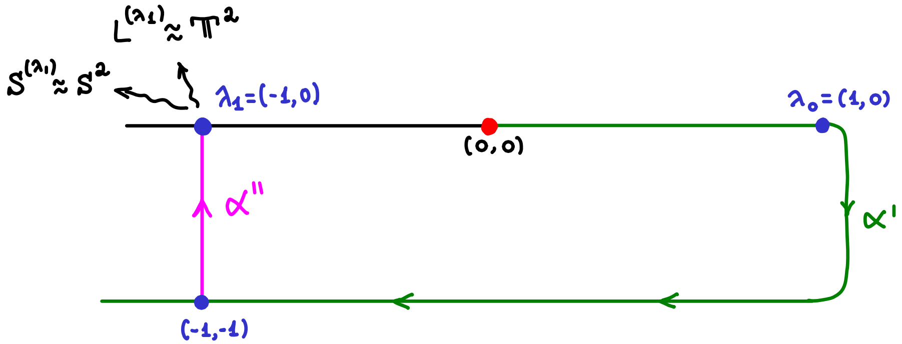

Fig. 23 The cobordism W with two negative ends, and the parallel transport of the sphere $L^{(\lambda_0)}$ to the fiber over λ_1

Transforming V to a negative-ended cobordism

In order to obtain a cobordism with only negative ends (as considered in the rest of the paper) we proceed as follows. Take the Lefschetz fibration $\pi : E \longrightarrow \mathbb{C}$ and $V \subset E$ as constructed above. Recall that $0 \in \mathbb{C}$ was the (single) critical value of π. Consider an embedded curve $\alpha' \subset \mathbb{R}^2$ starting at $(0, 0)$ as depicted (in green) in Fig. 23. For $p, q \in \alpha'$ we write $\alpha'_{p,q}$ for the part of α' that goes between p and q and α'_p the part of α' that starts at p and is unbounded towards the direction of the negative x-axis.

Now take the part of the cobordism V that lies over $(-\infty, 1] \times \mathbb{R} \subset \mathbb{R}^2$ and glue to its right-hand side the trail of the Lagrangian sphere $L^{(\lambda_0)} = V|_{(1,0)}$ along $\alpha'_{(1,0)}$. Denote the result by W. It is easy to see that W is a smooth Lagrangian cobordism with two negative ends. The lower end is a Lagrangian sphere and the upper end is a Lagrangian torus, both living inside symplectic manifolds that are symplectomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. See Fig. 23.

The Lefschetz fibration E is not tame. Therefore in order to apply the cone decomposition from Corollary 5.1.1 we need to identify fibers over different ends. To this end, denote by α'' the straight segment connecting $(-1, -1)$ to $\lambda_1 = (-1, 0)$. Denote by $\alpha = \alpha'|_{(1,0),(-1,-1)} * \alpha''$ the concatenation of the part of α' that runs between $(1, 0)$ and $(-1, -1)$ with α''. Denote by $\Pi_\alpha : E_{\lambda_0} \longrightarrow E_{\lambda_1}$ the parallel transport along α. Let $S^{(\lambda_1)} = \Pi_\alpha(L^{(\lambda_0)})$ be the parallel transport of the Lagrangian sphere $L^{(\lambda_0)}$ to the fiber $\Sigma^{(\lambda_1)} = E_{\lambda_1}$ of E over λ_1. See Fig. 23. By Corollary 5.1.1 we have in $D\mathcal{F}uk^*(\Sigma^{(\lambda_1)})$ an isomorphism:

$$S^{(\lambda_1)} \cong \mathrm{cone}\big(S_1 \otimes E \longrightarrow L^{(\lambda_1)}\big), \tag{42}$$

where $S_1 \subset \Sigma^{(\lambda_1)}$ is the vanishing cycle associated to the critical point of π over 0 and the path $\alpha'|_{(0,0),(-1,-1)} * \alpha''$. According to (28), the space E is $HF(\hat{S}_1, W)$, where $\hat{S}_1$ is the matching cycle emanating from z_1, which lies in a suitable extension of the fibration E (see Sect. 4.4.2).

In our case, it is not hard to see that $\hat{S}_1$ intersects W at a single point and the intersection is transverse. Therefore E is a 1-dimensional space. Applying $\phi^{(\lambda_1)}$ to (42) we now obtain the following isomorphism in $D\mathcal{F}uk^*(\mathbb{C}P^1 \times \mathbb{C}P^1)$:

$$\phi^{(\lambda_1)}(S^{(\lambda_1)}) \cong \mathrm{cone}\big(\phi^{(\lambda_1)}(S_1) \longrightarrow T\big).$$

By a result of Hind [11] all Lagrangian spheres in $\mathbb{C}P^1 \times \mathbb{C}P^1$ are Hamiltonian isotopic. Thus both $\phi^{(\lambda_1)}(S^{\lambda_1})$ and $\phi^{(\lambda_1)}(S_1)$ are Hamiltonian isotopic to the anti-diagonal S. It follows that in $D\mathcal{F}uk^*(\mathbb{C}P^1 \times \mathbb{C}P^1)$ we have the following isomorphism: $S \cong \mathrm{cone}\big(S \longrightarrow T\big)$. Rotating the exact triangle corresponding to this cone gives the following isomorphism in $D\mathcal{F}uk^*(\mathbb{C}P^1 \times \mathbb{C}P^1)$:

$$T \cong \mathrm{cone}\big(S \longrightarrow S\big). \tag{43}$$

Remarks a. The existence of an isomorphism of the type (43) is presumably well known in mirror symmetry and the theory of Fukaya categories. It could probably be derived also by the methods of [5,6] by the following construction whose details need to be worked out. Let S' be a Hamiltonian isotopic copy of S such that S, S' intersect transversely at two points. Perform a "figure-Y" surgery with ends corresponding to S and S' as described in [5, Section 6.1]. The result is a Lagrangian cobordism $V \subset \mathbb{R}^2 \times M$ with two negative ends being S and S' and one positive end being a torus T' which should be isotopic to T. For suitable choice of handles in the surgery the cobordism V should be monotone. The cone decomposition in (43) would now follow from the main results of [6].

b. Our work does not provide much information about the precise morphism $S \longrightarrow S$ from (43). It would be interesting to determine the precise map and also to figure out how (43) behaves with respect to grading (in this case a $\mathbb{Z}_2$-grading).

One can alter the above construction of E and V to obtain other Lefschetz fibrations, e.g. with more critical values and with other combinations of ends. This can be done for example by choosing the disk D to contain a different subset of the points in (41) or even none of them (subject to the requirement that D is invariant under c_H). More details can be found in the expanded version of this paper [3, Page 117].

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