THE GENUS AND THE FUNDAMENTAL GROUP OF HIGH DIMENSIONAL MANIFOLDS

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We consider the genus of an arbitrary compact connected manifold, geometrically defined in terms of separations properties of hypersurfaces, and characterise it exclusively in terms of the fundamental group. Related questions, explicit computations and 3-dimensional examples are also included.

INTRODUCTION

We consider the obvious generalization of the classical definition of the genus of a compact orientable surface; to be precise take the maximal number of disjoint, compact connected two-sided hypersurfaces, which do not disconnect the given manifold M. Our first main result characterizes the genus of M as the maximal size (= number of free generators) of a free quotient group of $\pi_1(M)$; this is accomplished in Theorem 1 (Section 1). Several classes of natural, geometrically meaningful, computational examples are explicitely worked out in Section 3.

The method of proof of Theorem 1 turns out to be suited for giving a satisfactory answer to a second equally natural question, namely: when the members of a finite system of integral codimension 1 homology classes of a closed oriented manifold M do admit disjoint realizations by submanifolds. The necessary and sufficient conditions are formulated in Theorem 2, also in terms of homomorphisms of $\pi_1(M)$ into free groups; the proof is given in Section 2.

The final section contains results related to the special properties of the genus of 3-manifolds.

To our knowledge, it seems that these highly natural questions have been ignored in the literature. Even more surprisingly, they may be handled by using rather elementary tools, as it will be seen. On the other hand, the relatively satisfactory degree of computability of the genus and its relevance to geometry give hope for a future use of it as a parameter in inductive arguments, for example in low-dimensional topology.

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DEFINITIONS AND PRELIMINARY REMARKS

In what follows, we shall consider the class V^n $(n \geqslant 1)$ of smooth n-dimensional manifolds M with boundary, which are compact and connected. For such an M, we shall also consider the class $V_i(M)$ of smooth codimension i neat compact submanifolds V of M, whose normal bundles are trivial. Denoting by $N(A)$ the number of connected components of A, we can make our first main definition.

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Definition 1. For $M \in V^n$ the *genus* of M, to be denoted by $C(M)$, is defined by $C(M) = \sup \{N(V) : V \in V_1(M) \text{ and } N(M - V) = 1\}$.

For $n = 2$, Bd $(M) = \emptyset$ and M orientable, we recover the classical definition of the genus of a surface (see e.g. [4], p. 205). The first elementary (but still very useful) properties of the genus are obtainable from the generalized Jordan theorem (see e.g. [2], p.261), if Bd$(M) = \emptyset$, plus a standard doubling argument, in the general case. We are now going to deduce them. (As a matter of notation, here and in the sequel, homology, cohomology and intersection numbers will be understood with $\mathbb{Z}_2$ coefficients, unless otherwise stated; z^*, respectively a_*, will denote Poincaré duals of homology, respectively cohomology classes.)

LEMMA 1. *Suppose* $M \in V^n$, Bd$(M) = \emptyset$ *and let* $V \in V_1(M)$. *Then* $N(M - V) = 1$ *if and only if the fundamental classes of the connected components of* V *represent linearly independent elements of* $H_{n-1}(M)$.

Proof. Write $V = V_1 \cup V_2 \cup \ldots \cup V_r$ (connected components). By the generalized Jordan theorem $N(M - V) = 1 + \dim (\ker (H_{n-1}(V) \to H_{n-1}(M)))$. The lemma follows.

LEMMA 2. *For* $M \in V^n$ *if* Bd $(M) = \emptyset$ *then* $C(M) \leqslant \dim(H_1(M))$. *In general* $C(M) < \infty$.

Proof. The first assertion is a direct consequence of the previous lemma, via Poincaré duality. To prove the second one, start with $V \in V_1(M)$, $N(M - V) = 1$ and double everything: more precisely let $\widetilde{M}$ be the double of M and let $\widetilde{V} \in V_1(M)$ be obtained by doubling the components of V with nonempty boundary. It follows that $N(\widetilde{M} - \widetilde{V}) = 1$ and $N(\widetilde{V}) = N(V)$, hence $N(V) \leqslant C(\widetilde{M})$ and consequently $C(M) < \infty$, by the boundaryless previous case.

The next result points out another convenient geometric signification of the genus. Let $M \in V^n$, Bd$(M) = \emptyset$ and $A \in V_1(M)$. Let $D(M, A) = N(A) - N(M - A) + 1$.

PROPOSITION 1. *We have* $C(M) = \max \{D(M, A) : A \in V_1(M)\}$.

Proof. We notice $N(A) = \dim (H_{n-1}(A))$ so $D(M, A) = \dim (\mathrm{Im}\,(i^A : H_{n-1}(A) \to H_{n-1}(M)))$, again by the Jordan theorem. If $D(M, A) = k$ then Lemma 1 and linear algebra imply that there exists $A' \subset A$, $A' \in V_1(M)$ with $N(A') = k$ and $N(M - A') = 1$. Consequently $D(M, A) \leqslant C(M)$. On the other side for $B \in V_1(M)$ with $N(M - B) = 1$ we have $D(M, B) = N(B)$. The stated equality follows.

Remark 1.a. In [1], using a different method, we prove the above equality even when Bd$(M) \neq \emptyset$; we will not need this fact in the following.

b. From the precedent proof it also results that when Bd$(M) = \emptyset$, $C(M) = \max \{\dim (\mathrm{Im}(i_*^V : H_{n-1}(V) \to H_{n-1}(M))) : V \in V_1(M)\}$.

1. THE GENUS AND THE FUNDAMENTAL GROUP

It is the moment to give our second definition. In the sequel F_r will denote the free group on r generators.

Definition 2. Let G be a group. We define the genus of G, to be denoted by $C(G)$, by $C(G) = \sup\{r : \textit{there exists an epimorphism } G \to F_r\}$.

LEMMA 3. *If G is finitely generated then $C(G) < \infty$.*

Proof. An epimorphism $G \to F_r$ abelianizes to an epimorphism $G_{ab} \to Z^r$, whence $r \leqslant \text{rank}\,(G_{ab}) < \infty$.

THEOREM 1. *For any $M \in V^n$ we have $C(M) \leqslant C(\pi_1(M))$ with equality holding if* $\mathrm{Bd}(M) = \emptyset$.

Proof. We will prove first that $C(M) \leqslant C(\pi_1(M))$. Let $A_i \in V_1(M)$ connected and disjoint, $1 \leqslant i \leqslant k$, with $N(M - \cup(A_i : 1 \leqslant i \leqslant k)) = 1$ where $k = C(M)$. Pick disjoint trivialized tubes of A_i, say $T_i = A_i \times [-1, +1]$, and perform a slightly modified Thom-Pontryagin construction to obtain a map f from M to a wedge of circles $\bigvee_k (S^1)$, by making k disjoint standard constructions and then glueing them together by throwing the complement of $\cup\,\mathrm{Int}(T_i)$ to the basepoint of the wedge. Choose a basepoint $x^0 \in M - \cup\,\mathrm{Int}(T_i)$ and a point $x_i \in A_i$, for any i. In order to deduce the desired inequality, we are going to show that $\pi_1(f)$ is epic. To this end we shall construct, for any $1 \leqslant i \leqslant k$, a loop g_i in M based at x^0, which covers via f the i-th circle of the wedge. The construction is possible because we know that $M - \cup\,\mathrm{Int}(T_i)$ is connected, and goes as follows: start from x^0 and first reach $x_i \times \{-1\}$, staying away from $\cup\,\mathrm{Int}(T_i)$, then linearly move from $x_i \times \{-1\}$ to $x_i \times \{+1\}$, and finally go back to x^0 still staying in $M - \cup\,\mathrm{Int}(T_i)$.

Let us prove now the opposite inequality. For a loop $g \subset M$ and $D \in V_1(M)$ we will denote by $\langle g, D\rangle$ their intersection number (mod 2) and by $[g]\cdot[D]$ their intersection number in terms of homology classes. Let $F : \pi_1(M) \to F_k$ be an epimorphism. Because $T = S_1 \vee S_2 \vee \ldots \vee S_k$ (the bouquet of k circles) is an Eilenberg MacLane space there exists a map $\bar{F} : M \to T$ such that $\pi_1(\bar{F}) = F$ (see e.g. [10]). Let $\{P\} = S_1 \cap \cap S_2 \cap \ldots \cap S_k$ and let $P_i \in S_i - \{P\}$ be regular values for $\bar{F}$ and let $A_i = \bar{F}^{-1}(P_i)$, $1 \leqslant i \leqslant k$. From the surjectivity of F it follows the existence of the loops g_i such that $F(\langle g_i\rangle) = \langle S_i\rangle$. (Here and in the sequel we will denote, for a loop g, by $\langle g\rangle$ its homotopy class and by $[g]$ its homology class.) We may suppose that these loops intersect transversely the A_i's. These loops verify $\langle g_i, A_j\rangle = \delta_{ij}$ (δ_{ij} is Kronecker symbol). Let us assume now that $\mathrm{Bd}(M) = \emptyset$. We have $[g_i]\cdot[A_j] = \delta_{ij}$. It follows that the homology classes $[A_1], \ldots, [A_k] \in H_{n-1}(M)$ are linearly independent. Using Proposition 1 and Remark 1.b. it follows that $C(M) \geqslant D(M, \cup(A_i : 1 \leqslant i \leqslant k)) \geqslant k$.

Remark 2.a. Using the doubling argument appearing in Lemma 2 and the result mentioned in Remark 1.a. it is easy to prove that if $\mathrm{Bd}(M) \neq \emptyset$ we also have $C(M) \geqslant C(\pi_1(M)) - N(\mathrm{Bd}(M)) + 1$. Consequently the stated equality holds even if M has nonempty but connected boundary.

b. Here is an extension of the "easy" part of Theorem 1.

Suppose that $\mathrm{Bd}(M) = \emptyset$ and let $V_1, V_2, \ldots, V_r \in V_1(M)$ be in general position and such that, their homology classes, $[V_1], \ldots, [V_r]$, are linearly independent in $H_{n-1}(M)$ and generate $H_{n-1}(\cup V_i)$. Then $C(\pi_1(M - \cup(V_i \cap V_j : i \neq j))) \geqslant r$.

Indeed, we may choose open tubes T_i around V_i and consider the manifold with boundary $M' = M - \cup(T_i \cap T_j : i \neq j)$ whose fundamental group is isomorphic to the one in our statement. Set $V'_i = V_i \cap M'$ and notice that $V'_i \cap V'_j = \emptyset$ for $i \neq j$. We claim now that it is enough to know that $V' = \cup V'_i$ does not disconnect M', if so, we are able to use the same arguments as in Theorem 1 and conclude that $C(\pi_1(M')) \geqslant r$. As far as this claim is concerned, first notice that $N(M' - V') = N(M - V)$, $V = \cup V_i$, and then use again the Jordan theorem.

c. We end this section with a generalization of Theorem 1. Let $M \in V^n$, Bd $(M) = \emptyset$, and let $A \in V_1(M)$. It is possible to define the genus of M with respect to A, $C(M, A) = \max \{D(M, B) : B \in V_1(M), B \cap A = \emptyset\}$. Using the same method as in Proposition 1 we easily obtain $C(M, A) = \max \{N(B) : B \in V_1(M), N(M - B) = 1, B \cap A = \emptyset\}$. The inclusion $A \subset M$ induces a morphism $i : \pi_1(A) \to \pi_1(M)$. Let $\langle i(\pi_1(A))\rangle$ be the normal subgroup generated by $i(\pi_1(A))$ in $\pi_1(M)$. It can be proved that we have $C(M,A) = C(\pi_1(M)/\langle i(\pi_1(A))\rangle)$. Because in the following we will not make any use of this fact we shall not prove it here. We mention only that the proof of the inequality $C(\pi_1(M)/\langle i(\pi_1(A))\rangle) \geqslant C(M,A)$ is immediate using the function constructed at the beginning of the proof of Theorem 1 and that the other inequality is proved in two steps: the first one is the same as in the proof of Theorem 1, while the second is carried as to modify the resulting submanifold in order to avoid A and thus requires some extra case and cut and paste techniques.

2. DISJOINT REALIZATIONS

We will now study the second problem announced in the introduction. Let $M \in V^n$ be oriented, $\mathrm{Bd}(M) = \emptyset$. For $D \in V_1(M)$ let $[D] \in H_{n-1}(M; \mathbb{Z})$ be its homology class.

THEOREM 2. *Given $a, b \in H_{n-1}(M; \mathbb{Z})$, there exists $A, B \in V_1(M)$ such that $A \cap B = \emptyset$, $[A] = a$, $[B] = b$ iff there exists a morphism $f : \pi_1(M) \to F_2$ such that $P_1 \circ f = a^*$ and $P_2 \circ f = b^*$ (here F_2 has the generators u_1 and u_2 and $P_j : F_2 \to Z$ is the morphism defined by $P_j(u_i) = \delta_{ij}$).*

Proof. Let $A, B \in V_1(M)$ be such that $A \cap B = \emptyset$, $[A] = a$, $[B] = b$. Let U_1, U_2 be tubular neighbourhoods of A respectively B such that $U_1 \cap U_2 = \emptyset$. We define $F : M \to S_1 V S_2$ as in the beginning of the proof of Theorem 1 (S_1, S_2 are circles). We have $A = F^{-1}(Q_1)$, $B = F^{-1}(Q_2)$, $Q_1 \in S_1 - S_2$, $Q_2 \in S_2 - S_1$. Let $F_i = \bar{P}_i \circ F$ where $\bar{P}_i : S_1 V S_2 \to S_i$ is a continuous map such that $\pi_1(\bar{P}_i) = P_i$. We notice that $[F^{-1}(Q_i)]^* = P_i \circ \pi_1(F)$ in $H^1(M; \mathbb{Z}) = [M, S^1] = \mathrm{Hom}(\pi_1(M), \mathbb{Z})$. We may take $f = \pi_1(F)$.

Now let $f : \pi_1(M) \to F_2$ be such that $P_1 \circ f = a^*$ and $P_2 \circ f = b^*$. Let $F : M \to S_1 V S_2$ be such that $\pi_1(F) = f$ and let Q_i be a regular value for $\bar{P}_i \circ F$ different from $\bar{P}_i(S_1 \cap S_2)$. We have $[(\bar{P}_1 \circ F)^{-1}(Q_1)] = a$,

$[(\bar{P}_2 \circ F)^{-1}(Q_2)] = b$. If $A = (\bar{P}_1 \circ F)^{-1}(Q_1)$, $B = (\bar{P}_2 \circ F)^{-1}$ then $A, B \in V_1(M)$, $[A] = a$, $[B] = b$ and $A \cap B = \emptyset$.

Remark 3. The preceding result admits an obvious extension for the case of $r \geqslant 2$ homology classes.

Clearly two linearly dependent codimension 1 homology classes may be represented by disjoint hypersurfaces. The preceding result helps to produce an interesting class of examples where this is the only possibility, namely manifolds with nilpotent fundamental group. Suppose indeed that $a_i^* \in H^1(M; \mathbb{Z})$ are given by $a_i^* = P_i \circ f$, $i = 1,2$, for some morphism $f: \pi_1(M) \to F_2$. But then the image of f, being simultaneously free and nilpotent, must be either trivial or Z, hence a_1^* and a_2^* must be linearly dependent. On the other hand, as it is well-known, we may consider the vanishing of the cup-product $a_1^* \cdot a_2^* \in H^2(M; \mathbb{Z})$ as a first obstruction to disjoint realizability of a_1 and a_2. On cohomologically simple examples as $M = S^1 \times S^1$, this is the only one obstruction. However, there are equally simple examples with two-step nilpotent fundamental group which indicate how much subtler the general decision could be. Take for instance $M^3 =$ associated circle bundle of the complex line bundle over $S^1 \times S^1$ with Chern class $c_1 = [\text{point}]^* \in H^2(S^1 \times S^1; \mathbb{Z})$. Consider the classes $[S^1 \times \text{point}]^*$, $[\text{point} \times S^1]^* \in H^1(S^1 \times S^1; \mathbb{Z})$ and denote by a_1^* and a_2^* their pull-back to $H^1(M^3; \mathbb{Z})$; they are linearly independent there, hence their disjoint realization is not possible, due to the nilpotence of $\pi_1(M^3)$. On the other hand $a_1^* \cdot a_2^*$ is the pull-back of c_1 and consequently is zero.

A natural question to ask for two homology classes which do not admit disjoint representatives is how "small" can be made their intersection. From the combinatorial point of view an answer is given by the following proposition.

PROPOSITION 2. *Let $M \in V^n$ be orientable with* $\mathrm{Bd}(M) = \emptyset$ *and let $a, b \in H_{n-1}(M; \mathbb{Z})$ be indivisible and linearly independent. There exists $A, B \in V_1(M)$ with transverse intersection such that $[A] = a$, $[B] = b$ and $N(A \cap B) \leqslant 1$.*

Proof. We fix an orientation for M and on $U \in V_1(M)$ we take the orientation induced from M and from the normal trivial bundle of U. Let $C_1, C_2 \in V_1(M)$ with $[C_1] = a$, $[C_2] = b$ be such that C_1, C_2 intersect transversely and let $E = C_1 \cap C_2$. Let g be a loop $g \subset C_i$ $i \in \{1,2\}$ which intersects transversely E. We will say that two points $P, Q \in g$ are neighbours if $P, Q \in E$ and there exists a connected component L of $g - \{P, Q\}$ such that $L \cap E = \emptyset$. For $P \in g \cap E$ let $s(P)$ be the Z-intersection number of g with E at P. We will say that g is reduced with respect to E if for each pair P, Q of neighbouring points such that $s(P)s(Q) < 0$, P and Q are placed in different connected components of E. Finally we will say that the loop g is full (with respect to E) if for each $P, Q \in g \cap E$ we have $s(P) = s(Q)$.

Let $g \subset C_1$ be a reduced loop which is not full. Then there exists $P, Q \in g \cap C_2$ and a connected component g' of $g - \{P, Q\}$ such that

$s(P)s(Q) < 0$, $g' \cap E = \emptyset$. Let W be the closure of a tubular neighbourhood of g' (W is diffeomorphic with $I \times D^{n-1}$, D^{n-1} being the $n-1$ dimensional disc and $I = [0,1]$) such that there exists two $n-1$ dimensional discs D_1, D_2 with $P \in D_1$; $Q \in D_2$; $D_1, D_2 \subset C_2$ and $W \cap (E - (D_1 \cup \cup D_2)) = \emptyset$. Let $C_1' \in V_1(M)$ be obtained by roundening the corners of $(C_1 \cup \mathrm{Bd}(W)) - (D_1 \cup D_2)$. From $s(p)s(Q) < 0$ we obtain that $[C_1'] = [C_1]$. We remark that C_1' intersects C_2 transversely and $N(C_1' \cap C_2) < N(E)$. If $N(E) \leqslant 1$ then any reduced loop is full. Obviously the preceding construction may be performed also on C_2. Applying this method repeatedly we obtain $D_1, D_2 \in V_1(M)$ with transverse intersection, $[D_1] = a$, $[D_2] = b$ and each loop $g \subset D_i$, $i = 1,2$, which is reduced is full (with respect to $D_1 \cap D_2 = F$). Let us suppose that there exists a connected component F_1 of F such that $N(D_1 - F_1) = 2$, $D_1 - F_1 = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$. If there exists F', F'' connected components of F which are not disconnecting D_1, with $F' \subset X_1$, $F'' \subset X_2$ then there exists a loop g_1 with $N(g_1 \cap F') = 1$, $N(g_1 \cap F'') = 1$, $\langle g_1, F_1 \rangle = 0$ and g_1 may be choosed to be reduced. Of course g_1 is not full. Consequently X_1 or X_2 contains only components of F that disconnect D_1. Hence there exists a connected component R of $D_1 - F$ such that $N(\bar{R} - R) = 1$ and Int $(\bar{R}) = R$. We will apply a typical cut and paste procedure. Let $K = \bar{R} - R$ and let W be the closure of a tubular neighbourhood of D_1 such that the connected component T of $W \cap D_2$ which contains K is diffeomorphic with $K \times [0,1]$, and $T \cup R_1 \cup R_2$ is diffeomorphic with $\mathrm{Bd}(R \times [0,1])$. Here $R_1, R_2 \subset W - (\mathrm{Int}\,(W) \cup D_2)$ are diffeomorphic with R and placed on different sides of R. Choosing the convenient orientations for R_1 and R_2 and roundening the corners of the set $(D_2 - T) \cup \cup R_1 \cup R_2$ we obtain a hypersurface $D_2' \in V_1(M)$ interesecting transversely D_1, with $[D_2'] = [D_2]$, $N(D_2' \cap D_1) < N(D_2 \cap D_1)$. Applying repeatedly this method the problem reduces to the case of $B_1, B_2 \in V_1(M)$ with transverse intersection $B_1 \cap B_2 = G$, each reduced loop $g \subset B_i$ $i = 1,2$ being full with respect to G and each component of G does not disconnect either B_1 or B_2. Moreover a and b being indivisible and linearly independent B_1 and B_2 may be supposed connected. In this case any two connected components G_1, G_2 of G disconnect B_i, $i = 1, 2$. Indeed if $G_1 \cup G_2$ does not disconnect B_i then there exists a loop $h \subset B_i$ such that $h \cap G_j = Y_j$, $j = 1,2$, $N(Y_j) = 1$ and $s(Y_1)s(Y_2) < 0$ (h intersects transversely G_1 and G_2). We may suppose h reduced but it follows that it is not full. Consequently any two connected components of G disconnect B_i, $i = 1,2$. By induction on $N(G)$ it results that there exist connected components $H_i \subset B_i - G$ such that $\bar{H}_1 - H_1 = \bar{H}_2 - H_2 = J$ and $N(J) = 2$. This fact enables us to apply a cut and paste procedure (we cut B_1 after J and we paste two pieces H', H'' diffeomorphic with H_2 and placed on opposite sides of B_2) which does not alter the homology classes of B_i, $i = 1,2$ (because we may choose for H' and H'' the convenient orientations). We notice that we obtain $B_1' \in V_1(M)$ such that $B_1' \cap B_2 = V \in V_2(M)$, $N(V) < N(G)$, $[B_1'] = a$. Applying repeatedly this procedure (also on B_2) we obtain the desired hypersurfaces.

3. EXAMPLES

In this section we shall consider the (geometric and group theoretic) genus from the computational point of view.

Example 1. Surfaces. Let $S \in V^2$, $\mathrm{Bd}(S) = \emptyset$. If S is orientable and diffeomorphic to the connected sum of g tori then $C(S) = g$. If S is not orientable and it is diffeomorphic to the connected sum of r projective planes then $C(S) = [r/2]$ (where if $r \in \mathbb{R}$ then $[r]$ is its entire part).

The inequality $C(S) \geqslant g$, respectively $C(S) \geqslant [r/2]$, is obvious, both from the geometric and group-theoretic point of view. For the reverse inequality we shall use a general useful majoration of the genus. For a finitely generated group and a coefficient field F we shall define following [8], the F isotropy index of G, to be denoted by $i_F(G)$, as the maximum dimension of subspaces, of $H^1(G\,;F)$ on which the bilinear form induced by the cup-product, $H^1(G\,;F) \otimes H^1(G\,;F) \to H^2(G\,;F)$ is trivial. It easily follows from the definition of the genus that we always have $C(G) \leqslant i_F(G)$. The same definition can be made for $i_F(X)$, where X is a finite connected complex, by just replacing G by X. Standard algebraic topology arguments show that $i_F(X) = i_F(\pi_1(X))$. When X is a surface we may use Poincaré duality (with $F = Q$ or $\mathbb{Z}_2$) to deduce, as in [8], that $i_F(X) \leqslant [\dim_F(H^1(X\,;F))/2]$.

We turn now to the non-empty boundary case. The general inequality provided by Theorem 1, namely $C(M) \leqslant C(\pi_1(M))$ turns out to be very useful here, becoming in fact an equality. To be specific, we have $C(\#_g T^2 - \coprod_b \mathrm{Int}(D^2)) = 2g-1+b$ $(g \geqslant 0, b > 0)$ and $C(\#_h P^2 - \coprod_b \mathrm{Int}(D^2)) = h-1+b$. To see this, notice first that the fundamental group of S is free on the stated number of generators. On the other hand the desired minoration of $C(S)$ is directly readable from the canonical combinatorial form of S.

The above example suggests that the genus is additive with respect to the connected sum of manifolds. In the following example we will prove this.

Example 2. Free product. If $M, N \in V^n$, $n \geqslant 3$, with $\mathrm{Bd}(M) = \mathrm{Bd}(N) = \emptyset$ then $C(M \# N) = C(M) + C(N)$.

Using Theorem 1 it results that it is enough to prove that for 2 finitely generated groups G_1 and G_2 we have $C(G_1 * G_2) = C(G_1) + C(G_2)$. Let $C(G_1) = n$ and $C(G_2) = m$. Clearly there exists an epimorphism $g : G_1 * G_2 \to F_n * F_m = F_{n+m}$. Let $f : G_1 * G_2 \to F_r$ be an epimorphism, then there exist two epimorphisms $f_1 : G_1 * G_2 \to f(G_1) * f(G_2)$ and $f_2 : f(G_1) * f(G_2) \to F_r$ such that $f_2 \circ f_1 = f$. Obviously $f(G_1)$, $f(G_2)$ are free and $f(G_1)$ is generated by less than n generators and $f(G_2)$ by less than m generators. Consequently $r \leqslant n + m$.

In [1] we prove this property by geometric methods.

The next example gives an estimation for the genus of product manifolds.

Example 3. Direct products. Let $M \in V^n$, $N \in V^m$, $n, m \geqslant 1$, $\mathrm{Bd}(M) = \mathrm{Bd}(N) = \emptyset$. We have $C(M \times N) = \max\{C(M), C(N)\}$.

Because $\pi_1(M \times N) = \pi_1(M) \times \pi_1(N)$ it is enough to prove that for two finitely generated groups G_1, G_2 we have $C(G_1 \times G_2) = \max\{C(G_1), C(G_2)\}$. Let $f : G_1 \times G_2 \to F_r$, $r \geqslant 2$ (the case $r \leqslant 1$ is easely dealt with) be an epimorphism. Let $g_1, g_2, \ldots, g_r$ be the generators of F_r. Let us suppose that for each $i \leqslant r$ we have $g_i = f((w_i, w_i'))$ with $w_i \in G_1 - \{1\}$, $w_i' \in G_2 - \{1\}$. Then $f((w_i, 1)) = g_i u_i$, $f((1, w_i')) = u_i^{-1}$ and u_i must commute with g_i. It results, g_i being a generator, that $u_i = g_i^{q_i}$. It follows that $f((w_i, 1)) = g_i^{s_i}$, $f((1, w_i')) = g_i^{r_i}$, $s_i, r_i \neq 0$. But then for $i \neq j$, $g_i^{s_i}$ and $g_j^{r_j}$ must commute, a contradiction. Consequently there exists $w \in G_1$ or $w' \in G_2$ and $i \leqslant r$ such that $f((w, 1)) = g_i$ or $f((1, w')) = g_i$. We will discuss only the first case, the second being similar. For each $h \in G_2$, $f((w, 1))$ and $f((1, h))$ must commute. Consequently $f((1, h)) = g_i^t$ so $f(1 \times G_2) \subset f(G_1 \times 1)$. It follows $r \leqslant C(G_1)$. The second case will give $r \leqslant C(G_2)$, so $C(G_1 \times G_2) \leqslant \max\{C(G_1), C(G_2)\}$. The opposite inequality is immediate.

Example 4. Central extensions and circle bundles over manifolds. Let $0 \to A \to G \to H \to 1$ be a central extension of finitely generated groups (for $A = Z$, G may geometrically arise as the fundamental group of a circle bundle manifold over a base manifold whose fundamental group is H; see also Example 6). We assert that $C(G) = C(H)$ except when $C(H) = 0$ and $C(G) = 1$. Obviously the inequality $C(G) \geqslant C(H)$ is valid in general. Suppose that $C(G) = r > 1$ and let $g : G \to F_r$ be an epimorphism. Since A lies in the center of G, $g(A)$ must lie in the center of F_r, which is trivial. That implies that g factorizes to an epimorphism $h : H \to F_r$ and consequently $r \leqslant C(H)$. The case $C(G) = 0$ is clear and when $C(G) = 1$ the only possible exception would be $C(H) = 0$. This situation may indeed occur, as the example $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$ shows.

Example 5. Semidirect products and mapping tori. Semidirect products of the form $1 \to H \to G \underset{s}{\rightleftarrows} F_r \to 1$ naturally appear. We shall consider here the case $r = 1$ (in this case G may geometrically arise as the fundamental group of the mapping torus of $h : M \to M$ where $\pi_1(M) = H$, and the action of $\mathbb{Z}$ on H in the resulting extension is given by $\pi_1(h)$; see also Example 6). We assert that if $r = 1$ then $C(G) \leqslant \max\{C(H), 1\}$ (compare with Example 3). Let $g : G \to F_l$ be epic. We know that G is generated by H and $s(1)$ and that $s(1)$ normalizes H; consequently F_l will be generated by $g(H) = F_m$ and $gs(1)$ (hence $l \leqslant m + 1$) and $gs(1)$ will normalize F_m. If either F_m or $gs(1)$ is trivial then the stated inequality is obvious; suppose then that $m \geqslant 1$ and $gs(1)$ generates an infinite cyclic subgroup. If $l \leqslant m$ we are done. Supposing that $l = m + 1$ we may construct an epimorphism $F_{m+1} \to F_l$ (which must necessarily be an isomorphism, (see [3], p. 205) and deduce that the free generator x_{m+1} of F_{m+1} normalizes the subgroup generated by the remaining free generators, a contradiction.

4. 3-MANIFOLDS

First let us notice that if $M \in V^3$, $\mathrm{Bd}(M) = \emptyset$ is an irreducible manifold which is not Haken then the free part of $H_1(M; Z)$ is trivial so $C(M) = 0$ (see [5] for some properties of Haken manifolds). Consequently using Example 2 it results:

COROLLARY 1. *If for $M \in V^3$ with $\mathrm{Bd}(M) = \emptyset$ we have $C(M) > 0$ then in the prime decomposition (see [6], [7]) of M at least one factor is Haken or is diffeomorphic with $S^1 \times \varphi S^2$ where φ preserves or not the orientation.*

COROLLARY 2. *If $M \in V^3$, $\mathrm{Bd}(M) = \emptyset$ and $h(M)$ is the Heegaard genus of M then $C(M) \leqslant h(M)$ and $C(M) = h(M)$ iff $\pi_1(M)$ is free (on $h(M)$ generators).*

Indeed Van Kampen's theorem applied for the Heegaard decomposition of M of minimal genus implies that there exists an epimorphism $f : F_t \to \pi_1(M)$, $t = h(M)$. It follows $C(\pi_1(M)) \leqslant h(M)$ and that $\pi_1(M)$ is free when $C(M) = h(M)$ (because in this case there exists an epimorphism $g : \pi_1(M) \to F_t$ and we argue as in Example 5).

Example 6. Seifert fibered manifolds, surface bundle over the circle and periodic monodromy. (We use Seifert manifolds as defined in [5].)

We start by coming back to the group-theoretic situation of Example 5. Let then $1 \to H \to G \underset{s}{\rightleftarrows} Z \to 0$ be a semidirect product extension of finitely generated groups. It is expectable that the upper bound we found for $C(G)$, namely $\max \times \{C(H), 1\}$, will not be sharp in general, simply because it does not involve the structural monodromy operator of the extension, to be denoted by u, $u \in \mathrm{Aut}(H)$. We are going to suppose here that u is periodic and derive a much more accurate evaluation of $C(G)$. To be more precise, define H_u = quotient group of H obtained by introducing the relations $u(x) = x$, $x \in H$. We assert that if u is periodic and $C(G) \neq 1$ then $C(G) = C(H_u)$. To see this, first notice the existence of an epimorphism $G \to H_u$, hence the inequality $C(G) \geqslant C(H_u)$ which is valid in general. Suppose then that $C(G) = r > 1$. An epimorphism $g : G \to F_r$ is given by a morphism $h : H \to F_r$ together with an element $y \in F_r$, with the property that $yh(x)y^{-1} = hu(x)$, $x \in H$, and such that F_r is generated by y and $\mathrm{Im}(h)$. Knowing that $u^p = \mathrm{id}$, we deduce that y^p centralizes $\mathrm{Im}(h)$, therefore $y^p = 1$ and then necessarily $y = 1$. This in turn implies that h is epic and factorizes to H_u, hence $r \leqslant C(H_u)$ and we are done.

As a concrete interesting class of examples consider an orientable surface group H, given as usual by the generators $x_1, y_1, \ldots, x_g, y_g$ ($g \geqslant 1$) with the relation $\Pi[x_i, y_i] = 1$, and the periodic automorphism u given by $u(x_1) = x_2, \ldots, u(x_g) = x_1$, $u(y_1) = y_2, \ldots, u(y_g) = y_1$. Denote by G the coresponding semidirect product and notice that H_u is given by the generators x and y with the relation $[x, y]^g = 1$; it easily follows that $C(H_u) = 1$. By our previous result $C(G) = 1$.

If $M \in V^3$, $\mathrm{Bd}(M) = \emptyset$ is a Seifert fibered manifold we have the following central extension $0 \to \mathrm{K} \to \pi_1(M) \to \pi_1(X) \to 1$ where $\pi_1(X)$ is the fundamental group of the base orbifold [9], and K is a cyclic group. From Example 4 we know that if $C(\pi_1(X)) \geqslant 1$ then $C(M) = C(\pi_1(X))$. But $\pi_1(X) = \langle a_1, b_1 \ldots, a_g, b_g, c_1, \ldots, c_t : c_i^{r_i} = 1, 1 \leqslant i \leqslant t, \Pi[a_i, b_i] \cdot c_1 \ldots \cdot c_t = 1\rangle$ $g, r_i, t \in \mathbb{N}^*$, if X is orientable and $\pi_1(X) = \langle a_1, \ldots, a_k, c_1, \ldots, c_s : c_i^{s_i} = 1, 1 \leqslant i \leqslant s, \Pi a_i^2 \cdot c_1 \ldots \cdot c_s = 1\rangle$ if X is not orientable. It is immediate that $C(\pi_1(X)) = C(\langle a, b_1, \ldots, a_g, b_g : \Pi[a_i, b_i] = 1\rangle)$ in the first case and $C(\pi_1(X)) = C(\langle a_1, \ldots, a_k : \Pi a_i^2 = 1\rangle)$ in the second. Consequently for a Seifert fibered manifold M with $\mathrm{Bd}(M) = \emptyset$ if the basis surface S has $C(S) \geqslant 1$ then $C(M) = C(S)$ and if $C(S) = 0$ then $C(M) \leqslant 1$. Conversely it follows that if $C(M) = l > 1$ then M admits Seifert fibrations with basis of genus l (i.e. C-invariant). There are 3 surfaces with $C = l > 2$. The orientable surface of genus l, the connected sum of $2l$ projective planes and the connected sum of $2l + 1$ projective planes.

Let $M \in V^3$, $\mathrm{Bd}(M) = \emptyset$. If M is the total space of a surface bundle over a circle we have the exact sequence $0 \to \pi_1(S) \to \pi_1(M) \to \mathbb{Z} \to 0$. Here S is the canonical fiber. From Example 5 we know that $1 \leqslant C(M) \leqslant \max\{C(S), 1\}$. In this case if the glueing map u is homotopically periodic ($\pi_1(u^{n}) = \mathrm{id}$) we also know that it is homotopic to a periodic homeomorphism (see [5] and [9] p. 443) and this implies that M admits a Seifert fibration structure. The preceding results help us to discover the possible basis for this structure.

For example if S is orientable of genus g and u is the periodic monodromy we already have considered (obviously it may be represented by a periodic homeomorphism) we know that $C(M) = 1$. On the other hand it is immediate to see that $rkH_1(M; \mathbb{Z}) = 3$. If X is the basis of a Seifert structure for M then necessarily the rank of $H_1(X; \mathbb{Z})$ is 2 or 3. If X is orientable then the rank must be two and the basis must be a torus. If X is not orientable then the basis must be $P^2 \# P^2 \# P^2$ or $P^2 \# P^2 \# P^2 \# P^2$; the second case is ruled out since it would imply that $C(M) \geqslant 2$.

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