PRIME POSSIBILITIES AND QUANTUM CHAOS

Andrew Granville

This is derived from a lecture given at the Museoin in New York City on February 6, 2002. This event was a fundraiser for M.S.R.I., and the intended audience for this talk was well-educated potential donors, who are not necessarily mathematically trained.

Introduction

I am a specialist in analytic number theory, and, in particular, the investigation of the distribution of prime numbers. This topic is one of the oldest in mathematics and has been intensively studied with the most modern methods of the time for one hundred and fifty years. Despite the age of the area, and the high quality of researchers, we still know depressingly little about many of the most fundamental questions. As one might expect in such an old and dignified field, progress in recent decades has been slow, and since the subject has been so well studied, even the seemingly most minor advances require deep, tough ideas and often great technical virtuosity.

But recently there has been extraordinary progress in our understanding from an unexpected direction. The ideas come from an area that seemed to have absolutely nothing to do with prime numbers—the mathematics of quantum physics.

I am a number theorist, untrained in physics; indeed, my undergraduate course in quantum physics left me more puzzled than enlightened. Due to the recent breakthrough in my own subject, I have had to go back and try to get a basic feel for the key developments in quantum physics. At the Museion I shared my limited understanding with the audience, discussing the disturbing consequences of quantum mechanics and the origins of the famous quote

"God does not play dice with the universe." — Albert Einstein

For a good layman's introduction, see *The Ghost in the Atom* [5]). Let me begin this article, though, with a subject I am much more at home in:

Primes: What are they and why do we need them?

This year, 2002, factors into primes as $2 \times 7 \times 11 \times 13$. Next year has a different factorization, and indeed, each whole number has its own way of being broken down into primes. "So what?" you might ask; but everyone uses the fact that factoring large numbers is difficult, in their everyday life ... Have you ever bought something on the web and been deluged by little screens that go on and on about "RSA cryptosystems"? That's number

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theory at work—what you input is held secure from unauthorized prying by the difficulty of factoring large numbers!

So primes are indeed worthy of careful study. Moreover, just as atoms are the building blocks of nature, so primes are the building blocks of numbers; therefore to study the theory of numbers, we must get a grip on primes. Next, I want to move on to a fundamental question in understanding primes.

How many primes are there? Up to a million? Up to a trillion? Up to any given point?

"I pondered this problem as a boy, in 1792 or 1793, and found that the density of primes around t is $1/\log t$, so that the number of primes up to a given bound x is approximately $\int_2^x dt/\log t$." — C.F.Gauss (1849)

Gauss, who was just 15 years old at the time of this finding, made his prediction by studying tables of primes up to three million. An extraordinary guess, as it turns out, which is amazingly accurate.

x	# of primes up to x	Overcount in Gauss's guess
10^8	5761455	754
10^9	50847534	1701
10^{10}	455052511	3104
10^{11}	4118054813	11588
10^{12}	37607912018	38263
10^{13}	346065536839	108971
10^{14}	3204941750802	314890
10^{15}	29844570422669	1052619
10^{16}	279238341033925	3214632
10^{17}	2623557157654233	7956589
10^{18}	24739954287740860	21949555
10^{19}	234057667276344607	99877775
10^{20}	2220819602560918840	223744644

Figure 1. Primes up to various x, and Gauss's prediction

Can we predict the size of the error in Gauss's guess? After a brief perusal of the table above we see that the error term is about half the length of the number of primes; that is, about the square root. In other words, we might guess that

$$\int_2^x \frac{dt}{\log t} - \#\text{ of primes up to } x$$

is bounded above by some function like $\sqrt{x}$. Another surprising feature of this data is that the error term is always positive, indicating that, at least in the data computed to date, Gauss's prediction is too large. This might lead us to think that we can introduce a secondary term which will give an even more accurate prediction, but this is not the case: The error term changes sign infinitely often, as was shown by Littlewood in 1914.

Trying to find out when the error term first goes negative is not easy. The first bound on the first such x (call it x_0) was given in 1933 by Skewes,

$$x_0 < 10^{10^{10^{34}}}.$$

For a long time this was recorded, in several places, as the largest number to have any significant meaning. This bound has been gradually whittled down to $x_0 < 1.39822 \times 10^{316}$, and persuasive arguments are given in [1] that this is indeed about the correct value of x_0! (See my forthcoming article [8] for more details.)

Gauss's statement can easily be modified to provide a probabilistic model for the primes (as was done in 1936 by Cramér [4]): let $X_3, X_4, \dots$ be a sequence of independent random variables with

$$\mathrm{Prob}(X_n = 1) = \frac{1}{\log n} \quad \text{and} \quad \mathrm{Prob}(X_n = 0) = 1 - \frac{1}{\log n}.$$

The sequence $\pi_3, \pi_4, \dots$, etc., where $\pi_n = 1$ if and only if n is prime, might be supposed to be a "typical" element of this probability space; and if a statement can be made about this space with probability 1, then it might be expected to be true of the primes (that is, for the sequence $\pi_3, \pi_4, \dots$). Certainly

$$\sum_{n \le x} X_n \sim \int_2^x \frac{dt}{\log t} \text{ as } x \to \infty \text{ with probability 1;}$$

that is, the expected value for the count of primes as given by the Gauss–Cramér model conforms with reality. Moreover in short intervals (which is more what Gauss was looking at)

$$\sum_{x < n \le x+y} X_n \sim \int_x^{x+y} \frac{dt}{\log t} \text{ as } x \to \infty \text{ with probability 1}$$

when y is a fixed small power of x. The second statistic people usually like to look at is the variance:

$$\text{mean} \left| \sum_{x < n \le x+y} X_n - \int_x^{x+y} \frac{dt}{\log t} \right|^2 .$$

Here we have a big surprise; it can be proved that the value predicted by the Gauss–Cramér model cannot be the variance for the counted primes! After Gauss's model worked so well before, the breakdown of the model for this question is quite unexpected, as noted by one famous number theorist:

> *"God may not play dice with the universe, but something strange is going on with the primes."* — Paul Erdős

Gauss's prediction is just that, it's not a proof of anything, and one would like a proof, after all. It turned out to be very difficult to find a method that would give a prediction for primes in a way that can be proved. When a method did come, it came from a quite unexpected direction.

Primes and music

We start with a seemingly unrelated topic—how does one transmit signals that are not "waves"? We've all heard the words "radiowaves" and "soundwaves," and indeed sound is transmitted in wave form, but the sound one makes doesn't seem very "wavy" to me; instead it sounds fractured, broken up, stopping and starting. How does that get converted into waves? As an example, we'll look at a gradually ascending line:

Fig. 2. The line $y = x - \frac{1}{2}$

If we approximate it with a wave, the closest we can come is something like:

Fig. 3. The wave $y = -\frac{1}{\pi}\sin 2\pi x$

The middle part is a good approximation to a straight line, but the approximation for $x < 1/4$, and $x > 3/4$ is poor. How can we fix that? The idea is to "add" a second wave to the first, this second wave going through two complete cycles in our interval rather than just one. By adding such a wave to that in Figure 3, we get an improved approximation. We can proceed like this, adding more and more waves, getting increasingly better approximations to the original straight line:

Fig. 4. The sum of one hundred carefully chosen sine waves

This is a good approximation to the original, though one can see, at the end points, that the approximation is not quite so good (this is an annoying and persistent problem known as the "Gibbs phenomena").

As one might guess from the pictures above, the more waves one allows, the better approximation one gets. For transmitting sound, maybe 100 sine waves will do; for data transfer, perhaps more. However, to get "perfect" transmission one would need infinitely many sine waves, which one gets by using the following formula:

$$x - \frac{1}{2} = -2 \sum_{n=1,2,3\ldots} \frac{\sin(2\pi n x)}{2\pi n} \qquad \text{for } 0 \leq x \leq 1.$$

Not of practical use (since we can't, in practice, add up infinitely many terms), but a gorgeous formula!

Riemann's revolutionary formula

The great geometer Riemann only wrote one paper that could be called number theory, but that one short memoir has had an impact lasting 140 years, and its ideas today define the subject we call analytic number theory. In our terms, Riemann's idea is simple, albeit rather surprising: *Try counting the primes as a sum of waves.* His precise formula is a bit too technical for this talk, but we can get a good sense of it from the following approximation:

$$(\dagger) \qquad \frac{\#\text{ of primes up to } x \; - \int_2^x \frac{dt}{\log t}}{\sqrt{x}} \approx -2 \sum_{\substack{\frac{1}{2}+i\gamma \text{ is a} \\ \text{zero of } \zeta(s)}} \frac{\sin(\gamma \log x)}{\gamma}$$

Notice that the left side of this formula is suggested by Gauss's guess: it is the error term when comparing Gauss's guess to the actual count for the number of primes up to x, divided by what appeared from our data to be about the actual size of the error term, namely $\sqrt{x}$.

The right side of the formula bears much in common with our formula for $x - 1/2$. It is a sum of sine functions with γ employed in two different ways in place of $2\pi n$: namely, inside the sine (the reciprocal of the "wavelength") and dividing into the sine (the reciprocal of the "amplitude"). We also get the "-2" factor in both formulae. However, the definition of the γ's here is much more subtle than just $2\pi n$ and needs some explanation:

The Riemann zeta-function $\zeta(s)$ is defined as

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$$

for complex number $s = \sigma + it$. The series only converges for sure when $\sigma > 1$; and it is not clear at first sight whether or not this is a genuine limitation on where we can define such a function. In fact, the beautiful theory of "analytic continuation" tells us that often there is a sensible definition of a function on all $s \in \mathbb{C}$ provided there is in part of $\mathbb{C}$; and this does apply to $\zeta(s)$. In other words $\zeta(s)$ can be defined in the whole complex plane (see [17] for details).

We are going to be interested in the "zeros of $\zeta(s)$"; that is, the values of s for which $\zeta(s)=0$. One can show

$$\zeta(-2)=\zeta(-4)=\zeta(-6)=\cdots=0$$

(which are called the "trivial zeros"), and that all other $s=\sigma+it$ with $\zeta(\sigma+it)=0$ satisfy $0\le\sigma\le 1$.

In Riemann's memoir he stated a remarkable hypothesis:

$$\text{If } \ \zeta(\sigma+i\gamma)=0 \text{ with } 0\le\sigma\le 1 \text{ then } \sigma=\frac{1}{2};$$

that is, the non-trivial zeros all lie on the line $Re(s)=1/2$. This leads to the definition of the γ's in our formula; these are the values of γ for which $\zeta(\frac{1}{2}+i\gamma)=0$. It has been proven that there are infinitely many such zeros, so you might ask how we add up this infinite sum? Simple, add up by order of ascending $|\gamma|$ values and it will work out.

The above formula, (†), holds if, and only if, Riemann's hypothesis holds. If it doesn't hold then there is a similar formula, but it is rather complicated and technically far less pleasant, since the coefficients $1/\gamma$, which are constants, get replaced by functions of x.

So we want Riemann's hypothesis to hold because it gives the formula above, and that formula is a delight. One of the great prime number specialists of our time notes :

"*That the distribution of primes can be so accurately represented in* [this way] *is absolutely amazing and incredibly beautiful. It tells of an arcane music and a secret harmony composed by the prime numbers*" — Enrico Bombieri (1992)

That is, it is like the formula for breaking sound down into sine waves. Thus, one can paraphrase the Riemann Hypothesis as:

"*The primes have music in them.*"

The Riemann Hypothesis—the evidence.

The Riemann Hypothesis.

$$\textit{All zeros of } \zeta(s) \textit{ with } 0\le Re(s)\le 1 \textit{ satisfy } Re(s)=\frac{1}{2}$$

Riemann's memoir (1859) did not contain any hints as to how he made this remarkable conjecture[1], and for many years this conjecture was held up as evidence of the heights one could attain by sheer intellect alone. It seemed as if Riemann had come to this very numerical prediction on the basis of some profound undisclosed intuition, rather than pedestrian calculation — the ultimate conclusion of the power of pure thought alone.

In 1929, many years after Riemann's death, the prominent number theorist Siegel heard that Riemann's widow had donated his scratch paper to the Göttingen University library.

[1]Riemann wrote:"It is very probable that all [Re(s)=1/2]. Certainly one could wish for a stricter proof here; I have temporarily set aside the search after some fleeting futile attempts."

It was quite an undertaking, deciphering Riemann's old notes, but Siegel uncovered several jewels. First he found a tremendously useful formula not quite fully developed by Riemann (so not included in his published memoir) which he now brought to flower (though it took him three years to produce a proof despite having the formula in front of him). Second, Siegel discovered pages of substantial calculations, including several of the lowest zeros calculated to several decimal places. So much for "pure thought alone".

There is a long history of computing zeros of $\zeta(s)$, and the very question is synonymous with several great events in the history of science. When the earliest computers were up and running, what was one of the first tasks set for them? Computing zeros of the Riemann zeta-function[2]. When the Clay Math Institute established seven million dollar prizes for problems to be solved in the new millenia, the Riemann Hypothesis headed the list (though, rest assured, there are far easier ways of earning a million dollars). By November of last year, the lowest ten billion zeros had been computed (by Stephan Wedeniwski of IBM Deutschland) and every last one of them lies on the 1/2-line (that is, is of the form $1/2+i\gamma$). This seems to be pretty good evidence for the truth of the Riemann hypothesis, but who knows? Perhaps the ten billion and first zero does not lie on the half line. Am I being too cautious? Maybe, but maybe not . . . remember Gauss's prediction for the count of primes doesn't get smaller than the actual count until we get out beyond 10^{316}, which is a lot further out than 10^{10} (ten billion).

My own view is that Riemann's formula, as discussed above, is far too beautiful not to be true; yes, I believe the primes have music in them.

The standard deviation for the count of primes—A new beginning.

In her 1976 Ph.D. thesis Julia Mueller, following up on a suggestion of her advisor, Pat Gallagher, revisited the old question of standard deviation[3] for the count of the number of primes (compared to Gauss's prediction). Remembering that the Gauss-Cramér model did not give a prediction that could possibly be correct, Mueller developed Riemann's approach to get a better idea, looking at the slightly more refined question of the distribution of primes in short intervals[4] around x (for example, between x and $x + x^{\delta}$ for values of δ between 0 and 1), and established an important link. Building on her work, Goldston and Montgomery made the remarkable discovery that a good understanding for the standard deviation is *equivalent* to a proper understanding of the spacing between pairs of zeros of $\zeta(s)$.

Riemann showed that understanding the count of primes is equivalent to knowledge of the zeros of $\zeta(s)$; and that the count is predictable from a beautiful natural formula if all the non-trivial zeros lie on the half-line. These new ideas went one big step further if the Riemann Hypothesis is true. A basic understanding of the size of the variation in the count of primes can be obtained by looking at pairs of zeros and the distance between them.

Assuming the Riemann Hypothesis, the zeros are of the form $\frac{1}{2} \pm i\gamma_1, \frac{1}{2} \pm i\gamma_2, \ldots$ with

$$0 \le \gamma_1 \le \gamma_2 \le \gamma_3 \le \ldots$$

[2]This computation, done on the Manchester University Mark 1 Computer, was Alan Turing's last publication.

[3]The standard deviation is the square root of the variance, so understanding one is tantamount to understanding the other.

[4]This is, in fact, closer to Gauss's original assertion.

up to height T (that is, the γ_n with $0 \le \gamma_n \le T$). We might ask how those γ_j are distributed on the line segment $[0, T]$. Do they look like randomly selected numbers on that interval? Or do they seem to adhere to some other pattern? In the diagram below we compare the data for some zeros of $\zeta(s)$ with other phenomena in mathematics and physics.

Fig. 5. Set of points from various distributions

It's not hard to see that the data for the γ_j's doesn't much look like randomly chosen numbers (a Poisson process). Indeed, in the distribution for randomly chosen values we see that the points do occasionally clump together (making them more-or-less indistinguishable), whereas the γ_j's do not seem to clump together anywhere like as often, and seem to be better spread out than random. If anything the γ_j's seem almost to repel one another.

Just a couple of years earlier, motivated by an entirely different question in number theory, Montgomery had wanted to understand this distribution, and so made a precise conjecture for understanding gaps between zeros.

Montgomery's Conjecture (1973). *The expected number of zeros in a gap of length T times the average gap, following a zero, is:*

$$\int_0^T \left\{ 1 - \left(\frac{\sin(\pi u)}{\pi u} \right)^2 \right\} du$$

If the zeros were like a random distribution then this value would simply be T; in fact, a careful examination of this conjecture reveals that it does affirm the repulsion of zeros we observed in the limited data above. For example, we expect a zero within $1/100$ of a given zero, 1 in 100 times for a randomly chosen zero, but only 1 in 911963 times for zeta-function zeros if Montgomery's prediction holds true.

Let us see how Montgomery's prediction compares with data collected by Andrew Odlyzko over the last fifteen years. In the graph below we are actually measuring "nearest neighbor spacings" that is the distribution of $(\gamma_{n+1} - \gamma_n)$/average spacing. The continuous line is Montgomery's prediction; the dots represent data collected by Odlyzko (a scatterplot).

Fig. 6. Data based on 79 million zeros at height 10^{20}

What a good fit! One surely believes Montgomery's prediction. In fact, Montgomery even proved his prediction is true in part (technically he showed that the "Fourier transform" of his distribution function is correct in a small range if the Riemann Hypothesis is true).

Quantum Mechanics enters the picture.

Scientific Progress can happen in strange ways. Sometimes it seems like the same revolutionary idea occured to two people at the same time, though they have not been in contact, without any obvious recent changes in the landscape of the subject. So why simultaneously? Chance meetings sometimes stimulate great new ideas, and so it was in our subject. Soon after developing his new outlook on the count of zeros, Montgomery passed through Princeton wanting, in particular, to discuss his idea with the two great analytic number theorists, Selberg and Bombieri, both at the Institute for Advanced Study. The Institute has a communal tea each working day, where people from different disciplines can and do socialize and discuss ideas of mutual interest. Freeman Dyson, the great mathematical physicist (though originally a number theorist), was at tea, and Montgomery explained to him what he was up to. Montgomery was taken aback to discover that Dyson knew very well the rather complicated function appearing in Montgomery's conjecture, and even knew it in the context of comparing gaps between points with the average gap. However — here's the amazing thing — it wasn't from number theory that Dyson knew this function but from quantum mechanics. It is precisely the function that Dyson himself had found a decade earlier when modelling energy levels in complex dynamical systems when taking a quantum physics viewpoint. It is now believed that the same statistics describe the energy levels of chaotic systems; in other words, quantum chaos!

Could this have been a coincidence? Surely not. Surely it was an indication of something lying much deeper? The questions beg answers and provide the starting point for much of the recent progress.

Mathematically speaking, the equations of quantum chaos are relatively simple to develop compared to those of prime number theory, and so much more was (and is) known about them. The Montgomery-Dyson observation cries out for mathematicians to develop further formulae for zeros of the Riemann zeta-function and to compare them with those of quantum chaos. The first things to look at were the models physicists had developed for comparing close-by zeros, not just two at a time, but also three at a time, four at a time, or even n at a time (the so-called "n-level correlations").

Although these led to obvious predictions for the zeros of $\zeta(s)$, showing these predictions to be to some extent correct was a major barrier, attempted by many but frustratingly difficult to accomplish ... It was more than twenty years until Rudnick and Sarnak in 1996 made the breakthrough and proved the analogy of Montgomery's result (assuming the Riemann Hypothesis, the Fourier transform of the predicted n-level correlation function is correct in a small range; in fact, the range directly analogous to Montgomery's range). Now number theorists had to believe that at least some of the predictions that could be made by analogy to quantum chaos had to be correct, and a flood of research ensued.

Also in 1996, two mathematical physicists, Bogolmony and Keating, re-derived the Montgomery-Dyson prediction (for n-level correlations without any restriction on the range) from a new angle (which was anticipated for the $n = 2$ case by [12]). They took a classic conjecture of analytic number theory, the Hardy-Littlewood version of the prime k-tuplets conjecture, and showed that this also led to the same conclusion. Now there was no room for doubt — these predictions have to be correct!

Mathematicians at play: The “Sarnak School”.

The Montgomery-Dyson predictions tell us that zeros of the Riemann zeta function “behave” much like numbers predicted in certain questions in quantum chaos. Although different chaotic systems have different quantum energy levels, one remarkable observation is that the energy levels that arise are distributed in one of only a handful of ways.

There are many different types of “zeta functions” that appear in number theory; not only in counting primes, but also in a fundamental way in algebraic, arithmetic, and analytical problems. For example, Wiles’s proof of Fermat’s Last Theorem is all about a certain kind of zeta function. All of these zeta functions share various properties with the original one: they have some easily identifiable “trivial” zeros; otherwise all other zeros lie in a “critical strip” (like $0 \leq Re(s) \leq 1$). To name one more property, the most important, we believe that all of their non-trivial zeros lie on some critical line (like $Re(s) = 1/2$), a “Riemann Hypothesis.” Sarnak became intrigued with determining whether the spacing between the zeros of other zeta functions were also predictable by these same handful of distributions from quantum chaos.

Together with Rubinstein they did large scale calculations and found excellent experimental agreement between the distributions of zeros of various zeta functions and the energy levels of various quantum chaotic systems. Then Katz and Sarnak thought to experiment with other, rather different, data of interest to number theorists; for example, how about the lowest zero for each zeta function? Should that be distributed according to one of these magical distributions? Experimental data implied a quantum chaotic model for this question and several others (see [9]). Finally they looked at analogies of the zeta functions that appear in algebraic geometry, a field far removed from quantum chaos. These have finitely many zeros, and the appropriate analogy to the Riemann Hypothesis is true for them (some optimists feel the proof(s) of this might point the way to a proof of the real Riemann Hypothesis; many of us have our doubts). Katz and Sarnak reasoned that since the Riemann Hypothesis is known to be true for these zeta functions (due to Deligne), perhaps they could go one step forward and actually prove the analogy to Montgomery’s pair correlation conjecture, or even the Montgomery-Dyson predictions.

In one of the most remarkable works of recent number theory, Katz and Sarnak did what they set out to do [10], using the results of Deligne in a perhaps unexpected and highly ingenious manner. Their four hundred page book is a landmark achievement: motivated by dubious forecasts from quantum chaos they proved a deep and profound result for zeta functions in an entirely unrelated field—lovely!

Physicists at play: The “Berry School”.

Just as a new generation of number theorists, led by Peter Sarnak, have learned to exploit these connections in new, exciting ways, so the next generation of mathematical physicists, led by Sir Michael Berry and his collaborators at the University of Bristol, have been taking new and more aggressive approaches to developing analogies between the two fields. Their basic attitude has been to go out on a limb, stretching analogies in a way that mathematicians would never dare. It's a quite different attitude, one that I find very appealing and a little shocking. They look at equations that they know can't really be formally justified and yet glean much useful information nonetheless.

The key developments come in a series of papers by Berry and Jon Keating and contain perhaps a road map for the future of the study of primes. Some of what they say may not be quite correct, but I'm sure it is close to the truth, and in several problems they make predictions where we number theorists had no idea how to proceed.

The ideas don't flow in just one direction either. The more cautious development in prime number theory allows for several rather precise formulae (such as the Riemann-Siegel formula alluded to earlier), which have helped to correct and modify less pedantically obtained (though analogous) formulae of quantum chaos.

The latest generation of mathematical physicists, led by Jon Keating and Nina Snaith are going one step further, and perhaps most usefully. They are targeting some of the biggest mysteries in the study of the Riemann zeta function; for instance, what's the largest it gets on a large interval of the half line? Proceeding with great care, they are making predictions about important problems on which prime number experts had had no idea how to proceed.

In summary, the more intuitive development of quantum chaos allows more fruitful predictions about the distribution of primes (and beyond). On the other hand the more cautious development of prime number theory leads to more accurate predictions in quantum chaos. This mutually beneficial interaction between two previously unrelated fields is an exciting new development and many researchers in both fields are now turning their attention to such questions. This is exactly what an institution like MSRI is perfect for: the time being ripe, we have the opportunity to bring these two communities together, physically, which would not be possible otherwise, to accelerate these developments.

Much ado about nothing?

Put aside all these developments for a moment, these generalizations, these exciting new formulae. What about the million dollar problem? Do any of these new ideas help us to get a better grip on the Riemann Hypothesis? Is there much chance now that this problem will finally succumb? Just a few years ago, one of the great analytic number theorists said:

> "*There have been very few attempts at proving the Riemann Hypothesis because nobody has had a really good idea about how to do it*" — Atle Selberg (1995)

An old idea of Hilbert and Polya for proving the Riemann Hypothesis is to find a quantum chaotic system in which every zero of the Riemann zeta function corresponds to an energy level of the system (they put this is in a somewhat different language). If such a quantum chaotic system exists, it must have several very special properties. Recently Berry and Keating gave even more restrictions on such a system (associating the primes to the periodic orcits of such a chaotic system), arguably pointing the way to finding it. This perhaps provoked our bold knight to say:

"*I have a feeling that the Riemann Hypothesis will be cracked in the next few years.*

I see the strands coming together." — Sir Michael Berry (2000)

It could be that Berry is correct though I suspect that the proof is still a long way off. Nonetheless these new findings are the most exciting in many years and promise, at the very least, a much better understanding of the Riemann zeta function.

Acknowledgements. *With thanks to Malcolm Adams, Bob Anderson, Steve Donnelly, Cari Gervin, Dan Goldston, Will Kazez, Jon Keating, Gene and Rebecca McCarthy and K. Soundararajan for their help in the preparation of the talk and article.*

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