Fragmentation pseudo-metrics

(based on work with P. Biran and E. Shelukhin)

I Triangulated categories (Verdier, Puppe; 1963)

$\mathcal{C}$ = category, additive, shift functor Σ

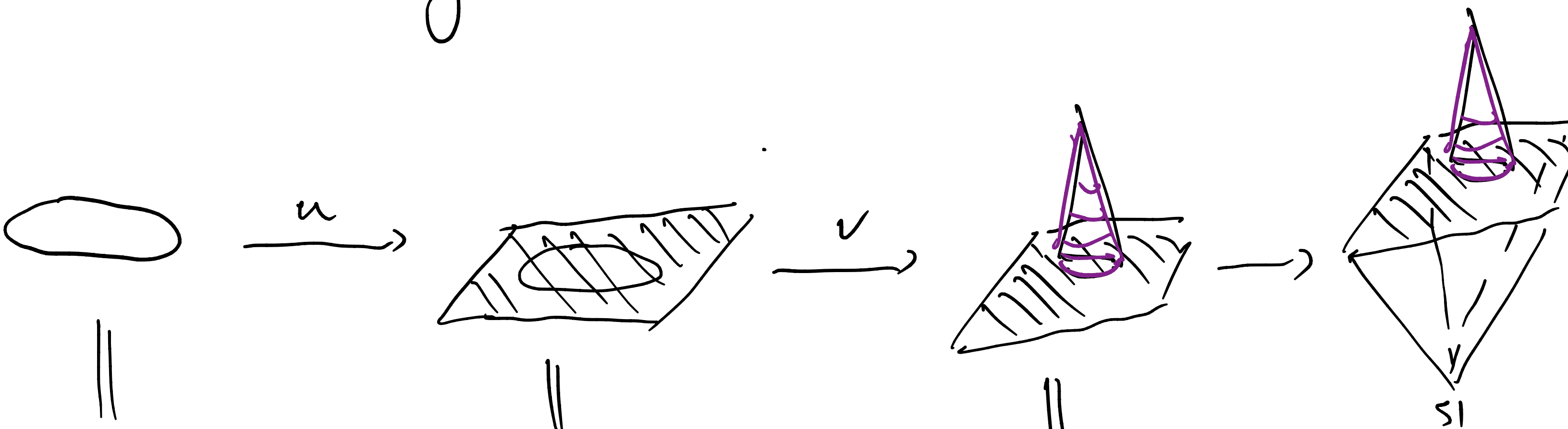
and a class of distinguished triangles:

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

subject to a number of axioms.

<u>Basic model</u>: htpy category of top spaces.

distinguished tr. model cone attachments



$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

II Gymnastics of iterated cone decompositions:

Fix $X \in Ob(\mathcal{C})$. $X_0 \dots X_k \in Ob(\mathcal{C})$. A cone-dec of X with linearization $(X_0, X_1, \dots X_k)$ is:

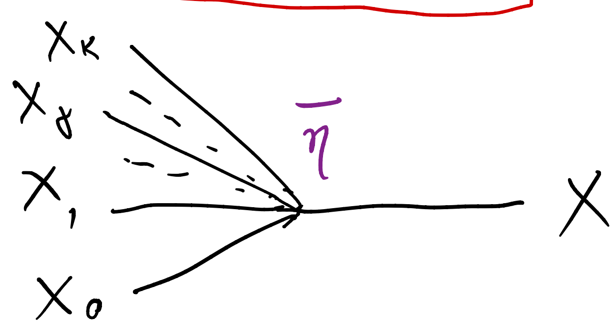
dist. tr
$$\left\{ \begin{array}{l} X_1 \xrightarrow{u_1} Y_0 \longrightarrow Y_1 \\ X_2 \xrightarrow{u_2} Y_1 \longrightarrow Y_2 \\ \qquad \vdots \\ X_k \xrightarrow{u_k} Y_{k-1} \longrightarrow Y_k \end{array} \right.$$

$$\boxed{X_0 = Y_0}$$

$$\boxed{Y_j = \mathrm{Cone}(u_j)}$$

$$\boxed{\eta : X \xrightarrow{\cong} Y_k}$$

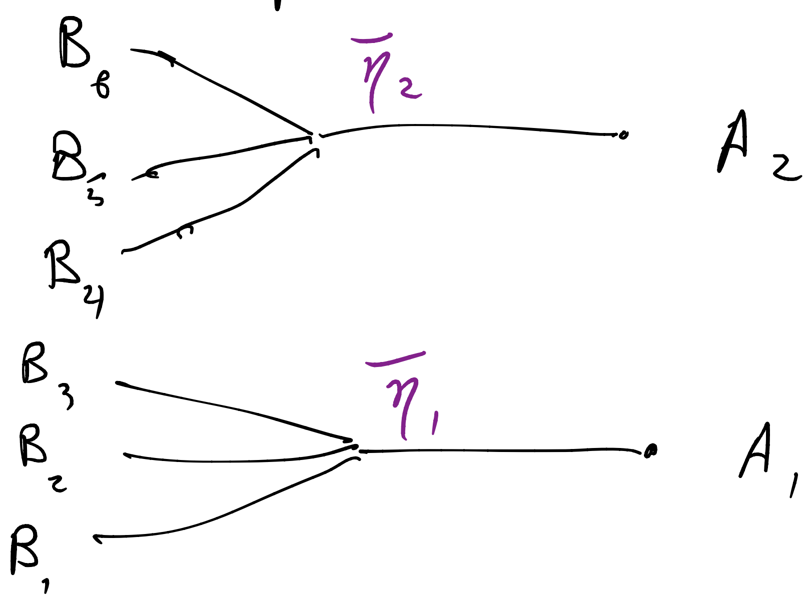
Represent such a structure :



$T^s\mathcal{C}$ = category associated to $\mathcal{C}$

objects: families $(A_1, \dots, A_m)$, $A_i \in \mathrm{Ob}(\mathcal{C})$.

morphisms: $\psi : (A_1, \dots A_m) \longrightarrow (B_1, \dots B_n)$ family of cone decompositions $\psi = (\bar{\eta}_1, \bar{\eta}_2 \dots \bar{\eta}_m)$



Composition corresponds to refinement of cone dec.

Definition: A weight on $\mathcal{C}$ is a map:

$$w : \mathrm{Mor}_{T^s\mathcal{C}} \longrightarrow [0, \infty) \text{ such that:}$$

i) $w(\varphi \circ \varphi') \leq w(\varphi) + w(\varphi')$

ii) $w(id) = 0$

Let $\mathcal{F} \subset \mathrm{Ob}(\mathcal{C})$; $X, Y \in \mathrm{Ob}(\mathcal{C})$

$$\boxed{\delta^{\mathcal{F}}(X,Y) = \inf \left\{ w(\varphi) \;\middle|\; \begin{array}{l} \varphi : X \longrightarrow (F_1, \ldots Y, \ldots F_k) \\ F_i \in \mathcal{F},\ \varphi \in \mathrm{Mor}_{T^s\mathcal{C}} \end{array} \right\}}$$

$$\boxed{d^{\mathcal{F}}(X,Y) = [\delta^{\mathcal{F}}(X,Y) + \delta^{\mathcal{F}}(Y,X)]/2 \text{ pseudo-metric}}$$

Remark

i) If $\mathcal{F}'$ is another family $\mathcal{F}' \subset Ob(\mathcal{C})$

$$\underbrace{d^{\mathcal{F},\mathcal{F}'} = \frac{d^{\mathcal{F}} + d^{\mathcal{F}'}}{2} \text{ pseudo-metric.}}$$

fragmentation p. metrics associated to family $\mathcal{F}$ (or $\mathcal{F},\mathcal{F}'$) and weight w

ii) w measures, in particular, all isomorphisms in $\mathcal{C}$.

V. Simple example: V = $\mathbb{Z}/2$ vector space with a norm $\| \ \|$.

$\mathcal{C}_V$: $\begin{cases} Ob = V, \ \# Mor(a,b) = 1, \ a \xrightarrow{b-a} b \\ \text{dist tr}: \ a \to b \to b+a \ ; \ \Sigma = id \end{cases}$

- all morphisms in $\mathcal{C}_V$ are isos.

- Cone dec of x with lin. $(x_1, x_2 \dots x_k)$:

$$x = x_1 + x_2 + \dots + x_k$$

- $w(a \to b) = \| b - a \|$ is a weight / $\mathcal{C}_V$.

Families $\mathcal{F}$: $\begin{cases} \mathcal{F}_0 = \phi \\ \mathcal{F}_1 = \{e_1, e_2 \,..\, e_k\} \text{ basis.} \\ \mathcal{F}_2 = \{e_1\} \\ \mathcal{F}_2' = \{e_1'\} \;, \; e_1' \neq e_1 \end{cases}$

- $d^{\mathcal{F}_0}(x, y) = \|y - x\|$ only c.d is the iso $x \to y$

$w(x \to y) = \|y - x\|.$

- $d^{\mathcal{F}_1}(x, y) = 0$ because $x = y$ + sum of basis el.

- $d^{\mathcal{F}_2}(x, y) = \inf \{ \|x - z\| \mid z \in \mathrm{Span}\langle y, e_1 \rangle \}$

- $d^{\mathcal{F}_2, \mathcal{F}_2'}$ non-degenerate.

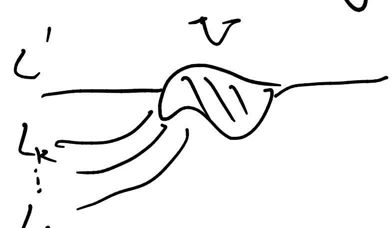
III Lagrangians

(M, ω) symplectic

- $\mathcal{L}ag^*(M)$ = class of Lag $L \hookrightarrow M$ (possibly immersed, $*$ = additional data).

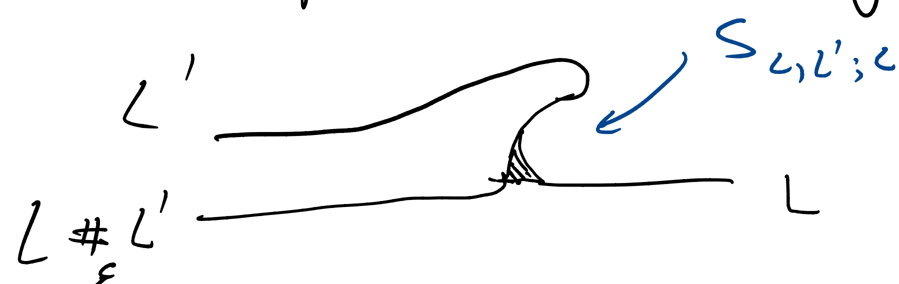
- $\mathcal{L}ag^*(\mathbb{C}\times M)$ = class of Lag cob $V \hookrightarrow \mathbb{C}\times M$ (possibly immersed).

- Organize as a category $\mathcal{C}ob^*(M)$:

$Mor(L, L') = \Big\{$ 

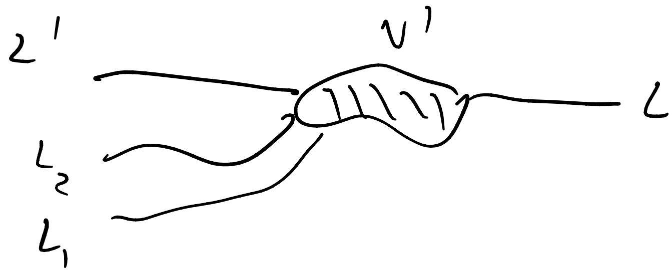
$L \; , \; V \in \mathcal{L}ag^*(\mathbb{C}\times M) \Big\} / \sim$

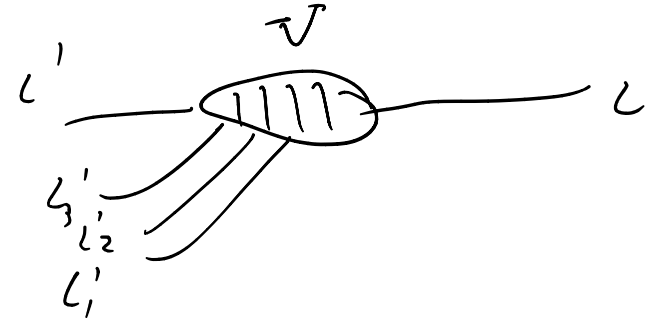
- $\mathcal{C}ob^{\alpha}(M)$ has surgery models (number of axioms). Mainly:
  - $\exists$ equivalence relation called cabling on $\mathrm{Mor}_{\mathcal{C}ob^{\alpha}(M)}$; $\sim$
  - $\forall$ morphism $\sim$ surgery morphism.

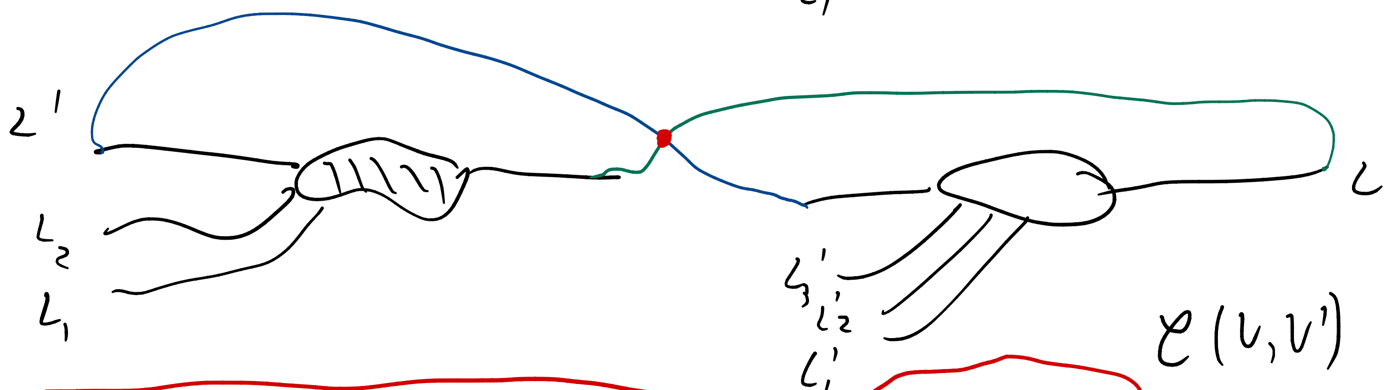


$c \subset L \pitchfork L'$

The cabling: $\mathcal{C}(V,V')$ for Two morphisms







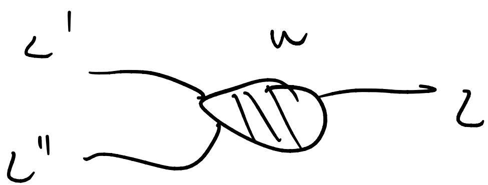
$\mathcal{C}(V,V')$

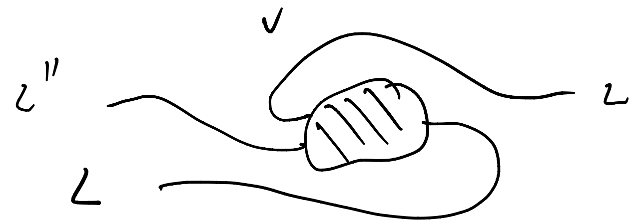
$$\boxed{V \sim V' \iff \mathcal{C}(V,V') \in \mathcal{L}ag^*(\mathbb{C}\times M)}$$

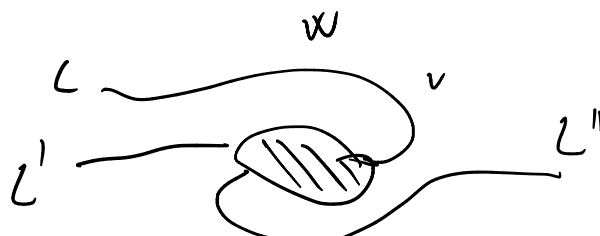
($\sim$ eq. rel. is part of the axioms)

Theorem (soft): If $\mathcal{C}ob^*(M)$ has surgery models then $\widehat{\mathcal{C}ob}^*(M) = \mathcal{C}ob^*(M)/\sim$ is triangulated.

Remark: i) Distinguished triangles are induced by:





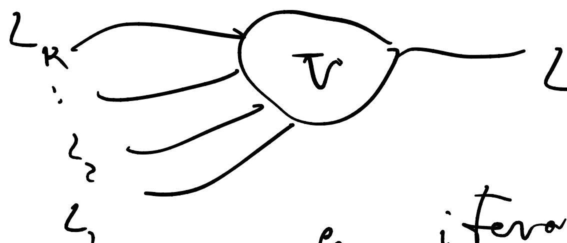


(basic example coming from surgery).

ii) Simple cob. are isos in $\widehat{\mathcal{C}ob}$.

Question: Any natural weights on $\widehat{Cob}^*(M)$?

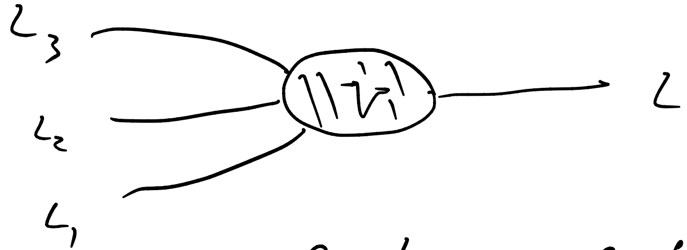
Proposition: In $\widehat{Cob}(M)$ for a V



one can associate an iterated cone-decomp of L with linearization $(L_1, L_2, \ldots, L_k)$.

Denote the resulting decomp: η_V.

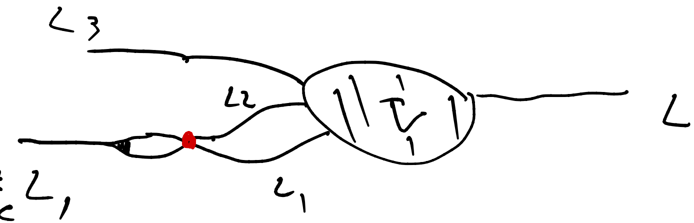
Proof:



Need to show $L \simeq \mathrm{Cone}(L_3 \to \mathrm{Cone}(L_2 \to L_1))$

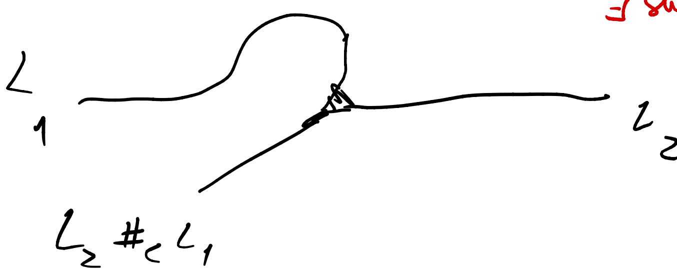
Step 1

$\exists\ \mathcal{C}(RV, S_{L_2,L_1;c}) =$ $L_2 \#_c L_1$

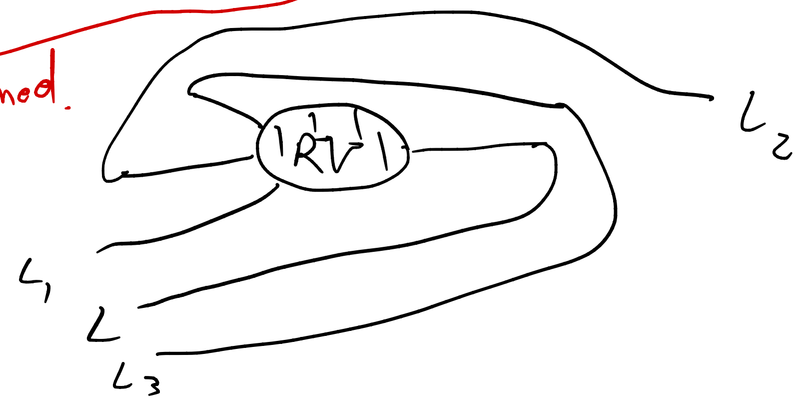


$\in \mathcal{L}og^*_0(\mathbb{C} \times M)$

$\exists$ surgery mod.

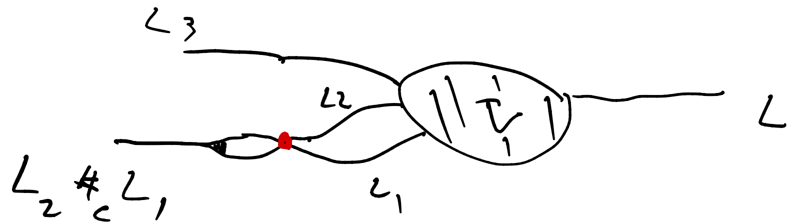


$\sim$

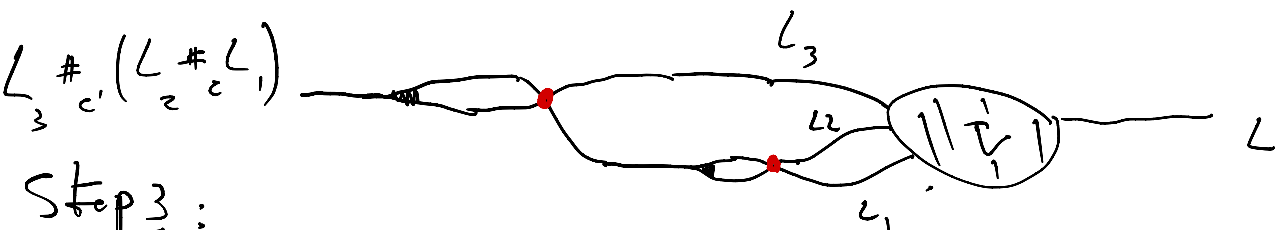


This operation is called braiding:

Step 2: I apply it again:



$\leadsto$



Step 3:

$L_3 \#_{c'} (L_2 \#_c L_1) = \mathrm{Cone}(L_3 \to \mathrm{Cone}(L_2 \to L_1))$

a simple cob is an iso. in $\widehat{\mathcal{C}ob}$

Using the proposition:

φ iterated cone decomp:

$$w^S(\varphi) = \inf\{ \mathcal{S}(V) \mid \bar{\eta}_V = \varphi \}$$

shadow weight of φ.

$$\mathcal{S}(V) = \text{Area of filling of } \pi(V)$$

$\pi : \mathbb{C} \times M \to \mathbb{C}$ projection.

$\Longrightarrow$ pseudo-metrics $d^{\mathcal{F},\mathcal{F}'}$ etc.

Theorem: When the classes $\mathcal{L}eg^{*}(M)$, $\mathcal{L}eg^{*}(\mathbb{C}\times M)$ are unobstructed (and with generic assumptions on the families $\mathcal{F},\mathcal{F}'$), the pseudo-metrics $d^{\mathcal{F},\mathcal{F}'}$ are non-degenerate.

Moreover, in this case, $\widehat{\mathcal{C}ob_e^{*}}(M)$ the subcategory generated by embedded Lagr is triangulated isomorphic to $D\mathcal{F}uk^{*}(M)$.

Proof of theorem goes through construction of functor:

$$\mathcal{F} : \widehat{\mathrm{Lab}}^*(M) \longrightarrow \mathrm{Mod}_{\mathrm{Fuk}^*(M)}$$

and uses the full force of Floer hlgy machinery.

Question (Eliashberg): If $d^{\mathcal{F},\mathcal{F}'}$ are non-degenerate on $\widehat{\mathrm{Lab}}^*(M)$, are $\mathrm{Lag}^*(M)$, $\mathrm{Lag}^*(\mathbb{C}\times M)$ un-obstructed?

IV Final comments.

1) Other weights coming from persistence modules (work in progress w. Jun Zhang)

2) Topology induced by $d^{\mathcal{F},\mathcal{F}'}$ (in progress by J. P. Chassé)

3) Sarkary models for Morse functions (P. Fontaine in progress).

4) Sarkary models on surfaces (D. Rathel-Fournier in progress)