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Negative values of truncations to $L(1,\chi)$

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Abstract. For fixed large x we give upper and lower bounds for the minimum of $\sum_{n\le x}\chi(n)/n$ as we minimize over all real-valued Dirichlet characters χ. This follows as a consequence of bounds for $\sum_{n\le x} f(n)/n$ but now minimizing over all completely multiplicative, real-valued functions f for which $-1\le f(n)\le 1$ for all integers $n\ge 1$. Expanding our set to all multiplicative, real-valued multiplicative functions of absolute value ≤ 1, the minimum equals $-0.4553\cdots+o(1)$, and in this case we can classify the set of optimal functions.

1. Introduction

Dirichlet's celebrated class number formula established that $L(1,\chi)$ is positive for primitive, quadratic Dirichlet characters χ. One might attempt to prove this positivity by trying to establish that the partial sums $\sum_{n\le x}\chi(n)/n$ are all non-negative. However, such truncated sums can get negative, a feature which we will explore in this note.

By quadratic reciprocity we may find an arithmetic progression $\pmod{4\prod_{p\le x}p}$ such that any prime q lying in this progression satisfies $\left(\frac{p}{q}\right)=-1$ for each $p\le x$. Such primes q exist by Dirichlet's theorem on primes in arithmetic progressions, and for such q we have $\sum_{n\le x}\left(\frac{n}{q}\right)/n=\sum_{n\le x}\lambda(n)/n$ where $\lambda(n)=(-1)^{\Omega(n)}$ is the Liouville function. Turán [6] suggested that $\sum_{n\le x}\lambda(n)/n$ may be always positive, noting that this would imply the truth of the Riemann Hypothesis (and previously Pólya had conjectured that the related $\sum_{n\le x}\lambda(n)$ is non-positive for all $x\ge 2$, which also implies the Riemann Hypothesis). In [**Has58**] Haselgrove showed that both the Turán and Pólya conjectures are false (in fact $x=72,185,376,951,205$ is the smallest integer x for which $\sum_{n\le x}\lambda(n)/n<0$, as was recently determined in [**BFM**]). We therefore know that truncations to $L(1,\chi)$ may get negative.

Let $\mathcal{F}$ denote the set of all completely multiplicative functions $f(\cdot)$ with $-1\le f(n)\le 1$ for all positive integers n, let $\mathcal{F}_1$ be those for which each $f(n)=\pm1$, and $\mathcal{F}_0$ be those for which each $f(n)=0$ or ±1. Given any x and any $f\in\mathcal{F}_0$ we may find a primitive quadratic character χ with $\chi(n)=f(n)$ for all $n\le x$ (again, by using

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quadratic reciprocity and Dirichlet's theorem on primes in arithmetic progressions) so that, for any $x \geq 1$,

$$\min_{\substack{\chi \text{ a quadratic} \\ \text{character}}} \sum_{n \leq x} \frac{\chi(n)}{n} = \delta_0(x) := \min_{f \in \mathcal{F}_0} \sum_{n \leq x} \frac{f(n)}{n}.$$

Moreover, since $\mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{F}$ we have that

$$\delta(x) := \min_{f \in \mathcal{F}} \sum_{n \leq x} \frac{f(n)}{n} \quad \leq \delta_0(x) \leq \quad \delta_1(x) := \min_{f \in \mathcal{F}_1} \sum_{n \leq x} \frac{f(n)}{n}.$$

We expect that $\delta(x) \sim \delta_1(x)$ and even, perhaps, that $\delta(x) = \delta_1(x)$ for sufficiently large x.

Trivially $\delta(x) \geq -\sum_{n \leq x} 1/n = -(\log x + \gamma + O(1/x))$. Less trivially $\delta(x) \geq -1$, as may be shown by considering the non-negative multiplicative function $g(n) = \sum_{d|n} f(d)$ and noting that

$$0 \leq \sum_{n \leq x} g(n) = \sum_{d \leq x} f(d) \left[\frac{x}{d}\right] \leq \sum_{d \leq x} \left(x \frac{f(d)}{d} + 1\right).$$

We will show that $\delta(x) \leq \delta_1(x) < 0$ for all large values of x, and that $\delta(x) \to 0$ as $x \to \infty$.

Theorem 1. *For all large x and all $f \in \mathcal{F}$ we have*

$$\sum_{n \leq x} \frac{f(n)}{n} \geq -\frac{1}{(\log \log x)^{\frac{3}{5}}}.$$

Further, there exists a constant $c > 0$ such that for all large x there exists a function $f(= f_x) \in \mathcal{F}_1$ such that

$$\sum_{n \leq x} \frac{f(n)}{n} \leq -\frac{c}{\log x}.$$

In other words, for all large x,

$$-\frac{1}{(\log \log x)^{\frac{3}{5}}} \leq \delta(x) \leq \delta_0(x) \leq \delta_1(x) \leq -\frac{c}{\log x}.$$

Note that Theorem 1 implies that there exists some absolute constant $c_0 > 0$ such that $\sum_{n \leq x} f(n)/n \geq -c_0$ for all x and all $f \in \mathcal{F}$, and that equality occurs only for bounded x. It would be interesting to determine c_0 and all x and f attaining this value, which is a feasible goal developing the methods of this article.

It would be interesting to determine more precisely the asymptotic nature of $\delta(x), \delta_0(x)$ and $\delta_1(x)$, and to understand the nature of the optimal functions.

Instead of completely multiplicative functions we may consider the larger class $\mathcal{F}^*$ of multiplicative functions, and analogously define

$$\delta^*(x) := \min_{f \in \mathcal{F}^*} \sum_{n \leq x} \frac{f(n)}{n}\ .$$

Theorem 2. *We have*

$$\delta^*(x) = \left(1 - 2\log(1+\sqrt{e}) + 4\int_1^{\sqrt{e}} \frac{\log t}{t+1} dt\right) \log 2 + o(1) = -0.4553\ldots + o(1).$$

If $f^* \in \mathcal{F}^*$ and x is large then

$$\sum_{n\le x} \frac{f^*(n)}{n} \ge -\frac{1}{(\log\log x)^{\frac{3}{5}}},$$

unless

$$\sum_{k=1}^{\infty} \frac{1+f^*(2^k)}{2^k} \ll (\log x)^{-\frac{1}{20}}.$$

Finally

$$\sum_{n\le x} \frac{f^*(n)}{n} = \delta^*(x) + o(1)$$

if and only if

$$\Big(\sum_{k=1}^{\infty} \frac{1+f^*(2^k)}{2^k}\Big)\log x + \sum_{3\le p\le x^{1/(1+\sqrt{e})}} \sum_{k=1}^{\infty} \frac{1-f^*(p^k)}{p^k} + \sum_{x^{1/(1+\sqrt{e})}\le p\le x} \frac{1+f^*(p)}{p} = o(1).$$

2. Constructing negative values

Recall Haselgrove's result [**Has58**]: there exists an integer N such that

$$\sum_{n\le N} \frac{\lambda(n)}{n} = -\delta$$

with $\delta > 0$, where $\lambda \in \mathcal{F}_1$ with $\lambda(p) = -1$ for all primes p. Let $x > N^2$ be large and consider the function $f = f_x \in \mathcal{F}_1$ defined by $f(p) = 1$ if $x/(N+1) < p \le x/N$ and $f(p) = -1$ for all other p. If $n \le x$ then we see that $f(n) = \lambda(n)$ unless $n = p\ell$ for a (unique) prime $p \in (x/(N+1), x/N]$ in which case $f(n) = \lambda(\ell) = \lambda(n) + 2\lambda(\ell)$. Thus

$$\begin{aligned}\sum_{n\le x} \frac{f(n)}{n} &= \sum_{n\le x} \frac{\lambda(n)}{n} + 2\sum_{x/(N+1)<p\le x/N} \frac{1}{p} \sum_{\ell\le x/p} \frac{\lambda(\ell)}{\ell} \\ &= \sum_{n\le x} \frac{\lambda(n)}{n} - 2\delta \sum_{x/(N+1)<p\le x/N} \frac{1}{p}.\end{aligned} \tag{2.1}$$

A standard argument, as in the proof of the prime number theorem, shows that

$$\sum_{n\le x} \frac{\lambda(n)}{n} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\zeta(2s+2)}{\zeta(s+1)} \frac{x^s}{s} ds \ll \exp(-c\sqrt{\log x}),$$

for some $c > 0$. Further the prime number theorem readily gives that

$$\sum_{x/(N+1)<p\le x/N} \frac{1}{p} \sim \log\Big(\frac{\log(x/N)}{\log(x/(N+1))}\Big) \asymp \frac{1}{N\log x}.$$

Inserting these estimates in (2.1) we obtain that $\delta(x) \le -c/\log x$ for large x (here $c \asymp \delta/N$), as claimed in Theorem 1.

Remark 2.1. In [**BFM**] it is shown that one can take $\delta = 2.0757641\cdots\cdot 10^{-9}$ for $N = 72204113780255$ and therefore we may take $c \approx 2.87\cdot 10^{-23}$.

3. The lower bound for $\delta(x)$

PROPOSITION 3.1. *Let f be a completely multiplicative function with $-1 \le f(n) \le 1$ for all n, and set $g(n) = \sum_{d|n} f(d)$ so that g is a non-negative multiplicative function. Then*

$$\sum_{n\le x} \frac{f(n)}{n} = \frac{1}{x}\sum_{n\le x} g(n) + (1-\gamma)\frac{1}{x}\sum_{n\le x} f(n) + O\Big(\frac{1}{(\log x)^{\frac{1}{5}}}\Big).$$

PROOF. Define $F(t) = \frac{1}{t}\sum_{n\le t} f(n)$. We will make use of the fact that $F(t)$ varies slowly with t. From [**GS03**, Corollary 3],we find that if $1 \le w \le x/10$ then

$$\Big||F(x)| - |F(x/w)|\Big| \ll \Big(\frac{\log 2w}{\log x}\Big)^{1-\frac{2}{\pi}} \log\Big(\frac{\log x}{\log 2w}\Big) + \frac{\log\log x}{(\log x)^{2-\sqrt{3}}}. \tag{3.1}$$

We may easily deduce that

$$\Big|F(x) - F(x/w)\Big| \ll \Big(\frac{\log 2w}{\log x}\Big)^{1-\frac{2}{\pi}} \log\Big(\frac{\log x}{\log 2w}\Big) + \frac{\log\log x}{(\log x)^{2-\sqrt{3}}} \ll \Big(\frac{\log 2w}{\log x}\Big)^{\frac{1}{4}}. \tag{3.2}$$

Indeed, if $F(x)$ and $F(x/w)$ are of the same sign then (3.2) follows at once from (3.1). If $F(x)$ and $F(x/w)$ are of opposite signs then we may find $1 \le v \le w$ with $|\sum_{n\le x/v} f(n)| \le 1$ and then using (3.1) first with $F(x)$ and $F(x/v)$, and second with $F(x/v)$ and $F(x/w)$ we obtain (3.2).

We now turn to the proof of the Proposition. We start with

$$\sum_{n\le x} g(n) = \sum_{d\le x} f(d)\Big[\frac{x}{d}\Big] = x\sum_{d\le x}\frac{f(d)}{d} - \sum_{d\le x} f(d)\Big\{\frac{x}{d}\Big\}. \tag{3.3}$$

Now

$$\begin{aligned}\sum_{d\le x} f(d)\Big\{\frac{x}{d}\Big\} &= \sum_{j\le x}\sum_{x/(j+1)<d\le x/j} f(d)\Big(\frac{x}{d} - j\Big)\\ &= \sum_{j\le \log x}\int_{x/(j+1)}^{x/j} \frac{x}{t^2}\sum_{x/(j+1)<d\le t} f(d)dt + O\Big(\frac{x}{\log x}\Big).\end{aligned}$$

From (3.2) we see that if $j \le \log x$, and $x/(j+1) < t \le x/j$ then

$$\sum_{x/(j+1)<d\le t} f(d) = \Big(t - \frac{x}{(j+1)}\Big)\frac{1}{x}\sum_{n\le x} f(n) + O\Big(\frac{x\log(j+1)}{j(\log x)^{\frac{1}{4}}}\Big).$$

Using this above we conclude that

$$\sum_{d\le x} f(d)\Big\{\frac{x}{d}\Big\} = \Big(\sum_{n\le x} f(n)\Big)\sum_{j\le\log x}\Big(\log\Big(\frac{j+1}{j}\Big) - \frac{1}{j+1}\Big) + O\Big(\frac{x(\log\log x)^2}{(\log x)^{\frac{1}{4}}}\Big). \tag{3.4}$$

Since $\sum_{j\le J}(\log(1+1/j) - 1/(j+1)) = \log(J+1) - \sum_{j\le J+1} 1/j + 1 = 1-\gamma+O(1/J)$, when we insert (3.4) into (3.3) we obtain the Proposition. ∎

Set $u = \sum_{p\le x}(1-f(p))/p$. By Theorem 2 of A. Hildebrand [**Hil87**] (with f there being our function g, $K = 2$, $K_2 = 1.1$, and $z = 2$) we obtain that

$$\frac{1}{x}\sum_{n\le x} g(n) \gg \prod_{p\le x}\Big(1-\frac{1}{p}\Big)\Big(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\dots\Big)\sigma_-\Big(\exp\Big(\sum_{p\le x}\frac{\max(0,1-g(p))}{p}\Big)\Big) + O(\exp(-(\log x)^{\beta})),$$

where β is some positive constant and $\sigma_-(\xi) = \xi\rho(\xi)$ with ρ being the Dickman function[1]. Since $\max(0, 1-g(p)) \le (1-f(p))/2$ we deduce that

$$\frac{1}{x}\sum_{n\le x} g(n) \gg (e^{-u}\log x)(e^{u/2}\rho(e^{u/2})) + O(\exp(-(\log x)^{\beta})) \\ \gg e^{-ue^{u/2}}(\log x) + O(\exp(-(\log x)^{\beta})), \tag{3.5}$$

since $\rho(\xi) = \xi^{-\xi+o(\xi)}$.

On the other hand, a special case of the main result in [**HT91**] implies that

$$\frac{1}{x}\Big|\sum_{n\le x} f(n)\Big| \ll e^{-\kappa u}, \tag{3.6}$$

where $\kappa = 0.32867\ldots$. Combining Proposition 3.1 with (3.5) and (3.6) we immediately get that $\delta(x) \ge -c/(\log\log x)^{\xi}$ for any $\xi < 2\kappa$. This completes the proof of Theorem 1.

Remark 3.2. The bound (3.5) is attained only in certain very special cases, that is when there are very few primes $p > x^{e^{-u}}$ for which $f(p) = 1 + o(1)$. In this case one can get a far stronger bound than (3.6). Since the first part of Theorem 1 depends on an interaction between these two bounds, this suggests that one might be able to improve Theorem 1 significantly by determining how (3.5) and (3.6) depend upon one another.

4. Proof of Theorem 2

Given $f^* \in \mathcal{F}^*$ we associate a completely multiplicative function $f \in \mathcal{F}$ by setting $f(p) = f^*(p)$. We write $f^*(n) = \sum_{d|n} h(d)f(n/d)$ where h is the multiplicative function given by $h(p^k) = f^*(p^k) - f(p)f^*(p^{k-1})$ for $k \ge 1$. Now,

$$\sum_{n\le x}\frac{f^*(n)}{n} = \sum_{d\le x}\frac{h(d)}{d}\sum_{m\le x/d}\frac{f(m)}{m} \\ = \sum_{d\le(\log x)^6}\frac{h(d)}{d}\sum_{m\le x/d}\frac{f(m)}{m} + O\Big(\log x\sum_{d>(\log x)^6}\frac{|h(d)|}{d}\Big). \tag{4.1}$$

Since $h(p) = 0$ and $|h(p^k)| \le 2$ for $k \ge 2$ we see that

$$\sum_{d>(\log x)^6}\frac{|h(d)|}{d} \le (\log x)^{-2}\sum_{d\ge 1}\frac{|h(d)|}{d^{\frac{2}{3}}} \ll (\log x)^{-2}. \tag{4.2}$$

[1]The Dickman function is defined as $\rho(u) = 1$ for $u \le 1$, and $\rho(u) = (1/u)\int_{u-1}^{u}\rho(t)dt$ for $u \ge 1$.

Further, for $d \le (\log x)^6$, we have (writing $F(t) = \frac{1}{t}\sum_{n\le t} f(n)$ as in section 3)

$$\sum_{x/d\le n\le x} \frac{f(n)}{n} = F(x) - F(x/d) + \int_{x/d}^{x} \frac{F(t)}{t} dt = \frac{\log d}{x}\sum_{n\le x} f(n) + O\Big(\frac{1}{(\log x)^{\frac15}}\Big),$$

using (3.2). Using the above in (4.1) we deduce that

$$\sum_{n\le x}\frac{f^*(n)}{n} = \Big(\sum_{n\le x}\frac{f(n)}{n}\Big)\sum_{d\le(\log x)^6}\frac{h(d)}{d} - \frac{1}{x}\sum_{n\le x} f(n)\sum_{d\le(\log x)^6}\frac{h(d)\log d}{d} + O\Big(\frac{1}{(\log x)^{\frac15}}\Big).$$

Arguing as in (4.2) we may extend the sums over d above to all d, incurring a negligible error. Thus we conclude that

$$\sum_{n\le x}\frac{f^*(n)}{n} = H_0\sum_{n\le x}\frac{f(n)}{n} + H_1\frac{1}{x}\sum_{n\le x} f(n) + O\Big(\frac{1}{(\log x)^{\frac15}}\Big),$$

with

$$H_0 = \sum_{d=1}^{\infty}\frac{h(d)}{d}, \qquad \text{and} \qquad H_1 = -\sum_{d=1}^{\infty}\frac{h(d)\log d}{d}.$$

Note that $H_0 = \prod_p (1 + h(p)/p + h(p^2)/p^2 + \dots) \ge 0$, and that $H_0, |H_1| \ll 1$.

We now use Proposition 3.1, keeping the notation there. We deduce that

$$\sum_{n\le x}\frac{f^*(n)}{n} = H_0\frac{1}{x}\sum_{n\le x} g(n) + \Big((1-\gamma)H_0 + H_1\Big)\frac{1}{x}\sum_{n\le x} f(n) + O\Big(\frac{1}{(\log x)^{\frac15}}\Big). \tag{4.3}$$

If $H_0 \ge (\log x)^{-\frac{1}{20}}$ then we may argue as in section 3, using (3.5) and (3.6). In that case, we see that $\sum_{n\le x} f^*(n)/n \ge -1/(\log\log x)^{\frac35}$. Henceforth we suppose that $H_0 \le (\log x)^{-\frac{1}{20}}$. Since

$$H_0 \asymp 1 + \frac{h(2)}{2} + \frac{h(2^2)}{2^2} + \dots \asymp 1 + \frac{f^*(2)}{2} + \frac{f^*(2^2)}{2^2} + \dots,$$

we deduce that (note $h(2) = 0$)

$$\sum_{k=2}^{\infty}\frac{2 + h(2^k)}{2^k} \asymp \sum_{k=1}^{\infty}\frac{1 + f^*(2^k)}{2^k} \ll (\log x)^{-\frac{1}{20}}. \tag{4.4}$$

This proves the middle assertion of Theorem 2.

Writing $d = 2^k\ell$ with ℓ odd,

$$\begin{aligned} H_1 &= -\sum_{\ell \text{ odd}}\frac{h(\ell)}{\ell}\sum_{k=0}^{\infty}\frac{h(2^k)}{2^k}(k\log 2 + \log\ell) \\ &= -\log 2\Big(\sum_{k=1}^{\infty}\frac{kh(2^k)}{2^k}\Big)\sum_{\ell \text{ odd}}\frac{h(\ell)}{\ell} + O((\log x)^{-\frac{1}{20}}) \\ &= 3\log 2\prod_{p\ge3}\Big(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \dots\Big) + O\Big(\frac{\log\log x}{(\log x)^{\frac{1}{20}}}\Big), \end{aligned}$$

where we have used (4.4) and that $\sum_{k=1}^\infty kh(2^k)/2^k = -3 + O(\log\log x/(\log x)^{\frac{1}{20}})$. Using these observations in (4.3) we obtain that

(4.5)
$$\sum_{n\le x}\frac{f^*(n)}{n} = H_0\frac{1}{x}\sum_{n\le x} g(n) + 3\log 2\prod_{p\ge 3}\Big(1+\frac{h(p)}{p}+\frac{h(p^2)}{p^2}+\dots\Big)\frac{1}{x}\sum_{n\le x} f(n) + o(1)$$
$$\ge 3\log 2\prod_{p\ge 3}\Big(1+\frac{h(p)}{p}+\frac{h(p^2)}{p^2}+\dots\Big)\frac{1}{x}\sum_{n\le x} f(n) + o(1).$$

Let $r(\cdot)$ be the completely multiplicative function with $r(p)=1$ for $p\le \log x$, and $r(p)=f(p)$ otherwise. Then Proposition 4.4 of [**GS01**] shows that

$$\frac{1}{x}\sum_{n\le x} f(n) = \prod_{p\le\log x}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{f(p)}{p}\Big)^{-1}\frac{1}{x}\sum_{n\le x} r(n) + O\Big(\frac{1}{(\log x)^{\frac{1}{20}}}\Big).$$

Since $f(2) = -1 + O(H_0)$ we deduce from (4.5) and the above that

$$\sum_{n\le x}\frac{f^*(n)}{n}\ge \log 2\prod_{p\ge 3}\Big(1-\frac{1}{p}\Big)\Big(1+\frac{f^*(p)}{p}+\frac{f^*(p^2)}{p^2}+\dots\Big)\frac{1}{x}\sum_{n\le x} r(n)+o(1). \tag{4.6}$$

One of the main results of [**GS01**] (see Corollary 1 there) shows that

$$\frac{1}{x}\sum_{n\le x} r(n)\ge 1-2\log(1+\sqrt{e})+4\int_1^{\sqrt{e}}\frac{\log t}{t+1}dt+o(1) = -0.656999\ldots+o(1), \tag{4.7}$$

and that equality here holds if and only if

$$\sum_{p\le x^{1/(1+\sqrt{e})}}\frac{1-r(p)}{p}+\sum_{x^{1/(1+\sqrt{e})}\le p\le x}\frac{1+r(p)}{p} = o(1). \tag{4.8}$$

Since the product in (4.6) lies between 0 and 1 we conclude that

$$\sum_{n\le x}\frac{f^*(n)}{n}\ge\Big(1-2\log(1+\sqrt{e})+4\int_1^{\sqrt{e}}\frac{\log t}{t+1}dt\Big)\log 2+o(1), \tag{4.9}$$

and for equality to be possible here we must have (4.8), and in addition that the product in (4.6) is $1+o(1)$. These conditions may be written as

$$\sum_{3\le p\le x^{1/(1+\sqrt{e})}}\sum_{k=1}^\infty\frac{1-f^*(p^k)}{p^k}+\sum_{x^{1/(1+\sqrt{e})}\le p\le x}\frac{1-f^*(p)}{p} = o(1).$$

If the above condition holds then, by (3.5), $\sum_{n\le x} g(n)\gg x\log x$ and so for equality to hold in (4.5) we must have $H_0 = o(1/\log x)$. Thus equality in (4.9) is only possible if

$$\Big(\sum_{k=1}^\infty\frac{1+f^*(2^k)}{2^k}\Big)\log x+\sum_{3\le p\le x^{1/(1+\sqrt{e})}}\sum_{k=1}^\infty\frac{1-f^*(p^k)}{p^k}+\sum_{x^{1/(1+\sqrt{e})}\le p\le x}\frac{1-f^*(p)}{p} = o(1).$$

Conversely, if the above is true then equality holds in (4.5), (4.6), and (4.7) giving equality in (4.9). This proves Theorem 2.

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