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On Hajós' Conjecture

(Minimum cycle-partitions of the edge-set of Eulerian graphs)

by

Andrew Granville and Alexandros Moisiadis

Queen's University, Kingston, Ontario, Canada

Abstract

Let G be a simple, Eulerian graph of order n and $c = c(G)$ be the minimum number of edge-disjoint cycles needed to partition the edge-set $E(G)$. In this paper we prove that $c \le \lfloor (n-1)/2 \rfloor$, for all such graphs with maximum degree $\Delta = \Delta(G) \le 4$.

Introduction

In this paper we shall only consider graphs that are finite and simple.

It is a well-known fact that the edge-set of an Eulerian graph can be partitioned into simple edge-disjoint cycles. If the graph is of order n, Hajós conjectured (see [1], [2]) that $c \le \lfloor n/2 \rfloor$. Dean [3] proved that Hajós' conjecture is equivalent to the statement "If G is a simple Eulerian graph of order n then $c = c(G) \le \lfloor (n-1)/2 \rfloor$".

Let K_ℓ be the complete graph of order ℓ and T_m, m even, be the complete graph of order m minus the edges of a perfect matching. It is well-known (see eg. [2]) that $c(K_{2k+1}) = k = \lfloor ((2k+1)-1)/2 \rfloor$ for $k \ge 1$ and $c(T_{2k}) = k-1 = \lfloor (2k-1)/2 \rfloor$ for $k \ge 2$, where in both cases all cycles are Hamiltonian.

In this paper we prove the following.

Theorem 1: For every Eulerian graph G of order n and maximum degree $\Delta = \Delta(G) \le 4$, $c(G) \le \lfloor (n-1)/2 \rfloor$.

The theorem for graphs with Eulerian components follows immediately from the above.

CONGRESSUS NUMERANTIUM 56(1987), pp.183-187

Theorem 2: For every integer $n \geq 3$ and every integer i, $1 \leq i \leq \lfloor (n-1)/2 \rfloor$, there exist Eulerian graphs G_i of order n with maximum degree ≤ 4 such that $c(G_i) = i$.

Theorem 2 shows that the bound $\lfloor (n-1)/2 \rfloor$ is best possible for Eulerian graphs with maximum degree ≤ 4 for every order $n \geq 3$.

Notation. By $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$, $\mathcal{C}(G)$ we will denote the vertex set, edge-set, maximum degree, minimum degree and a cycle-partition of the edge-set, of a graph G respectively. Also if v, w are vertices of G then $v \sim w$ ($v \not\sim w$) will mean that v is (is not) adjacent to w, (v, w) will denote the edge between v and w, $N_G(v)$ the neighbourhood of v in G and $d_G(v)$ the degree of v in G.

Let Δ be the smallest positive integer for which there exists Eulerian graph G, with $\Delta(G) = \Delta$ and $c(G) > \lfloor (|V(G)| - 1)/2 \rfloor$. Let $\mathcal{H}$ be the set of Eurlerian graphs G with $\Delta(G) = \Delta$, $c(G) > \lfloor (|V(G)| - 1)/2 \rfloor$ where $|V(G)| = n$ is minimal and then $|E(G)|$ is minimal.

The following lemmas describe some of the properties that such graphs possess.

Lemma 1: If $G \in \mathcal{H}$ the G is 2-connected.

Proof: Suppose that G has a cut-vertex v. Let $G_1, G_2, \ldots, G_k$, $k \geq 2$ denote the v-components of G of order $0 < n_i = |V(G_i)|$, $1 \leq i \leq k$, where $n = \sum_{i=1}^{k} n_i - (k-1)$, $V(G_i) \cap V(G_j) = \{v\}$ and $E(G_i) \cap E(G_j) = \phi$ for all $i \leq i < j \leq k$. Now, by taking $\mathcal{C}(G) = \bigcup_{i=1}^{k} \mathcal{C}(G_i)$ we see that

$$c(G) = \sum_{i=1}^{k} c(G_i) \leq \sum_{i=1}^{k} \lfloor (n_i-1)/2 \rfloor \leq \lfloor (n-1)/2 \rfloor .$$

Lemma 2: If $G \in \mathcal{H}$ then G contains at most one vertex of degree two.

Proof: Suppose $v, w \in V(G)$ where $d_G(v) = d_G(w) = 2$. As G is 2-connected (by lemma 1) there exists a cycle C containing both v and w (by Menger's theorem). Let G' be the graph formed by removing $E(C)$ and vertices v and w from G . By taking $\mathcal{C}(G) = C \cup \mathcal{C}(G')$ we see that $c(G) \le 1 + c(G') \le 1 + \lfloor(n-3)/2\rfloor = \lfloor(n-1)/2\rfloor$.

Lemma 3: If $G \in \mathcal{K}$ and $v \in V(G)$ with $d_G(v) = 2$ and $N_G(v) = \{a, b\}$ then $a \sim b$.

Proof: Suppose $a \nsim b$ in G . Form the graph G' by adding (a,b) to the graph induced in G by $V(G) \setminus \{v\}$. Then $E(G')$ has a cycle-partition $\mathcal{C}(G')$ into $t \le \lfloor(n-2)/2)\rfloor$ cycles. Let C be the cycle in $\mathcal{C}(G')$ which contains the edge (a,b). Replacing the edge (a,b) in C by the path $a - v - b$ we obtain a cycle-partition of $E(G)$ into $t \le \lfloor(n-2)/2\rfloor$ cycles.

Lemma 4: If $G \in \mathcal{K}$ and $v \in V(G)$ such that $d_G(v) = 4$, $N_G(v) = \{a, b, c, d\}$ and $a \sim b$ then $c \sim d$.

Proof: Suppose that $a \sim b$ and $c \nsim d$ in G . Let G' be the graph formed from G by deleting the edges (c,v) and (v,d) and adding the edge (c,d). Then $d_{G'}(v) = 2$, G' is Eulerian of order n where $|E(G')| < |E(G)|$ and so $E(G')$ has a cycle-partition $\mathcal{C}(G')$ into $t \le \lfloor(n-1)/2\rfloor$ cycles. Let C be the cycle in $\mathcal{C}(G')$ that contains the edge (c,d). Then C must also contain vertex v (and so the path $a - v - b$) else by replacing the edge (c,d) in C by the path $c - v - d$ we obtain a cycle-partition of $E(G)$ into $t \le \lfloor(n-1)/2\rfloor$ cycles. Let D be the cycle in $\mathcal{C}(G')$ that contains the edge (a,b). By replacing the edge (c,d) by the path $c - v - d$, and the path $a - v - b$ by the edge (a,b) in C ; and by replacing the edge (a,b) by the path $a - v - b$ in D we obtain a cycle-partition of $E(G)$ into $t \le \lfloor(n-1)/2\rfloor$ cycles.

Lemma 5: If $\Delta = 4$ and $G \in \mathcal{K}$ then any edge lies on at most one triangle in G .

Proof: Suppose the edge (v,w) be on triangles $\{a,v,w\}$ and $\{b,v,w\}$ in G. If $N_G(v) = \{w,a,b,c\}$ then, by lemma 4, $c \sim a$ and $c \sim b$ as $w \sim b$ and $w \sim a$: If $N_G(w) = \{v,a,b,d\}$ then, by lemma 4, $d \sim a$ and $d \sim b$ as $v \sim b$ and $v \sim a$.

Thus, either $c = d$, in which case $a \sim b$, as $w \sim c$, and so $G = K_5$; or $c \neq d$, in which case $c \sim d$ as $v \sim w$ and $N_G(b) = \{c,v,w,d\}$, and so $G = T_6$.

Proof of Theorem 1: If $\Delta = 2$ then, as all cycles have length ≥ 3, the cycle partition of G contains at most $\lfloor n/3 \rfloor \leq \lfloor (n-1)/2 \rfloor$ cycles. So assume $\Delta = 4$ and $G \in \mathcal{H}$.

Suppose there exists a triangle $\{u,v,w\}$ in G. By Lemma 2 we may assume w has degree 4 and, by lemma 1, there exists a path from w to u or v in $G \setminus \{(u,w), (v,w)\}$. Let P be a shortest such path and assume it ends at v (so that u does not lie on P and v has degree 4)

Let $N_G(v) = \{u,w,a,b\}$ and $N_G(w) = \{u,v,c,d\}$ and suppose b and c lie on P (so that a and d do not lie on P, as P is a shortest such path). Let C be the cycle formed by adding the edge (v,w) to P. As $a \neq d$ and u is not adjacent to a or d (by lemma 5), we can form an Eulerian graph G' from G by replacing the paths $a - v - u$ and $d - w - u$ by the edges (a,u) and (d,u), and by removing the cycle C and the now isolated vertices v and w. Then there exists a cycle partition $\mathcal{C}(G')$ of G' of order $t \leq \lfloor (n-3)/2 \rfloor$. But, by replacing the edges (a,u) and (d,u) by the paths $a - v - u$ and $d - w - u$, and by adding the cycle C, we get a cycle partition $\mathcal{C}(G)$ of G of order $1+t \leq \lfloor (n-1)/2 \rfloor$.

Thus G has girth $g \geq 4$ and so, by lemma 3, is 4-regular. But then G has order $n \geq 8$: Dirac in [4] showed that if H is a simple 2-connected graph with minimum degree δ and order $n \geq 2\delta$, then H contains a cycle of length at least 2δ, and so G contains a cycle of length $\ell \geq 8$. Thus $c(G) \leq 1 + \lfloor (2n-\ell)/g \rfloor \leq 1 + \lfloor (2n-8)/4) \rfloor \leq \lfloor (n-2)/2 \rfloor$.

Proof of Theorem 2:

Case 1: $n = 2k$, $k \geq 2$. First we construct the graph G_{k-1}. Let C_1 be a 4-cycle and $C_2, C_3, \ldots, C_{k-1}$ be $(k-2)$ 3-cycles. Arrange the cycles in sequence $C_1, C_2, \ldots, C_{k-1}$ and for each $2 \leq j \leq k-2$ identify one vertex, say v_{j-1}, of C_j with a vertex of C_{j-1} and another vertex v_j of C_j with a vertex of C_{j+1}, thus creating a 1-connected graph with $(k-2)$ cut-vertices each of degree 4 and $(k+2)$ vertices each of degree 2. Thus $c(G_{k-1}) = \sum_{j=1}^{k-1} c(C_j) = k-1$. For $1 \leq i \leq k-2$ we obtain the graphs G_i from G_{i+1} by removing the edges of the cycle C_{i+1} and the two resulting isolated vertices, and by performing 2 elementary subdivisions on an edge of C_1, thus transforming the originally 4-cycle C_1 into a $(2(k-i+1))$-cycle. Clearly $|V(G_i)| = 2k$, $\Delta(G_i) \leq 4$ and $c(G_i) = i$.

Case 2: $n = 2k+1$, $k \geq 1$. Take k 3-cycles and proceed as in case 1.

Remark: It would be interesting to know whether the bound $c \leq \lfloor (n-1)/2 \rfloor$ is sharp for 4-regular graphs: We are aware of only two 4-regular graphs, namely K_5 and T_6, with the property $c = \lfloor (n-1)/2 \rfloor$.

References:

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