Extreme Values of $|\zeta(1+it)|$

Andrew Granville and K. Soundararajan

Département de Mathématiques
et Statistique,
Université de Montréal,
CP 6128 succ Centre-Ville,
Montréal, QC H3C 3J7, Canada.
andrew@dms.umontreal.ca

and

Department of Mathematics,
University of Michigan,
Ann Arbor, Michigan 48109, USA.
ksound@umich.edu

Dedicated to Professor K. Ramachandra on his 70th birthday

1 Introduction

Improving on a result of J.E. Littlewood, N. Levinson [3] showed that there are arbitrarily large t for which $|\zeta(1+it)| \geq e^\gamma \log_2 t + O(1)$. (Throughout $\zeta(s)$ is the Riemann-zeta function, and $\log_j$ denotes the j-th iterated logarithm, so that $\log_1 n = \log n$ and $\log_j n = \log(\log_{j-1} n)$ for each $j \geq 2$.) The best upper bound known is Vinogradov's $|\zeta(1+it)| \ll (\log t)^{2/3}$.

Littlewood had shown that $|\zeta(1+it)| \lesssim 2e^\gamma \log_2 t$ assuming the Riemann Hypothesis, in fact by showing that the value of $|\zeta(1+it)|$ could be closely approximated by its Euler product for primes up to $\log^2(2+|t|)$ under this assumption. Under the further hypothesis that the Euler product up to $\log(2+|t|)$ still serves as a good approximation, Littlewood conjectured that $\max_{|t|\leq T} |\zeta(1+it)| \sim e^\gamma \log_2 T$, though later he wrote in [5] (in connection with a q-analogue): "*there is perhaps no good reason for believing ... this hypothesis*".

Our Theorem 1 evaluates the frequency with which such extreme values are attained; and if this density function were to persist to the end of the viable range then this implies the conjecture that

$$\max_{t\in[T,2T]} |\zeta(1+it)| = e^\gamma(\log_2 T + \log_3 T + C_1 + o(1)), \tag{1}$$

for some constant C_1. In fact it may be that $C_1 = C + 1 - \log 2$, where

$$C = \int_0^2 \log I_0(t) \frac{dt}{t^2} + \int_2^\infty (\log I_0(t) - t)\frac{dt}{t^2} = -.3953997,$$

and $I_0(t) := \mathbb{E}(e^{\mathrm{Re}(tX)}) = \sum_{n=0}^\infty (t/2)^{2n}/n!^2$ is the Bessel function (with X a random variable equidistributed on the unit circle). In Theorem 2 we show that there are

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arbitrarily large t for which $|\zeta(1+it)| \geq e^\gamma(\log_2 t + \log_3 t - \log_4 t + O(1))$, which improves upon Levinson's result but falls a little short of our conjecture.

Levinson also showed that $1/|\zeta(1+it)| \geq \frac{6e^\gamma}{\pi^2}(\log_2 t - \log_3 t + O(1))$ for arbitrarily large t. Theorem 1 exhibits even smaller values of $|\zeta(1+it)|$ and determines their frequency. Extrapolating Theorem 1 we are also led to conjecture that

$$\max_{t\in[T,2T]} 1/|\zeta(1+it)| = \frac{6e^\gamma}{\pi^2}(\log_2 T + \log_3 T + C_1 + o(1));$$

but only succeed in proving that $1/|\zeta(1+it)| \geq \frac{6e^\gamma}{\pi^2}(\log_2 t - O(1))$ for arbitrarily large t. K. Ramachandra [6] has obtained results analogous to Levinson's in short intervals, and R. Balasubramanian, Ramachandra and A. Sankaranarayanan [1] have considered extreme values of $|\zeta(1+it)|^{e^{i\theta}}$ for any $\theta \in [0, 2\pi)$.

To be more precise let us define, for $T, \tau \geq 1$,

$$\Phi_T(\tau) := \frac{1}{T}\text{meas}\left\{t \in [T, 2T] : \quad |\zeta(1+it)| > e^\gamma \tau\right\},$$
$$\text{and } \Psi_T(\tau) := \frac{1}{T}\text{meas}\left\{t \in [T, 2T] : \quad |\zeta(1+it)| < \frac{\pi^2}{6e^\gamma \tau}\right\}.$$

Theorem 1. *Let T be large. Uniformly in the range $1 \ll \tau \leq \log_2 T - 20$ we have*

$$\Phi_T(\tau) = \exp\left(-\frac{2e^{\tau-C-1}}{\tau}\left(1 + O\left(\frac{1}{\tau^{\frac{1}{2}}} + \left(\frac{e^\tau}{\log T}\right)^{\frac{1}{2}}\right)\right)\right),$$

where c is a positive constant. The same asymptotic also holds for $\Psi_T(\tau)$.

With a judicious application of the pigeonhole principle we can exhibit even larger values of $|\zeta(1+it)|$, indeed of almost the same quality as the conjectured 1.

Theorem 2. *For large T the subset of points $t \in [0, T]$ such that*

$$|\zeta(1+it)| \geq e^\gamma\left(\log_2 T + \log_3 T - \log_4 T - \log A + O(1)\right)$$

has measure at least $T^{1-\frac{1}{A}}$, uniformly for any $A \geq 10$.

One can also establish results analogous to Theorems 1 and 2 for the distribution of values of $|L(1,\chi)|$ where χ ranges over all non-trivial characters modulo a large prime p (see section 7 for further details). In fact Theorems 1 and 2 hold almost verbatim, just changing T to p. If one also averages over p in a dyadic interval $P \leq p \leq 2P$ then one can obtain asymptotics for the distribution function in the wider range $1 \ll \tau \leq \log_2 P + \log_3 P - O(1)$ (which we expect is the full range, up to the explicit value of the "$O(1)$").

As in [2] we can compare the distribution of $\zeta(1+it)$ with that of an appropriate probabilistic model. Let $X(p)$ denote independent random variables uniformly distributed on the unit circle, for each prime p. We extend X multiplicatively to all integers n: that is set $X(n) = \prod_{p^\alpha \| n} X(p)^\alpha$. We wish to compare the distribution of values of $\zeta(1+it)$ with the distribution of values of the random Euler products $L(1,X) := \prod_p (1 - X(p)/p)^{-1}$ (these products converge with probability 1). Now define

$$\Phi(\tau) = \text{Prob}\left(|L(1,X)| \geq e^{\gamma}\tau\right) \text{ and } \Psi(\tau) = \text{Prob}\left(|L(1,X)| \leq \tfrac{\pi^2}{6e^{\gamma}\tau}\right).$$

By the same methods one can show that $\Phi(\tau)$ and $\Psi(\tau)$ satisfy the same asymptotic as $\Phi_T(\tau)$ as in Theorem 1, but for arbitrary τ (see the remarks immediately after the proof of Theorem 1).

2 Preliminaries

We collect here some standard facts on $\zeta(s)$ which will be used later.

Lemma 1. *Let $y \geq 2$ and $|t| \geq y+3$ be real numbers. Let $\frac{1}{2} \leq \sigma_0 < 1$ and suppose that the rectangle $\{z : \sigma_0 < Re(z) \leq 1,\ |Im(z) - t| \leq y+2\}$ is free of zeros of $\zeta(z)$. Then for any $\sigma_0 < \sigma \leq 2$ and $|\xi - t| \leq y$ we have*

$$|\log\zeta(\sigma + i\xi)| \ll \log|t| \log(e/(\sigma - \sigma_0)).$$

Further for $\sigma_0 < \sigma \leq 1$ we have

$$\log\zeta(\sigma+it) = \sum_{n=2}^{y} \frac{\Lambda(n)}{n^{\sigma+it}\log n} + O\left(\frac{\log|t|}{(\sigma_1 - \sigma_0)^2} y^{\sigma_1 - \sigma}\right),$$

where we put $\sigma_1 = \min\left(\sigma_0 + \frac{1}{\log y}, \frac{\sigma+\sigma_0}{2}\right)$.

Proof: The first assertion follows from Theorem 9.6(B) of Titchmarsh [8]. In proving the second assertion we may plainly suppose that $y \in \mathbb{Z} + \frac{1}{2}$. Then Perron's formula gives, with $c = 1 - \sigma + \frac{1}{\log y}$,

$$\begin{aligned} \frac{1}{2\pi i}\int_{c-iy}^{c+iy} \log\zeta(\sigma+it+w)\frac{y^w}{w}dw &= \sum_{n=2}^{y} \frac{\Lambda(n)}{n^{\sigma+it}\log n} + O\left(\frac{1}{y}\sum_{n=1}^{\infty} \frac{y^c}{n^{\sigma+c}} \frac{1}{|\log(y/n)|}\right) \\ &= \sum_{n=2}^{y} \frac{\Lambda(n)}{n^{\sigma+it}\log n} + O(y^{-\sigma}\log y). \end{aligned} \tag{3}$$

We now move the line of integration to the line $\text{Re}(w) = \sigma_1 - \sigma < 0$. Our hypothesis ensures that the integrand is regular over the region where the line is moved, except for a simple pole at $w = 0$ which leaves the residue $\log\zeta(\sigma+it)$. Thus the left side of (3) equals $\log\zeta(\sigma+it)$ plus

$$\frac{1}{2\pi i}\left(\int_{c-iy}^{\sigma_1-\sigma-iy} + \int_{\sigma_1-\sigma-iy}^{\sigma_1-\sigma+iy} + \int_{\sigma_1-\sigma+iy}^{c+iy}\right)\log\zeta(\sigma+it+w)\frac{y^w}{w}dw \ll \frac{\log|t|}{(\sigma_1-\sigma_0)^2}y^{\sigma_1-\sigma},$$

upon using the first part of the Lemma. □

Using Lemma 1 we shall show that most of the time we may approximate $\zeta(s)$ by a short Euler product.

Lemma 2. *Let $\frac{1}{2} < \sigma \leq 1$ be fixed and let T be large. Let $T/2 \geq y \geq 3$ be a real number. The asymptotic*

$$\log \zeta(\sigma + it) = \sum_{n=2}^{y} \frac{\Lambda(n)}{n^{\sigma+it} \log n} + O\left(y^{(\frac{1}{2}-\sigma)/2} \log^3 T\right)$$

holds for all $t \in (T, 2T)$ except for a set of measure $\ll T^{5/4-\sigma/2} y (\log T)^5$.

Proof: This follows upon using the zero-density result $N(\sigma_0, T) \ll T^{3/2-\sigma_0}(\log T)^5$ (see Theorem 9.19 A of [8]) and appealing to Lemma 1 (taking $\sigma_0 = (1/2+\sigma)/2$ there). □

3 Approximating $\zeta(1+it)$ by a short Euler product

Lemma 3. *Suppose $2 \leq y \leq z$ are real numbers. Then for arbitrary complex numbers $x(p)$ we have*

$$\frac{1}{T}\int_T^{2T} \left| \sum_{y \leq p \leq z} \frac{x(p)}{p^{it}} \right|^{2k} dt \ll \left(k \sum_{y \leq p \leq z} |x(p)|^2 \right)^k + T^{-\frac{2}{3}} \left(\sum_{y \leq p \leq z} |x(p)| \right)^{2k}$$

for all integers $1 \leq k \leq \log T/(3 \log z)$.

Proof: The quantity we seek to estimate is

$$\sum_{\substack{p_1,\ldots,p_k \\ y \leq p_j \leq z}} \sum_{\substack{q_1,\ldots,q_k \\ y \leq q_j \leq z}} \overline{x(p_1)\cdots x(p_k)} x(q_1)\cdots x(q_k) \frac{1}{T}\int_T^{2T} \left(\frac{p_1 \cdots p_k}{q_1 \cdots q_k} \right)^{it} dt.$$

The diagonal terms $p_1 \cdots p_k = q_1 \cdots q_k$ contribute

$$\ll k! \left(\sum_{y \leq p \leq z} |x(p)|^2 \right)^k.$$

If $p_1 \cdots p_k \neq q_1 \cdots q_k$ then as both quantities are below $z^k \leq T^{\frac{1}{3}}$ we have that

$$\frac{1}{T}\int_T^{2T} \left(\frac{p_1 \cdots p_k}{q_1 \cdots q_k} \right)^{it} dt \ll \frac{1}{T|\log(p_1 \cdots p_k / q_1 \cdots q_k)|} \ll T^{-\frac{2}{3}}.$$

Hence the off diagonal terms contribute $\ll T^{-\frac{2}{3}} (\sum_{y \leq p \leq z} |x(p)|)^{2k}$, proving the Lemma. □

Define $\zeta(s;y) := \prod_{p\le y}(1-p^{-s})^{-1}$.

Proposition 1. *Let T be large and let $\log T(\log_2 T)^4 \ge y \ge e^2\log T$ be a real number. Then*

$$\zeta(1+it)=\zeta(1+it;y)\left(1+O\left(\frac{\sqrt{\log T}}{\sqrt{y}\log_2 T}\right)\right)$$

for all $t\in(T,2T)$ except for a set of measure at most $T\exp(-\log T/50\log_2 T)$.

Proof: Setting $z=(\log T)^{100}$ we deduce from Lemma 2 that $\zeta(1+it)=\zeta(1+it;z)(1+O(1/\log T))$ for all $t\in(T,2T)$ except for a set of measure at most $T^{4/5}$. Using Lemma 3 with $k=[\log T/(300\log_2 T)]$ and $x(p)=1/p$ we get that

$$\frac{1}{T}\int_T^{2T}\left|\sum_{y\le p\le z}\frac{1}{p^{1+it}}\right|^{2k}dt \ll \left(k\sum_{y\le p\le z}\frac{1}{p^2}\right)^k + T^{-\frac{2}{3}}\left(\sum_{y\le p\le z}\frac{1}{p}\right)^{2k}$$
$$\ll \left(\frac{\log T}{y}\right)^k\left(\frac{1}{10\log y}\right)^{2k}+T^{-\frac{1}{3}},$$

and so

$$\left|\sum_{y\le p\le z}\frac{1}{p^{1+it}}\right| \le \frac{\sqrt{\log T}}{\sqrt{y}\log y}$$

for all $t\in[T,2T]$ except for a set of measure $\le T\exp(-\log T/49\log_2 T)$. The Proposition thus follows, by combining the above estimates, since

$$\zeta(1+it;y)=\zeta(1+it;z)\exp\left(-\sum_{y\le p\le z}\left(\frac{1}{p^{1+it}}+O\left(\frac{1}{p^2}\right)\right)\right).$$

□

4 Moments of short Euler products

In this section we show how to evaluate large moments of the short Euler products obtained in §3. Below, for any complex number z, $d_z(n)$ will denote the z-th divisor function. That is, $d_z(n)$ is the n-th Dirichlet series coefficient of $\zeta(s)^z$.

Theorem 3. *Let $\log T(\log_2 T)^4 \ge y \ge e^2\log T$ be a real number. Let $z=\delta k$ where $\delta=\pm1$ and $2\le k\le \log T/(e^{10}\log(y/\log T))$ is an integer. Then*

$$\frac{1}{T}\int_T^{2T}|\zeta(1+it;y)|^{2z}dt = \sum_{\substack{n=1\\ p|n \implies p\le y}}^{\infty}\frac{d_z(n)^2}{n^2}\left(1+O\left(\exp\left(-\frac{\log T}{2(\log_2 T)^4}\right)\right)\right)$$

$$= \prod_{p\le k}\left(1-\frac{\delta}{p}\right)^{-2k\delta}\exp\left(\frac{2k}{\log k}\left(C+O\left(\frac{k}{y}+\frac{1}{\log k}\right)\right)\right).$$

Throughout this section let z, y, k, δ be as in Theorem 3. If $k \le 10^6$ then we divide $[1,y]$ into the intervals $I_0 = [k,y]$ and $I_1 = [1,k)$ and take here $J := 1$. If $k > 10^6$ then we define $J := [4\log_2 k/\log 2]+1$ and divide $[1,y]$ into the $J+1$-intervals $I_0 = [k,y]$, $I_j = [k/2^j, k/2^{j-1})$ for $1 \le j \le J-1$, and $I_J = [1, k/2^J) \subset [1, k/(\log k)^4]$. Given a subset R of the index set $\{0,\ 1,\ \ldots,\ J\}$ we define $\mathcal{S}(R)$ to be the set of integers n whose prime factors all lie in $\cup_{r\in R} I_r$. We also define

$$\zeta(s;R) := \prod_{p\in\cup_{r\in R}I_r}\left(1-\frac{1}{p^s}\right)^{-1} = \sum_{n\in\mathcal{S}(R)}\frac{1}{n^s}.$$

Proposition 2. *Let R be any subset of $\{0,\ldots,J\}$. Then we have that*

$$\frac{1}{T}\int_T^{2T}|\zeta(1+it;R)|^{2z}dt = \sum_{n\in\mathcal{S}(R)}\frac{d_z(n)^2}{n^2}\left(1+O\left(\exp\left(-\frac{\log T}{2(\log_2 T)^4}\right)\right)\right).$$

Note that the first part of Theorem 3 follows from the case $R = \{0,1,\ldots,J\}$. While this is the case of interest for us, the formulation of Proposition 2 is convenient for our proof which is based on induction on the cardinality of R.

Lemma 4. *For any prime p we have*

$$\sum_{a=0}^{\infty}\frac{d_z(p^a)^2}{p^{2a}} = I_0\left(\frac{2k}{p}\right)\exp(O(k/p^2)),$$

where I_0 denotes the I-Bessel function. Also

$$\left(1-\frac{\delta}{p}\right)^{-2k\delta} \ge \sum_{a=0}^{\infty}\frac{d_z(p^a)^2}{p^{2a}} \ge \frac{1}{50}\min\left(1,\frac{p}{k}\right)\left(1-\frac{\delta}{p}\right)^{-2k\delta},$$

so that if $\mathcal{P}$ is any subset of the primes $\le y$ then, uniformly,

$$\sum_{\substack{n\ge 1\\ p|n \implies p\in\mathcal{P}}}\frac{d_z(n)^2}{n^2} \ge T^{O(1/\log_2 T)}\prod_{p\in\mathcal{P}}\left(1-\frac{\delta}{p}\right)^{-2k\delta}.$$

Proof: Since

$$\sum_{a=0}^{\infty}\frac{d_z(p^a)^2}{p^{2a}} = \int_0^1\left|1-\frac{e(\theta)}{p}\right|^{-2z}d\theta = \int_0^1\exp(O(k/p^2))\exp\left(2\frac{z}{p}\cos(2\pi\theta)\right)d\theta$$

we obtain the first assertion. The upper bound in the second statement follows since $|1-e(\theta)/p|^{-\delta} \le (1-\delta/p)^{-\delta}$. When $p > k$ we have that $(1-\delta/p)^{-2k\delta} \le$

$(1-1/\max(2,k))^{-2k} \le 16$ and so the lower bound follows in this case. When $p \le k$ consider only θ such that $e(\theta)$ lies on the arc $(\delta e^{-ip/(10k)}, \delta e^{ip/(10k)})$. For such θ we may check that $|1-e(\theta)/p|^{-2k\delta} \ge (1-\delta/p)^{-2k\delta}(1-1/(25k))^k \ge \frac{4}{5}(1-\delta/p)^{-2k\delta}$ from which the lower bound in this case follows.

Now

$$\prod_{\substack{k<p\le y\\ p\in\mathcal{P}}} \left(1-\frac{\delta}{p}\right)^{-2k\delta} \le \exp\left(O\left(\sum_{k<p\le y}\frac{k}{p}\right)\right) \ll \left(\frac{\log y}{\log k}\right)^{O(k)} \ll T^{O(1/\log_2 T)},$$

and

$$\sum_{\substack{n\ge 1\\ p|n \implies p\in\mathcal{P}}} \frac{d_z(n)^2}{n^2} > \sum_{\substack{n\ge 1\\ p|n \implies p\le k \text{ and } p\in\mathcal{P}}} \frac{d_z(n)^2}{n^2} \ge \prod_{\substack{p\le k\\ p\in\mathcal{P}}} \frac{p}{50k}\left(1-\frac{\delta}{p}\right)^{-2k\delta},$$

which together imply the third assertion by the prime number theorem. □

Lemma 5. *Suppose* $0\le r\le J$ *and put* $M_0 := T^{\frac{1}{5}}$ *and* $M_r = T^{\frac{1}{5r^2}}$ *for* $r\ge 1$*. Then we have that*

$$\sum_{\substack{m\in\mathcal{S}(\{r\})\\ m\ge M_r}} \frac{2^{\omega(m)}}{m} \sum_{\ell\in\mathcal{S}(\{r\})} \frac{|d_z(m\ell)d_z(\ell)|}{\ell^2} \le \left(\sum_{\ell\in\mathcal{S}(\{r\})}\frac{d_z(\ell)^2}{\ell^2}\right)\exp\left(-\frac{\log T}{(\log_2 T)^4}\right).$$

Proof: Denote the left side of the estimate in Lemma 5 by N_r and let

$$D_r = \sum_{\ell\in\mathcal{C}(\{r\})}\frac{d_z(\ell)^2}{\ell^2}.$$

For any $1\ge\alpha>0$ we have

$$\begin{aligned} N_r &\le M_r^{-\alpha}\sum_{m\in\mathcal{C}(\{r\})}\frac{2^{\omega(m)}}{m^{1-\alpha}}\sum_{\ell\in\mathcal{C}(\{r\})}\frac{|d_z(m\ell)d_z(\ell)|}{\ell^2}\\ &= M_r^{-\alpha}\prod_{p\in I_r}\left(\sum_{a=0}^{\infty}\frac{|d_z(p^a)|^2}{p^{2a}} + 2\sum_{u=1}^{\infty}\frac{1}{p^{u(1-\alpha)}}\sum_{a=0}^{\infty}\frac{|d_z(p^a)d_z(p^{u+a})|}{p^{2a}}\right). \end{aligned} \tag{4}$$

We record two bounds for the pth term of the product in (4): Firstly

$$\begin{aligned} \sum_{a=0}^{\infty}\frac{|d_z(p^a)|^2}{p^{2a}} + 2\sum_{u=1}^{\infty}\frac{1}{p^{u(1-\alpha)}}\sum_{a=0}^{\infty}\frac{|d_z(p^a)d_z(p^{u+a})|}{p^{2a}} &\le 2\sum_{a=0}^{\infty}\frac{|d_z(p^a)|}{p^{a(1+\alpha)}}\sum_{u=-a}^{\infty}\frac{|d_z(p^{u+a})|}{p^{(u+a)(1-\alpha)}}\\ &= 2\left(1-\frac{\delta}{p^{1-\alpha}}\right)^{-\delta k}\left(1-\frac{\delta}{p^{1+\alpha}}\right)^{-\delta k}. \end{aligned} \tag{5}$$

Secondly, since $|d_z(p^{u+a})| \le |d_z(p^a)||d_z(p^u)|$,

$$\begin{aligned}\sum_{a=0}^{\infty} \frac{|d_z(p^a)|^2}{p^{2a}} + 2\sum_{u=1}^{\infty} \frac{1}{p^{u(1-\alpha)}} \sum_{a=0}^{\infty} \frac{|d_z(p^a)d_z(p^{u+a})|}{p^{2a}} &\le \sum_{a=0}^{\infty} \frac{|d_z(p^a)|^2}{p^{2a}} \left(1 + 2\sum_{u=1}^{\infty} \frac{|d_z(p^u)|}{p^{u(1-\alpha)}}\right) \\ &\le \sum_{a=0}^{\infty} \frac{|d_z(p^a)|^2}{p^{2a}} \left(2\left(1-\frac{\delta}{p^{1-\alpha}}\right)^{-\delta k} - 1\right). \qquad (6)\end{aligned}$$

Now consider the case $r = 0$ and note that $k \le p$ for all $p \in I_0$. Here we use the bound (6) in (4). We choose $\alpha = 1/(10\log_2 T)$ and note that for $p \in I_0$, $2(1-\delta/p^{1-\alpha})^{-\delta k} - 1 \le 2(1-e^{1/9}/p)^{-k} - 1 \le e^{4k/p}$. Hence we get that

$$\begin{aligned}N_0 &\le D_0 \exp\left(-\frac{\log M_0}{10\log_2 T} + 4k\sum_{k\le p\le y} \frac{1}{p}\right) \\ &\le D_0 \exp\left(-\frac{\log M_0}{10\log_2 T} + \frac{4k}{\log k}\sum_{k\le p\le y} \frac{\log p}{p}\right).\end{aligned}$$

Now $\sum_{k\le p\le y} \log p/p \le \log(25y/k)$ (see Theorem I.1.7 of Tenenbaum [7]) and recall that $k \le \log T/(e^{10}\log(y/\log T))$ and that $M_0 = T^{1/5}$. The bound in the lemma then follows in this case.

Suppose now that $r \ge 1$ so that $p \le k$ for all $p \in I_r$. Here we use the bound (5) in (4). We take $\alpha = 1/(10\cdot 2^{r/2}\log(ek))$ and note that for $p \le k$,

$$\begin{aligned}\left(1-\frac{\delta}{p^{1-\alpha}}\right)^{-\delta}\left(1-\frac{\delta}{p^{1+\alpha}}\right)^{-\delta}\left(1-\frac{\delta}{p}\right)^{2\delta} &\le \left(1-\frac{p(p^{\alpha}+p^{-\alpha}-2)}{(p-1)^2}\right)^{-1} \\ &\le \exp\left(\frac{\log^2 p}{10\cdot 2^r p\log^2(ek)}\right).\end{aligned}$$

Using also the lower bound in Lemma 4 we obtain that

$$N_r \le D_r \exp\left(-\frac{\log M_r}{10\cdot 2^{r/2}\log(ek)} + \sum_{p\in I_r}\left(\log\frac{100k}{p} + \frac{k\log p}{10\cdot 2^r p\log(ek)}\right)\right). \qquad (7)$$

If $1 \le r \le J-1$ then we deduce that

$$\begin{aligned}N_r &\le D_r \exp\left(-\frac{\log M_r}{10\cdot 2^{r/2}\log(ek)} + \sum_{k/2^r\le p\le k/2^{r-1}} (r+5)\right) \\ &\le D_r \exp\left(-\frac{\log M_r}{10\cdot 2^{r/2}\log(ek)} + \frac{8(r+5)k}{2^r\log(ek)}\right)\end{aligned}$$

and since $\log M_r = (\log T)/(5r^2)$ this gives $N_r \le D_r \exp(-\log T/(\log_2 T)^4)$ for large T. If $r = J$ and $k \le 10^6$ then the Lemma follows at once from (7). If $r = J$ and $k > 10^6$ then (7) gives that

$$\begin{aligned} N_r &\le D_r \exp\left(-\frac{\log M_J}{10\cdot 2^{J/2}\log(ek)} + \sum_{p\le k/(\log k)^4}\left(\log\frac{100k}{p} + \frac{k\log p}{10\cdot 2^J p\log(ek)}\right)\right) \\ &\le D_r \exp\left(-\frac{\log M_J}{10\cdot 2^{J/2}\log(ek)} + O\left(\frac{\log T}{(\log_2 T)^4}\right)\right), \end{aligned}$$

which proves the Lemma in this case. □

Proof of Proposition 2 : We prove Proposition 2 by induction on the cardinality of R. The case when $R = \emptyset$ is clear and suppose the Proposition holds for all proper subsets of R. We expand

$$|\zeta(1+it;R)|^{2z} = \sum_{\substack{m_r,n_r\in\mathcal{S}(\{r\}) \\ \text{for all } r\in R}} \prod_{r\in R}\left(\frac{d_z(m_r)d_z(n_r)}{m_r n_r}\right)\left(\frac{\prod_{r\in R} m_r}{\prod_{r\in R} n_r}\right)^{it}.$$

Set $u_r = m_r n_r/(m_r,n_r)^2$. Using inclusion-exclusion we decompose the above as

$$\sum_{\substack{bm_r,n_r\in\mathcal{S}(\{r\}),\text{ and} \\ u_r\le M_r \text{ for all } r\in R}} + \sum_{\substack{W\subset R \\ W\ne\emptyset}} (-1)^{|W|-1} \sum_{\substack{m_r,n_r\in\mathcal{S}(\{r\}) \\ \text{for all } r\in R,\text{ and} \\ u_w>M_w \text{ for all } w\in W}} \tag{8}$$

with M_w as in Lemma 5.

First let us consider the contribution of the first sum in (8). This gives

$$\sum_{\substack{m_r,n_r\in\mathcal{S}(\{r\}),\text{ and} \\ u_r\le M_r \text{ for all } r\in R}} \prod_{r\in R}\left(\frac{d_z(m_r)d_z(n_r)}{m_r n_r}\right)\frac{1}{T}\int_T^{2T}\left(\prod_{r\in R}\frac{m_r}{n_r}\right)^{it} dt. \tag{9}$$

If we reduce $\prod_{r\in R} m_r/n_r$ to lowest terms then both the numerator and denominator would be bounded by $\prod_r u_r \le \prod_{r\in R} M_r \le T^{\frac{(1+\pi^2/6)}{5}} \le T^{\frac{3}{5}}$. Thus if $\prod_{r\in R} m_r/n_r \ne 1$ then

$$\frac{1}{T}\int_T^{2T}\left(\frac{\prod_{r\in R} m_r}{\prod_{r\in R} n_r}\right)^{it} dt \ll \frac{1}{T|\log\prod_r m_r/n_r|} \ll T^{-\frac{2}{5}}.$$

Hence we obtain that the expression in (9) equals

$$\sum_{\substack{m_r=n_r\in\mathcal{S}(\{r\}) \\ \text{for all } r\in R}} \prod_{r\in R}\left(\frac{d_z(m_r)}{m_r}\right)^2 + O\left(T^{-\frac{2}{5}} \sum_{\substack{m_r,n_r\in\mathcal{S}(\{r\}) \\ \text{for all } r\in R}} \prod_{r\in R}\left(\frac{|d_z(m_r)d_z(n_r)|}{m_r n_r}\right)\right).$$

The main term above is $\sum_{n\in\mathcal{S}(R)} d_z(n)^2/n^2$. The error term is $\ll T^{-\frac{2}{5}}\prod_{p\in\cup_{r\in R}I_r}(1-\delta/p)^{-2k\delta}$ and using the lower bound of Lemma 4 this is $\ll T^{-\frac{1}{3}}\sum_{n\in\mathcal{S}(R)} d_z(n)^2/n^2$. Thus the contribution of the first term in (8) is

$$(1+O(T^{-\frac{1}{3}}))\sum_{n\in\mathcal{S}(R)}\frac{d_z(n)^2}{n^2}. \tag{10}$$

Now we consider the contribution of the second term in (8). This gives

$$\sum_{\substack{W\subset R\\ W\neq\emptyset}}(-1)^{|W|-1}\sum_{\substack{m_w,n_w\in\mathcal{S}(\{w\}),\text{ and }\\ u_w>M_w\text{ for all }w\in W}}\prod_{w\in W}\left(\frac{d_z(m_w)d_z(n_w)}{m_w n_w}\right)$$
$$\times\frac{1}{T}\int_T^{2T}\left(\frac{\prod_{w\in W}m_w}{\prod_{w\in W}n_w}\right)^{it}|\zeta(1+it;R-W)|^{2z}dt,$$

which is bounded in magnitude by

$$\sum_{\substack{W\subset R\\ W\neq\emptyset}}\sum_{\substack{m_w,n_w\in\mathcal{S}(\{w\}),\text{ and }\\ u_w>M_w\text{ for all }w\in W}}\prod_{w\in W}\left(\frac{|d_z(m_w)d_z(n_w)|}{m_w n_w}\right)\frac{1}{T}\int_T^{2T}|\zeta(1+it;R-W)|^{2z}dt.$$

By the induction hypothesis we see that

$$\frac{1}{T}\int_T^{2T}|\zeta(1+it;R-W)|^{2z}dt\ll\sum_{n\in\mathcal{S}(R-W)}\frac{d_z(n)^2}{n^2},$$

while from Lemma 5 (with $m=u_w$ and $\ell=(m_w,n_w)$ so that $d_z(m\ell)d_z(\ell)=d_z(m_w)d_z(n_w)$; and note that the number of pairs m_w,n_w which give rise to a given pair ℓ,m is exactly $2^{\omega(m)}$) we deduce that

$$\sum_{\substack{m_w,n_w\in\mathcal{S}(\{w\})\\ u_w>M_w}}\frac{|d_z(m_w)d_z(n_w)|}{m_w n_w}\leq\sum_{n\in\mathcal{S}(\{w\})}\frac{d_z(n)^2}{n^2}\exp\left(-\frac{\log T}{(\log_2 T)^4}\right).$$

From these estimates it follows that the contribution of the second term in (8) is

$$\ll|R|\sum_{n\in\mathcal{S}(R)}\frac{d_z(n)^2}{n^2}\exp\left(-\frac{\log T}{(\log_2 T)^4}\right).$$

Combining this with (10) we obtain Proposition 2. □

Proof of Theorem 3: In view of Proposition 2 it remains only to prove that

$$\sum_{\substack{n=1\\ p|n\implies p\leq y}}^{\infty}\frac{d_z(n)^2}{n^2}=\prod_{p\leq k}\left(1-\frac{\delta}{p}\right)^{-2k\delta}\exp\left(\frac{2k}{\log k}\left(C+O\left(\frac{k}{y}+\frac{1}{\log k}\right)\right)\right). \tag{11}$$

Using the first part of Lemma 4 for $p \geq \sqrt{k}$ and the second part for $p < \sqrt{k}$ we see that

$$\sum_{\substack{n=1 \\ p|n \implies p \leq y}}^{\infty} \frac{d_z(n)^2}{n^2} = \prod_{p<\sqrt{k}} \left(1 - \frac{\delta}{p}\right)^{-2k\delta} \prod_{\sqrt{k} \leq p \leq y} I_0\left(\frac{2k}{p}\right) \exp(O(\sqrt{k})).$$

Since $\log I_0(t) = O(t^2)$ for $0 \leq t \leq 2$ we have by the prime number theorem and partial summation that

$$\begin{aligned}\sum_{k \leq p \leq y} \log I_0\left(\frac{2k}{p}\right) &= \frac{2k}{\log k} \int_{2k/y}^{2} \log I_0(t) \frac{dt}{t^2} + O\left(\frac{k}{\log^2 k}\right) \\ &= \frac{2k}{\log k} \int_0^2 \log I_0(t) \frac{dt}{t^2} + O\left(\frac{k^2}{y \log k} + \frac{k}{\log^2 k}\right).\end{aligned}$$

Since $\log I_0(t) = t + O(\log t)$ for $t \geq 2$ we obtain by the prime number theorem and partial summation that

$$\begin{aligned}&\sum_{\sqrt{k} \leq p \leq k} \left(\log I_0\left(\frac{2k}{p}\right) + 2k\delta \log\left(1 - \frac{\delta}{p}\right)\right) \\ &\qquad = \frac{2k}{\log k} \int_2^{\infty} (\log I_0(t) - t) \frac{dt}{t^2} + O\left(\frac{k}{\log^2 k}\right).\end{aligned}$$

These estimates prove (11) and so Theorem 3 follows. □

5 Proof of Theorem 1

Let $\log T (\log_2 T)^4 \geq y \geq e^2 \log T$, and let $T\Phi_T(\tau; y)$ denote the measure of points $t \in [T, 2T]$ for which $|\zeta(1 + it; y)| \geq e^{\gamma}\tau$. Taking $z = k$ for an integer $3 \leq k \leq \log T/(e^{10} \log(y/\log T))$ in Theorem 3 and using Mertens' theorem $\prod_{p \leq k}(1 - 1/p)^{-1} = e^{\gamma} \log k + O(1/\log^2 k)$ we get that

$$\begin{aligned}2k \int_0^{\infty} \Phi_T(t; y) t^{2k-1} dt &= \frac{1}{T} \int_T^{2T} e^{-2k\gamma} |\zeta(1 + it; y)|^{2k} dt \\ &= (\log k)^{2k} \exp\left(\frac{2k}{\log k}\left(C + O\left(\frac{k}{y} + \frac{1}{\log k}\right)\right)\right). \qquad (12)\end{aligned}$$

Now

$$\begin{aligned}\int_0^{\infty} \Phi_T(t; y) dt &= e^{-\gamma}(1/T) \int_T^{2T} |\zeta(1 + it; y)| dt \\ &\leq e^{-\gamma}((1/T) \int_T^{2T} |\zeta(1 + it; y)|^4 dt)^{1/4} \ll 1\end{aligned}$$

by Theorem 3; so, by Hölder's inequality,

$$\int_0^\infty \Phi_T(t;y)t^a dt \le \left(\int_0^\infty \Phi_T(t;y)dt\right)^{1-a/b} \left(\int_0^\infty \Phi_T(t;y)t^b dt\right)^{a/b}$$
$$\ll \left(\int_0^\infty \Phi_T(t;y)t^b dt\right)^{a/b}$$

for $a < b$. While (12) at present holds only for integer values of k, we may interpolate to non-integer value $\kappa \in (k-1,k)$ by taking $a = 2k-3,\ b = 2\kappa - 1$ and then $a = 2\kappa - 1,\ b = 2k-1$ in the last inequality to obtain

$$\left(\int_0^\infty \Phi_T(t;y)t^{2k-3}dt\right)^{\frac{2\kappa-1}{2k-3}} \ll \int_0^\infty \Phi_T(t;y)t^{2\kappa-1}dt \ll \left(\int_0^\infty \Phi_T(t;y)t^{2k-1}dt\right)^{\frac{2\kappa-1}{2k-1}},$$

and so we get (12) for κ by substituting (12) for $k-1$ and k into this equation.

Suppose $1 \ll \tau \le \log_2 T - 20 - \log_2(y/\log T)$ and select $\kappa = \kappa_\tau$ such that $\log\kappa = \tau - 1 - C$. Let $\epsilon > 0$ be a bounded parameter to be fixed shortly and put $K = \kappa e^\epsilon$. Observe that

$$2\kappa \int_{\tau+\epsilon}^\infty \Phi_T(t;y)t^{2\kappa-1}dt \le 2\kappa(\tau+\epsilon)^{2\kappa-2K} \int_{\tau+\epsilon}^\infty \Phi_T(t;y)t^{2K-1}dt$$
$$\le (\tau+\epsilon)^{2\kappa(1-e^\epsilon)} \left(2K\int_0^\infty \Phi_T(t;y)t^{2K-1}dt\right).$$

Using (12) we observe that

$$2K\int_0^\infty \Phi_T(t;y)t^{2K-1}dt$$
$$= \left((\log\kappa+\epsilon)\exp\left(\frac{C}{\log\kappa}\left(1+O\left(\frac{1}{\log\kappa}+\frac{\kappa}{y}\right)\right)\right)\right)^{2K}$$
$$= \exp\left(\frac{2\kappa(\epsilon e^\epsilon + C(e^\epsilon-1))}{\log\kappa} + O\left(\frac{\kappa}{\log^2\kappa}+\frac{\kappa^2}{y\log\kappa}\right)\right)$$
$$\times (\log\kappa)^{2\kappa(e^\epsilon-1)}\int_0^\infty \Phi_T(t;y)t^{2\kappa-1}dt.$$

We conclude from the last two displayed equations

$$2\kappa\int_{\tau+\epsilon}^\infty \Phi_T(t;y)t^{2\kappa-1}dt = \exp\left(\frac{2\kappa}{\log\kappa}(1+\epsilon-e^\epsilon) + O\left(\frac{\kappa}{\log^2\kappa}+\frac{\kappa^2}{y\log\kappa}\right)\right)$$
$$\times \int_0^\infty \Phi_T(t;y)t^{2\kappa-1}dt.$$

Choose $\epsilon = c(1/\tau + (\log T)/y)^{\frac{1}{2}}$ for a suitable constant $c > 0$, so that for large τ (and hence large κ),

$$\int_{\tau+\epsilon}^\infty \Phi_T(t;y)t^{2\kappa-1}dt \le \frac{1}{100}\int_0^\infty \Phi_T(t;y)t^{2\kappa-1}dt,$$

say. A similar argument reveals that

$$\int_0^{\tau-\epsilon} \Phi_T(t;y)t^{2\kappa-1}dt \le \frac{1}{100}\int_0^{\infty} \Phi_T(t;y)t^{2\kappa-1}dt.$$

Combining these two assertions with (12) for κ we obtain

$$\int_{\tau-\epsilon}^{\tau+\epsilon} \Phi_T(t;y)t^{2\kappa-1}dt = (\log\kappa)^{2\kappa}\exp\left(\frac{2\kappa C}{\log\kappa}(1+O(\epsilon^2))\right).$$

Since Φ_T is a non-increasing function we deduce that the left side above is

$$\ge \Phi_T(\tau+\epsilon;y)\tau^{2\kappa}\exp(O(\kappa\epsilon/\tau)), \qquad \text{and} \qquad \le \Phi_T(\tau-\epsilon;y)\tau^{2\kappa}\exp(O(\kappa\epsilon/\tau)).$$

It follows that

$$\Phi_T(\tau+\epsilon;y) \le \exp\left(-(2+O(\epsilon))\frac{e^{\tau-1-C}}{\tau}\right) \le \Phi_T(\tau-\epsilon;y),$$

and hence that uniformly in $\tau \le \log_2 T - 20 - \log_2(y/\log T)$ we have

$$\Phi_T(\tau;y) = \exp\left(-\frac{2e^{\tau-1-C}}{\tau}(1+O(\epsilon))\right). \tag{13}$$

From Proposition 1 we know that

$$\Phi_T(\tau) = \Phi_T(\tau+O(\epsilon);y) + O(\exp(-\log T/50\log_2 T))$$

for $\tau \ll \log_2 T$; and so from (13) we deduce that uniformly in $\tau \le \log_2 T - 20 - \log(y/\log T)$ we have

$$\Phi_T(\tau) = \exp\left(-\frac{2e^{\tau-1-C}}{\tau}(1+O(\epsilon))\right) + O\left(\exp\left(-\frac{\log T}{50\log_2 T}\right)\right).$$

Taking $y = \min(\tau\log T, (\log^2 T)/e^{10+\tau})$ above we easily obtain Theorem 1 for Φ_T. The argument for Ψ_T is analogous, using $z=-k$ in Theorem 3. □

One finds, using the first part of Lemma 4 and the observation that $\log I_0(2k/p) \ll k^2/p^2$ for $p > k$, that

$$\begin{aligned}\mathbb{E}(|L(1,X)|^{2z}) = \sum_{n\ge1}\frac{d_z(n)^2}{n^2} &= \sum_{\substack{n=1\\ p|n \implies p\le y}}^{\infty}\frac{d_z(n)^2}{n^2}\exp\left(O\left(\frac{k^2}{y\log y}\right)\right)\\ &= \prod_{p\le k}\left(1-\frac{\delta}{p}\right)^{-2k\delta}\exp\left(\frac{2k}{\log k}\left(C+O\left(\frac{k}{y}+\frac{1}{\log k}\right)\right)\right),\end{aligned}$$

the last line following as in the proof of Theorem 3. With this estimate we can proceed precisely as in the proof of Theorem 1 to obtain the analagous estimate.

6 Large values of $|\zeta(1+it)|$: Proof of Theorem 2

Let T be large and put $y = \log T \log_2 T/(4B\log_3 T)$ for some $B \geq 5$, and $\delta = 1/[\log_2 T]^4$. Let $\| z \|$ denote the distance of z from the nearest integer.

Lemma 6. *For any real t_0 there is a positive integer $m \leq T^{\frac{1}{B}}$ such that for each prime $p \leq y$ we have $\| (mt_0 \log p)/2\pi \| \leq \delta$.*

Proof: This follows from Dirichlet's theorem on Diophantine approximation (see for example §8.2 of [8]) since $1/\delta$ is an integer and $(1/\delta)^{\pi(y)} \leq T^{\frac{1}{B}}$, by the prime number theorem. □

Lemma 7. *For any real t_1 there is a positive integer $n \leq [\log_2 T]^2$ for which*

$$\mathrm{Re} \sum_{y \leq p \leq \exp((\log T)^{10})} \frac{1}{p^{1+int_1}} \geq -\frac{10}{\log_2 T}.$$

Proof: Let $K(x) = \max(0, 1-|x|)$ and note that $\sum_{l=-L}^{L} K(l/L)e^{ilt}$ (the Fejer kernel) is non-negative for all positive integers L and all t. It follows therefore that

$$\sum_{j=-[\log_2 T]^2}^{[\log_2 T]^2} K\left(\frac{j}{[\log_2 T]^2}\right) \sum_{y \leq p \leq \exp((\log T)^{10})} \frac{1}{p^{1+ijt_1}} \geq 0.$$

Hence we obtain that

$$\begin{aligned} \mathrm{Re} \sum_{j=1}^{[\log_2 T]^2} K\left(\frac{j}{[\log_2 T]^2}\right) \sum_{y \leq p \leq \exp((\log T)^{10})} \frac{1}{p^{1+ijt_1}} &\geq -\frac{1}{2} \sum_{y \leq p \leq \exp((\log T)^{10})} \frac{1}{p} \\ &\geq -5\log_2 T. \end{aligned}$$

The lemma follows at once. □

Proof of Theorem 2 : For $T^{\frac{1}{10}} \leq |t| \leq T$ one has

$$\log \zeta(1+it) = -\sum_{p \leq \exp((\log T)^{10})} \log\left(1 - \frac{1}{p^{1+it}}\right) + O\left(\frac{1}{\log T}\right).$$

(One can prove this, arguing as in the proof of the prime number theorem, by noting that $(1/2i\pi)\int_{(c)} \log\zeta(1+it+w)(x^w/w)dw$ with $x = \exp((\log T)^{10})$ and $c > 0$ gives the main term of the right side by Perron's formula, and by shifting the contour to the left of 0, but enclosing a region free of zeros of $\zeta(s)$, we get residue $\log\zeta(1+it)$ from the simple pole at $w = 0$, and the error term from the remaining integral.)

Combining Lemmas 6 and 7 (with $t_1 = mt_0$) we see that for any $t_0 \in [T^{1/10}, T]$ there exists an integer ℓ (where $\ell = mn$) with $1 \le \ell \le T^{\frac{1}{B}}[\log_2 T]^2$ such that $\| (\ell t_0 \log p)/2\pi \| \le 1/[\log_2 T]^2$ for each prime $p \le y$, and such that

$$\mathrm{Re} \sum_{y \le p \le \exp((\log T)^{10})} \frac{1}{p^{1+i\ell t_0}} \ge -\frac{10}{\log_2 T}.$$

We deduce therefore that

$$\begin{aligned} |\zeta(1+i\ell t_0)| &\ge \prod_{p \le y} \left(1 - \frac{1}{p} + O\left(\frac{1}{p(\log_2 T)^2}\right)\right)^{-1} \left(1 + O\left(\frac{1}{\log_2 T}\right)\right) \\ &\ge e^\gamma(\log_2 T + \log_3 T - \log_4 T - \log A + O(1)), \end{aligned}$$

using the prime number theorem, where $A = 1/(2/B + 3\log_2 T/\log T)$.

We use the above procedure with $t_0 = T_0, T_0+1, T_0+2, \ldots, T_0+U_0$ where $T_0 = [T^{1-1/B}/3[\log_2 T]^2]$ and $U_0 = [T^{1-2/B}/7[\log_2 T]^4]$. Let ℓ_i be as above so $\ell_i \le T^{1/B}[\log_2 T]^2$ and thus $\tau_i = \ell_i(T_0+i) \le T/2$. We claim that $|\tau_i - \tau_j| \ge 1$ if $i \ne j$ for if not then evidently $\ell_i \ne \ell_j$ (else $1 \le |(T_0+j)-(T_0+i)| = |\tau_j - \tau_i|/\ell_i < 1$), so that

$$T_0 \le |(\ell_i - \ell_j)T_0| \le |\tau_i - \tau_j| + |i\ell_i - j\ell_j| < 1 + U_0 T^{1/B}[\log_2 T]^2,$$

which is false. Now each $|\zeta(1+i\tau_j)| \ge e^\gamma(\log_2 T + \log_3 T - \log_4 T - \log A + O(1))$. Since $|\zeta'(1+it)| \ll \log^2 T$ for $1 \le |t| \le T$ we see that for any $|\alpha| \le 1/\log^2 T$ we have that $|\zeta(1+i\tau_j+i\alpha)| = |\zeta(1+i\tau_j)| + O(\alpha \log^2 T) = |\zeta(1+i\tau_j)| + O(1)$. Thus the measure of $t \in [0,T]$ with $|\zeta(1+it)| \ge e^\gamma(\log_2 T + \log_3 T - \log_4 T - \log A + O(1))$ is at least $2U_0/\log^2 T$, proving Theorem 2. □

7 The analogous results for *L*-functions at 1

By analogous methods one can prove:

Theorem 4. *Let q be a large prime.*

(i) *The proportion of characters $\chi \pmod q$ for which $|L(1,\chi)| > e^\gamma \tau$ is*

$$\exp\left(-\frac{2e^{\tau-C-1}}{\tau}\left(1 + O\left(\frac{1}{\tau^{\frac{1}{2}}} + \left(\frac{e^\tau}{\log q}\right)^{\frac{1}{2}}\right)\right)\right), \tag{14}$$

uniformly in the range $1 \ll \tau \le \log_2 q - 20$. The same asymptotic also holds for the proportion of characters $\chi \pmod q$ for which $|L(1,\chi)| < \pi^2/6e^\gamma\tau$.

(ii) *There are at least $q^{1-1/A}$ characters $\chi \pmod q$ such that*

$$|L(1,\chi)| \ge e^\gamma(\log_2 q + \log_3 q - \log_4 q - \log A + O(1)),$$

for any $A \ge 10$.

If, in addition, we vary over all characters $\chi \pmod q$ and all primes $Q \le q \le 2Q$, then we can get a good estimate for the distribution function of $|L(1,\chi)|$ in almost the entire viable range. Thus we may prove that the proportion of $|L(1,\chi)| \ge e^\gamma \tau$ is (14) for the range $1 \le \tau \le \log_2 Q + \log_3 Q - 100$, but now with the error term "$(e^\tau/(\log Q \log_2 Q))^{\frac12}$" in place of "$(e^\tau/\log q)^{\frac12}$" (and a corresponding result holds for $1/|(6/\pi^2)L(1,\chi)|$).

The broad outline of the proof is the same, though now replacing $\log T$ by $\log Q \log_2 Q$, so that $\log Q(\log_2 Q)^4 \ge y \ge e^2 \log Q \log_2 Q$ and the range for k becomes $2 \le k \le \log Q \log_2 Q/(e^{10}\log(y/(\log Q\log_2 Q)))$. The result follows easily from the following analogy to Theorem 3,

$$\frac{1}{\pi(Q)}\sum_{q\le Q}\frac{1}{\varphi(q)}\sum_{\chi \pmod q}|L(1,\chi;y)|^{2z} = \prod_{p\le k}\left(1-\frac{\delta}{p}\right)^{-2k\delta} \times \exp\left(\frac{2k}{\log k}\left(C_1 + O\left(\frac{k}{y}+\frac{1}{\log k}\right)\right)\right),$$

and an appropriate development of Lemma 4, where $L(1,\chi;y) := \prod_{p\le y}(1-\chi(p)/p)^{-1}$. The above estimate, though, is proved rather more easily than Theorem 3. Since $L(1,\chi;y)^z = \sum_{n\in\mathcal{S}(y)} d_z(n)\chi(n)/n$, and $L(1,\overline{\chi};y)^z = \sum_{m\in\mathcal{S}(y)} d_z(m)\overline{\chi}(m)/m$ where $S(y)$ is the set of integers all of whose prime factors are $\le y$, the left side of this equation equals

$$\sum_{m,n\in\mathcal{S}(y)}\frac{d_z(m)d_z(n)}{mn}\left\{\frac{1}{\pi(Q)}\sum_{q\le Q}\frac{1}{\varphi(q)}\sum_{\chi \pmod q}\chi(m)\overline{\chi}(n)\right\}.$$

The term in $\{\}$ equals $1 - \#\{q\le Q:\ q|mn\}/\pi(Q)$ if $m=n$, and is $\le \#\{q \le Q : q|m-n\}/\pi(Q)$ if $m \ne n$. Therefore our sum is

$$\sum_{n\in\mathcal{S}(y)}\frac{d_z(n)^2}{n^2} + O\left(\frac{1}{\pi(Q)}\left(\sum_{m\in\mathcal{S}(y)}\frac{|d_z(m)|\log 2m}{m}\right)^2\right).$$

Now $\log 2n \ll k^2 + n^{1/k}$ so that

$$\begin{aligned}\sum_{n\in\mathcal{S}(y)}\frac{|d_z(n)|}{n}\log 2n &\ll k^2\prod_{p\le y}\left(1-\frac{\delta}{p}\right)^{-\delta k} + \prod_{p\le y}\left(1-\frac{\delta}{p^{1-1/k}}\right)^{-\delta k}\\ &\ll \prod_{p\le y}\left(1-\frac{\delta}{p}\right)^{-\delta k}\left(k^2 + \exp\left(O\left(k\sum_{p\le y}\frac{p^{1/k}-1}{p}\right)\right)\right)\\ &\ll (\log Q)^{O(1)}\prod_{p\le y}\left(1-\frac{\delta}{p}\right)^{-\delta k},\end{aligned}$$

and the claimed estimate follows from Lemma 4.

References

[1] R. Balasubramanian, K. Ramachandra, and A. Sankaranarayanan, *On the frequency of Titchmarsh's phenomenon for* $\zeta(s)$*-VIII*, Proc. Ind. Acad. Sci. **102** (1992), 1-12.

[2] A. Granville and K. Soundararajan, *The distribution of values of* $L(1, \chi_d)$, Geometric and Funct. Anal **13** (2003), 992-1028.

[3] N. Levinson, Ω*-theorems for the Riemann zeta-function*, Acta Arith. **20** (1972) 319–332.

[4] J.E. Littlewood, *On the function* $1/\zeta(1+it)$, Proc. London Math. Soc, **27** (1928) 349-357.

[5] J.E. Littlewood, *On the class number of the corpus* $P(\sqrt{-k})$, Proc. London Math. Soc, **27** (1928) 358-372.

[6] K. Ramachandra, *paper On the frequency of Titchmarsh's phenomenon for* $\zeta(s)$*-VII*, Ann. Acad. Sci. Fenn. **14** (1989) 27-40.

[7] G. Tenenbaum. *Introduction to analytic and probabilistic number theory*, Cambridge Studies in Advanced Mathematics **46**, Cambridge University Press, Cambridge (1995).

[8] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford University Press, Oxford; Second edition, revised by D.R. Heath-Brown (1986).

[9] I.M. Vinogradov, *A new estimate for* $\zeta(1+it)$, Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958) 161–164.