Limitations to the Equi-distribution of Primes IV

John Friedlander * **and Andrew Granville** **

Abstract: We construct polynomials of given degree which take either significantly more or significantly less prime values than expected.

1. Introduction.

Gauss observed that "the density of primes around x is $1/\log x$", which allowed him to predict the Prime Number Theorem:

$$\pi(x) \sim \int_2^x \frac{dt}{\log t} \quad \text{as} \quad x \to \infty, \tag{1}$$

where $\pi(x)$ is the number of primes $\leq x$. Similarly, one might predict that the asymptotic formula

$$\pi(x+y) - \pi(x) \sim y/\log x \quad \text{as} \quad x \to \infty \tag{2}$$

should hold for any y larger than $\log^{2+\varepsilon} x$ (and for other such predictions, see [2]). However, in 1985, Maier [8] showed that (2) fails when y is any fixed power of $\log x$. Similarly, under the belief that the primes should be equally distributed amongst those arithmetic progressions $a \pmod q$ with a coprime to q, one predicts the Prime Number Theorem for arithmetic progressions:

$$\pi(x; q, a) \sim \frac{\pi(x)}{\phi(q)} \tag{3}$$

as $x \to \infty$ for all $(a, q) = 1$, where $\pi(x; q, a)$ is the number of primes $\leq x$ that are $\equiv a \pmod q$ and ϕ is Euler's function. One might also expect (3) to hold uniformly for x larger than $\phi(q)\log^{2+\varepsilon} q$; however, in [3], (3) was shown to fail for all sufficiently large primes q, for some a coprime to q, when x/q is any fixed power of $\log q$. Moreover, in [5], (3) was shown to fail for any fixed non-zero integer a, for infinitely many q with x/q any fixed power of $\log q$.

---

* Partially supported by NSERC grant A5123
** Partially supported by NSF grant DMS-8610730

More generally, it has been conjectured [1,7] that the number of prime values assumed by any irreducible polynomial $F(X)$ is given by the formula

$$\pi_F(x) \;=\; \{C_F + o(1)\}\frac{x}{\log|F(x)|} \tag{4}$$

as $x \to \infty$, where $\pi_F(x)$ is the number of integers $n \le x$ for which $|F(n)|$ is prime and C_F is the following, rather complicated, constant:

$$C_F = \prod_{p \text{ prime}} \left(1 - \frac{\omega_F(p)}{p}\right) \Big/ \left(1 - \frac{1}{p}\right) \tag{5}$$

and $\omega_F(p)$ counts the number of integers n, in the range $1 \le n \le p$, for which $F(n) \equiv 0 \pmod{p}$. (It is easy to deduce both (1) and (3) from (4).) Elementary results on prime ideals guarantee that the product in (5) indeed converges if the primes are taken in ascending order.

As with $\pi(x)$ and $\pi(x; q, a)$, one might have expected that the asymptotic formula (4) holds whenever x is some fixed power of $\log|F|$ or larger, where $|F|$ is the product of the absolute values of the non–zero coefficients of $F(X)$. We prove here that this is false for infinitely many polynomials of any given degree.

Theorem 1. *Let d be a fixed positive integer and let $N \ge 2$ be real. There exist infinitely many irreducible polynomials $F(X)$ of degree d with non-negative integer coefficients, such that for some $\delta_N > 0$ depending only on N, and some $x > \log^N|F|$, we have*

$$\left|\pi_F(x) - C_F\frac{x}{\log|F(x)|}\right| \;>\; \delta_N C_F \frac{x}{\log|F(x)|}. \tag{6}$$

We should like to get both upper and lower bounds for π_F as was done in the case of linear polynomials. There is some technical difficulty with this however and we only prove

Theorem 2. *Assume that the Riemann Hypothesis holds for the Dedekind zeta–functions of number fields. Let d be a fixed positive integer and let $N \ge 2$ be real. Then, for some $\delta_N > 0$ and all $x > x_0(d, N)$ there exist irreducible polynomials $F_\pm$ of degree d with non-negative coefficients such that*

$$\begin{aligned} x \;&<\; \log^N|F_\pm| \le 2x, \\ \pi_{F_+}(x) \;&\ge\; (1+\delta_N)C_{F_+}\frac{x}{\log|F_+(x)|}, \\ \textit{and}\quad \pi_{F_-}(x) \;&\le\; (1-\delta_N)C_{F_-}\frac{x}{\log|F_-(x)|}. \end{aligned}$$

It seems likely that the above assumption of the Riemann Hypothesis can be substantially weakened, by replacing it with a zero density theorem for the zeros of the Dedekind zeta–functions of a certain family of number fields. Such a theorem might well be susceptible to current techniques.

We mention also that one can show by an almost identical argument, the curious unconditional result that if either of the bounds, for π_F, of Theorem 2 fails to hold then for some F the asymptotic formula (4) already fails at the much "safer" value

$$x = \exp((\log |F|)^{\frac{1}{2}}).$$

2. Good and bad moduli.

Fix $c > 0$ and call the modulus q *good* if $L(s, \chi)$ has no real zeros β with $\beta > 1 - c/\log q$ for each Dirichlet character $\chi \pmod q$ (and call q *bad* otherwise). A result of Gallagher [6] implies that, for some c, if q is good and $(a, q) = 1$ then

$$\pi(X + x; q, a) - \pi(X; q, a) \sim \pi(x)/\phi(q)$$

provided that $\log q = o(\log x)$ and $x \gg X$. An immediate consequence of [5] Proposition 2 is

Lemma 1. *Choose $c > 0$ sufficiently small. For all sufficiently large y and z with $\log y \leq z^{1/2}$ and $P(z) := \prod_{p<z} p$, there exists an integer $q > z$ such that either*

(i) q divides n for every bad modulus $nP(z)$ with $n \leq y$, or

(ii) q divides $P(z)$, $q \leq z^2$, $q/\phi(q) = 1 + O(1/\log\log z)$ and q divides n for every bad modulus $nP(z)/q$ with $n \leq y$.

3. Sieving with most of the small primes.

Define $\Phi_q(x, z)$ to be the number of integers $\leq x$, free of prime factors $< z$ which do not divide q. In Lemma 2 below we will see how this function may be estimated smoothly in terms of Buchstab's function, $\omega(u)$, which equals 0 for $u \leq 1$, and is determined by the equation

$$u\omega(u) \;=\; 1 \;+\; \int_0^{u-1} \omega(t) \, \mathrm{d}t \qquad \text{for } u > 1.$$

We shall need the following well–known properties of $\omega(.)$ (see [8]): First that $1 \geq \omega(u) \geq 1/2$ for $u > 1$, and second that $\omega(u) - e^{-\gamma}$ tends to 0 as $u \to \infty$ (where γ is Euler's constant), changing sign at least once in every unit interval. As a consequence we may deduce that for each $N \geq 1$ there exist $M_\pm$ in the range $N < M_\pm < N + 1$ such that $\omega(M_+) > e^{-\gamma} > \omega(M_-)$: note that $M_\pm \leq \frac{9}{5}N$ for each $N \geq 2$.

An immediate consequence of [3] Lemma 5 (see also [4], end of §3) is

Lemma 2. *The estimate*

$$\Phi_q(z^M, z) \;=\; \prod_{p|q, p<z} \left(\frac{p}{p-1}\right) \; \frac{z^M}{\log z} \; \left\{\omega(M) + O\left(\frac{\log\log z}{\log z}\right)\right\}$$

holds uniformly for $z \geq 2$, $M \geq 2$, and integers q with no more than $\log^2 z$ distinct prime factors $< z$.

4. An irreducibility criterion.

We shall use the following criterion in the proofs of the Theorems:

Lemma 3. *Fix the integer $d \geq 2$. The polynomial $F(X) := aX^d + bX + 1$ is irreducible in $\mathbf{Z}[X]$, for any integers a and b satisfying $|a| < |b| - 1$.*

Proof: Noting that $|az^d| < |bz+1|$ for z on the unit circle, we may apply Rouché's Theorem to deduce that $F(X)$ and $bX+1$ have the same number of zeros inside the unit circle. Therefore $F(X)$ has exactly one zero, call it α, inside the unit circle, and the rest are outside the unit circle.

So now suppose that $F(X) = G(X)H(X)$ is reducible, where $H(\alpha) = 0$. But $|G(0)|$ equals the absolute value of the leading term of $G(X)$ (which is a non–zero integer, and so of absolute value ≥ 1) times the product of the absolute values of the roots of $G(X)$ (which are all > 1); thus $|G(0)| > 1$. On the other hand $G(0)$ divides $F(0) = 1$, giving a contradiction.

(Thanks are due to a referee for the simplification, given above, of our original proof of Lemma 3.)

5. The Proof of the Theorem 1.

Select $M = M_\pm$ as in Section 3, so that $N < M_\pm < 9N/5$ and $\omega(M_-) < e^{-\gamma} < \omega(M_+)$. Given z, let $y = \exp(z^{9/10})$ and $R = \exp(z^{M/N})$. Pick q as in Lemma 1, let $q_0 = 1$ if (i) holds, q if (ii) holds, and take $P = P(z)/q_0$.

For each integer r, let $F_r(X) = PX^d + (rP+1)X + 1$. If z is sufficiently large then $F_r(X)$ is irreducible for each $R < r \leq 2R$, by Lemma 3. Moreover

$$\log|F_r| = \log R + 2\log P + O(1) = z^{M/N} + O(z) \tag{7}$$

whenever $R < r \leq 2R$. Now, for any given x,

$$\sum_{r\sim R} \pi_{F_r}(x) = \sum_{r\sim R} \sum_{\substack{n\leq x \\ F_r(n) \text{ is prime}}} 1 = \sum_{n\leq x} \sum_{\substack{r\sim R \\ F_r(n) \text{ is prime}}} 1$$

$$(8) \qquad = \sum_{\substack{n\leq x \\ (n+1,P)=1}} \pi(F_0(n)+2RPn; Pn, F_0(n)) - \pi(F_0(n)+RPn; Pn, F_0(n))$$

as $(Pn, F_0(n)) = (P, n+1)$. Here $r \sim R$ denotes summation over r in the range $R < r \leq 2R$.

A significant restriction on the bad moduli Pn in (8) is given by Lemma 1 and we estimate these restricted terms using the Brun–Titchmarsh Theorem:

$$\pi(F_0(n)+2RPn; Pn, F_0(n)) - \pi(F_0(n)+RPn; Pn, F_0(n)) \ll \frac{RPn}{\phi(Pn)\log R}.$$

For the remaining moduli Pn, which are all good, we may use Gallagher's Theorem (see Section 2), to get

$$\pi(F_0(n)+2RPn; Pn, F_0(n)) - \pi(F_0(n)+RPn; Pn, F_0(n)) \sim \frac{RPn}{\phi(Pn)\log R},$$

if $(P, n+1) = 1$.

Substituting these estimates into (8) we obtain

$$\sum_{r\sim R} \pi_{F_r}(x) = \{1+o(1)\} \sum_{\substack{n\leq x \\ (n+1,P)=1}} \frac{RPn}{\phi(Pn)\log R} + O\left(\sum_{\substack{n\leq x \\ (n+1,P)=1 \\ q|n}} \frac{RPn}{\phi(Pn)\log R} \right)$$

$$\sim R\frac{x}{\log R} \left(\Phi_{q_0}(x,z) \Big/ x \prod_{\substack{p<z \\ p\nmid q_0}} \left(1-\frac{1}{p}\right) \right)$$

$$(9) \qquad \sim R\frac{x}{\log R} e^{\gamma} \omega \left(\frac{\log x}{\log z} \right).$$

Here we have used the fact that $q > z$, Lemma 1, and then Lemma 2, for the last two equations; see also §5 of [5].

Now, if for each $r \sim R$, (6) fails to hold for $x = x_+ = z^{M_+}$, then

$$\left| \sum_{r\sim R} (\pi_r(x_+) - C_r \frac{x_+}{\log R}) \right| < 2\delta_N \frac{x_+}{\log R} \sum_{r\sim R} C_r$$

where, for ease of notation, we have written $\pi_F = \pi_r$ and $C_F = C_r$ for $F = F_r$. By (9), if $\delta_N < \frac{1}{6}$,

$$(1-3\delta_N)Re^{\gamma}\omega(M_+) < \sum_{r\sim R} C_r < (1+3\delta_N)Re^{\gamma}\omega(M_+).$$

Similarly, if for each $r \sim R$, (6) fails to hold for $x = x_- = z^{M_-}$, then

$$(1-3\delta_N)Re^{\gamma}\omega(M_-) < \sum_{r\sim R} C_r < (1+3\delta_N)Re^{\gamma}\omega(M_-)$$

which is a contradiction if δ_N is chosen so small that $(1+3\delta_N)\omega(M_-) < (1-3\delta_N)\omega(M_+)$, for example, any $\delta_N < \frac{e^{\gamma}}{12}(\omega(M_+) - \omega(M_-))$. This completes the proof of Theorem 1.

6. Proof of Theorem 2.

We give the proof for F_+, the proof for F_- being entirely analogous. We choose $M = M_+$ as before and, for given x, we define $z = x^{1/M}$. We choose $y = \exp(z^{9/10})$ and $R = \exp(z^{M/N})$ as in the proof of Theorem 1. By the Riemann Hypothesis (for quadratic fields) the bad moduli of section 2 do not occur so we may take $q_0 = 1$, $P = P(z)$. We recall from the proof of Theorem 1:

$$\sum_{r\sim R} \pi_r(x) \ \sim\ R\,\frac{x}{\log R}\, e^{\gamma}\omega(M). \tag{9$'$}$$

In section 7, we will show, assuming appropriate Riemann Hypotheses, that

$$\sum_{r\sim R} C_r \sim R, \tag{10}$$

for the constants defined by (5). From (7), (9)$'$, and (10),

$$\sum_{r\sim R} \pi_r(x) \sim e^{\gamma}\omega(M) \sum_{r\sim R} C_r \frac{x}{\log |F_r(x)|},$$

so that for some $r \sim R$ we have

$$\pi_r(x) \ge (e^{\gamma}\omega(M) + o(1))C_r \frac{x}{\log |F_r(x)|}.$$

For this value of r we take $F_+ = F_r$, obtaining the result for any δ_N in the range $0 < \delta_N < e^{\gamma}\omega(M) - 1$. This completes the proof of Theorem 2, subject to the demonstration of (10).

7. On the sum $\sum_{r\sim R} C_r$

Throughout this section fix $\varepsilon > 0$ and let $2 \le N < M < 9N/5$, $P = P(z)$ and $R = \exp(z^{M/N})$; the implied constants throughout may depend on d and ε.

For each $r \sim R$, fix a number field K_r, obtained by adjoining a zero β of F_r to the rationals, and define $\Delta = \Delta_r$ to be the absolute value of the discriminant of K_r, and $\omega_r(p) = \omega_{F_r}(p)$. Let $\omega_r(p) + \alpha(p)$ be the number of prime ideals of K_r having norm p, so that $|\alpha(p)| \le d$, and $\alpha(p) = 0$ for all primes p that do not divide ΔP. Now

$$C_r = \prod_p \left(1 - \frac{\omega_r(p)}{p}\right) \Big/ \left(1 - \frac{1}{p}\right) \; = \sum_{n\ge 1} \frac{a_n}{n}$$

where $a_n = \sum_{m|n} \mu(m)\omega_r(m)$ and $\omega_r(m) = \prod_{p|m} \omega_r(p)$. Therefore the Dirichlet series

$$A(s) := \sum_{n\ge 1} \frac{a_n}{n^s} \; = \; \frac{\zeta(s)}{\zeta_K(s)} \; A_1(s)A_2(s),$$

where ζ_K is the Dedekind zeta–function of $K = K_r$, $A_1(s) = \prod_{p|\Delta P}\left(1 - \frac{\alpha(p)}{p^s}\right)$, and $A_2(s)$ is a correctional factor corresponding in part to the prime ideals of K having norm p^j, $j \ge 2$, and in part to the ratio $\left(1 - \frac{\omega_r(p)+\alpha(p)}{p^s}\right) \Big/ \left(1 - \frac{\omega_r(p)}{p^s}\right)\left(1 - \frac{\alpha(p)}{p^s}\right)$ for each prime p dividing ΔP. Therefore

$$A_2(s) \ll 1 \quad \text{for} \quad \sigma \ge \frac{1}{2} + \varepsilon, \tag{11}$$

and, as $|\alpha(p)| \le d$,

$$\begin{aligned} A_1(s) \; &\ll \; \exp\left(d \sum_{p|\Delta P} \frac{1}{p^\sigma}\right) \; \ll \; \exp\left(2d \sum_{p<\log(\Delta P)} \frac{1}{p^\sigma}\right) \\ &\ll \; \exp(cd(\log(\Delta P))^{1-\sigma}), \end{aligned} \tag{12}$$

for $\sigma \ge \varepsilon$.

As $K_r = \mathbf{Q}(\beta^{-1})$, and because the reciprocal polynomial of F_r is monic, we can bound Δ by

$$\Delta \; \le \; |\text{discriminant } X^d F_r(X^{-1})| \; \le \; (3dRP^2)^d, \tag{13}$$

which follows from:

Lemma 4. *The discriminant of an irreducible polynomial* $f(X) = X^d + bX^{d-1} + a$ *is*

$$(-1)^{\binom{d}{2}} a^{d-2}(ad^d + b^d(1-d)^{d-1}).$$

Proof. If $f(X) = \prod_{i=1}^d (X-\beta_i)$ then $\prod_{i=1}^d \beta_i = (-1)^d a$, and

$$f'(\beta_i) = d\beta_i^{d-1} + (d-1)\beta_i^{d-2} b = -d\beta_i^{d-2}\left(\frac{b(1-d)}{d} - \beta_i\right)$$

for each i. Thus the discriminant, Δ_f, is given by

$$\begin{aligned}(-1)^{\binom{d}{2}}\Delta_f &= \prod_{i=1}^d f'(\beta_i) = (-d)^d((-1)^d a)^{d-2} f\left(\frac{b(1-d)}{d}\right) \\ &= a^{d-2}\{b^d(1-d)^d + bdb^{d-1}(1-d)^{d-1} + ad^d\} = a^{d-2}\{ad^d + b^d(1-d)^{d-1}\},\end{aligned}$$

giving Lemma 4.

Returning to the main argument, we write

$$\sum_{r\sim R} C_r = \sum_{r\sim R}\sum_{n\geq 1}\frac{a_n}{n} = \sum_{r\sim R}\sum_{n\leq x}\frac{a_n}{n} + \sum_{r\sim R}\sum_{n>x}\frac{a_n}{n} = S_1 + S_2, \tag{14}$$

where x is to be chosen; note that x has a different meaning than in the earlier sections.

Now, for given m and $n \pmod m$, the congruence $F_r(n) \equiv 0 \pmod m$ is linear in r; therefore, the number of integers $r \sim R$, for which $F_r(n) \equiv 0 \pmod m$, equals $(R/m + O(1))(P,m)$ if both $(n,m) = 1$ and (P,m) divides $n+1$, and equals zero otherwise. Thus

$$\begin{aligned}\sum_{r\sim R}\omega_r(m) &= (P,m)\left\{\frac{R}{m} + O(1)\right\} \sum_{\substack{n \pmod m \\ (n,m)=1 \\ n+1\equiv 0 \pmod{(P,m)}}} 1 \\ &= (P,m)\left\{\frac{R}{m} + O(1)\right\}\frac{m}{(P,m)}\prod_{\substack{p\mid m\\ p\nmid P}}\left(1-\frac{1}{p}\right) = R\prod_{\substack{p\mid m\\ p\nmid P}}\left(1-\frac{1}{p}\right) + O(m),\end{aligned}$$

by the Chinese Remainder Theorem. Therefore

$$\begin{aligned}S_1 &= \sum_{r\sim R}\sum_{n\leq x}\frac{1}{n}\sum_{m\mid n}\mu(m)\omega_r(m) = \sum_{n\leq x}\frac{1}{n}\sum_{m\mid n}\mu(m)\sum_{r\sim R}\omega_r(m) \\ &= R\sum_{n\leq x}\frac{1}{n}\sum_{m\mid n}\mu(m)\prod_{\substack{p\mid m\\ p\nmid P}}\left(1-\frac{1}{p}\right) + O(x).\end{aligned}$$

Now m has $\le \log x/\log z$ prime factors $\ge z$, as it is $\le x$, and so

$$1 \ \ge\ \prod_{\substack{p\mid m\\ p\nmid P}} \left(1-\frac{1}{p}\right) \ \ge\ \left(1-\frac{1}{z}\right)^{\frac{\log x}{\log z}} \ \ge\ 1+O\left(\frac{\log x}{z\log z}\right).$$

Therefore

$$(15)\qquad S_1 \ = \ R+O(x)+O\left(\frac{R\log x}{z\log z}\sum_{n\le x}\frac{1}{n}\sum_{m\mid n}1\right) \ = \ R+O(x)+O\left(\frac{R\log^2 x}{z\log z}\right).$$

We estimate S_2 by contour integration, under the assumption that the Riemann Hypothesis holds for the Dedekind zeta–functions of each field $K=K_r$ with $r\sim R$ (though an appropriate zero density theorem would suffice). Actually, with this assumption, we estimate not only the sum S_2, but the individual sums $\sum_{n>x} a_n/n$: From [10, Lemma 3.12] we have

$$\sum_{n\le x}\frac{a_n}{n} = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} A(s+1)\frac{x^s}{s}\mathrm{d}s + O\left(\frac{x^\varepsilon}{T}\right),$$

for $\sigma = 1/\log x,\ 1\le T\le x$. Now, shifting the contour to $\sigma=-\sigma_0$ (which remains to be chosen), we pass a simple pole at the origin with residue $\sum_{n\ge 1} a_n/n$. Therefore

$$(16)\ \sum_{n>x}\frac{a_n}{n} \ = \ -\frac{1}{2\pi i}\left(\int_{\sigma-iT}^{-\sigma_0-iT} + \int_{-\sigma_0-iT}^{-\sigma_0+iT} + \int_{-\sigma_0+iT}^{\sigma+iT}\right) A(s+1)\frac{x^s}{s}\mathrm{d}s \ + \ O\left(\frac{x^\varepsilon}{T}\right).$$

For $\sigma\ge\frac{1}{2}+\varepsilon,\ \tau=|t|+2$, we have, on the Riemann Hypothesis for $\zeta(s)$,

$$\zeta(s) \ \ll \ \exp((\log\tau)^{2(1-\sigma)+\varepsilon})$$

by Theorem 14.2 of [10]. Similarly using the argument in the proof of that theorem (take $(s-1)\zeta_K$ in case $|t|\le 2$) together with the bound $|\zeta_K(s)| \ \ll \ (\Delta\tau^d)^A/|s-1|$, cf. [9, Theorem 4] valid for $\sigma\ge\frac{1}{2}$, we have, for $\sigma\ge\frac{1}{2}+\varepsilon$, on the Riemann Hypothesis for $\zeta_K(s)$,

$$1/\zeta_K(s) \ \ll \ \exp((\log(\Delta\tau^d))^{2(1-\sigma)+\varepsilon}).$$

These bounds together with (11), (12) and (13) show that the horizontal integrals in (16) are

$$\ll \ T^{-1}\exp(cd(\log(dRPx))^{2\sigma_0+\varepsilon}),$$

for a suitable c, while the vertical integral is

$$\ll \ \ x^{-\sigma_0}\exp(cd(\log(dRPx))^{2\sigma_0+\varepsilon}).$$

Choose $T = x^{\sigma_0}$ and $\sigma_0 = \varepsilon$ so that $S_2 \ll Rx^{-\varepsilon}\exp(2cd(\log R)^{3\varepsilon})$ for $R > dPx$. This, together with (14) and (15), gives

$$\frac{1}{R}\sum_{r\sim R} C_r - 1 \ll \frac{\log^2 x}{z\log z} + \frac{x}{R} + \frac{\exp((2cd\log R)^{3\varepsilon})}{x^{\varepsilon}}. \tag{17}$$

Recall that $R = \exp(z^{M/N})$ where $N < M < 9N/5$: therefore, if we choose $x = \exp(z^{1/6})$ and $\varepsilon < 1/100$ then

$$\sum_{r\sim R} C_r = R(1 + O(z^{-\frac{1}{2}})),$$

which gives (10), and completes the proof of Theorem 2.

References

[1] P.T. Bateman and R.A. Horn, *Primes represented by irreducible polynomials in one variable,* Proc. Symp. Pure Math **(8)** 119-132, AMS (Providence) 1965.

[2] H. Cramér, *On the order of magnitude of the difference between consecutive prime numbers,* Acta Arith. **2** (1936) 23-46.

[3] J. Friedlander and A. Granville, *Limitations to the equi–distribution of primes* I, Ann. Math. **129** (1989), 363–382.

[4] J. Friedlander, A. Granville, A. Hildebrand, and H. Maier, *Oscillation theorems for primes in arithmetic progressions and for sifting functions*, to appear in J. Amer. Math. Soc.

[5] J. Friedlander and A. Granville, *Limitations to the equi–distribution of primes* III, to appear in Comp. Math.

[6] P.X. Gallagher, *A large sieve density estimate near* $\sigma = 1$, Invent. Math. **11** (1970), 329–339.

[7] G.H. Hardy and J.E. Littlewood, *Some problems on partitio numerorum III. On the expression of a number as a sum of primes,* Acta Math. **44** (1923) 1-70.

[8] H. Maier, *Primes in short intervals*, Michigan Math. J. **32** (1985), 221–225.

[9] H. Rademacher, *On the Phragmén-Lindelöf theorem and some applications,* Math. Zeit. **72** (1959) 192-204.

[10] E.C. Titchmarsh, *The Theory of the Riemann Zeta-function,* Oxford, 1951.

Scarborough College, University of Toronto, Scarborough, Ontario, M1C 1A4 CANADA.

School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA.