CATEGORIFICATION OF SEIDEL'S REPRESENTATION

BY

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ABSTRACT

Two natural symplectic constructions, the Lagrangian suspension and Seidel's quantum representation of the fundamental group of the group of Hamiltonian diffeomorphisms, $Ham(M)$, with (M, ω) a monotone symplectic manifold, admit categorifications as actions of the fundamental groupoid $\Pi(Ham(M))$ on a cobordism category recently introduced in [BC14] and, respectively, on a monotone variant of the derived Fukaya category. We show that the functor constructed in [BC14] that maps the cobordism category to the derived Fukaya category is equivariant with respect to these actions.

1. Introduction

Let (M^{2n}, ω) be a symplectic manifold which is closed or tame at infinity.

The main purpose of this paper is to relate two important, basic constructions in symplectic topology: Lagrangian suspension [Arn80a, Arn80b], [Aud85], [Pol01] and Seidel's representation $S : \pi_1(Ham(M)) \to QH(M)$ in its Lagrangian version [Sei97], [H10], [HLL11].

Our first remark is that, after convenient "categorification", these two constructions can be naturally viewed as two actions of the fundamental groupoid $\Pi(Ham(M))$ on a cobordism category $\mathcal{C}ob_0^d(M)$ (introduced in [BC13]) and, respectively, on the derived Fukaya category $D\mathcal{F}uk^d(M)$ of M. We then show that these two actions are interchanged by the functor

$$\mathcal{F} : \mathcal{C}ob_0^d(M) \to D\mathcal{F}uk^d(M)$$

introduced in [BC13] and [BC14]. In fact, we prove a slightly more refined statement:

THEOREM A: *The following diagram of action bifunctors commutes:*

$$
(1) \quad
\begin{array}{ccc}
\Pi(Ham(M)) \times \mathcal{C}ob_0^d(M) & \xrightarrow{\ \ \Sigma\ \ } & \mathcal{C}ob_0^d(M) \\
{\scriptstyle \mathrm{id} \times \widetilde{\mathcal{F}}} \downarrow & & \downarrow {\scriptstyle \widetilde{\mathcal{F}}} \\
\Pi(Ham(M)) \times T^S D\mathcal{F}uk^d(M) & \xrightarrow{\ \ \widetilde{S}\ \ } & T^S D\mathcal{F}uk^d(M)
\end{array}
$$

The category $T^S D\mathcal{F}uk^d(M)$ is a category associated to $D\mathcal{F}uk^d(M)$ by a purely algebraic process [BC14]. The morphisms in this category reflect the ways in which the objects in $D\mathcal{F}uk^d(M)$ can be decomposed by iterated exact triangles in $D\mathcal{F}uk^d(M)$ (which, we recall, is a triangulated category). The functor $\widetilde{\mathcal{F}}$ — constructed in [BC14] — is a refinement of $\mathcal{F}$ that "sees" the triangulated structure of $D\mathcal{F}uk^d(M)$. The decorations ^{d} and $_0$ come from certain constraints that need to be imposed on the class of Lagrangians in use so that all these notions are well defined.

In the diagram, Σ represents the action of $\Pi(Ham(M))$ on $\mathcal{C}ob_0^d(M)$ and is an extension of Lagrangian suspension. The action of $\Pi(Ham(M))$ on $T^S D\mathcal{F}uk^d(M)$ is denoted by $\widetilde{S}$ and is a refinement, first to the derived Fukaya category $D\mathcal{F}uk^d(M)$ and then to $T^s D\mathcal{F}uk^d(M)$, of the Lagrangian Seidel representation.

There is yet another perspective on the commutativity in (1) that is based on the equivalent definition of the action of a (strict) monoidal category $\mathcal{M}$ on a category $\mathcal{C}$ as a strict monoidal functor $\mathcal{M} \to \mathcal{E}nd(\mathcal{C}, \mathcal{C})$. The commutativity of diagram (1) is then equivalent to the commutativity of the bottom square of diagram (2) below:

COROLLARY B: *The following diagram of categories and functors commutes:*

(2)
$$
\begin{array}{ccc}
\pi_1(Ham(M)) & \xrightarrow{\quad S \quad} & QH(M)^* \\
{\scriptstyle i} \downarrow & & \downarrow {\scriptstyle *} \\
\Pi(Ham(M)) & \xrightarrow{\quad \widetilde{S} \quad} & \mathcal{E}nd(T^S D\mathcal{F}uk^d(M)) \\
{\scriptstyle \Sigma} \downarrow & & \downarrow {\scriptstyle \widetilde{\mathcal{F}}^*} \\
\mathcal{E}nd(\mathcal{C}ob_0^d(M)) & \xrightarrow{\quad \widetilde{\mathcal{F}}_* \quad} & fun(\mathcal{C}ob_0^d(M), T^S D\mathcal{F}uk^d(M))
\end{array}
$$

The categories and functors in the top square are strict monoidal as is the functor Σ.

Here $QH(M)^*$ are the invertible elements in the quantum homology of M, the functor S is Seidel's representation [Sei97] viewed as a monoidal functor and the action $*$ is a refinement of the module action of quantum homology on Lagrangian Floer homology [BC09]. The functors $\widetilde{\mathcal{F}}_*$ and $\widetilde{\mathcal{F}}^*$ are induced respectively by composition and pre-composition with $\widetilde{\mathcal{F}}$.

Remark 1.0.1: A simpler form of diagram (2) is yet another commutative diagram of monoidal categories and functors, of the same shape, but with the more familiar $D\mathcal{F}uk^d(M)$ taking the place of $T^S D\mathcal{F}uk^d(M)$. Further, $QH(M)$ replaces $QH(M)^*$ in the upper right corner and $\mathcal{F}$, a linearization of the functor $\widetilde{\mathcal{F}}$, takes the place of $\widetilde{\mathcal{F}}$ in this simplified diagram. In this form, the action $*$ is closely related to a map sometimes called the *closed-open map*, $\mathcal{CO}$. More precisely, $*$ is the composition

(3) $\quad QH(M) \xrightarrow{\mathcal{CO}} HH(\mathcal{F}uk^d(M), \mathcal{F}uk^d(M))$
$$
= H(\mathrm{Mor}_{\mathcal{E}nd(\mathcal{F}uk^d(M))}(\mathrm{id}, \mathrm{id})) \to \mathcal{E}nd(D\mathcal{F}uk^d(M)).
$$

Here $HH(-,-)$ stands for Hochschild co-homology [Sei]. A definition of this co-homology is provided by the equality in (3) above. $\mathcal{E}nd(\mathcal{F}uk^d(M))$ stands

for the A_∞-category of endofunctors of $\mathcal{F}uk^d(M)$ and the last arrow in the composition is the restriction of the canonical functor $H(\mathcal{E}nd(\mathcal{F}uk^d(M)) \to \mathcal{E}nd(D\mathcal{F}uk^d(M))$. This simplified diagram is of interest in itself even if it does not take into account explicitly the triangulation of $D\mathcal{F}uk^d(M)$.

The plan of the paper is as follows. In §2 we describe the ingredients in diagram (2). This part consists mostly of recalls but there are also a few constructions that seem not to be present explicitly in the literature and thus we spend more time on these points. In particular, we show that the usual notion of Lagrangian suspension extends to the action Σ on the cobordism category. We also give a few details to justify why the closed-open map gives rise to an action.

The main part of the paper is in §3. In §3.1 we verify the commutativity in diagram (1) and thus we prove Theorem A. There are two main ingredients in this proof. The first consists in some algebraic manipulations based on the properties of the functor $\widetilde{\mathcal{F}}$ from [BC14]. The second ingredient — that first appeared in the first author's thesis [Cha] — is the main geometric novelty brought by the paper. Its content is as follows. Given two Lagrangians L, L' (under the usual technical constraints) and a path $\mathbf{g}_t$ of Hamiltonian diffeomorphisms there is a natural "moving boundary" morphism

$$CF(g_0(L), L') \to CF(g_1(L), L')$$

that induces in homology Seidel's Lagrangian morphism. On the other hand, the Lagrangian suspension of L with respect to $\mathbf{g}$ is a cobordism from $g_0(L)$ to $g_1(L)$. The functor $\widetilde{\mathcal{F}}$ associates to such a cobordism another morphism $CF(g_0(L), L') \to CF(g_1(L), L')$. The key geometric point is that these two morphisms are chain-homotopic.

In §3.2 we sketch the proof of the commutativity of the top square in diagram (2). This commutativity is known in various settings by experts but we include the argument here so as to complete the proof of Corollary B.

The last section, §4, mainly contains a few examples. We show how to use the commutativity of (2) to produce examples of cobordisms with identical ends but that are not horizontally Hamiltonian isotopic. In particular, we produce Lagrangians $L \subset M$ and cobordisms $V \subset \mathbb{C} \times M$ diffeomorphic to $\mathbb{R} \times L$, that coincide with $\mathbb{R} \times L$ away from a compact set, but are not Hamiltonian isotopic to $\mathbb{R} \times L$ even through isotopies that are not compactly supported and

"slide" the ends along themselves. Thus, we produce examples L so that the set $\mathrm{Mor}_{\mathcal{C}ob_0^d}(L, L)$ is not trivial. In our examples L is a real Lagrangian in a toric symplectic manifold and we can also provide a lower bound on the size of $\mathrm{Mor}_{\mathcal{C}ob_0^d}(L, L)$. Finally, for the convenience of the reader, in §4.2 we make explicit the equivalent descriptions of the Seidel morphism that appear in a variety of sources in the literature as well as in the paper itself.

2. The ingredients in diagram (2).

2.1. THE CATEGORIES IN THE DIAGRAM.

2.1.1. *The Hamiltonian diffeomorphisms group.* The Hamiltonian diffeomorphisms group of M is denoted by $Ham(M)$ and $\pi_1(Ham(M))$ is its fundamental group; the base point is the identity. The group $\pi_1(Ham(M))$ is viewed here as a category with one element and so that each element of the group is a morphism from that element to itself. This category is monoidal in an obvious way. The fundamental groupoid of $Ham(M)$, denoted by $\Pi(Ham(M))$, is a category having as objects the Hamiltonian diffeomorphisms of M and the morphisms are homotopy classes of paths in $Ham(M)$ relating two such diffeomorphisms.

2.1.2. *Quantum Homology.* The category $QH(M)^*$ has a single object and its morphisms are the invertible elements in the quantum homology of M. The composition is the quantum product.

2.1.3. *The cobordism category.* This subsection contains a summary of the construction of the category $\mathcal{C}ob_0^d(M)$ following [BC14].

A. Monotonicity assumptions. All families of Lagrangian submanifolds in our constructions have to satisfy a monotonicity condition in a uniform way as described below. Given a Lagrangian submanifold $L \subset M$, let

$$\omega : \pi_2(M, L) \to \mathbb{R} \ , \ \mu : \pi_2(M, L) \to \mathbb{Z}$$

be the morphisms given respectively by integration of ω and by the Maslov index. The Lagrangian L is **monotone** if there exists a positive constant $\rho > 0$ so that for all $\alpha \in \pi_2(M, L)$ we have $\omega(\alpha) = \rho\mu(\alpha)$ and the minimal Maslov number $N_L = \min\{\mu(\alpha) : \alpha \in \pi_2(M, L) \,, \ \mu(\alpha) > 0\}$ satisfies $N_L \geq 2$. We work at all times over $\mathbb{Z}_2$ as ground ring.

For a closed, connected, monotone Lagrangian L there is an associated invariant $d_L \in \mathbb{Z}_2$ given as the number (mod 2) of J-holomorphic disks of Maslov index 2 going through a generic point $P \in L$ for J a generic almost complex structure that is compatible with ω.

A family of Lagrangian submanifolds L_i, $i \in I$, is uniformly monotone if each L_i is monotone and the following condition is satisfied: there exists $d \in K$ so that for all $i \in I$ we have $d_{L_i} = d$ and there exists a positive real constant ρ so that the monotonicity constant of L_i equals ρ for all $i \in I$. All the Lagrangians used in the paper will be assumed monotone and, similarly, the Lagrangian families will be assumed uniformly monotone. For $d \in \mathbb{Z}_2$ and $\rho > 0$, we let $\mathcal{L}_d(M)$ be the family of closed, connected Lagrangian submanifolds $L \subset M$ that are monotone with monotonicity constant ρ and with $d_L = d$.

 B. Cobordism. The plane $\mathbb{R}^2$ is endowed with the symplectic structure $\omega_{\mathbb{R}^2} = dx \wedge dy$, $(x, y) \in \mathbb{R}^2$. The product $\widetilde{M} = \mathbb{R}^2 \times M$ is endowed with the symplectic form $\omega_{\mathbb{R}^2} \oplus \omega$. We denote by $\pi : \mathbb{R}^2 \times M \to \mathbb{R}^2$ the projection. For a subset $V \subset \mathbb{R}^2 \times M$ and $S \subset \mathbb{R}^2$ we let $V|_S = V \cap \pi^{-1}(S)$.

Definition 2.1.1: Let $(L_i)_{1 \leq i \leq k_-}$ and $(L'_j)_{1 \leq j \leq k_+}$ be two families of closed Lagrangian submanifolds of M. We say that these two (ordered) families are Lagrangian cobordant, $(L_i) \simeq (L'_j)$, if there exists a smooth compact cobordism $(V; \coprod_i L_i, \coprod_j L'_j)$ and a Lagrangian embedding $V \subset ([0, 1] \times \mathbb{R}) \times M$ so that for some $\epsilon > 0$ we have

(4)
$$V|_{[0, \epsilon) \times \mathbb{R}} = \coprod_i ([0, \epsilon) \times \{i\}) \times L_i,$$
$$V|_{(1-\epsilon, 1] \times \mathbb{R}} = \coprod_j ((1 - \epsilon, 1] \times \{j\}) \times L'_j \ .$$

The manifold V is called a Lagrangian cobordism from the Lagrangian family (L'_j) to the family (L_i); see Figure 1. We denote it by $V : (L'_j) \rightsquigarrow (L_i)$ or $(V; (L_i), (L'_j))$.
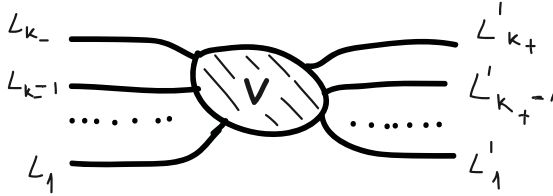
Figure 1. A cobordism $V : (L'_j) \rightsquigarrow (L_i)$ projected on $\mathbb{R}^2$.

A cobordism is called **monotone** if

$$V \subset ([0,1] \times \mathbb{R}) \times M$$

is a monotone Lagrangian submanifold.

We often view cobordisms as embedded in $\mathbb{R}^2 \times M$. Given a cobordism $V \subset ([0,1] \times \mathbb{R}) \times M$ as in Definition 2.1.1, we can extend trivially its negative ends towards $-\infty$ and its positive ends to $+\infty$ thus getting a Lagrangian $\overline{V} \subset \mathbb{R}^2 \times M$ and we will in general not distinguish between V and $\overline{V}$.

Remark 2.1.2: The notion of Lagrangian cobordism was introduced by Arnold [Arn80a, Arn80b] in a slightly less general form than above. Flexible aspects were discussed in early work of Eliashberg [Eli84] and Audin [Aud85]. The first indication that rigidity is also relevant to this notion of cobordism appeared in the work of Chekanov [Che97]. See [BC13] for more background material.

Two Lagrangian cobordisms V, $V' \subset \mathbb{C} \times M$ are called **horizontally** isotopic if there is an ambient Hamiltonian isotopy ψ_t (not necessarily compactly-supported) so that $\psi_0 = \mathbb{1}$, $\psi_1(V) = V'$ and each ψ_t slides along the ends of V with bounded speed. In particular, the ends of V and $\psi_t(V)$ coincide away from a set $K \times M$ with $K \subset \mathbb{C}$ compact for all $t \in [0,1]$ (see [BC14] for a more explicit form of this definition).

C. The category $\mathcal{C}ob_0^d(M)$. Consider first the category $\widetilde{\mathcal{C}ob}_0^d(M)$. Its objects are families

$$(L_1, L_2, \ldots, L_r)$$

with $r \geq 1$, $L_i \in \mathcal{L}_d(M)$ so that additionally:

i. For all $1 \leq i \leq r$, if L_i is non-void, then it is non-narrow (recall that a monotone Lagrangian is non-narrow if its quantum homology $QH(L)$ with $\Lambda = \mathbb{Z}_2[t, t^{-1}]$ coefficients does not vanish).

ii. For all $1 \leq i \leq r$ we have that the morphism

(5) $$\pi_1(L_i) \xrightarrow{i_*} \pi_1(M)$$

induced by the inclusion $L_i \subset M$ vanishes.

Remark 2.1.3: In case the first Chern class c_1 and ω are proportional as morphisms defined on $H_2(M; \mathbb{Z})$ (and not only on $\pi_2(M)$), then it is enough to assume in (5) that the image of i_* is torsion.

We denote by $\mathcal{L}_d^*(M)$ the Lagrangians in $\mathcal{L}_d(M)$ that are non-narrow and additionally satisfy (5). Similarly, we denote by $\mathcal{L}_d(\mathbb{C} \times M)$ the Lagrangians in $\mathbb{C} \times M$ that are uniformly monotone with the same $d_V = d$ and the same monotonicity constant ρ and we let $\mathcal{L}_d^*(\mathbb{C} \times M)$ be those Lagrangians $V \in \mathcal{L}_d(\mathbb{C} \times M)$ so that $\pi_1(V) \to \pi_1(\mathbb{C} \times M)$ is null.

We proceed to define the morphisms in $\widetilde{\mathcal{C}ob}_0^d(M)$. For any two horizontal isotopy classes of cobordisms $[V]$ and $[U]$ with $V : (L_j') \rightsquigarrow (L_i)$ (as in Definition 2.1.1) and $U : (K_s') \rightsquigarrow (K_r)$, we define the sum $[V] + [U]$ to be the horizontal isotopy class of a cobordism $W : (L_j') + (K_s') \rightsquigarrow (L_i) + (K_r)$ so that $W = V \coprod \widetilde{U}$ with $\widetilde{U}$ a suitable translation up the y-axis of a cobordism horizontally isotopic to U so that $\widetilde{U}$ is disjoint from V.

The morphisms in $\widetilde{\mathcal{C}ob}_0^d(M)$ are now defined as follows. A morphism

$$[V] \in \mathrm{Mor}\big((L_j')_{1 \leq j \leq S}, (L_i)_{1 \leq i \leq T}\big)$$

is a horizontal isotopy class that is written as a sum $[V] = [V_1] + \cdots + [V_S]$ with each $V_j \in \mathcal{L}_d^*(\mathbb{C} \times M)$ a cobordism from the Lagrangian family formed by the *single* Lagrangian L_j' and a subfamily $(L_{r(j)}, \ldots, L_{r(j)+s(j)})$ of the (L_i)'s, so that $r(j) + s(j) + 1 = r(j+1)$. In other words, V decomposes as a union of V_i's each with a single positive end but with possibly many negative ones — see Figure 2. We will often denote such a morphism by $V : (L_j') \longrightarrow (L_i)$.

The composition of morphisms is induced by concatenation followed by a rescaling to reduce the "width" of the cobordism to the interval $[0, 1]$.

We consider here the void set as a Lagrangian of arbitrary dimension. There is a natural equivalence relation among the objects of this category: it is induced

by the relations

$$(6) \qquad\qquad (L, \emptyset) \sim (\emptyset, L) \sim (L).$$

There is also a corresponding equivalence relation for morphisms (we refer again to [BC14] for details).
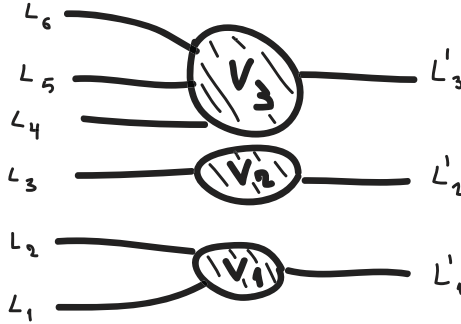

Figure 2. A morphism $V : (L'_1, L'_2, L'_3) \longrightarrow (L_1, \ldots, L_6)$, $V = V_1 + V_2 + V_3$, projected to $\mathbb{R}^2$.

The category $\mathcal{C}ob_0^d(M)$ is the quotient of $\widetilde{\mathcal{C}ob}_0^d(M)$ with respect to the equivalence relations mentioned above (obviously, they are applied to objects as well as to morphisms).

Remark 2.1.4: The quotient $\mathcal{C}ob_0^d(M)$ has a more complicated definition than $\widetilde{\mathcal{C}ob}_0^d(M)$ but it presents the advantage to avoid redundancies related to the presence of multiple empty ends (these are ends consisting of the empty Lagrangian). In particular, the morphisms in $\mathcal{C}ob_0^d(M)$ are represented by unique horizontal isotopy classes of cobordisms V (not necessarily connected) so that the empty Lagrangian can appear as a positive (or negative) end of V only if there is a single positive end (or, respectively, a single negative one). Moreover, $\mathcal{C}ob_0^d(M)$ is strictly monoidal.

2.1.4. *The categories $D\mathcal{F}uk^d(M)$ and $T^S D\mathcal{F}uk^d(M)$.* We again recall a construction detailed in [BC14].

A. The derived Fukaya category. We review here very briefly a few ingredients in the construction of this category. This construction is presented in detail in [Sei08] and is summarized in the precise monotone context we use here in

[BC14]. In the same monotone setting it has also appeared in the recent work of Sheridan [She]. We emphasize that we do not complete with respect to idempotents.

The derived Fukaya category $D\mathcal{F}uk^d(M)$ is a triangulated completion of the Donaldson category $\mathcal{D}on^d(M)$. The objects of $\mathcal{D}on^d(M)$ are Lagrangians $L \in \mathcal{L}_d^*(M)$ and the morphisms are $\mathrm{Mor}_{\mathcal{D}on^d(M)}(L, L') = HF(L, L')$. Here $HF(L, L')$ is the Floer homology of L and L'. This is the homology of the Floer complex $(CF(L, L'), d)$ that we consider here without grading and over $\mathbb{Z}_2$. Assuming that L and L' are transverse, the vector space $CF(L, L')$ has as basis the intersection points $L \cap L'$. The differential d counts J-holomorphic strips $u : \mathbb{R} \times [0, 1] \to M$ so that $u(\mathbb{R} \times \{0\}) \subset L$ and $u(\mathbb{R} \times \{1\}) \subset L'$ and so that $\lim_{s \to -\infty} u(s, t) = x \in L \cap L'$, $\lim_{s \to +\infty} u(s, t) = y \in L \cap L'$ (J is a generic almost complex structure on M). Monotonicity together with condition (5) are used to show the finiteness of sums in the expression of d. Uniform monotonicity is again used to show that $d^2 = 0$. The main geometric step in the construction of $D\mathcal{F}uk^d(M)$ is the construction of the Fukaya A_∞-category $\mathcal{F}uk^d(M)$. We will not review here the definition of this A_∞-category; we refer instead again to [Sei08]. It suffices to say that the objects in this A_∞-category are again Lagrangians in $\mathcal{L}_d^*(M)$. At least naively, the morphisms $\mathrm{Mor}_{\mathcal{F}uk^d(M)}(L, L') = CF(L, L')$ (this is naive as morphisms have to be defined even if L and L' are not transverse). The structural maps are so that $\mu^1 = d =$ the Floer differential and, for $k > 1$, μ^k is defined by counting J-holomorphic polygons with $k+1$ sides. These polygons have k "inputs" asymptotic to successive intersection points $x_1 \in L_1 \cap L_2$, $x_2 \in L_2 \cap L_3$, ... $x_k \in L_k \cap L_{k+1}$ and one "exit" asymptotic to $y \in L_1 \cap L_{k+1}$ (this is again naive for the same reason as before: these operations have to be defined for all families $L_1, \ldots, L_{k+1}$ and not only when L_i, L_{i+1}, etc., are transverse). Again condition (5) and monotonicity are used to show that the sums appearing in the definition of the μ^k's are finite. One then considers the category of modules over the Fukaya category

$$mod(\mathcal{F}uk^d(M)) := fun(\mathcal{F}uk^d(M), Ch^{opp})$$

where Ch^{opp} is the opposite of the dg-category of chain complexes. This A_∞-category of modules is also actually a dg-category and is triangulated in the A_∞-sense with the triangles being inherited from the triangles in Ch (where they correspond simply to the usual cone-construction). At the same time,

there is a Yoneda embedding $\mathcal{Y} : \mathcal{F}uk^d(M) \to mod(\mathcal{F}uk^d(M))$, the functor associated to an object $L \in \mathcal{L}_d^*(M)$ being $CF(-, L)$. The derived Fukaya category $D\mathcal{F}uk^d(M)$ is the homology category associated to the triangulated completion of the image of the Yoneda embedding inside $mod(\mathcal{F}uk^d(M))$.

Remark 2.1.5 (The pullback functor): The following property of module categories will be useful later on. Given two A_∞ categories, $\mathcal{A}$ and $\mathcal{B}$, the respective module categories are related by an A_∞ pullback functor $* : fun(\mathcal{A}, \mathcal{B}) \to fun(mod(\mathcal{B}), mod(\mathcal{A}))$. This is defined as follows (see also [Sei08, §**(1k)**)] as well as [BC14, Appendix A]).

A morphism in a functor A_∞ category, $fun(\mathcal{C}, \mathcal{C}')$, (such as, for instance, $mod(\mathcal{A})$, $mod(\mathcal{B})$, $fun(\mathcal{A}, \mathcal{B})$) relating two functors $U, U' : \mathcal{C} \to \mathcal{C}'$ is, by definition, a pre-natural transformation T. This consists of a couple (T^0, T') where T^0 is a collection of morphisms $T_L^0 \in C(U(L), U'(L))$ for each object L of $\mathcal{C}$ and T' is a collection of multilinear maps

$$(T')^d : C(L_0, L_1) \otimes \cdots \otimes C(L_{d-1}, L_d) \to C(U(L_0), U'(L_d))$$

defined for each family $(L_0, \ldots, L_d)$ of objects of $\mathcal{C}$ and $d \geq 1$. Such a collection is called an extended multilinear map; we denote by $C(-, -)$ the morphisms in the respective categories.

Let $F \in fun(\mathcal{A}, \mathcal{B})$. Define $F^* : mod(\mathcal{B}) \to mod(\mathcal{A})$ on objects by $F^*\phi = \phi \circ F$, where $\circ$ is the composition of A_∞ functors (this is the obvious composition of mltilinear maps — see again [BC14, Appendix A] for the notation). By the definition of a functor between A_∞ categories (this is again an extended multilinear map), we also need to define a sequence of multilinear maps

$$(F^*)^d : C_{mod(\mathcal{B})}(\phi_0, \phi_1) \otimes \cdots \otimes C_{mod(\mathcal{B})}(\phi_{d-1}, \phi_d) \to C_{mod(\mathcal{A})}(F^*\phi_0, F^*\phi_d).$$

We set $(F^*)^d = 0$ when $d \geq 2$ and we are left with defining

$$(F^*)^1 : C_{mod(\mathcal{B})}(\phi_0, \phi_1) \to C_{mod(\mathcal{A})}(F^*\phi_0, F^*\phi_1)$$
$$\eta = (\eta^0, \eta') \mapsto F^*\eta = (F^*\eta^0, F^*\eta') \ .$$

For every $L \in \mathrm{Ob}(\mathcal{A})$ we put $(F^*\eta^0)_L := \eta_{F(L)}^0 \in C_{Ch^{opp}}(F^*\phi_0(L), F^*\phi_1(L))$ and we define $F^*\eta'$ to be the extended multilinear map defined by $\eta' \circ F$. Consider now $T \in \mathrm{Mor}_{fun(\mathcal{A}, \mathcal{B})}(F, G)$, $T = (T^0, T')$, a pre-natural transformation from F to G. We need to define a pre-natural transformation T^* from F^* to G^*. We set $(T^*)' = 0$. Further, for each $\phi \in \mathrm{Ob}(mod(\mathcal{B}))$ the element $(T^*)_\phi^0 \in$

$C_{mod(\mathcal{A})}(F^*\phi, G^*\phi)$ is a pre-natural transformation $(T^*)^0_\phi = \chi = (\chi^0, \chi')$ defined as follows. For $L \in \mathrm{Ob}(\mathcal{A})$, set $\chi^0_L := \phi(\eta^0_L) \in C_{Ch^{opp}}(\phi(F(L)), \phi(G(L)))$ and $\chi' := \phi \circ \star \circ (F, \eta, G)$ (see once more [BC14, Appendix A] for the notation). Sketchily, this composition is

$$\phi \circ \star \circ (F, \eta, G)(\cdots) = \sum \phi(F(\cdots), \ldots, F(\cdots), \eta(\cdots), G(\cdots), \ldots, G(\cdots)).$$

Using the fact that module categories are dg-categories, direct computation shows that the pull-back so defined is indeed a functor of A_∞ categories and, moreover, if T is a natural transformation (see [Sei08] for the definition), then so are χ and T^*.

B. The category $T^S D\mathcal{F}uk^d(M)$. Given any triangulated category $\mathcal{C}$, the associated category $T^S\mathcal{C}$ has morphisms that reflect the various ways to decompose objects in $\mathcal{C}$ by iterated exact triangles.

A cone decomposition of length k of an object $A \in \mathcal{C}$ is a sequence of exact triangles:

$$T^{-1}X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Y_{i+1} \xrightarrow{w_i} X_i$$

with $1 \leq i \leq k$, $Y_{k+1} = A$, $Y_1 = 0$. (Note that $Y_2 \cong X_1$.) The sequence $(X_1, \ldots, X_k)$ is called the **linearization** of the cone decomposition. There is an obvious equivalence relation among cone decompositions. The category $T^S\mathcal{C}$, called the **category of (stable) triangle (or cone) resolutions over $\mathcal{C}$** has as objects finite, ordered families $(x_1, x_2, \ldots, x_k)$ of objects $x_i \in \mathcal{Ob}(\mathcal{C})$.

The morphisms in $T^S\mathcal{C}$ are simplest to describe when defined on a family formed by a single object $x \in \mathcal{Ob}(\mathcal{C})$ and target $(y_1, \ldots, y_q)$, $y_i \in \mathcal{Ob}(\mathcal{C})$. For this, consider triples (ϕ, a, η), where $a \in \mathcal{Ob}(\mathcal{C})$, $\phi : x \to T^s a$ is an isomorphism (in $\mathcal{C}$) for some index s and η is a cone decomposition of the object a with linearization $(T^{s_1}y_1, T^{s_2}y_2, \ldots, T^{s_{q-1}}y_{q-1}, y_q)$ for some family of indices $s_1, \ldots, s_{q-1}$. A morphism $\Psi : x \longrightarrow (y_1, \ldots, y_q)$ is an equivalence class of triples (ϕ, a, η) as before up to a natural equivalence relation (reflecting the equivalence of cone decompositions). We now define the morphisms between two general objects. A morphism

$$\Phi \in \mathrm{Mor}_{T^S\mathcal{C}}((x_1, \ldots x_m), (y_1, \ldots, y_n))$$

is a sum $\Phi = \Psi_1 \oplus \cdots \oplus \Psi_m$ where $\Psi_j \in \mathrm{Mor}_{T^S\mathcal{C}}(x_j, (y_{\alpha(j)}, \ldots, y_{\alpha(j)+\nu(j)}))$, and $\alpha(1) = 1$, $\alpha(j+1) = \alpha(j) + \nu(j) + 1$, $\alpha(m) + \nu(m) = n$. The sum $\oplus$ means here the obvious concatenation of morphisms. With this definition this category is

strict monoidal, the unit element being given by the void family. We again refer to [BC14] for more ample details.

The indexes s_i, s do not play any role in this paper as we work in an ungraded context.

There is a natural projection functor $\mathcal{P} : T^S\mathcal{C} \to \mathcal{C}$ that sends, on objects, $(x_1, \ldots, x_k) \to x_k$. On morphisms, we first define $\mathcal{P}$ for a simple morphism $\Psi : x \to (y_1, \ldots, y_q)$ represented by a triple $\Psi = (\phi, a, \eta)$ as above. Denote by $T^{-1}x_k \to Z_k \to a \xrightarrow{\delta} x_k$ the last exact triangle in the cone decomposition η. In this case, $\mathcal{P}(\Psi) = \delta \circ \phi$. The value of $\mathcal{P}$ on a more general sum $\Phi = \Psi_1 \oplus \Psi_2 \oplus \cdots \oplus \Psi_m$ — again as above — is $\mathcal{P}(\Phi) = \mathcal{P}(\Psi_m)$.

Remark 2.1.6: a. It is useful to notice that for two objects of $T^S\mathcal{C}$ that coincide with single objects $a, a' \in \mathcal{Ob}(\mathcal{C})$, the morphisms from a to a' in $T^S\mathcal{C}$ are the **isomorphisms** from a to a' in $\mathcal{C}$. In particular, not all the morphisms in $\mathcal{C}$ appear as morphisms in $T^S\mathcal{C}$.

b. We will need further the following simple remark. Assume that $\mathcal{H} : \mathcal{C} \to \mathcal{C}$ is a functor that preserves the triangulated structure. It is then easy to see that $\mathcal{H}$ admits a lift $\hat{\mathcal{H}} : T^S\mathcal{C} \to T^S\mathcal{C}$. Further, assuming that H' is another such functor and that $\eta : \mathcal{H} \to \mathcal{H}'$ is a natural isomorphism, then η induces a natural transformation $\hat{\eta} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}'$.

2.1.5. *The endomorphism category.* For completeness, given a category $\mathcal{X}$, we recall the (strict) monoidal structure on the endofunctor category $\mathcal{End}(\mathcal{X})$; see also [Mac98]. In brief, a category C is strict monoidal if its objects as well as its morphisms can be multiplied via a bifunctor $C \otimes C \to C$. This multiplication is required to be associative and there has to be a unit object. A functor is (strict) monoidal if it preserves the multiplication and the unit. For non-strict monoidal categories the multiplication is only associative up to a natural isomorphism and similarly for the axioms verified by the unit.

On objects of $\mathcal{End}(\mathcal{X})$, multiplication is simply the composition of functors. On morphisms, given two natural transformations $T_i \in \text{Mor}(F_i, G_i)$, $i = 1, 2$, the product $T_1 \times T_2 \in \text{Mor}(F_2 \circ F_1, G_2 \circ G_1)$ is defined as follows. For any pair of objects X, Y and any morphism $f : X \to Y$, we have the following commutative

diagrams:

$$
\begin{array}{ccc}
F_i(X) & \xrightarrow{\;F_i(f)\;} & F_i(Y) \\
\Big\downarrow{\scriptstyle T_i} & & \Big\downarrow{\scriptstyle T_i} \\
G_i(X) & \xrightarrow[G_i(f)]{} & G_i(Y)
\end{array}
$$

We apply the functor F_2 to the diagram associated to T_1 and then use T_2 in the following fashion:

$$
\begin{array}{ccc}
F_2(F_1(X)) & \xrightarrow{\;F_2(F_1(f))\;} & F_2(F_1(Y)) \\
\Big\downarrow{\scriptstyle F_2(T_1)} & & \Big\downarrow{\scriptstyle F_2(T_1)} \\
F_2(G_1(X)) & \xrightarrow{\;F_2(G_1(f))\;} & F_2(G_1(Y)) \\
\Big\downarrow{\scriptstyle T_2} & & \Big\downarrow{\scriptstyle T_2} \\
G_2(G_1(X)) & \xrightarrow{\;G_2(G_1(f))\;} & G_2(G_1(Y))
\end{array}
$$

We then take $T_2 \circ F_2(T_1)$ to be the desired natural transformation. Simply put, we follow the natural transformations $F_2 \circ F_1 \xrightarrow{T_1} F_2 \circ G_1 \xrightarrow{T_2} G_2 \circ G_1$. Clearly, we could as well have taken $F_2 \circ F_1 \xrightarrow{T_2} G_2 \circ F_1 \xrightarrow{T_1} G_2 \circ G_1$ but because the square below commutes,

$$
\begin{array}{ccc}
F_2(F_1(X)) & \xrightarrow{\;F_2(T_1)\;} & F_2(G_1(X)) \\
\Big\downarrow{\scriptstyle T_2} & & \Big\downarrow{\scriptstyle T_2} \\
G_2(F_1(X)) & \xrightarrow{\;G_2(T_1)\;} & G_2(G_1(X))
\end{array}
$$

this does not make any difference.

2.2. THE FUNCTORS IN THE DIAGRAM.

2.2.1. *The functors in the top square.* The inclusion $i : \pi_1(M) \to \Pi(Ham(M))$ is viewed here as an inclusion of categories and is easily seen to be monoidal.

a. The functor S is Seidel's "standard" representation [Sei97]. One possible definition is as follows. Fix a loop of Hamiltonian diffeomorphisms $\mathbf{g} = \{g_t\}_{t \in S^1}$. Assume that $\mathbf{g}$ is induced by the Hamiltonian vector field X^G associated to a time-dependent Hamiltonian $G : S^1 \times M \to \mathbb{R}$. Additionally, let $H : S^1 \times M \to \mathbb{R}$ be another time-dependent Hamiltonian and let $H'(t, x) =$

$G(t, g_t(x)) + H(t, g_t(x))$. Consider an almost complex structure (possibly time-dependent) J that is compatible with ω. Consider also the almost complex structure $\tilde{J} = (g_t)_*(J)$. Assume that J is so that both (H, J) and $(H', \tilde{J})$ are regular in the sense that the Floer complexes $CF(H, J)$ and $CF(H', \tilde{J})$ are both defined. Recall that there is a chain morphism $\psi : CF(H', \tilde{J}) \to CF(H, J)$ that is unique up to homotopy, induces an isomorphism in homology and is associated to a generic homotopy $(H_\tau, J_\tau) : (H', \tilde{J}) \simeq (H, J)$, $\tau \in [0, 1]$ of the data. A 1-periodic orbit $\gamma(t)$ of H transforms into a 1-periodic orbit of H' under the transformation $\gamma(t) \to \gamma'(t) = g_t^{-1}(\gamma(t))$. Moreover, let $u : \mathbb{R} \times S^1 \to M$ be a solution to Floer's equation $\partial_s u + J\partial_t u + \nabla H(t, u) = 0$ and let $v(s, t) = (g_t)^{-1}(u(s, t))$. It is easy to see that v verifies the equation $\partial_s v + \tilde{J}\partial_t v + \nabla H'(t, v) = 0$ (the gradient is taken in each equation with respect to the metric associated to the respective almost complex structure). As the solutions of Floer's equation form the moduli spaces used to define the differential in the Floer complex, one concludes that the map $\gamma \to \gamma'$ provides a chain level isomorphism (also called the naturality isomorphism) $N : CF(H, J) \to CF(H', \tilde{J})$. We thus obtain that the morphism $N \circ \psi$ is an endomorphism of $HF(H, J) \cong QH(M)$. It is not hard to see that this morphism is a module morphism over the quantum ring $QH(M)$ and that it is invertible. Thus it is completely determined by its value $(N \circ \psi)([M]) \in QH(M)^*$ on the unit $[M] \in QH(M)$ of quantum homology. This specific element $(N \circ \psi)([M])$ will be denoted by $S(\mathbf{g})$ and it depends only on the homotopy class of $\mathbf{g}$. The resulting application $[\mathbf{g}] \in \pi_1(Ham(M)) \to S(\mathbf{g}) \in QH(M)^*$ is a group morphism known as Seidel's representation [Sei97].

Remark 2.2.1: In Seidel's paper [Sei97] the actual transformation used is rather $\gamma(t) \to g_t(\gamma(t))$. Thus, our class $S(\mathbf{g})$ is actually the inverse of the class there.

b. The functor $\widetilde{S}$. The main part of the construction appears in [Sei08]. First, one constructs a category $\mathcal{F}uk^d(M)^{free}$ on which $Ham(M)$ acts freely. This category has as objects pairs (L, g) with $g \in Ham(M)$ and $L \in \mathcal{L}_d^*(M)$. One then picks for $g = \mathrm{id}$ all the almost complex structures, Floer and perturbation data needed to define the Fukaya category $\mathcal{F}uk^d(M)$. With these fixed choices we now denote the resulting A_∞ category by $\mathcal{F}uk^d(M, 0)$. The objects of this category are (L, id). This data is then transported by g: the Lagrangians by $L \to g(L)$, but also the almost complex structures as well as all the auxiliary data. This defines Fukaya categories $\mathcal{F}uk^d(M, g)$ with objects (L, g).

The multiplications in $\mathcal{F}uk^d(M)^{free}$ are constructed so as to extend those on $\{\mathcal{F}uk^d(M,g)\}, g \in Ham(M)$ in the sense that each one of $\mathcal{F}uk^d(M,g)$ is a full and faithful subcategory of $\mathcal{F}uk^d(M)^{free}$ and in such a way that the multiplications are equivariant under the action of $Ham(M)$. This is possible because the action of $Ham(M)$ is free on objects. To fix notation, if $h \in Ham(M)$ the action is $h(L,g) = (h(L), h \circ g)$. Thus, to each element $g \in Ham(M)$ we can associate an obvious functor $\widetilde{g} : \mathcal{F}uk^d(M)^{free} \to \mathcal{F}uk^d(M)^{free}$. Given two elements $g_0, g_1 \in Ham(M)$ together with a path $\mathbf{g} = \{g_t\}_{t \in [0,1]}, g_t \in Ham(M)$ that joins them, there is an associated natural transformation $\xi_{\mathbf{g}} : \widetilde{g}_0 \to \widetilde{g}_1$ that is constructed in [Sei08] (it appears there only when $g_0 = $ id but the construction is the same in the general case). The definition of $\xi_{\mathbf{g}}$ is needed further in the paper so we review it shortly here. The natural transformation $\xi_{\mathbf{g}}$ is a collection of multi-linear maps defined for any $k + 1$ objects of $\mathcal{F}uk^d(M)^{free}$, $(L_1, h_1), \ldots, (L_{k+1}, h_{k+1})$:

$$\xi_{\mathbf{g}}^k : CF((L_1, h_1), (L_2, h_2)) \otimes \cdots \otimes CF((L_k, h_k), (L_{k+1}, h_{k+1})) \to$$
$$CF(g_0(L_1, h_1), g_1(L_{k+1}, h_{k+1}))$$

together with some elements $\xi_{\mathbf{g}}^0 \in CF(g_0(L,h), g_1(L,h))$ that are defined for each (L, h). We summarize the construction of these maps next. It is not difficult to verify that, with the definitions below, $\xi_{\mathbf{g}}$ is indeed a natural transformation and not only a pre-natural one by the same argument as that given in [Sei08, §(10d)].

We describe first the maps $\xi_{\mathbf{g}}^k$ for $k \geq 1$. They are defined as follows. Consider $(k + 1)$-pointed stable disks S'_r endowed with an interior marked point $z = z_{S'_r}$ (r is here a parameter moving inside the appropriate Deligne–Mumford–Stasheff associahedron). This marked point is used to stabilize the disk in the source space (see also [Sei08, §(10d)]) and to allow for a way to parametrize the choices of perturbation data compatible with splitting and gluing. The $k + 1$ boundary punctures are ordered clockwise. We denote by $C_j \subset \partial S'_r$ the connected components of $\partial S'_r$ indexed so that C_1 goes from the exit to the first entry, C_j goes from the $(j - 1)$-th entry to the j-th, $1 \leq j \leq k$, and C_{k+1} goes from the k-th entry to the exit. Up to reparametrization we may assume that the point $z_{S_r} = 0$ and that the exit is identified with $(1, 0)$. Each of the other k punctures, z_j, $1 \leq j \leq k$ corresponds to an entry and, in this writing, is given by $z^j = e^{-2i\pi s_{j+1}}$ for a well defined $s_j \in [0, 1]$. We put $s_1 = 0$ and

$s_{k+1} = 1$ and we let $\mathbf{s} = \mathbf{s}_{S'_r} = \{0 = s_1 \leq s_2 \leq \cdots \leq s_{k+1}\}$. The map $\xi_{\mathbf{g}}^k$ counts perturbed J-holomorphic polygons, as in the definition of the A_∞-structure $\mathcal{F}uk^d(M)$, but subject to moving boundary conditions controlled by $\mathbf{s}$: if $z \in C_j$, then $u(z)$ moves along $g_s(L_j)$ for $s_j \leq s \leq s_{j+1}$ so that, with the parametrization above, $u(e^{-2i\pi s}) \in g_s(L_j)$. The asymptotic conditions along the strip like ends of S_r are so that for the j-th input the asymptotic limit belongs to $CF(g_{s_{j+1}}(L_j, h_j), g_{s_{j+1}}(L_{j+1}, h_{j+1}))$ and the output corresponds to an asymptotic limit in $CF(g_0(L_1, h_1), g_1(L_{k+1}, h_{k+1}))$. Notice that the complex $CF(g_{s_{j+1}}(L_j, h_j), g_{s_{j+1}}(L_{j+1}, h_{j+1}))$ is the same as $CF((L_j, h_j), (L_{j+1}, h_{j+1}))$ because of the construction of the category $\mathcal{F}uk^d(M)^{free}$.

The elements $\xi_{\mathbf{g}}^0 \in CF(g_0(L, h), g_1(L, h))$ are given by counting curves with domain a stable disk with a single boundary puncture — identified with $(1, 0)$ — and again together with an interior marked point identified with 0. We will continue to denote a general such curve by S'_r. To have a better intuition of the definition of $\xi_{\mathbf{g}}^0$ assume for a moment that $h = id$ and that $g_0(L)$ and $g_1(L)$ are in general position (if this is not the case we use, as always, Hamiltonian perturbations). Then $\xi_{\mathbf{g}}^0$ equals a sum: $\sum n_i x_i$ where the x_i's are intersection points of $g_0(L) \cap g_1(L)$ and $n_i \in \mathbb{Z}$ counts the elements in the moduli space consisting of J-holomorphic curves $u : S'_r \to M$ so that (assymptotically) $u(1, 0) = x_i$ and, on the boundary, $u(e^{-2i\pi t}) \in g_t(L)$ (as always in this type of "counting" definitions, we restrict to the components of the relevant moduli space that are 0-dimensional; monotonicity and Gromov compactness imply that the sum is finite).

Of course, for this construction to succeed one also needs to show that the choices of interior marked points $z_{S'_r}$ together with the additional choices specific to the definition of an A_∞-category (strip-like ends, etc.) can all be made in a coherent way with respect to gluing and splitting in the Deligne–Mumford–Stasheff associahedron. This is achieved as in [Sei08]. Moreover, it is easy to see that the fact that we are working in the monotone context and no longer in the exact one, as in [Sei08], can be dealt with by the methods in [BC09].

We now consider the category of functors $fun(\mathcal{F}uk^d(M)^{free}, Ch^{opp})$ and we see from Remark 2.1.5 that the functors $\widetilde{g}$ induce functors

$$\hat{g} : fun(\mathcal{F}uk^d(M)^{free}, Ch^{opp}) \to fun(\mathcal{F}uk^d(M)^{free}, Ch^{opp})$$

given by composition and the natural transformation $\xi_{\mathbf{g}}$ induces a natural transformation $\hat{\xi}_{\mathbf{g}} : \hat{g}_0 \to \hat{g}_1$. It is easily seen that the correspondence $g \to \hat{g}$, $\mathbf{g} \to \hat{\xi}_{\mathbf{g}}$

provides an action of $\Pi(Ham(M))$ on $Hfun(\mathcal{F}uk^d(M)^{free}, Ch^{opp})$. In particular, $[\hat{\xi}_{\mathbf{g}}]$ — the homology image of $\hat{\xi}_{\mathbf{g}}$ — only depends on the homotopy class (with fixed ends) of $\mathbf{g}$ and the natural transformations $[\hat{\xi}_{\mathbf{g}}]$ are natural isomorphisms. The inclusions $\mathcal{F}uk^d(M,g) \hookrightarrow \mathcal{F}uk^d(M)^{free}$ are all quasi-equivalences so that we can pull-back this action to $Hfun(\mathcal{F}uk^d(M), Ch^{opp})$ as described in [Sei08]. We will not change the notation of the various functors, natural transformations, etc., after this pull-back.

The functors $\hat{g}_0$ and $\hat{g}_1$ preserve exact triangles and, given that $[\hat{\xi}_{\mathbf{g}}]$ is a natural isomorphism, we deduce from Remark 2.1.6 that the action of $\Pi(Ham(M))$ on $Hfun(\mathcal{F}uk^d(M), Ch^{opp})$ induces an action $\widetilde{S}$ of $\Pi(Ham(M))$ on $T^S D\mathcal{F}uk^d(M)$.

Remark 2.2.2: It will be useful in the following to have an alternative description of the homology classes $[\xi_{\mathbf{g}}^0] \in HF(g_0(L), g_1(L))$. They can be described as follows: we consider a morphism

(7) $$\phi_{\mathbf{g}} : CF(g_0(L), g_0(L)) \to CF(g_0(L), g_1(L))$$

defined by counting Floer strips $u : \mathbb{R} \times [0,1] \to M$ with $u(\mathbb{R} \times \{0\}) \subset g_0(L)$ and with $u(s \times \{1\}) \in g_{\psi(s)}(L)$ with $\psi : \mathbb{R} \to [0,1]$ an appropriate function that is increasing, null at $-\infty$ and 1 at $+\infty$. Then

(8) $$[\xi_{\mathbf{g}}^0] = \phi_{\mathbf{g}}(PSS([g_0(L)]))$$

where $[g_0(L)]$ is the unit in $QH(g_0(L))$. The morphism $PSS : QH(g_0(L)) \to HF(g_0(L), g_0(L))$ is the PSS-isomorphism, $[g_0 L]$ is the unit in $QH(g_0(L))$. The class $PSS([g_0(L)])$ is itself the unit in the ring $HF(g_0(L), g_0(L))$ and sometimes, in case no confusion is possible, to shorten notation we will omit PSS from the notation of this homology class.

The proof of identity (8) is an exercise. Indeed, since $\xi_{\mathbf{g}}^0$ does not depend on the parametrization of $\mathbf{g}$, we may choose the path $\mathbf{g}$ such that $g_s = g_0$ for $0 \leq s \leq 2/3$. Fix now a Morse function $f : g_0(L) \to \mathbb{R}$ so that f has a single maximum that we denote by w. In the pearl complex [BC] of f this maximum represents $[g_0(L)]$. With this choice, for a generic metric on $g_0(L)$, the curves $u : S'_r \to M$ counted by $\xi_{\mathbf{g}}^0$ are in bijection with the pairs (u, γ) where $\gamma : (-\infty, 0] \to g_0(L)$ is a negative flow line of f with origin at w and so that $\gamma(0) = u(-1, 0) \in u(S'_r)$. Indeed, by our choice of parametrization for $\mathbf{g}$, $u(-1, 0) \in g_0(L)$ and thus, for a generic metric on $g_0(L)$, there is precisely one such flow line for each map u (again, we restrict to 0-dimensional moduli spaces for these counts so that there are actually a finite number of maps u to worry

about). The algebraic count of all these pairs (u, γ) is precisely the definition of the moving boundary PSS morphism (for instance, see [BC13]), hence it is equal to $PSS([g_0(L)])$ in homology.

c. We now describe the action $*$. On objects, it associates to the unique object in $QH(M)^*$ the identity functor of $T^S D\mathcal{F}uk^d(M)$. Given an element $\alpha \in QH(M)^*$ we need to explain how we can associate to it a natural transformation $\hat{\eta}_\alpha$ from the identity to the identity in $T^S D\mathcal{F}uk^d(M)$.

By Remark 2.1.6, to define such a transformation it is enough to produce a natural isomorphism of the identity in $D\mathcal{F}uk^d(M)$. The natural transformation $\hat{\eta}_\alpha$ is easy to describe on the Donaldson category. For this, fix $L \in \mathcal{L}_d^*(M)$. Then $\hat{\eta}_\alpha$ associates to L the homology class $\eta_\alpha(L) = \alpha * [L] \in QH(L) \cong HF(L, L)$. Here $*$ is the module action $QH(M) \otimes QH(L) \to QH(L)$. The properties of this module action (as seen, for instance, in [BC]) show that this definition induces a natural transformation of the identity in the Donaldson category. We now describe briefly the refinement of this definition at the level of the Fukaya category.

First, one defines a natural transformation $\eta_\alpha : \mathrm{id}_{\mathcal{F}uk^d(M)} \to \mathrm{id}_{\mathcal{F}uk^d(M)}$ of A_∞ functors by using moduli spaces of perturbed J-holomorphic polygons S'_r with one interior marked point $z_{S'_r}$ as were used at the point b above. The relevant moduli spaces are formed by curves $u : S'_r \to M$ that satisfy the usual boundary conditions $u(C_i) \subset L_i$ and are so that if c_α is a cycle representing the homology class α — for instance, this could be a union of unstable manifolds of an appropriate Morse function on M — then $u(z_{S'_r}) \in c_\alpha$, thus yielding multilinear maps $\eta_\alpha^k : CF(L_1, L_2) \otimes \cdots \otimes CF(L_k, L_{k+1}) \to CF(L_1, L_{k+1})$. For each L, the morphism $\eta_L^0 \in CF(L, L)$ is defined by counting curves $u : S'_r \to (M, L)$, where S'_r is a stable disk with one marked point on the boundary and one marked point in the interior. The boundary point is mapped to an element of $CF(L, L)$ and the interior point $z_{S'_r}$ satisfies $u(z_{S_r}) \in c_\alpha$.

As a consequence of Remark 2.1.5, we obtain that η_α induces a natural transformation $\hat{\eta}_\alpha$ of the identity of $fun(\mathcal{F}uk^d(M), Ch^{opp})$. We then transport all this structure to the homological category. It is not difficult to show that the resulting natural transformation $[\hat{\eta}_\alpha]$ does not depend on the choices used in the construction. It is then shown that, additionally, for $\alpha, \beta \in QH(M)^*$, we have $[\hat{\eta}_{\alpha*\beta}] = [\hat{\eta}_\alpha] \circ [\hat{\eta}_\beta]$. This identity is an obvious generalization of the properties of the module action of $QH(M)$ on Lagrangian quantum homology

as explained, for instance, in [BC] §5.3. The proof uses moduli spaces of disks with boundary punctures as before but with two internal marked points with the constraint that, up to re-parametrization, one of the internal marked points is at $(0,0)$, the output corresponds to a boundary puncture at $(1,0)$ and the second internal marked point belongs to the segment $(0,1) \times \{0\} \subset D^2$.

All the resulting natural transformations are natural isomorphisms because the elements α are, by definition, invertible in the ring $QH(M)$. It then follows that $[\hat{\eta}_\alpha]$ induces a natural isomorphism $\bar{\eta}_\alpha$ of the identity on $D\mathcal{F}uk^d(M)$ and, further, a natural transformation, also denoted by $\bar{\eta}_\alpha$, on $T^S D\mathcal{F}uk^d(M)$. Moreover, $\bar{\eta}_{\alpha * \beta} = \bar{\eta}_\alpha \circ \bar{\eta}_\beta$. This provides the action of $QH(M)^*$ on $T^S D\mathcal{F}uk^d(M)$.

2.2.2. *The functor $\widetilde{\mathcal{F}}$.* There is first an auxiliary functor $\mathcal{F}$ that will be needed here and that we now recall from [BC13]. This functor is a sort of linearization of $\widetilde{\mathcal{F}}$. It is defined on a simplified version of $\mathcal{C}ob_0^d(M)$ that is denoted by $S\mathcal{C}ob_0^d(M)$. The objects of $S\mathcal{C}ob_0^d(M)$ are single Lagrangians $L \in \mathcal{L}_d^*(M)$ and the morphisms from L to L' are horizontal isotopy classes of cobordisms $V : L \rightsquigarrow (L_1, \ldots, L_{k-1}, L')$, $V \in \mathcal{L}_d^*(\mathbb{C} \times M)$, where $k \in \mathbb{N}$ and $L_i \neq \emptyset$ if $k > 0$ with the understanding that if $k = 0$, then $V : L \rightsquigarrow L'$.

The functor

$$\mathcal{F} : S\mathcal{C}ob_0^d(M) \to D\mathcal{F}uk^d(M)$$

is the identity on objects. For a cobordism V giving a morphism between L and L' in $S\mathcal{C}ob_0^d(M)$, the morphism $\mathcal{F}(V) \in HF(L, L') = \mathrm{Mor}_{D\mathcal{F}uk}(L, L')$ is the image of the unity in $HF(L, L)$ (induced by the fundamental class of L) through a morphism $\phi_V : HF(L, L) \to HF(L, L')$. The morphism ϕ_V is given by counting Floer strips in $\mathbb{R}^2 \times M$ with boundary conditions along V on one side and along $\gamma \times L$ on the other side, $\mathcal{F}(V) = \phi_V([L])$. Here $\gamma \subset \mathbb{R}^2$ and V are as in Figure 3, with $L' = L_k$. In case the cobordism V is elementary — which means that L is the single positive end of V and L' is its single negative end — then ϕ_V as well as $\mathcal{F}(V)$ are isomorphisms.

The functor

$$\widetilde{\mathcal{F}} : \mathcal{C}ob_0^d(M) \to T^S D\mathcal{F}uk^d(M)$$

is constructed in [BC14]. Here we only need three of its properties:

    i. $\widetilde{\mathcal{F}}$ coincides with the identity on objects.
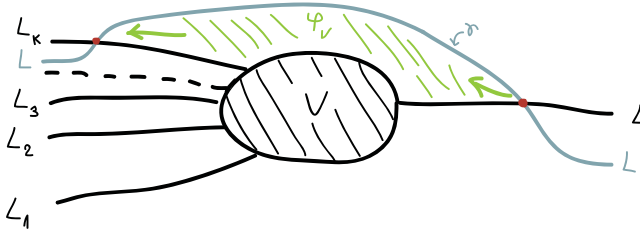   ii. $\widetilde{\mathcal{F}}$ is monoidal.

Figure 3. A cobordism $V \subset \mathbb{R}^2 \times M$ with a positive end L and with $L' = L_k$ together with the projection of the J-holomorphic strips that define the morphism ϕ_V.

iii. For a morphism $V : L \to L'$, $V \in \mathrm{Mor}_{\mathcal{C}ob_0^d(M)}(L, L')$ (that is, a morphism associated to an elementary cobordism) the value of $\widetilde{\mathcal{F}}$ coincides with the value on V of the linearized functor $\mathcal{F}$:

$$\widetilde{\mathcal{F}}(V) = \mathcal{F}(V) \ .$$

To make explicit the identity at iii recall that the value $\widetilde{\mathcal{F}}(V)$ is by definition a triple (ϕ, a, η), where $\phi : L \to a$ is an isomorphism in $D\mathcal{F}uk^d(M)$ and η is a cone decomposition of a in just one stage

$$T^{-1}L' \to 0 \to a \xrightarrow{\delta} L' \ .$$

In other words $\delta : a \to L'$ is an isomorphism that identifies L' and a. Thus $\widetilde{\mathcal{F}}(V)$ reduces to giving an isomorphism $\bar{\phi} : L \to L'$ in $D\mathcal{F}uk^d(M)$ which can then be written as $\bar{\phi} = \delta \circ \phi$ (conversely, for any such isomorphism, one can take $a = L'$, $\delta = \mathrm{id}$). As seen above, because V is elementary, $\mathcal{F}(V)$ is precisely such an isomorphism $\in \mathrm{Mor}_{D\mathcal{F}uk^d(M)}(L, L') = HF(L, L')$.

2.2.3. *The functor Σ.* We first fix some notation. A Lagrangian cobordism $V : L \to (L_i)$ is written in coordinates as:

$$V \subset T^*[0, 1] \times M \ , \quad p \mapsto (t(p), y(p), \pi_M(p)) \ .$$

We recall that the symplectic form on $T^*[0, 1] \times M$ is $\bar{\omega} = \omega_0 \oplus \omega$. The coordinates in $T^*[0, 1] = [0, 1] \times \mathbb{R}$ are (t, y).

Given a path $\mathbf{g} = \{g_t\}_{t \in [0,1]}, g_t \in Ham(M)$ generated by a time-dependent Hamiltonian function $G : [0, 1] \times M \to \mathbb{R}$, we get a symplectomorphism

$$\Phi_G : T^*[0, 1] \times M \to T^*[0, 1] \times M, \quad (t, y, p) \mapsto (t, y + G_t(g_t(p)), g_t(p)).$$

By restricting this map to the cobordism V, we get a new Lagrangian embedding
(9)
$$\Phi_G : V \to T^*[0,1] \times M, \quad p \mapsto (t(p), y(p) + G_{t(p)}(g_{t(p)}(\pi_M(p))), g_{t(p)}(\pi_M(p))) \ .$$

We will assume here and below that, after a possible reparametrization, the path $\mathbf{g}$ is constant close to each one of its ends. Moreover, the Hamiltonian G is supposed to be so that G vanishes near the ends of the interval $[0,1]$ in the sense that for some very small ϵ and any $m \in M$, $G(t,m) = 0$ for $t < \epsilon$ and $t > 1 - \epsilon$. It follows that $V^G = \Phi_G(V) : g_1(L) \to (g_0(L_i))$ is a cobordism. It will be called the Lagrangian suspension of V by the path $\mathbf{g}$. This agrees with standard terminology when V is the trivial cobordism [Pol01].

PROPOSITION 2.2.3: *The Lagrangian suspension extends to an action of $\Pi(Ham(M))$ on $\mathcal{C}ob_0^d(M)$.*

Proof. We start with the following geometric statement, which also shows that the suspension is independent of the parametrization of the path $\mathbf{g}$.

LEMMA 2.2.4: *Suppose $\mathbf{f} = (f_t)$ and $\mathbf{g} = (g_t)$ are paths in $Ham(M)$ that are induced, respectively, by Hamiltonians G and G' as above. Further, assume that $\mathbf{f}$ and $\mathbf{g}$ are homotopic in $Ham(M)$ relative to their endpoints. Then the associated suspensions V^G and $V^{G'}$ are horizontally Hamiltonian isotopic for any cobordism V.*

Proof of Lemma 2.2.4. Let $h : [0,1] \times [0,1] \to Ham(M)$ denote a homotopy so that $h(0,t) = f_t$ and $h(1,t) = g_t$ (the homotopy parameter is s). Up to reparametrization we may assume that the paths of Hamiltonian diffeomorphisms $h_s(t)$, $t \in [0,1]$ (s fixed and t varies) and $h_t(s)$, $s \in [0,1]$ (t fixed and s varies) are constant near their ends for all $s, t \in [0,1]$. We now consider two families of Hamiltonian vector fields induced by h:

$$X_{s,t}(h(s,t)(m)) = \frac{\partial}{\partial t}(h(s,t)(m)), \quad Y_{s,t}(h(s,t)(m)) = \frac{\partial}{\partial s}(h(s,t)(m)).$$

In particular, $X_{0,t}$ is the Hamiltonian vector field of G and $X_{1,t}$ is the Hamiltonian vector field of G'.

Let $H, F : [0,1] \times [0,1] \times M \to \mathbb{R}$ be associated one parametric families of Hamiltonians so that $\omega(-, X_{s,t}) = dH_{s,t}(-)$ and $\omega(-, Y_{s,t}) = dF_{s,t}(-)$. The notation here is $H_{s,t}(m) = H(s,t,m)$ (and similarly for F) and we are taking exterior derivative in M and keeping s, t fixed. We also assume that $H_0 = G$

and $H_1 = G'$ (this is not restrictive because both G and G' vanish near their ends). Moreover, for each fixed s, the Hamiltonian $H(s, -, -)$ vanishes close to the ends of the interval $[0, 1]$ and the same is true for each fixed t for the Hamiltonian $F(-, t, -)$. Consider the following map:

(10) $$\Psi : [0, 1] \times T^*[0, 1] \times M \to T^*[0, 1] \times M$$

$$(s, t, y, m) \mapsto (t, y + H(s, t, h(s, t)(m)), h(s, t)(m))$$

It is easy to check that each map Ψ_s, obtained from Ψ by keeping s fixed, is symplectic. In particular, the restriction of Ψ to any cobordism V is a Lagrangian isotopy from V^G to $V^{G'}$. Moreover, given that $H(s, -, -)$ is vanishing near the ends of $[0, 1]$ for each $s \in [0, 1]$, to verify that $\Psi|_V$ is a horizontal isotopy it only remains to prove that Ψ is Hamiltonian. Define for each $s \in [0, 1]$,

$$\alpha_s(-) = \bar{\omega}\Big(\Psi_*\Big(\frac{\partial}{\partial s}\Big), -\Big) \in \Omega^1(T^*[0, 1] \times M) \ .$$

Thus, we need to show that α_s is exact for each s. Let

$$\bar{F} : [0, 1] \times T^*[0, 1] \times M \to \mathbb{R} \ ; \ (s, t, y, m) \mapsto F(s, t, m).$$

Denote by D the exterior derivative on $T^*[0, 1] \times M$. We want to note that $D\bar{F}_s = -\alpha_s$. For this we compute

$$\Psi_*\Big(\frac{\partial}{\partial s}\Big) = \Big(0, \frac{\partial H}{\partial s} + dH\Big(\frac{\partial h}{\partial s}\Big), \frac{\partial h}{\partial s}\Big) = \Big(0, \frac{\partial H}{\partial s} + dH(Y), Y\Big) = \Big(0, \frac{\partial F}{\partial t}, Y\Big),$$

the last equality coming from the identity $\frac{\partial H}{\partial s} + \{H, F\} = \frac{\partial F}{\partial t}$ (our convention for the Poisson bracket being $\{A, B\} = \omega(X^B, X^A)$). We immediately deduce $\alpha_s(\frac{\partial}{\partial y}) = D\bar{F}(\frac{\partial}{\partial y}) = 0$. Further, $\alpha_s(\frac{\partial}{\partial t}) = -\frac{\partial F_s}{\partial t} = -D\bar{F}(\frac{\partial}{\partial t})$. Finally, for $\xi \in TM$, we have $\alpha_s(\xi) = -dF_s(\xi) = -D\bar{F}(\xi)$ which proves the claim. ∎

It follows from the Lemma that the horizontal isotopy class of V^G only depends on the horizontal isotopy class of V and on the homotopy class (with fixed endpoints) of $\mathbf{g}$. We will denote this horizontal isotopy class by $[V]^{\mathbf{g}}$, in other words, $[V]^{\mathbf{g}} = [V^G]$. By a slight abuse of notation we will generally denote by $V^{\mathbf{g}}$ the cobordism V^G in case the choice of the Hamiltonian G is not significant.

We now proceed to define a monoidal functor

$$\Sigma : \Pi(Ham(M)) \to \mathcal{E}nd(\mathcal{C}ob_0^d(M)).$$

Let $g \in Ob(\Pi(Ham(M)))$ and set

$$\Sigma(g) : \mathcal{C}ob_0^d(M) \longrightarrow \mathcal{C}ob_0^d(M)$$

$$\left\{ \begin{array}{c} L \in Ob \\ [V] \in \mathrm{Mor} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} g(L) \\ [g(V)] := [(\mathrm{id} \times g)(V)] \end{array} \right\}$$

This is well defined and a functor by Lemma 2.2.4. Given a path of Hamiltonian diffeomorphisms, now viewed as a morphism $\mathbf{g} : g_0 \to g_1$ in $\Pi(Ham(M))$ we need to define a natural transformation $\Sigma(\mathbf{g})$ between the two functors $\Sigma(g_0)$ and $\Sigma(g_1)$. For each Lagrangian $L \in \mathcal{O}b(\mathcal{C}ob_0^d(M))$ we define

$$\Sigma(\mathbf{g})(L) = [[0,1] \times L]^{\mathbf{g}^{-1}} .$$

Here $\mathbf{g}^{-1}$ is the path g_{1-t} (the term $1 - t$ is due to the fact that morphisms in $\mathcal{C}ob$ go from right to left). For further use we will put more generally $\Sigma(\mathbf{g})(V) = [V]^{\mathbf{g}^{-1}}$.

Given a cobordism $V : L \to (L_i)$, we need to show that the following diagram commutes:

(11)

$$
\begin{array}{ccc}
g_0(L) & \xrightarrow{\ g_0(V)\ } & (g_0(L_i)) \\
\Big\downarrow{\scriptstyle \Sigma(\mathbf{g})(L)} & \searrow{\scriptstyle \Sigma(\mathbf{g})(V)} & \Big\downarrow{\scriptstyle \coprod_i \Sigma(\mathbf{g})(L_i)} \\
g_1(L) & \xrightarrow[\ g_1(V)\ ]{} & (g_1(L_i))
\end{array}
$$

Consider two functions $a_j(t) : [0,1] \to [0,1]$, $j = 0,1$ that are smooth, surjective and so that $a_0(t) = 1$ for $\frac{1}{2} \le t \le 1$ and $a_1(t) = 0$ for $0 \le t \le \frac{1}{2}$. We use these functions to reparametrize the path $\mathbf{g}$. We put $g_t^j = g_{a_j(t)}$ and $\mathbf{g}^j = (g_t^j)$. We also need two cobordisms V_0 and V_1 horizontally isotopic to V and so that $V_0|_{[0,\frac{1}{2}] \times \mathbb{R} \times M} = \bigcup_i [0,\frac{1}{2}] \times \{i\} \times L_i$ and $V_1|_{[\frac{1}{2},1] \times \mathbb{R} \times M} = [\frac{1}{2}, 1] \times \{1\} \times L$. By Lemma 2.2.4 we have $\Sigma(\mathbf{g})(V) = \Sigma(\mathbf{g}^0)(V_0) = \Sigma(\mathbf{g}^1)(V_1)$. The commutativity of the diagram now follows by noticing that $\Sigma(\mathbf{g}^0)(V_0)$ represents the composition $(\coprod \Sigma(\mathbf{g})(L_j)) \circ g_0(V)$ and $\Sigma(\mathbf{g}^1)(V_1)$ represents the composition $g_1(V) \circ \Sigma(\mathbf{g})(L)$.

Next, we show that Σ is a functor. Given $\mathbf{f}, \mathbf{g}$ two morphisms in $\Pi(Ham(M))$ such that $f_1 = g_0$, we have the following composition of natural transformations:

$$f_0(L) \xrightarrow{\Sigma(\mathbf{f})(L)} f_1(L) \xrightarrow{\Sigma(\mathbf{g})(L)} g_1(L) \ .$$

Clearly, $\Sigma(\mathbf{g})(L) \circ \Sigma(\mathbf{f})(L) = [[0,1] \times L]^{(\mathbf{f}\#\mathbf{g})^{-1}}$ where $\mathbf{f}\#\mathbf{g}$ is the concatenation of the paths $\mathbf{f}$ and $\mathbf{g}$. Given that in $\Pi(Ham(M))$ the composition of morphisms is so that $\mathbf{f}\#\mathbf{g} = \mathbf{g} \circ \mathbf{f}$, we also have $\Sigma(\mathbf{g} \circ \mathbf{f})(L) = [[0,1] \times L]^{(\mathbf{f}\#\mathbf{g})^{-1}}$.

Finally, we need to show that Σ is monoidal. Let $\mathbf{f}, \mathbf{g}$ be two morphisms in $\Pi(Ham(M))$. By the definition of the multiplication rule for natural transformations in §2.1.5, Σ is monoidal if the following diagram of cobordisms commutes:

$$
\begin{array}{ccc}
g_0 f_0(L) & \xrightarrow{g_0(\Sigma(\mathbf{f})(L))} & g_0 f_1(L) \\
& \searrow^{\Sigma(\mathbf{g}\cdot\mathbf{f})(L)} & \downarrow{\Sigma(\mathbf{g})(f_1(L))} \\
& & g_1 f_1(L)
\end{array}
$$

where $\mathbf{g}\cdot\mathbf{f}$ denotes the path $g_t f_t \in Ham(M)$. Notice that $\mathbf{g}\cdot\mathbf{f}$ is homotopic, with fixed end-points, to the composition $(\mathbf{g} \cdot f_1) \circ (g_0 \cdot \mathbf{f})$ so that the commutativity above follows from Lemma 2.2.4 and this concludes the proof. ∎

3. Commutativity of diagrams (1) and (2).

3.1. PROOF OF THEOREM A. As mentioned in the introduction, the commutativity of diagram (1) is equivalent to the commutativity of the diagram below:

$$
(12) \quad
\begin{array}{ccc}
\Pi(Ham(M)) & \xrightarrow{\widetilde{S}} & \mathcal{E}nd(T^S D\mathcal{F}uk^d(M)) \\
\Sigma \downarrow & & \downarrow \widetilde{\mathcal{F}}^* \\
\mathcal{E}nd(\mathcal{C}ob_0^d(M)) & \xrightarrow{\widetilde{\mathcal{F}}_*} & fun(\mathcal{C}ob_0^d(M), T^S D\mathcal{F}uk^d(M))
\end{array}
$$

On objects this commutes in the sense that for any $g \in Ham(M)$ the following diagram of functors commutes:

$$
\begin{array}{ccc}
\mathcal{C}ob_0^d(M) & \xrightarrow{\widetilde{\mathcal{F}}} & T^S D\mathcal{F}uk^d(M) \\
\Sigma(g) \downarrow & & \downarrow \widetilde{S}(g) \\
\mathcal{C}ob_0^d(M) & \xrightarrow{\widetilde{\mathcal{F}}} & T^S D\mathcal{F}uk^d(M)
\end{array}
$$

This is immediate because g acts on both $\mathcal{C}ob_0^d(M)$ and on $T^S D\mathcal{F}uk^d(M)$ by "translating" all the data by the Hamiltonian diffeomorphism g.

To prove the commutativity of (12) it remains to show that for any path $\mathbf{g} = (g_t)_{t\in[0,1]}$ in $Ham(M)$ the natural transformations $\widetilde{S}(\mathbf{g})$ and $\Sigma(\mathbf{g})$ are related by

$$(13) \qquad \widetilde{S}(\mathbf{g}) \circ \widetilde{\mathcal{F}} = \widetilde{\mathcal{F}} \circ \Sigma(\mathbf{g}) .$$

Given that $\widetilde{\mathcal{F}}$ is monoidal this is sufficient to verify for single Lagrangians $L \in \mathcal{O}b(\mathcal{C}ob_0^d(M))$. Recall that $\widetilde{\mathcal{F}}$ acts as the identity on objects and that on elementary cobordisms (those that have a single positive and a single negative end) it coincides with the linearization $\mathcal{F}$ of $\widetilde{\mathcal{F}}$. It follows that the proof reduces to showing that for all Lagrangians L as before we have

$$(14) \qquad \widetilde{S}(\mathbf{g})(L) = \mathcal{F}(\Sigma(\mathbf{g})(L)).$$

To shorten notation we now put

$$V = \Sigma(\mathbf{g})(L) = ([0,1] \times L)^{\mathbf{g}^{-1}}.$$

Recall that $\mathcal{F}(V) \in HF(g_0(L), g_1(L)) = \mathrm{Mor}_{D\mathcal{F}uk}(g_0(L), g_1(L))$ is defined as $\mathcal{F}(V) = \phi_V([g_0(L)])$ where $[g_0(L)] \in HF(g_0(L), g_0(L))$ is the unit. The morphism

$$\phi_V : CF(g_0(L), g_0(L)) \to CF(g_0(L), g_1(L))$$

was recalled in §2.2.2 (see also [BC13]).

As discussed in Remark 2.2.2, the term $\widetilde{S}(\mathbf{g})(L)$ coincides with the homology class $[\xi_{\mathbf{g}}^0] = \phi_{\mathbf{g}}([g_0(L)])$ where

$$\phi_{\mathbf{g}} : CF(g_0(L), g_0(L)) \to CF(g_0(L), g_1(L))$$

is the moving boundary morphism (7). Thus, to finish the proof it is enough to show the following result that first appeared in [Cha].

LEMMA 3.1.1: *With the notation above ϕ_V and $\phi_{\mathbf{g}}$ induce the same morphism in homology.*

Proof of Lemma 3.1.1. We will prove that for adequate choices of Floer data and other auxiliary data, the morphisms ϕ_V and $\phi_{\mathbf{g}}$ are chain homotopic.

Consider a one-parameter family of paths $h(s,t) := g_{s(1-t)} \in Ham(M)$, $s \in [0,1]$. This gives a homotopy between the constant path $\mathbf{g}_0 = (g_0)$ and the path $\mathbf{g}$. Obviously, the 1 endpoint of this homotopy is not fixed. We now follow
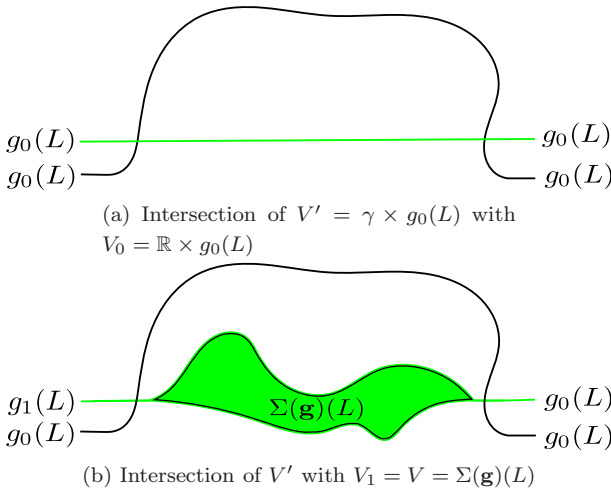
the general scheme in the proof of Lemma 2.2.4 to define a global Hamiltonian isotopy $\Psi : [0,1] \times T^*[0,1] \times M \to T^*[0,1] \times M$,

$$\Psi(s,t,y,m) = (t, y + H(s,t,h(s,t)(m)), h(s,t)(m))$$

as in (10). After an appropriate reparametrization of $h(s,t)$ we can again assume that $H(s,-,-) : [0,1] \times M \to \mathbb{R}$ vanishes for t sufficiently close to the ends of the interval $[0,1]$. The same argument as in the proof of Lemma 2.2.4 shows that Ψ is indeed Hamiltonian. The isotopy $\Psi = \{\Psi_s\}_{s \in [0,1]}$ verifies:

- $\Psi_0 = \mathrm{id} \times g_0$;
- $\Psi_1([0,1] \times L) = \Sigma(\mathbf{g})(L)$;
- $\Psi_s|_{\{1\} \times \{0\} \times M} = g_0$;
- $\Psi_s|_{\{0\} \times \{0\} \times M} = g_s$.

In particular, $\Psi|_{[0,1] \times L}$ is not horizontal. We now denote $V_s = \Psi_s([0,1] \times L)$ so that $V_0 = [0,1] \times g_0(L)$, $V_1 = V = \Sigma(\mathbf{g})(L)$. We also extend all these cobordisms trivially to $\mathbb{C} \times M$. Recall that $\pi : \mathbb{C} \times M \to \mathbb{C}$ is the projection. The set $\pi(\bigcup_{s \in [0,1]} V_s)$ is contained in a compact $K \subset T^*[0,1]$ so that there exists a curve γ as in the picture below, with $\gamma \cap \mathbb{R} \times \{0\} = \{(-1,0)\} \cup \{2,0)\} = \gamma \cap \pi(V_s)$ for all $s \in [0,1]$, and so that γ intersects $\mathbb{R} \times \{0\}$ transversally. The two points $Q = (-1,0)$ and $P = (2,0)$ will be referred to as the "bottlenecks" (see [BC14]).

(a) Intersection of $V' = \gamma \times g_0(L)$ with $V_0 = \mathbb{R} \times g_0(L)$

(b) Intersection of V' with $V_1 = V = \Sigma(\mathbf{g})(L)$

We let $V' = \gamma \times g_0(L)$ and consider the Floer complexes

$$\mathcal{C}_1 = (CF(V', V_0), D_1) \quad \text{and} \quad \mathcal{C}_2 = (CF(V', V), D_2) \ .$$

The complex $\mathcal{C}_2$ is defined as in [BC13] and [BC14] by first choosing Floer data for the pairs $(g_0(L), g_0(L))$ and $(g_0(L), g_1(L))$ in M. This data is used in the fibers of π over the bottlenecks and is then extended to the pairs (V', V_0) and (V', V_1), away from the bottlenecks, by using an almost complex structure on $\mathbb{C} \times M$ so that the projection $\pi : \mathbb{C} \times M \to \mathbb{C}$ is holomorphic outside of K. For $\mathcal{C}_1$ we use the same procedure but additionally it is easy to see that one can work with an almost complex structure on $M \times \mathbb{C}$ so that π is globally holomorphic. The construction here is in fact simpler than that in [BC13] because all the pairs (V', V_s) are cylindrically distinct at infinity. By the same arguments as in [BC13] we deduce that the form of the two differentials appearing here is

$$D_1 = \begin{pmatrix} d_0 & 0 \\ 1 & d_0 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} d_0 & 0 \\ \phi_V & d_1 \end{pmatrix},$$

where d_0 is the differential in $CF(g_0(L), g_0(L))$ and d_1 is the differential in $CF(g_0(L), g_1(L))$.

We now define a chain morphism

$$\eta_\Psi : \mathcal{C}_1 \to \mathcal{C}_2$$

by counting finite energy Floer trajectories $u(s,t) : \mathbb{R} \times [0,1] \to \mathbb{C} \times M$ whose planar projection starts in P and ends in Q and so that u is subject to moving boundary conditions:

$$u(s,0) \in V_0, \qquad u(s,1) \in V_{\alpha(s)},$$

where $\alpha : \mathbb{R} \to [0,1]$ is a smooth increasing function equal to 0 for s sufficiently small and equal to 1 for s sufficiently large. This construction is perfectly similar to many others in [BC13] but for completeness we review it rapidly here. We define the relevant moduli space by again using an almost complex structure that projects holomorphically away from a compact set. An application of the open mapping theorem together with the fact that V_s is independent of s around the bottlenecks shows rapidly that the projections of the curves u onto $\mathbb{C}$ remain inside a compact set $\subset \mathbb{C}$. Combined with the fact that the Hamiltonians $H_{s,t}$ have a finite variation on M, we obtain a priori energy bounds and this implies the compactness of the relevant moduli space, up to Floer splitting. Finally,

regularity is achieved by standard perturbative methods (see possibly [BC14]). In summary, we deduce that the map η_Ψ is a chain map given by a matrix:

$$\eta_\Psi : CF(g_0(L), g_0(L))_P \oplus CF(g_0(L), g_0(L))_Q$$
$$\to CF(g_0(L), g_0(L))_P \oplus CF(g_0(L), g_1(L))_Q$$

(15)
$$\begin{pmatrix} 1 & 0 \\ \mu & \phi_{\mathbf{g}} \end{pmatrix}.$$

The indexes P and Q in this formula indicate to which one of the two bottlenecks correspond the respective complexes. The one fact that is remarkable here is that $\phi_{\mathbf{g}}$ appears in the lower right corner of this matrix — this is due directly to the moving boundary definition of $\phi_{\mathbf{g}}$. The equation $D_2\eta_\Psi = \eta_\Psi D_1$ implies that μ is a chain homotopy between ϕ_V and $\phi_{\mathbf{g}}$ and concludes the proof. ∎

3.2. THE COROLLARY B: COMMUTATIVITY OF THE TOP SQUARE IN (2). We only provide a sketch of the proof of the commutativity of the top square in (2) as the result is known by experts. A variant of this commutativity, with the Donaldson category of M in the place of $T^S D\mathcal{F}uk^d(M)$, is contained in [HLL11].

PROPOSITION 3.2.1: *The top square in diagram* (2) *commutes.*

Proof. Fix $\mathbf{g} = \{g_t\}$, $g_t \in Ham(M)$, a loop of Hamiltonian diffeomorphisms. Thus, $g_0 = g_1 = $ id. By the description in §2 of the various functors involved, the desired commutativity follows if we show that the two natural transformations $\xi_{\mathbf{g}}$ and $\eta_{S[\mathbf{g}]}$ of $\mathrm{id}|_{\mathcal{F}uk^d(M)}$ have the same image in the homology category $Hfun(\mathcal{F}uk^d(M), \mathcal{F}uk^d(M))$,

(16)
$$[\xi_{\mathbf{g}}] = [\eta_{S[\mathbf{g}]}] \in Hfun(\mathcal{F}uk^d(M), \mathcal{F}uk^d(M)) .$$

The outline of the proof of (16) is as follows. We first rewrite both $[\xi_{\mathbf{g}}]$ and $[\eta_{S[\mathbf{g}]}]$ in terms of moduli spaces formed of curves u and, respectively, v defined on surfaces S_r'' with $k + 1$ boundary punctures and one internal puncture and appropriate asymptotic conditions and boundary conditions (these conditions are different for the u's compared to those of the v's). Then, in a second step, we relate the curves u and v by a geometric "naturality transformation" induced by the map $u(z) \to g_{a(z)}^{-1}(u(z))$, where $a(-)$ is an appropriate map $a = a_{S_r''} : S_r'' \to [0, 1]$.

To proceed we first define the punctured surfaces S_r'' in more detail. They are just as the surfaces S_r' used in §2.2.1 b, except that the point $z_{S_r'}$ is replaced by a puncture. We will use the S_r'''s as domains of curves $u : S_r'' \to M$ that satisfy the usual perturbed J-holomorphic equation and boundary conditions but with an additional property relative to the internal puncture. More precisely we will assume that, in the same way as strip-like ends are fixed for the boundary punctures, there is a universal choice of cylindrical-like ends around the internal punctures. The purpose of the choice of cylindrical-like ends is the following. Suppose that $K : S^1 \times M \to \mathbb{R}$ is a generic periodic Hamiltonian and assume J is an almost complex structure so that (K, J) is regular in the sense of Hamiltonian Floer homology. We can then write the correct perturbed J-holomorphic equation for $u : S_r'' \to M$ by using the strip-like ends around the boundary punctures, as well as the Floer data as in the construction of the Fukaya category and using the Hamiltonian Floer data (K, J) around the internal puncture. We use this setup to define two pre-natural transformations of the identity on $\mathcal{F}uk^d(M)$ associated to a Floer cycle $\beta \in CF(K, J)$. The first one is denoted by $\tilde{\eta}_\beta$ and is defined as follows. We first define

$$\tilde{\eta}^k : CF(L_1, L_2) \otimes \ldots \otimes CF(L_k, L_{k+1}) \otimes CF(K, J) \to CF(L_1, L_{k+1}),$$

given by counting curves u as before with boundary conditions so that $u(C_i) \subset L_i$ where $L_1, \ldots, L_{k+1} \in \mathcal{O}b(\mathcal{F}uk^d(M))$ and with asymptotic conditions corresponding to the generators of $CF(L_i, L_{i+1})$ for the i-th boundary puncture and to a generator of $CF(K, J)$ for the internal puncture. We then put $\tilde{\eta}_\beta = \tilde{\eta}(-, \ldots, -, \beta)$.

The second pre-natural transformation is denoted $\tilde{\xi}_{\beta, \mathbf{g}}$ and its definition involves also a fixed loop inside $Ham(M)$, $\mathbf{g} = \{g_t\}$, $t \in [0, 1]$, with $g_0 = g_1 = \mathrm{id}$. The definition is exactly as in the case of $\tilde{\eta}$ with the difference that the boundary conditions are now "moving", as in the definition of $\xi_\mathbf{g}$ in §2.2.1. In other words, in this case we have that along the segment C_j of the boundary of S_r'', $u(e^{-2i\pi s}) \in g_s(L_j)$ for $s_j \leq s \leq s_{j+1}$ (if we use a parametrization with the output at $(1, 0)$ and the internal puncture at $(0, 0)$).

Next we compare $\xi_\mathbf{g}$ and $\tilde{\xi}_{PSS([M]), \mathbf{g}}$ where $[M]$ is the fundamental class of M represented as the maximum of a Morse function $f : M \to \mathbb{R}$ and $PSS([M]) \in CF(K, J)$ is the image of this maximum by the PSS morphism (we assume a generic metric fixed on M so that the negative gradient of f is Morse–Smale). To compare these two transformations we notice that the moduli spaces of curves

$u : S'_r \to M$ used to define $\xi_{\mathbf{g}}$ can be thought as well as being pairs (u, l) where l is a negative gradient flow line of f that joins the maximum of f to the point $u(z_{S'_r})$. Because of this we can use the PSS-method to define some interpolating moduli spaces that define a pre-natural transformation τ with the property that $\mu^1(\tau) = \xi_{\mathbf{g}} - \tilde{\xi}_{PSS([M]),\mathbf{g}}$ in the functor A_∞-category $fun(\mathcal{F}uk^d(M), \mathcal{F}uk^d(M))$ (to be more precise we should work here with the free category $\mathcal{F}uk^d(M)^{free}$, but we leave these details to the reader). In other words, the homology classes $[\xi_{\mathbf{g}}]$ and $[\tilde{\xi}_{PSS([M]),\mathbf{g}}]$ coincide.

We proceed in a perfectly similar way to show that, when $\alpha \in C(f)$ is a cycle, then η_α is homologous to $\tilde{\eta}_{PSS(\alpha)}$.

To end the proof we need to show that

$$(17) \qquad [\tilde{\xi}_{PSS([M]),\mathbf{g}}] = [\tilde{\eta}_{PSS(S(\mathbf{g}))}].$$

From the discussion above we know already that, in homology, both natural transformation do not depend on the choice of Hamiltonian K.

To show (17), denote the moduli spaces used in the definition of $\tilde{\xi}_{PSS([M]),\mathbf{g}}$ by $\mathcal{M}_K$ and the moduli spaces used in the definition of $\tilde{\eta}_{PSS(S(\mathbf{g}))}$ by $\mathcal{M}'_K$. Certainly, these moduli spaces depend on various data and choices that we omit from the notation. The important point is that we will define a homeomorphism $\phi : \mathcal{M}_K \to \mathcal{M}'_{H'}$ where $H'(t, x) = G(t, g_t(x)) + K(t, g_t(x))$ and G generates $\mathbf{g}$, so that our transformation transports admissible data on one side to admissible data on the other. We will denote the elements of $\mathcal{M}_K$ by u and the elements of $\mathcal{M}'_{H'}$ by u'. The expression of $\phi(u)$ is

$$(18) \qquad \phi(u)(z) = g^{-1}_{a_{S''_r}(z)}(u(z)),$$

where $a_{S''_r} : S''_r \to [0, 1]$ are functions so that:

    i. On each boundary component C_i of $\partial S''_r$ the moving boundary conditions along $g_s(L_i)$ are transformed to fixed ones along L_i.

    ii. The asymptotic orbit γ corresponding to the internal puncture is transformed by $\gamma(s) \to g^{-1}_s(\gamma(s))$.

Of course, these maps $a_{S''_r}$ have to be defined also in a way coherent with gluing and splitting of the Deligne–Mumford–Stasheff associahedron. It is easy to see that such a system of functions $a_{S''_r}$ can be found. For instance, assuming that S''_r is a disk with $k + 1$ boundary punctures parametrized so that the output is at the point $(1, 0)$ and the internal puncture is at the point $(0, 0)$, then we can take $a_{S''_r}(re^{-2i\pi s}) = s$. It is well-known that a transformation as in (18) takes

elements in the moduli space $\mathcal{M}$ to elements in $\mathcal{M}'$ which are defined in terms of the "transported" almost complex structures, Floer data, etc. In view of the definition of $S(\mathbf{g})$ from §2.2.1, this shows that, with choices of data related by a transformation as in (18), the two natural transformations $\tilde{\xi}_{PSS([M]),\mathbf{g}}$ and $\tilde{\eta}_{PSS(S(\mathbf{g}))}$ agree even at the chain level and this implies (17).   ∎

4. Some calculations and examples.

The main purpose of this section, in §4.1, is to provide a class of Lagrangians L so that the monoid $\mathrm{Mor}_{\mathcal{C}ob_0^d}(L, L)$ (the operation is concatenation of cobordisms) is nontrivial and to get some "lower bound" estimate for its size. Notice that the morphisms in $\mathcal{C}ob_0^d$ from L to L are represented by elementary cobordisms (L is viewed here as a family with a single element).

Along the way, we will make explicit some of the maps in diagram (2). Moreover, in §4.2 we summarize seven various descriptions (from the literature and/or this paper) of the Lagrangian Seidel morphism in its simplest form which is the Donaldson category version of $\widetilde{S}$ from diagram (2), for a fixed Lagrangian L.

4.1. TORIC CALCULATIONS. Our ambient manifold M^{2n} is, all along this subsection, toric, Fano with minimal Chern number at least 2. We denote by G_M the subgroup of the group of units of $QH(M)$ ($QH(M)$ is ungraded here) generated by the divisors associated to the maximal faces of the Delzant polytope (the definition will be made explicit in §4.1.1).

A toric manifold (M, ω) admits a canonical anti-symplectic involution τ : $M \to M$ that leaves the moment map invariant and that maps orbits of the torus action to orbits. The fixed point set of the involution is the *real* Lagrangian $L_{\mathbb{R}} := \mathrm{Fix}(\tau)$. It is not difficult to see [Hau13] that our assumption that M is Fano with minimal Chern class at least two implies that $L_{\mathbb{R}}$ is monotone.

COROLLARY C: *With the notation and under the assumptions above we have:*

  i. *Any monotone, non-narrow Lagrangian $L \subset M$ (thus, so that $QH(L) \neq 0$) with $N_L \geq 3$ has the property that $\mathrm{Mor}_{\mathcal{C}ob_0^d}(L, L)$ is nontrivial.*
  ii. *The image of the monoid morphism*

$$\mathcal{F}_{L_{\mathbb{R}}} : \mathrm{Mor}_{\mathcal{C}ob_0^d}(L_{\mathbb{R}}, L_{\mathbb{R}}) \to QH(L_{\mathbb{R}})^*$$

— *obtained by restriction from the bottom line of diagram* (2) — *contains a group isomorphic to* G_M.

To exemplify point ii, we immediately deduce from the definition of G_M that for the standard real projective space $\mathbb{R}P^n \subset \mathbb{C}P^n$ the monoid $\mathrm{Mor}_{\mathcal{C}ob_0^d}(\mathbb{R}P^n, \mathbb{R}P^n)$ contains an element u so that $\{1, u, \ldots, u^i, \ldots, u^n\}$ are pairwise distinct.

The proof is based on the specialization of diagram (2) to a single Lagrangian $L \in \mathcal{L}_d^*$ as below:

$$
(19) \qquad
\begin{array}{ccc}
\pi_1(Ham(M)) & \xrightarrow{\quad S \quad} & QH(M)^* \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle *[L]} \\
\Pi(Ham(M)) & \xrightarrow{\quad \widetilde{S}(-)_L \quad} & HF(g_0(L), g_1(L)) \\
\Big\downarrow{\scriptstyle \Sigma} & & \Big\downarrow{\scriptstyle id} \\
\mathrm{Mor}_{\mathcal{C}ob_0^d(M)}(g_0(L), g_1(L)) & \xrightarrow{\quad \mathcal{F}_L \quad} & HF(g_0(L), g_1(L))
\end{array}
$$

Here g_0, respectively g_1, are the ends of the path $\mathbf{g} \in \Pi(Ham(M))$. We will only restrict to examples with $g_0 = id$ and $g_1(L) = L$. In this case $HF(g_0(L), g_1(L))$ is canonically identified with $HF(L, L) = QH(L)$ — this explains the vertical arrow originating in the upper-most right corner. In a different context, Lagrangian loops of this type have been also studied by Akveld-0Salamon [AS01]. The morphism $\mathcal{F}_L$ associates to each cobordism $V : L \rightsquigarrow L$ a quantum homology class $\phi_V([L]) \in HF(L, L) = QH(L)$ and was described explicitly at the beginning of section §2.2.2. Given that in our case V is elementary, the image of $\mathcal{F}_L$ is included in the invertibles of $QH(L)$.

Besides diagram (19), as we shall see, the proof of Corollary C follows rapidly from recent results of Haug [Hau13] and Hyvrier [Hyv] as well as older work of McDuff-Tolman [MT06]. We will review the ingredients needed from these works in the first two subsections below, §4.1.1 and §4.1.2. The proof of Corollary C appears in §4.1.3.

In the arguments that follow we make use of the identification $QH(L) \cong HF(L, L)$ via the PSS isomorphism and in the calculations below we work with coefficients in the Novikov ring $\Lambda = \mathbb{Z}_2[t, t^{-1}]$, $\deg t = -1$. This is legitimate because we only consider a single Lagrangian at a time. Similarly, the coefficient

ring for the quantum homology of the ambient symplectic manifold will be the subring of even powers $\Lambda_{ev} = \mathbb{Z}_2[t^2, t^{-2}]$. In case we need to distinguish the ungraded quantum homology from the graded one, we indicate explicitly the coefficients: $QH(M; \mathbb{Z}_2)$ stands for the ungraded version and $QH(M; \Lambda_{ev})$ stands for the graded one. One can obviously pass from the graded version to the ungraded one by setting $t = 1$ and forgetting the degrees.

4.1.1. *Elements of toric topology.* We review rapidly some toric basic facts following closely McDuff–Salamon's presentation from [MS04], Chapter 11.4. Our manifold is endowed with an effective Hamiltonian action of an n-torus and there is an associated moment map $\mu : M \to (\mathbb{R}^n)^*$ whose image, the moment polytope, is determined by vectors $v_i \in \mathbb{R}^n \cap \mathbb{Z}^n$, $i = 1, \ldots, d$, and real numbers $\{a_i\}_{i=1}^d$ via

$$\Delta := \mu(M) = \{f \in (\mathbb{R}^n)^* \mid f(v_i) \geq a_i\}.$$

The codimension one faces of Δ are assumed non-empty and are given by

$$F_i = \{f \in \Delta \mid f(v_i) = a_i\}.$$

Denote by $[D_i] \in H_{2n-2}(M)$ the fundamental class of the submanifold $D_i := \mu^{-1}(F_i)$.

The quantum homology ring of M is generated by the classes $[D_i]$:

$$(20) \qquad QH(M; \Lambda_{ev}) \cong \frac{\mathbb{Z}_2[[D_1], \ldots, [D_d]][t^2, t^{-2}]}{(P(\Delta), SR_Q(\Delta))}.$$

We refer to [MS04] for the description of the ideals $P(\Delta)$ and $SR_Q(\Delta)$.

The classes $[D_i]$ are easily seen to be invertible. We denote by G_M the subgroup they generate inside the group of invertible elements $QH(M; \mathbb{Z}_2)^* \subset QH(M; \mathbb{Z}_2)$.

The relation with Seidel's representation is provided by a result of McDuff–Tolman [MT06]: there is a family of loops $\mathbf{G}_i \in \pi_1(Ham(M))$ generated by Hamiltonians $\mu_{v_i} : M \to \mathbb{R}$, $\mu_{v_i}(m) := \mu(m)(v_i)$, whose Seidel elements are given by

$$S(\mathbf{G}_i) = [D_i] t^{-2}.$$

For any Lagrangian $L \in \mathcal{L}_d^*$ we deduce from diagram (19) that

$$(21) \qquad \mathcal{F}_L(([0,1] \times L)^{\mathbf{G_i}^{-1}}) = [D_i] * [L].$$

4.1.2. *Properties of $L_\mathbb{R}$.* The intersection $L_\mathbb{R} \cap D_i$ is non-transverse but it is a codimension one submanifold x_i of $L_\mathbb{R}$. Set $[x_i] := [L_\mathbb{R} \cap D_i] \in H_{n-1}(L_\mathbb{R})$.

Haug [Hau13] showed that there is an isomorphism of rings that doubles the degree and is defined by

$$\beta_\mathbb{R} : QH_*(L_\mathbb{R}; \Lambda) \to QH_{2*}(M; \Lambda_{ev})$$

$$[x_i] \mapsto [D_i]$$

$$t \mapsto t^2.$$

The $QH(M)$-module structure on $QH(L_\mathbb{R})$ has been computed by Hyvrier [Hyv] and is completely determined by

$$[D_i] * [L_\mathbb{R}] = [x_i]^2,$$

since the ring structures on $QH(M)$ and $QH(L_\mathbb{R})$ are generated respectively by $[D_i] \otimes \Lambda_{ev}$ and $[x_i] \otimes \Lambda$. Hyvrier [Hyv] also shows that there is a Lagrangian analogue of the loops $\mathbf{G}_i$, given by the paths $\mathbf{g}_i(t) := \mathbf{G}_i(t/2)$. These paths still satisfy $\mathbf{g}_i(1)(L_\mathbb{R}) = L_\mathbb{R}$ and he shows that the associated relative Seidel elements are given by

$$\widetilde{S}(\mathbf{g}_i)(L_\mathbb{R}) = [x_i]t^{-1}.$$

4.1.3. *Proof of Corollary C.* For the first point we notice that equation (21) directly implies the claim as soon as we show that

$$[D_i] * [L] \neq [L]t^2 \in QH(L; \Lambda).$$

But in view of the assumption that $N_L \geq 3$ and given that $|t| = -1$, by possibly using the pearl model for Lagrangian quantum homology, this inequality is immediate.

We proceed to show the second point. Let $w = Y_1^{i_1} \cdots Y_d^{i_d}$ be a word written in the letters Y_i, $1 \leq i \leq d$. We view Y_i as $Y_i = x_i t^{-1}$. We associate to this word an element in $\mathrm{Mor}_{Cob_0^d}(L_\mathbb{R}, L_\mathbb{R})$ as follows. We put $V_0 = [0,1] \times L_\mathbb{R}$ and write

$$\Xi(w) = (V_0^{\mathbf{g}_d^{-1}})^{*i_d} * \cdots * (V_0^{\mathbf{g}_1^{-1}})^{*i_1}.$$

Here $*$ stands for concatenation. In short, $\Xi(w)$ is obtained by concatenating from right to left the suspensions $V_0^{\mathbf{g}_i^{-1}}$ in the number and order prescribed by the word w. The calculations in §4.1.2 together with the commutativity of diagram (19) imply that $\mathcal{F}_{L_\mathbb{R}}(\Xi(w)) = [w] \in QH(L_\mathbb{R}; \mathbb{Z}_2)$ where $[w]$ is now

written in the $[x_i]$'s. Given that $QH(L_\mathbb{R})$ is isomorphic to $QH(M)$ as a ring, we deduce that the image of $\mathcal{F}_{L_\mathbb{R}}$ contains the subgroup $\beta_\mathbb{R}^{-1}(G_M)$. ∎

4.2. The many flavors of the relative Seidel morphism. We shall discuss seven different, yet homologically equivalent, definitions of the relative Seidel morphism. Let $\mathbf{g}$ be a path of Hamiltonian diffeomorphisms with ends g_0 to g_1. We concentrate on a single Lagrangian $L \in \mathcal{L}_d^*$. Each one of the seven definitions gives a different description of the same homology class $\in HF(g_0(L), g_1(L))$ which is called the Lagrangian (or relative) Seidel element.

A. The first definition — originating in [Sei08] — and recalled in §2.2.1 is defined as the element $\xi_\mathbf{g}^0 \in CF(g_0(L), g_1(L))$ given by counting curves u with a single boundary puncture mapped to an intersection point of $g_0(L)$ and $g_1(L)$, satisfying the moving boundary condition $u(e^{-2\pi it}) \in g_t(L)$.

B. In Remark 2.2.2 we gave an alternative definition of the element at A as $\phi_\mathbf{g}(PSS([g_0(L)]))$, where $\phi_\mathbf{g} : CF(g_0(L), g_0(L)) \to CF(g_0(L), g_1(L))$ is the moving boundary morphism.

C. The chain morphism $\phi_\mathbf{g}$ from point B is chain homotopic — as is shown in Lemma 3.1.1 — to the "cobordism" morphism $\phi_{\Sigma(\mathbf{g})(L)} : CF(g_0(L), g_0(L)) \to CF(g_0(L), g_1(L))$ as constructed in §2.2.2.

D. When $g_0 = id$, there is a definition of a morphism, chain equivalent to the one at B, based on the same type of naturality transformation as described in §2.2.1 but starting from Lagrangian Floer homology — with the differential counting "semitubes". This morphism was used in [Lec08].

E. Given a Morse–Smale function $f : L \to \mathbb{R}$, we denote by $\mathcal{C}(f)$ the associated pearl complex (we omit the almost complex structure and the Riemannian metric from the notation) [BC09]. Assuming that $g_0 = id$ and $g_1(L) = L$, one then defines $\widetilde{S}_\mathcal{P}(\mathbf{g}) : \mathcal{C}(f) \to \mathcal{C}(f)$ by counting pearly trajectories with the property that exactly one J-holomorphic disc verifies a moving boundary condition $u(e^{-2\pi it}) \in g_{a(t)}(L)$. The function $a : [0,1] \to [0,1]$ is monotone, satisfies $a|_{[0,5/8]} = 0$ and $a|_{[7/8,1]} = 1$ (this reparametrization is chosen so that the disc

moves along $g_t(L)$ on its upper half part). It was shown in [Cha] that this morphism is chain-homotopic to the morphism at point B.

F. Still assuming that $g_0 = $ id and $g_1(L) = L$, Hu and Lalonde [H10] considered a Hamiltonian fibration $M \to P \to D^2$, defined using $\mathbf{g}$, in which the mapping torus $N := \{g_t(L)\}$ appears as a Lagrangian submanifold. The restriction of this fibration to ∂D^2 gives by construction $L \to N \to S^1$. By using holomorphic sections $(D^2, S^1) \to (P, N)$, they defined a morphism $\mathcal{C}(f) \to \mathcal{C}(f)$ that induces an isomorphism in Lagrangian quantum homology. It was also shown in [Cha] that this morphism coincides in homology with $[\widetilde{S}_{\mathcal{P}}(\mathbf{g})]$. The proof of this fact uses a result of [AS01], in which they prove that such holomorphic sections correspond to disks in M with moving boundary condition $u(e^{-2\pi it}) \in g_t(L)$, satisfying a Hamiltonian perturbation of the $\overline{\partial}_J$-equation. One can then count pearly trajectories using exactly one such disc, and a homotopy sending the perturbation to zero shows that this gives the same count as $\widetilde{S}_{\mathcal{P}}(\mathbf{g})$ in homology.

G. Finally, when $\mathbf{g}$ is a loop based at the identity, the Lagrangian Seidel element described before descends from the Seidel element of $\mathbf{g}$ in M via the quantum module action so that, using, for instance, the description at B (which is possibly the simplest), we have

$$\phi_{\mathbf{g}}([L]) = S(\mathbf{g}) * [L] .$$

The identity F= D was first shown in [HLL11]. It is obviously reflected in the commutativity of the top of the diagram (2).

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