DON'T BE SEDUCED BY THE ZEROS!

Andrew Granville

1. The explicit formula. Once we know that there are infinitely many primes it is natural to ask how many there are up to x. By studying tables of the primes up to 3×10^6 Gauss understood, as a boy of 15, that the primes occur with density $\frac{1}{\log x}$ at around x, and so the number of primes up to x is approximately $\int_2^x \frac{dt}{\log t}$. This formula is fairly cumbersome, and can be simplified by weighting each prime p with $\log p$. Then Gauss's guesstimate predicts that $\sum_{p \le x} \log p$ is approximately x. So far, all primes up to 10^{23} have been calculated, and the error term never much exceeds $\sqrt{x}$.

How can we approach Gauss's conjecture? We can identify all the composite numbers (and hence all the primes) in $(\sqrt{x}, x]$ by test-dividing by the primes up to $\sqrt{x}$. This is the sieve of Eratosthenes. Nobody has found a way to use this, or any other sieve procedure, to accurately estimate the number of primes up to x. Indeed there is no successful approach known based on simple intuitive reasoning.

In a nine page memoir written in 1859, Riemann outlined an extraordinary plan to attack the elementary question of counting prime numbers using deep ideas from complex function theory. His approach begins with the *Riemann zeta-function*, $\zeta(s) := \sum_{n \ge 1} 1/n^s$. This can be extended, in a unique way, to a function that is analytic in the whole complex plane (except at $s = 1$, where it has a pole of order 1). With this analytic continuation, Riemann gave the following remarkable *explicit formula*:

$$\sum_{\substack{p \text{ prime} \\ p^m \le x \\ m \ge 1}} \log p = x - \sum_{\rho:\ \zeta(\rho)=0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}.$$

If, as Riemann hypothesized, all zeros ρ of $\zeta(\rho) = 0$ have real part $\le \frac{1}{2}$, then each $|x^\rho| \le x^{1/2}$, and one can deduce that the error term in Gauss's guesstimate does not exceed $3\sqrt{x} \log x$.

The *Riemann Hypothesis* is far from proved, but we can deduce more from the explicit formula. For example, fix $1 > \beta > 1/2$. Then all zeros of $\zeta(s)$ satisfy $\mathrm{Re}(s) < \beta$ iff $\left|\sum_{p \le x} \log p - x\right| \le C_\beta x^\beta$. This is unproved, but the *prime number theorem* (that the number of primes up to x is about $x/\log x$) was established by Hadamard and de la Vallée Poussin in 1896, by establishing that there are no zeros of $\zeta(s)$ very close to the 1-line. Using the explicit formula, we can *reformulate* many different problems about

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primes as problems about zeros of zeta-functions, which we can approach through analysis. Mathematicians love to build bridges between apparently disconnected fields, hoping to get a better perspective of both.

These observations are so seductive that they have stimulated most research into the distribution of prime numbers ever since. Moreover, there are many other good questions about prime numbers, number fields, finite fields, curves, and varieties, which can be recast in terms of appropriate zeta-functions, so there is no end to what such methods can investigate.

2. Other approaches. Given this tautology between primes and zeros, no lesser authorities than Hardy, Ingham, and Bohr asserted that it is impossible to find an *elementary proof* of the prime number theorem, a feat achieved, however, by Selberg and Erdős in the late 1940s. (Ingham's brilliant Math Review shows how zeta functions lurk just beneath the surface of their work, so that the avoidance of zeros seems more as a clever trick than a fundamentally new proof.) There are many other important results about prime numbers whose proofs do not revolve around zeta functions, for instance theorems involving gaps between consecutive primes. Nonetheless, these proofs tend to use whatever tools are needed, including information gathered from zeta function methods, as well as sieve methods, so they tend to be viewed as *ad hoc*.

New and quite different techniques have recently achieved great results where zeta function methods fail to yield much, in the wonderful work of Green and Tao on primes in arithmetic progressions, as well as their recent theorems, with Tammy Ziegler, on a wide variety of prime patterns.

3. The pretentious approach. A *multiplicative function* f is one for which $f(mn) = f(m)f(n)$ whenever m and n are coprime integers. Important examples include n^{it} for fixed $t \in \mathbb{R}$, $\chi(n)$ where χ is a Dirichlet character, and others that appear in arithmetic as a consequence of the Chinese Remainder Theorem, as well as $\mu(n)$, defined by $\mu(p) = -1$ and $\mu(p^k) = 0$ for all $k \geq 2$, for all primes p. One can show, in an elementary way, that the prime number theorem holds if and only if the mean value of $\mu(n)$ up to N, tends to 0 as $N \to \infty$.

If we restrict to multiplicative functions satisfying $|f(n)| \leq 1$ for all n, when does the mean value of $f(n)$ *not* tend to 0? An obvious example is 1, or any example much like 1 (i.e., when we perturb the value at each prime by just a small amount). Another example is n^{it} since the mean value is approximately $\frac{1}{N}\int_0^N u^{it}dt = \frac{N^{it}}{1+it}$; in this case we see that the mean value does not tend to a limit as $N \to \infty$, but rather rotates around a circle of radius $1/\sqrt{1+t^2}$. Halász's great 1971 theorem proves that these are essentially all the examples: If the mean value of f does not tend to 0 then f looks a lot like n^{it} for some t, that is f *pretends* to be n^{it}. Halász's proof involves Dirichlet series to the right of 1 and Parseval's identity, but doesn't use analytic continuation.

Soundararajan and I have improved known results in analytic number theory using Halász's ideas. We worked on the size of character sums, the sizes of L-function values, least non-residues, and convexity problems for L-functions. Most recently Soundararajan and Holowinsky completed the proof of Arithmetic Quantum Unique Ergodicity. Halász's theorem is bound to be a better tool to study more general analytic problems than classical analytic methods since the Dirichlet series arising from the given multiplicative function does not need to be analytic (which is the main point of using zeta-functions).

Linnik's Theorem states that there exist constants $c, L > 0$, such that if $(a, q) = 1$ there is a prime $\equiv a \pmod q$ that is $< cq^L$. Previous proofs have been difficult and important. In November 2009, Friedlander and Iwaniec presented a new proof, based on sieve methods, for the first time entirely avoiding zeros of zeta functions. This method inspired Soundararajan and I to further develop an idea we had for a *pretentious large sieve*, yielding what is surely the shortest and technically easiest proof of Linnik's Theorem (though bearing much in common with an earlier proof of Elliott.)

More importantly, our work on Linnik's Theorem revealed that we could prove all of the basic results of analytic number theory *without ever using analytic continuation*. In the past year we have been developing this new approach. Our goal is to reprove the key results in Davenport's *Multiplicative Number Theory* and Bombieri's *Le Grand Crible* using only "pretentious methods." The past semester I taught the first ever "pretentious introduction to analytic number theory" in Montréal, and 40 junior researchers have signed up to participate in an AMS Mathematical Research Community this summer and we hope will go on and further develop our methods.

It is believed that the L in Linnik's Theorem can be taken to be any number > 1; the current record is 5.2. We do not yet know what our method will yield, but we await a talented, energetic researcher who will advance our ideas and beat the current record!