Lagrangian cobordism and Fukaya categories.

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Symplectic manifolds and Lagrangian submanifolds.

(M^{2n}, ω) symplectic $\Leftrightarrow$ ω 2-form, $d\omega = 0$, ω non-degenerate.

$L^n \hookrightarrow M$ submanifold - in this talk, compact, closed.

$$L \text{ Lagrangian} \iff \omega|_L \equiv 0 \ .$$

Pairs $L \hookrightarrow (M, \omega)$ appear in classical mechanics, string theory, algebraic geometry, complex analysis etc...

Definition (*Arnold '80*)

(M,ω) symplectic manifold; $(L_1,\ldots,L_k)$, $(L'_1,\ldots,L'_{k'})$ two families of closed, connected Lagrangian submanifolds $\subset M$.
A Lagrangian cobordism:

$$V:(L_i)\to(L'_j) \text{ is a Lagrangian } V\subset(\mathbb{C}\times M,\omega_0\oplus\omega) \text{ so that}$$

$$V|_{[1,\infty)\times\mathbb{R}\times M}=\cup_i[1,\infty)\times\{i\}\times L_i$$
$$V|_{(-\infty,0]\times\mathbb{R}\times M}=\cup_j(-\infty,0]\times\{j\}\times L'_j\ .$$

If $\pi:\mathbb{C}\times M\to\mathbb{C}$ is the projection, $\pi(V)$ looks like this:

More generally, may assume that V is emebdded in the total space E of a Lefschetz fibration $\pi : E \to \mathbb{C}$ of generic fibre (M, ω).

Form a group: $\Omega^{*}_{Lag}(M; E) = \mathbb{Z}_2 < L \subset M, \ \mathrm{Lagrangian}^{*} > / \mathcal{R}_{cob}$.

Relations $\mathcal{R}_{cob}$ generated by:

$L_1 + \ldots L_k = 0 \ \text{ if } \exists \ \ V : \emptyset \to (L_1, \ldots L_k), \ \ V \subset E \ \ \text{Lag. cobordism}$

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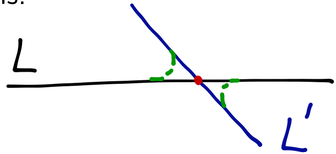
$-^{*}$ means that the Lagrangians and the cobordisms are restricted.

$* = e$ *exact* $\subset$ $* = m$ *monotone* $\subset$ $* = g$ *general*
most rigid ... *most flexible*

If $E = \mathbb{C} \times M$ (the trivial fibration) the notation is $\Omega^{*}_{Lag}(M)$ (we omit E).

a. The Gromov *h*-principle.(*Gromov*, *Eliashberg* '80's)
Lagrangian *immersions* are governed by the *h*-principle: algebraic topological criteria suffice to decide whether a smooth map can be perturbed to a Lagrangian immersion.

b. Lagrangian Surgery. (*Lalonde-Sikorav*, *Polterovich* '91) May assume only double points and these can be removed via surgery $\Rightarrow$ *embedded* Lagrangians.



Remark

By surgery *immersed* cobordism $\rightsquigarrow$ *embedded* cobordism $\Rightarrow$ "general" cobordism is flexible $\Rightarrow$ $\Omega^{g}_{Lag}(M)$ are computable ($M = \mathbb{C}^n$ *Audin* '85 and *Eliashberg* '84) by alg. top. methods.

Rigidity: Lagrangian intersections, $HF(-,)$, $D\mathcal{F}uk(-)$.

Class of Lagrangians to study: $Lag^*(M)$ (will omit $*$ from now on).

Pointwise $L \in Lag(M)$ is given as

$$L = \cup_{L'}\{L' \cap L \ : \ L' \in Lag(M)\}$$

May assume here $L' \pitchfork L \quad \rightsquigarrow$

$$L \ \equiv \ \cup_{L'}\mathbb{Z}_2 < L' \cap L > \quad \text{we put} \quad CF(L', L) := \mathbb{Z}_2 < L' \cap L >$$

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Coherence among the vector spaces $CF(-, L)$ provided by holomorphic curves.

Gromov '85: (M, ω) symplectic $\Rightarrow \exists\, J : TM \to TM$ almost complex structure compatible with ω ($\Leftrightarrow\ J^2 = -Id,\ \omega(-, J-)$ is a Riemannian metric).

J a. c. structure $\Rightarrow$ Cauchy-Riemann operator:

$$\bar{\partial}_J(-) = \frac{1}{2}\big[\frac{\partial}{\partial s}(-) + J\frac{\partial}{\partial t}(-)\big]\ .$$

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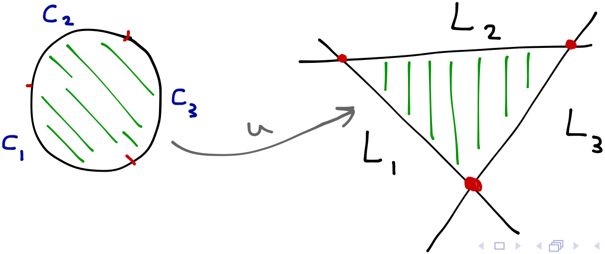
Fix $L_1, \ldots, L_{k+1} \in Lag(M)$. Let D^2_{k+1} be the 2-disk with $k+1$-boundary punctures. Write $\partial D^2_{k+1} = C_1 \cup \ldots C_{k+1}$.

$$\mathcal{M}(J; L_1, \ldots, L_{k+1}) = \{u : D^2_{k+1} \to M : \bar\partial_J u = 0\ ,\ u(C_i) \subset L_i\}$$



Assuming regularity (this requires additional constraints) $\mathcal{M}$ is a manifold $\Rightarrow$ admits Gromov compactification as manifold with boundary $\Rightarrow$ various invariants (up to quasi-isomorphism independent of J etc).

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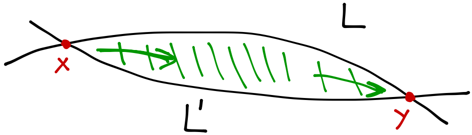
Example 1. Floer Homology (*Floer* '88 continued by *Hofer*, *Salamon*, *Oh*, *Fukaya*, *Fukaya-Oh-Ohta-Ono* etc): $L, L' \in Lag(M)$, $L' \pitchfork L$.

$$CF(L', L) = \mathbb{Z}_2 < L' \cap L > \quad \text{with differential}$$

$$d : CF(L', L) \to CF(L', L), dx = \sum \#(\mathcal{M}(J; L', L; x, y))\ y$$

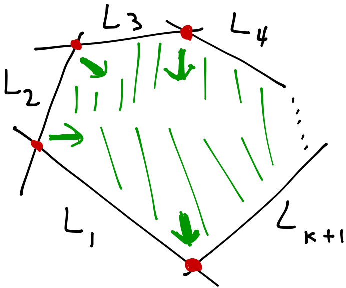
x, y are the assymptotic limits of the image of the two punctures: x is the entry, y the exit.



Structure of the compactification $\Rightarrow d^2 = 0$ (as in Morse theory).

Example 2. The Fukaya category $\mathcal{F}uk(M)$ (*Donaldson* '93, *Fukaya* '95, made rigorous by *Seidel* '06) is:

- An A_∞ category.
- Objects: $L \in Lag(M)$.
- Morphisms: $\mathrm{Mor}(L_1, L_2) = CF(L_1, L_2)$.
- Multiplications μ^k count elements in $\mathcal{M}(J; L_1, \ldots, L_{k+1})$; μ^1 coincides with the Floer differential.



$L \in Lag(M)$ was viewed - naively - as $L \equiv \cup_{L'} CF(L', L)$.

Improved version: view L as an A_∞ functor (called the Yoneda functor of L):

$$\mathcal{Y}_L : \mathcal{F}uk(M) \to Ch^{op} \ , \ \mathcal{Y}_L(L') = CF(L', L)$$

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A_∞ category $\mathcal{A} \Rightarrow Fun(\mathcal{A}, Ch^{op})$ is an A_∞ category that is *triangulated*.

Exact triangles in $Fun(\mathcal{A}, Ch^{op})$ come from the sequences in Ch^{op} of form:

$$C \xrightarrow{\phi} C' \to C'' \text{ with } C'' = cone(\phi) \ .$$

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Therefore

$$Fun(\mathcal{F}uk(M), Ch^{op})$$

is triangulated.

Kontsevich '97 : Let

$$\mathcal{F}uk(M)^{\wedge} \subset Fun(\mathcal{F}uk(M), Ch^{op})$$

be the *triangulated completion* of the Yoneda functors

$$\mathcal{Y}_L \in Fun(\mathcal{F}uk(M), Ch^{op}) \ , \ L \in Lag(M); \ \mathcal{Y}_L(L') = CF(L', L) \ .$$

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Remark

- $D\mathcal{F}uk(M)$ is a triangulated category in the usual sense.
- $\mathcal{O}b(D\mathcal{F}uk(M))$ are of two types: geometric that correspond to $\mathcal{Y}_L$'s and "non-geometric" given as iterated cones of geometric objects.

$D\mathcal{F}uk(M)$ triangulated $\Rightarrow$

- can decompose objects by means of iterated exact triangles.

- Grothendieck group

$$K_0(D\mathcal{F}uk(M)) = \mathbb{Z}_2 < O \in \mathcal{O}b(D\mathcal{F}uk(M)) > /\mathcal{R}' \ .$$

Relations $\mathcal{R}'$ are generated by:

$$A \to B \to C \text{ exact triangle} \Rightarrow A - B + C \in \mathcal{R}' \ .$$

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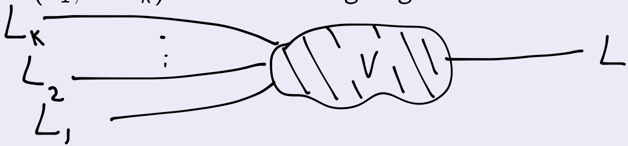
Key problems:

- Understand geometrically the exact triangles in $D\mathcal{F}uk(M)$.
- Give a geometric interpretation to the objects of $D\mathcal{F}uk(M)$ that are not of type $\mathcal{Y}_L$.

Theorem (*Biran - C.* '11,'13,'15)

$V : L \to (L_1, \dots L_k)$ *monotone Lagrangian cobordism.*



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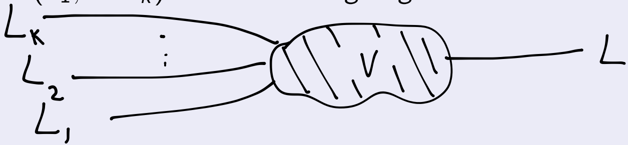

i. $V \subset \mathbb{C} \times M \Rightarrow$ *in $D\mathcal{F}uk(M)$ there are exact triangles:*

$$L_i \to X_i \to X_{i+1} \;\text{ with }\; X_1 = 0,\; L \cong X_{k+1}\;.$$

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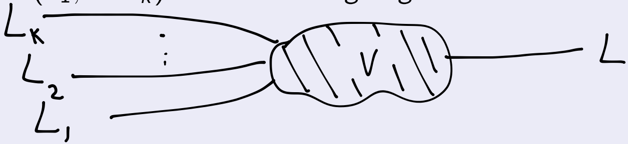

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$$L_i \to X_i \to X_{i+1} \text{ with } X_1 = 0, \; L \cong X_{k+1} \; .$$

ii. $V \subset E$ *for some Lefschetz fibration* $\pi : E \to \mathbb{C}$ *of generic fibre* (M, ω), *with* r *singularities. For appropriate choices of vanishing cycles* $S_j, 1 \leq j \leq r$, $\exists$ *vector spaces* E_j *and additional exact triangles:*

$$S_j \otimes E_j \to X_{k+1+j} \to X_{k+j+2} \text{ with } L \cong X_{k+r+2}$$

Remark

a. One type of exact triangle in $D\mathcal{F}uk(M)$ was discovered by *Seidel '03*:

$$\tau_S L \to L \to S \otimes HF(S, L)$$

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b. Another exact triangle noticed by *Fukaya-Oh-Ohta-Ono '05*. Assume $L_1 \cap L_2 = P$ and let $L_1 \# L_2 = \text{ surgery of } L_1, L_2 \text{ at } P$. The surgery exact sequence is:

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d. There are no other known geometric constructions for exact triangles in $D\mathcal{F}uk$.

Theorem implies that monotone cobordism (by contrast to general cobordism) is very rigid.

For instance, $V : L \to L'$ monotone $\Rightarrow$

$$HF(N, L) \cong HF(N, L') \ , \forall \ \ N \in Lag(M)$$

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If V as before is even *exact*, then (under mild constraints)

$$V \approx L \times [0, 1] \ \ \text{(smoothly)}$$

(*Suarez* '15, related work by *Tanaka* '14).

Remark

Before 2010 the only indication that monotone Lagrangian cobordism is rigid appeared in a paper of *Chekanov '97*.

Return to cobordism groups.

Corollary *(Biran-C.)*

Fix $\pi : E \to \mathbb{C}$ *a Lefschetz fibration.*
$\exists$ *(non-trivial) group morphism :*

$$\hat{\mathcal{F}} : \Omega^{m}_{Lag}(M; E) \longrightarrow K_0(D\mathcal{F}uk(M)) / < \text{vanishing cycles} > \ .$$

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Corollary (*Biran-C.*)

Fix $\pi : E \to \mathbb{C}$ a Lefschetz fibration.
$\exists$ (non-trivial) group morphism :

$$\hat{\mathcal{F}} : \Omega^{m}_{Lag}(M; E) \longrightarrow K_0(D\mathcal{F}uk(M))/ < vanishing\ cycles > \ .$$

Focus on $E = \mathbb{C} \times M$.

$$\hat{\mathcal{F}} : \Omega^{m}_{Lag}(M) \to K_0(D\mathcal{F}uk(M))$$

is given by $L \to [L] \in K_0(D\mathcal{F}uk(M))$.

Theorem $\Rightarrow$ the cobordism relations translate to exact triangles in $D\mathcal{F}uk(M) \Rightarrow \hat{\mathcal{F}}$ well-defined.

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e. Understanding geometrically the exact triangles in $D\mathcal{F}uk(M)$ is related to understanding $Ker(\hat{\mathcal{F}})$.

Approaches to $Ker(\hat{\mathcal{F}})$ (Biran - C. '13, '15):

- there is an *algebraic cobordism* group $\Omega^m_{Alg}(M)$ which is a quotient of $\Omega^m_{Lag}(M)$ and:

$$\Omega^m_{Alg}(M) \cong K_0(D\mathcal{F}uk(M)) \ .$$

$\Omega^m_{Alg}(M)$ is obtained by completing the cobordism relations in the same way that $Lag(M)$ is completed to $\mathcal{O}b(D\mathcal{F}uk(M))$.

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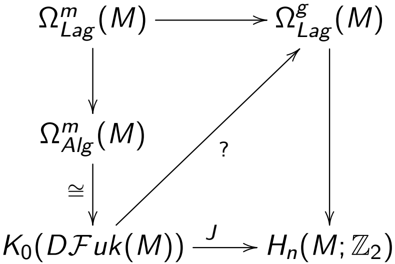
- there is a categorification of $\hat{\mathcal{F}}$: $\Omega^m_{Lag}(M)$ is replaced by a (Lagrangian) cobordism category ; $K_0(D\mathcal{F}uk(M))$ is replaced by an (enrichement) of $D\mathcal{F}uk(M)$ and $\hat{\mathcal{F}}$ is replaced by a functor $\mathcal{F}$.

All cobordism groups together:



Map J is non-trivial to construct. It would be great to be able to construct a diagonal lift of J. In all cases:

- Ω^g_{Lag} - computable by flexibility.

- Ω^m_{Alg} - computable by mirror symmetry (when available).

All cobordism groups together:

$$
\begin{array}{ccc}
\Omega^m_{Lag}(M) & \longrightarrow & \Omega^g_{Lag}(M) \\
\downarrow & & \\
\Omega^m_{Alg}(M) & \overset{?}{\nearrow} & \downarrow \\
\cong \downarrow & & \\
K_0(D\mathcal{F}uk(M)) & \xrightarrow{J} & H_n(M;\mathbb{Z}_2)
\end{array}
$$

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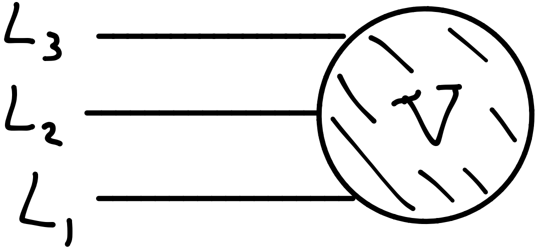
Current work/speculation:
"non geometric" objects of $D\mathcal{F}uk(M)$ $\leftrightsquigarrow$ *immersed* Lagrangians.
If so, diagonal lift of J should follow.....

Some ideas used in the proof of the Theorem.

a. An example in $\mathbb{C} \times M$.

Consider a cobordism $V : \emptyset \to (L_1, L_2, L_3)$, $V \subset \mathbb{C} \times M$.



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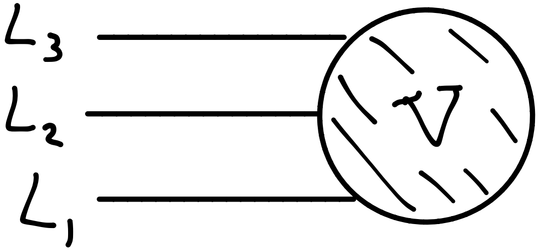

We need to show - forgetting the higher structures - for each $N \in Lag(M)$:

$$[Cone(CF(N, L_3) \xrightarrow{\phi_2} Cone(CF(N, L_2) \xrightarrow{\phi_1} CF(N, L_1)))] \simeq 0$$

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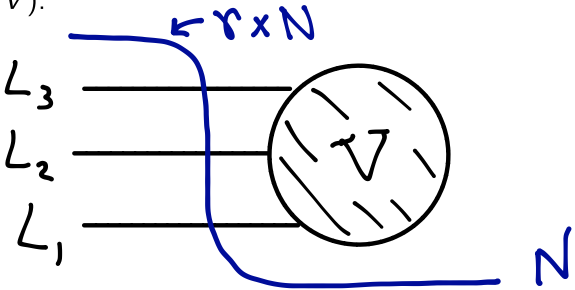
- Consider $CF(\gamma \times N, V)$:

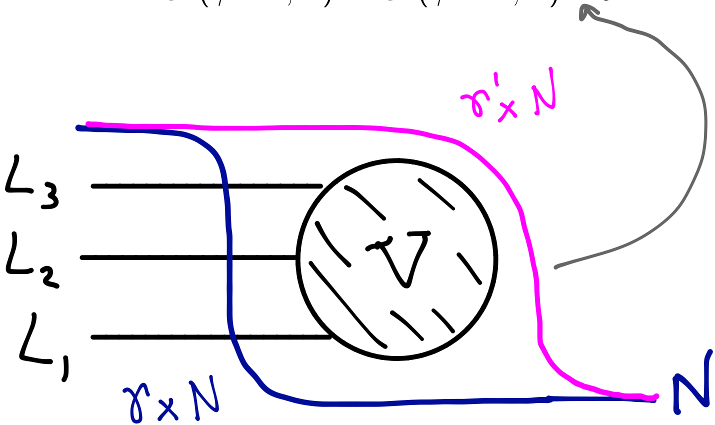


We intend to show two things:

$$CF(\gamma \times N, V) \overset{(1)}{=}$$

$$= [Cone(CF(N, L_3) \xrightarrow{\phi_2} Cone(CF(N, L_2) \xrightarrow{\phi_1} CF(N, L_1)))]$$

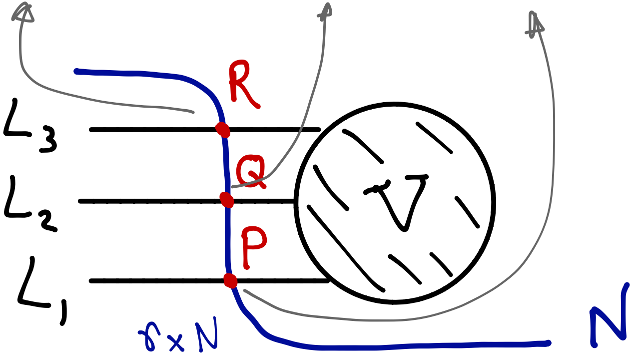
$$CF(\gamma \times N, V) \overset{(2)}{\simeq} CF(\gamma' \times N, V) = 0$$



Remains to show:

$$CF(\gamma \times N, V) \overset{(1)}{=}$$

$$= [Cone(CF(N, L_3) \xrightarrow{\phi_2} Cone(CF(N, L_2) \xrightarrow{\phi_1} CF(N, L_1)))]$$

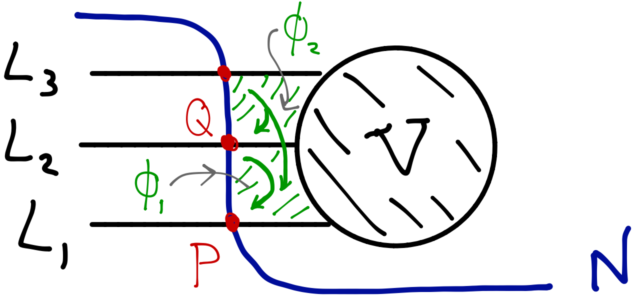


First we identify the complexes $CF(N, L_i)$.

$$CF(\gamma \times N, V) \overset{(1)}{=}$$

$$= [Cone(CF(N, L_3) \xrightarrow{\phi_2} Cone(CF(N, L_2) \xrightarrow{\phi_1} CF(N, L_1)))]$$

Finally, we identify the maps ϕ_1, ϕ_2:



- ϕ_1 is given by the strips from Q to P.

- cone structure follows from the fact that strips can only "go down" ! (use special almost c. structures so that $\pi : \mathbb{C} \times M \to \mathbb{C}$ is holomorphic and the open mapping th....).

b. An example in a non-trivial Lefschetz fibration.

$\pi : E \to \mathbb{C}$ Lefschetz with a single singularity of critical value $0 \in \mathbb{C}$; $V : \emptyset \to (L, \tau_S L)$; S vanishing cycle.

b. An example in a non-trivial Lefschetz fibration.

$\pi : E \to \mathbb{C}$ Lefschetz with a single singularity of critical value $0 \in \mathbb{C}$; $V : \emptyset \to (L, \tau_S L)$; S vanishing cycle.

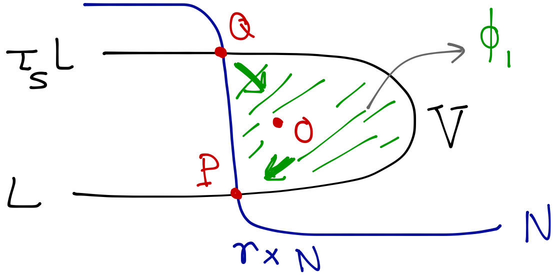
Need to show that there is a vector space E so that for all $N \in Lag(M)$:

$$[Cone(CF(N,S) \otimes E \xrightarrow{\psi_1} Cone(CF(N,\tau_S L) \xrightarrow{\phi_1} CF(N,L)))] \simeq 0$$

Equivalently:

$$\exists \quad E, \phi_1 \quad \text{so that} \quad Cone(\phi_1) \simeq CF(N,S) \otimes E$$

As in the case of the trivial fibration we consider the cobordisms V and $\gamma \times N$.



The map ϕ_1 is given by the strips going down from Q to P as before $\Rightarrow$

$$Cone(\phi_1) \simeq CF(\gamma \times N, V)\ .$$

Put $E = HF(S, L)$. Want to show:

$$CF(\gamma \times N, V) \simeq CF(N, S) \otimes HF(S, L).$$

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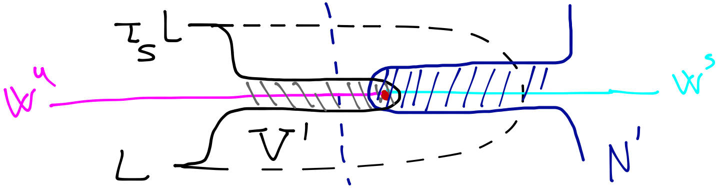
$$CF(\gamma \times N, V) \simeq CF(N, S) \otimes HF(S, L).$$

- $f = Re(\pi) : E \to \mathbb{R}$ is Morse; the unstable manifold of the critical point of f is the thimble over $(-\infty, 0]$; the stable manifold is the thimble over $[0, \infty)$.

- $grad(f)$ is also Hamiltonian. Will use it to stretch $\gamma \times N \rightsquigarrow N'$ in the direction $grad(f)$ and $V \rightsquigarrow V'$ in the direction $-grad(f)$.



Put $E = HF(S, L)$. Want to show:

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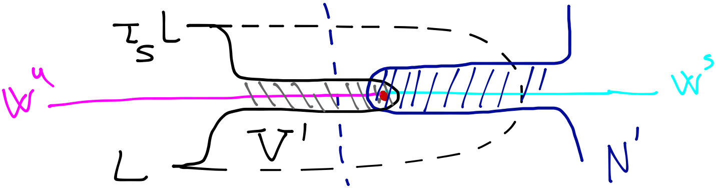

We have $CF(\gamma \times N, V) \simeq CF(N', V')$.

To end, the key is that the intersection points $N' \cap V'$ are in bijection with $(N \cap S) \times (S \cap L)$....