

Oscillation Theorems for Primes in Arithmetic Progressions and for Sifting Functions

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I. Introduction

1. Primes in Arithmetic Progressions

For $x \geq 2$ real, q a positive integer, and a an integer coprime to q , let θ and Δ be defined by

$$(1.1) \quad \theta(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log p = \frac{x}{\phi(q)}(1 + \Delta(x; q, a)),$$

where ϕ is Euler's function. The prime number theorem for arithmetic progressions is equivalent to the statement that $\Delta(x; q, a) = o(1)$ holds, as $x \rightarrow \infty$, for fixed q and a . The function Δ thus provides a measure for the regularity of distribution of primes in arithmetic progressions.

The main focus in the study of primes in arithmetic progressions has so far been on estimating $\Delta(x; q, a)$ from above uniformly for all, or almost all, $q \leq Q$, with Q as large as possible. The two fundamental theorems in this connection (cf. [Da]) are the Siegel-Walfisz theorem and the Bombieri-Vinogradov theorem. It follows from the former that the bound

$$(1.2) \quad |\Delta(x; q, a)| \ll (\log x)^{-A}$$

holds uniformly for $q \leq (\log x)^B$, for any fixed positive constants A and B . The latter states, in one form, that

$$(1.3) \quad \sum_{q \leq Q} \frac{1}{\phi(q)} \max_{(a, q)=1} |\Delta(x; q, a)| \ll (\log x)^{-A}$$

holds for any $A > 0$ with $Q = \sqrt{x}(\log x)^{-B}$, where B is a suitable constant depending on A . As a consequence, the estimate (1.2) is true for all but $o(Q)$ moduli $q \leq Q$, provided $Q \leq \sqrt{x}(\log x)^{-C}$ with a sufficiently large constant C .

It is generally expected that these results are far from being best-possible. In fact, under the Generalized Riemann Hypothesis, (1.2) is known to hold for all $q \leq \sqrt{x}(\log x)^{-B}$ and all a coprime with q , if B is a sufficiently large constant. However, even this conditional result is probably not optimal, and stronger conjectures about the size of $\Delta(x; q, a)$ and the range of validity of estimates like (1.2) or (1.3) have been put forward.

Perhaps the best known of these conjectures is the Elliott-Halberstam conjecture [E-H] according to which the Bombieri-Vinogradov estimate is valid for $Q = x^{1-\varepsilon}$ and possibly even $Q = x(\log x)^{-B}$ with a sufficiently large constant B ; see [E-H, p. 62] for the precise formulation. This conjecture was at least in part motivated by probabilistic considerations; in fact, Elliott [E 1] had shown earlier that, in an appropriate sense, "almost all" primelike sequences of positive integers satisfy a Bombieri-Vinogradov type estimate (1.3) with $Q = x(\log x)^{-B}$, $B = B(A)$.

A corresponding conjecture for the estimate of individual terms $\Delta(x; q, a)$ has been made by Montgomery [Mo]. It implies that

$$(1.4) \quad |\Delta(x; q, a)| \ll_{\varepsilon} (q/x)^{1/2-\varepsilon} \log x$$

holds uniformly for $q \leq x$, for any given $\varepsilon > 0$. A weaker form of (1.4) may be given as

$$(1.5) \quad |\Delta(x; q, a)| \ll (q/x)^{1/2} x^{\varepsilon}.$$

The latter estimate is only of interest for $q \leq x^{1-\varepsilon}$, while (1.4) remains non-trivial for moduli as large as x divided by a fixed power of $\log x$.

The strongest forms of both of these conjectures were recently disproved by Friedlander and Granville [F-G]. They showed that for any $A > 0$ there exist arbitrarily large values of x and integers $q \leq x(\log x)^{-A}$ and $(a, q) = 1$, for which $|\Delta(x; q, a)| \gg 1$, thus disproving Montgomery's conjecture (1.4). Moreover, such lower bounds hold for most moduli q , so that (1.2) even fails when averaged over a suitable range for q . As a consequence, the Bombieri-Vinogradov type estimate (1.3) cannot be valid with Q as large as x divided by any fixed power of $\log x$.

The main purpose of the present paper is to extend these results by showing that the mentioned conjectures fail already for moduli that are significantly smaller than x divided by any fixed power of $\log x$. Specifically, we shall prove that (1.4) fails to hold for all moduli q as small as $x \exp(-(\log x)^{1/5-\delta})$, for most moduli q as small as $x \exp(-(\log x)^{1/3-\delta})$, and for many moduli q as small as $x \exp(-(\log x)^{1/2-\delta})$, for any fixed $\delta > 0$, if the parameter ε in (1.4) is sufficiently small. We shall also show that, given any positive constant A , the estimate (1.3) is false for all sufficiently large x with $Q = x \exp\{-(1-\varepsilon)A(\log_2 x)^2/\log_3 x\}$, where here, as in the rest of the paper, $\log_k x$ denotes the k times iterated logarithm.

We shall state our main results in three theorems, Theorems A1 to A3 below, which we shall prove by rather different methods. The first method is closest in spirit to the argument of Friedlander and Granville [F-G]. It has its roots in a paper of Maier [Ma], in which he showed that the asymptotic formula $\pi(x+y) - \pi(x) \sim y/\log x$ fails to hold when $y = (\log x)^\lambda$, for any fixed positive λ . The second method is a modification of the first, while the third is new and quite different from the other two approaches.

Common to all three methods is that they ultimately depend on oscillation results for certain sifting functions, a typical example being the well-known function

$$\Phi(y, z) = \#\{n \leq y : p|n \Rightarrow p \geq z\}.$$

Such oscillation results turn out to be quite difficult to prove. While it is relatively easy to obtain upper bounds for the oscillations of functions like $\Phi(y, z)$ thereby showing that these functions behave, to some extent, in a regular manner, it is much harder to exhibit irregularities of such functions. In Chapter II we shall prove three such oscillation theorems, Theorems B1 to B3, which will then, in Chapter III, be applied to prove Theorems A1 to A3, respectively. Two of these theorems depend on a further and more fundamental result, Theorem C, which gives an "asymptotic formula" for the amount of oscillations of the function $\Phi(y, z)$. This theorem is quite difficult to prove and will take up the bulk of Chapter II.

2. The Main Theorems

We shall consider here only the error function $\Delta(x; q, a)$ defined in (1.1). It is perhaps more customary to deal with the functions $\tilde{\Delta}, \bar{\Delta}$ defined by

$$\pi(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} 1 = \frac{\text{li } x}{\phi(q)} (1 + \tilde{\Delta}(x; q, a)),$$

and

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n) = \frac{x}{\phi(q)} (1 + \bar{\Delta}(x; q, a)).$$

In fact, although we shall find Δ slightly simpler to work with, similar results hold for $\tilde{\Delta}, \bar{\Delta}$.

Our first theorem gives lower bounds for the oscillations of the function $\Delta(x; q, a)$ in the case of moduli q that do not have too many small prime factors.

Theorem A1. *Let $\varepsilon > 0$. Then for all $q \geq q_0(\varepsilon)$ and all x satisfying*

$$(2.1) \quad \sum_{\substack{p|q \\ p < \log q}} 1 < \frac{3}{2} \log_3 q,$$

$$(2.2) \quad q (\log q)^{1+\varepsilon} < x \leq q \exp((\log q)^{5/11-\varepsilon})$$

there exist numbers x_{\pm} with $x/2 < x_{\pm} \leq 2x$ and integers a_{\pm} coprime with q such that

$$(2.3) \quad \Delta(x_+; q, a_+) \geq y^{-(1+\varepsilon)\delta_1(x,y)},$$

$$(2.4) \quad \Delta(x_-; q, a_-) \leq -y^{-(1+\varepsilon)\delta_1(x,y)},$$

where $y = x/q$ and $\delta_1(x, y) = \log(\log y / \log_2 x) / \log(\log x / \log y)$.

The right-hand side of (2.3) and (2.4) is $\gg 1$ if $y = (\log x)^{O(1)}$, and is $\gg y^{-(1+\varepsilon)\alpha/(1-\alpha)}$ if $y = \exp((\log x)^\alpha)$ with $0 < \alpha \leq 1/3$. The theorem thus shows that Montgomery's conjecture (1.4) is incorrect for moduli q as small as $x \exp(-(\log x)^{1/3-\delta})$, for any given positive δ .

The lower bounds in (2.3) and (2.4) as well as the range (2.2) for x are essentially the limits of what can be achieved by our method of proof. In connection with the lower bound in (2.2) it should be pointed out that for $x < (1-\varepsilon)\phi(q)\log q$ large lower bounds for Δ follow trivially from the fact that there are more reduced classes modulo q than there are primes $\leq x$. As far as the upper bound is concerned, we note that the theorem is not of much interest in the range $q < x \exp(-(\log x)^{1/3+\varepsilon})$, since then $\delta_1(x, y) > 1/2$ and the lower bound in (2.3) and (2.4) thus becomes smaller than the bounds predicted by conjectures like (1.4). Under the assumption of the Generalized Riemann Hypothesis the exponent $\delta_1(x, y)$ can be reduced to $\log(\log y / \log_2 x) / \log_2 x$.

Theorem A1 sharpens and extends to a much larger range the corresponding result of Friedlander and Granville [F-G], and represents the analogue for arithmetic progressions of a recent result of Hildebrand and Maier [H-M] on irregularities in the distribution of primes in short intervals. This latter result asserts that bounds of the type (2.3) and (2.4) hold with the same function δ_1 for all sufficiently large x and $2 \leq y \leq \exp(A(\log x)^{1/3})$, if $\Delta(x_{\pm}; q, a_{\pm})$ is replaced by $\Delta(x_{\pm}, y)$ for suitable x_{\pm} , $x \leq x_{\pm} \leq 2x$, where

$$\Delta(x, y) = \frac{1}{y} \sum_{x < p \leq x+y} \log p - 1$$

and A is a positive constant.

The proof of Theorem A1 combines the methods of [F-G] and [H-M]. The basic idea, which is essentially due to Maier [Ma], is to consider the occurrence of primes in a rectangular array

$$rP + qs$$

where r and s each run independently through the integers in certain intervals. Here q is the modulus of the progression in which we are interested. Thus, for each fixed r and varying s , we are considering the primes in a progression modulo q and trying to prove that this number is not always close to what we "expect". On the other hand, for each fixed s , and varying r , we are considering the primes in an arithmetic progression modulo P . Here P is chosen much smaller than q so that the number of such primes can be accurately estimated. This estimate, which rests on the Linnik-Gallagher prime number theorem, then reduces the problem to counting the number of s satisfying $(s, P)=1$. Now, if this number is more (or

less) than the expected number, then there will be correspondingly more (or less) primes than expected in some progression modulo q .

Thus one is led to study oscillations of sifting functions of the type $\#\{s \leq y : (s, P) = 1\}$. In the case P is of the form $P = \prod_{p < z} p$ these functions reduce to the well-known function $\Phi(y, z)$. Oscillation theorems for $\Phi(y, z)$ have been given in [Ma] (see also [Iw, p. 211]) for the range $\log y \ll \log z$, and in [H-M] for the range $\log y \ll z^{1/2-\varepsilon}$. However, these results would not be sufficient for the proof of Theorem A1, and we will therefore prove here a new oscillation result, Theorem B1 below, which is much sharper than the previous results.

The above argument requires that $(q, P) = 1$ and thus, when P is chosen as above as the product of all primes $< z$, applies only to moduli q having no prime factors $< z$. For more general q it is necessary to modify the choice of P , and hence the definition of $\Phi(y, z)$, to exclude the small prime factors of q . Provided q does not have too many small prime divisors, the modified function can be shown to exhibit essentially the same oscillatory behavior. This is the genesis of the condition (2.1). It is possible to deduce from the Turán-Kubilius inequality that almost all integers q have $(1+o(1)) \sum_{p < \log q} (1/p) = (1+o(1)) \log_3 q$ distinct prime factors less than $\log q$ and so almost all integers satisfy (2.1) (see, e. g., Chapter 4 in Elliott [E 2]).

The condition (2.1) could probably be relaxed to some extent, but in the above approach it cannot be completely removed without sacrificing the quality of the result. Our next theorem gives a result that is somewhat weaker than the estimates of Theorem A1, but has the advantage of being valid for all q . This result is obtained by taking P in the above argument to be a product of primes chosen from an appropriate short interval.

Theorem A2. *Let $\varepsilon > 0$. There exist $N(\varepsilon) > 0$ and $q_0 = q_0(\varepsilon) > 0$ such that for any $q > q_0$ and any x with*

$$(2.5) \quad q (\log q)^{N(\varepsilon)} < x \leq q \exp((\log q)^{1/3})$$

there exist numbers x_{\pm} with $x/2 < x_{\pm} \leq 2x$ and integers a_{\pm} coprime with q such that

$$(2.6) \quad \Delta(x_+; q, a_+) \geq \frac{1}{\log^5 x} y^{-(1+\varepsilon)\delta_2(x,y)},$$

$$(2.7) \quad \Delta(x_-; q, a_-) \leq -\frac{1}{\log^5 x} y^{-(1+\varepsilon)\delta_2(x,y)},$$

where $y = x/q$ and $\delta_2(x, y) = 3 \log(\log y / \log_2 x) / \log(\log x \log y)$.

The factor $1/\log^5 x$ is significant only if $\log y \ll (\log_2 x)^2$. If two-sided estimates are not required and only a lower bound for $|\Delta|$ is sought, then this factor may be replaced by $\log y / (\log_2 x)^2$; see the remarks after the proof.

As compared with Theorem A1 the exponent δ_2 is somewhat worse than δ_1 , being less than $1/2$ only for $y < \exp(\log^{1/5-\varepsilon} x)$. (Assuming the Generalized Riemann Hypothesis this too may be improved.) However, Theorem A2 has the advantage of being valid for all q and has the additional advantage that the required oscillation result, Theorem B2 (see Sections 3 and 4), is much easier to prove.

We next turn to lower bounds for averages over $\Delta(x; q, a)$. As almost all q satisfy the hypothesis of Theorem A1, we obtain immediately from Theorem A1 the following result.

Corollary 2.1. For all sufficiently large x and $2x \exp(-(\log x)^{5/11-\varepsilon}) \leq Q \leq x/2$ we have

$$(2.8) \quad \sum_{q \leq Q} \frac{1}{\phi(q)} \max_{x' \leq x} \max_{(a,q)=1} |\Delta(x'; q, a)| \gg y^{-(1+\varepsilon)\delta_1(x,y)},$$

where $y = x/Q$.

Theorem A2 yields a similar, but slightly weaker result. Stronger results may be obtained by bounding from below averages of the type

$$(2.9) \quad \sum_{\substack{q < Q \\ (q,a)=1}} \frac{1}{\phi(q)} |\Delta(x; q, a)| = \frac{1}{x} \sum_{\substack{q < Q \\ (q,a)=1}} \left| \theta(x; q, a) - \frac{x}{\phi(q)} \right|$$

or

$$(2.10) \quad \sum_{\substack{q \sim Q \\ (q,a)=1}} \frac{1}{\phi(q)} \Delta(x; q, a) = \frac{1}{x} \sum_{\substack{q \sim Q \\ (q,a)=1}} \left(\theta(x; q, a) - \frac{x}{\phi(q)} \right),$$

where a is a fixed non-zero integer and $q \sim Q$ means $Q < q \leq 2Q$. Such averages have been considered in recent work of Fouvry [Fo], and Bombieri, Friedlander, and Iwaniec [B-F-I], where non-trivial upper bounds have been given for (2.9) and (2.10) in ranges beyond the known range of validity of the Bombieri-Vinogradov theorem in its original form (1.3). Thus, it is of interest to obtain lower bounds for these quantities similar to the bound (2.8), and, of course, such bounds provide, a fortiori, a lower bound for the left-hand side of (2.8). Our third theorem gives such a result.

Theorem A3. Let $\varepsilon > 0$. Then for some $c > 0$, for all $x > x_0(\varepsilon)$ and all Q satisfying

$$(2.11) \quad x \exp(-c\sqrt{\log x}) < Q \leq x (\log x)^{-(1+\varepsilon)},$$

there exist Q_{\pm} with $Q/2 < Q_{\pm} \leq 2Q$ and integers a_{\pm} such that

$$(2.12) \quad \sum_{\substack{q \sim Q_+ \\ (q,a_+)=1}} \frac{1}{\phi(q)} \Delta(x; q, a_+) \geq \frac{1}{\log_2 x} y^{-(1+\varepsilon)\delta_3(x,y)},$$

$$(2.13) \quad \sum_{\substack{q \sim Q_- \\ (q,a_-)=1}} \frac{1}{\phi(q)} \Delta(x; q, a_-) \leq -\frac{1}{\log_2 x} y^{-(1+\varepsilon)\delta_3(x,y)},$$

where $y = x/Q$ and $\delta_3(x, y) = \log(\log y / \log_2 x) / \log_2 x$.

We remark that δ_3 is smaller, and hence better, than the exponents δ_1 and δ_2 from Theorems A1 and A2, and indeed is the same exponent achieved by the method of Theorem A1 on the assumption of the Generalized Riemann Hypothesis. The result remains of interest in the entire range $y < \exp((\log x)^{1/2-\varepsilon})$ and with more effort could be somewhat sharpened so that the bound remained larger than $y^{1/2}$ in the range $y < \exp(c\sqrt{\log x})$.

The factor $1/\log_2 x$ in (2.12) and (2.13) is significant only for $\log y \ll \log_2 x \log_3 x$. In fact, for this latter range it can also be shown that this factor may be dropped, if one uses the method that gives Theorem A1, provided that the sum $\sum_{\substack{q \sim Q \\ (q,a_{\pm})=1}}$ is replaced by the sum $\sum_{q \sim Q} \max_{(a,q)=1}$ (respectively $\sum_{q \sim Q} \min_{(a,q)=1}$).

Since $y^{\delta_3(x,y)} = (\log x)^{(1+o(1))A}$ if $\log y \sim A(\log_2 x)^2 / \log_3 x$ ($A > 0$ fixed), we deduce from Theorem A3 the following result, which sets a limit on the range of Q for which Bombieri-Vinogradov type estimates for averages of the form (2.9) may hold.

Corollary 2.2. *Let $\varepsilon > 0$, $A > 0$. For every $x > x_0(\varepsilon, A)$ there exists an integer a such that for all Q with*

$$(2.14) \quad Q \geq x \exp(-(1 - \varepsilon)A (\log_2 x)^2 / \log_3 x)$$

we have

$$(2.15) \quad \sum_{\substack{q < Q \\ (q, a) = 1}} \left| \theta(x; q, a) - \frac{x}{\phi(q)} \right| \geq \left| \sum_{\substack{q \sim Q/2 \\ (q, a) = 1}} \left(\theta(x; q, a) - \frac{x}{\phi(q)} \right) \right| > x (\log x)^{-A}.$$

The method of proof of Theorem A3 is rather different from that of Theorems A1, A2. It is based on the following simple idea. We consider primes $p \sim x$, and for given a we wish to study $p \equiv a \pmod{q}$ on average over q . This leads to a double sum over p and q . Writing $p - a = qr$ the double sum can be transformed to a double sum over p and r .

Since our values for q are almost as large as x , the complementary factor r is quite small and the transformed sum deals with primes in progressions where the modulus r is sufficiently small that analytic methods can be successfully applied. The resulting estimates lead to a sum which, in the special case that a is of the form $\prod_{p < z} p$, can be shown to have an oscillatory behavior which leads to the conclusion in a manner similar to that of the earlier methods.

In contrast to the earlier methods this one offers the advantage of leading to a stronger result (at least when x/q is not too small) and also of not requiring for its proof the rather deep Linnik-Gallagher theorem. However this method seems to be incapable of dealing with individual arithmetic progressions and thus cannot even give, as far as one can see, a new proof of Maier's original result [Ma].

In the case where $Q > x(\log x)^{-N}$ for some fixed N , Theorem A3 and the Corollary recover Theorem 1 of [F-G]. As there we deduce lower bounds for sums $\sum_q |\text{error}|$, by bounding below the smaller quantity $|\sum_q \text{error}|$. The integers a produced by the proof grow as x does. We believe that this is inevitable, at least for bounding $|\sum_q \text{error}|$. That this is true, at least for Q in the range (2.14), is seen from the following result which slightly refines earlier theorems of Bombieri-Friedlander-Iwaniec [B-F-I] and of Fouvry [Fo]. The proofs of these results are based on the same simple switching principle described above in connection with the proof of Theorem A3.

Proposition 2.1. *Fix $A > 0$. Then, for $2 \leq Q \leq \frac{1}{2}x$, $0 < |a| < (\log x)^A$, we have*

$$(2.16) \quad \left| \sum_{\substack{q \sim Q \\ (q, a) = 1}} \left(\theta(x; q, a) - \frac{x}{\phi(q)} \right) \right| \ll_A \tau(a) Q \log(x/Q) + x (\log x)^{-A} + Q \log |a|.$$

Here the last term on the right may be deleted unless a is prime.

Thus, when Q satisfies both (2.11) and (2.14), then (2.12) and (2.13) cannot hold with numbers a_{\pm} in the range $0 < |a_{\pm}| < (\log x)^A$.

We conclude this section with some speculation as to what might be the right upper bound for $\Delta(x; q, a)$ and formulate the following

Hypothesis. *Let $\varepsilon > 0$. For all q, a , and x with $q \geq q_0(\varepsilon)$, $(a, q) = 1$, $x \geq q(\log q)^{1+\varepsilon}$, we have*

$$(2.17) \quad |\Delta(x; q, a)| \ll y^{-(1-\varepsilon)\delta(x, y)}$$

where $y = x/q$ and

$$(2.18) \quad \delta(x, y) = \min \left(\frac{1}{2}, \frac{\log(\log y / \log_2 x)}{\log_2 x} \right) = \min \left(\frac{1}{2}, \delta_3(x, y) \right).$$

It is equally reasonable to expect that the upper bound given by (2.17) and (2.18) holds for the short interval analogue where $\Delta(x; q, a)$ is replaced by $\Delta(x, y) = y^{-1}(\theta(x + y) - \theta(x) - y)$. These hypotheses carry the implication that the results we obtain on average over q (Theorem A3) and, if GRH be assumed, the results we obtain here for individual progressions and those given for short intervals in [H-M], are essentially best possible.

As already indicated, the proof of Theorem Aj ($1 \leq j \leq 3$) is dependent upon a Theorem Bj which demonstrates an oscillatory behavior for a certain sifting function of the type $\sum_{\substack{n \leq y \\ (n, P)=1}} f(n)$. Those theorems will be stated and proved in Chapter II. The proofs of Theorems B1 and B3 depend on a further Theorem C whose proof will occupy the greater part of the paper. The final Chapter III gives the derivation of the results stated above.

II. Oscillation Theorems for Sifting Functions

3. The Statements

In this section we shall state our principal results on oscillations of $\Phi(y, z)$ and certain related functions. These results will be proved in Sections 4-10.

We first introduce some notation. For real numbers $y > 0$ and $z \geq 2$ and positive integers k we set

$$\begin{aligned} \Phi_k(y, z) &= \#\{n \leq y : p|n, p \nmid k \Rightarrow p \geq z\} \\ &= y \prod_{\substack{p < z \\ p|k}} \left(1 - \frac{1}{p}\right) (1 + r_k(y, z)), \end{aligned}$$

and let $\Phi(y, z) = \Phi_1(y, z)$, $r(y, z) = r_1(y, z)$. We further define a weighted form $\bar{r}^{(p)}(y, z)$ of r by

$$\bar{r}^{(p)}(y, z) = \int_{y/2}^y r^{(p)}(t, z) \frac{dt}{t},$$

where $r^{(p)}(y, z)$ is defined by

$$\sum_{\substack{n \leq y \\ p'|n \Rightarrow p' \geq z, p' \neq p}} \frac{n}{\phi(n)} = xC \prod_{\substack{p' < z \\ \text{or } p' = p}} \left(1 - \frac{1}{p'}\right) (1 + r^{(p)}(y, z)),$$

with

$$C = \prod_{\substack{p' \geq z \\ p' \neq p}} \left(1 + \frac{1}{p'(p' - 1)}\right),$$

and we set $\bar{r}(y, z) = \bar{r}^{(2)}(y, z)$.

Throughout this and the following sections we shall work with pairs of real numbers (y, z) lying in a range of the type

$$(R_\alpha) \quad z \geq z_0, \quad y = z^u, \quad u_0 \leq u \leq z^{1-\alpha},$$

where α will be a given positive real number, and z_0 and u_0 suitable sufficiently large constants, possibly depending on α . Given y and z , we define u and \mathcal{L} by

$$(3.1) \quad u = \log y / \log z, \quad \mathcal{L} = \log z.$$

For $u > 0$ we define ξ_u as the positive solution of the equation

$$e^{\xi_u} = u \xi_u + 1.$$

It is easy to see that ξ_u satisfies

$$(3.2) \quad \xi_u = \log u + \log_2 u + O\left(\frac{\log_2 u}{\log u}\right) \quad (u \geq 3)$$

and

$$(3.3) \quad \frac{d}{du} \xi_u = \frac{1}{u} + O\left(\frac{1}{u \log u}\right) \quad (u \geq 3)$$

(cf. [H-T, Lemma 1]).

We can now state the two oscillation theorems for the functions r_k and $\bar{r}^{(p)}$ that will be required for the proof of Theorems A1 and A3.

Theorem B1. *Let $\varepsilon > 0$ and (y, z) be in the range $(R_{1/6+\varepsilon})$ with $u_0 = 1 + \varepsilon$, and let $u = \log y / \log z$. There exist numbers z_{\pm} satisfying*

$$(3.4) \quad \min\left(\frac{1}{2}z, z^{1-\frac{2}{u+2}}\right) \leq z_{\pm} \leq z$$

such that for every positive integer k satisfying

$$(3.5) \quad \sum_{\substack{p|k \\ p \leq z}} 1 \leq 2 \log_2 z$$

and for some y_{\pm} (which may depend on k) satisfying

$$(3.6) \quad y\left(1 - \frac{1}{\log y}\right) \leq y_{\pm} \leq y$$

there hold

$$(3.7) \quad \begin{cases} r_k(y_+, z_+) \geq \exp\{-u\xi_u - c_1 u\}, \\ r_k(y_-, z_-) \leq -\exp\{-u\xi_u - c_1 u\}, \end{cases}$$

where c_1 is a positive constant depending at most on ε .

Theorem B3. *Let $\varepsilon > 0$ and (y, z) be in $(R_{1/6+\varepsilon})$ with $u_0 = 1 + \varepsilon$. There exist z_{\pm} satisfying (3.4) such that for every prime p and for some y_{\pm} (which may depend on p) satisfying (3.6) the inequalities (3.7) hold with r_k replaced by $\bar{r}^{(p)}$.*

In what follows we shall for the most part be concerned with the condition $u > u_0$ for a large unspecified u_0 . How this assumption is relaxed to $u \geq 1 + \varepsilon$ is indicated at the end of this section. This weaker condition is actually necessary for the proofs of Theorems A1 and A3 in that $\log^{1+\varepsilon}$ in (2.3) and (2.12) would otherwise have to be replaced by $\log^{u_0+\varepsilon}$.

The above results are quite sharp. It can be shown that, in the range (R_{ε}) , $r_k(y, z)$ is bounded from above by $O(\exp(-u\xi_u + O(u)))$ (cf. [H-M, Lemma 2] for the case $k = 1$). Thus, the lower bound (3.7) is best-possible aside from the value of the constant c_1 . Moreover, the localization of z_{\pm} cannot be much improved. Indeed, it will be clear from Theorem C and Proposition 3.2 below that the numbers $u_{\pm} = \log y / \log z_{\pm}$ cannot be better localized than within an interval of length 1. On the other hand, the localization of y_{\pm} is probably not best-possible. In fact, under the Riemann Hypothesis, (3.7) can be shown to hold in the

range $(R_{1/2+\varepsilon})$ for any $\varepsilon > 0$ with $y_{\pm} = y$ (cf. the remarks following Corollary 3.1 below). However, for our applications, this deficiency is inconsequential.

We remark that, instead of keeping y (essentially) fixed and varying z in order to obtain extreme values of $r_k(y, z)$ or $\bar{r}^{(p)}(y, z)$, we could have instead fixed z and varied y . Specifically, it can be shown that, under the conditions of Theorem B1, the estimate (3.7) holds with $z_{\pm} = z$ and suitable numbers y_{\pm} satisfying $y \leq y_{\pm} \leq yz^2$. A similar estimate, but with a much weaker localization of y_{\pm} , was obtained in [H-M] for the case of $r(y, z)$. This was achieved by bounding from below certain weighted averages over the function $f(u) = |r(z^u, z)|^2$. This method, while being relatively simple, however is not capable of yielding results of the quality of Theorems B1 or B3 which will be needed for the proof of Theorems A1 and A3. It will therefore be necessary to use a different and more complicated approach.

The restriction (3.5) in Theorem B1 is responsible for the corresponding restriction (2.1) in Theorem A1. In order to deal with moduli k containing many small prime factors it is natural to consider the analogue to $\Phi(y, z)$ obtained by counting those $n \leq y$ all of whose prime factors lie in a given interval. In general this should be more difficult to estimate but, in the case that the interval is rather short, a fairly precise oscillation theorem may be obtained by combinatorial means. The following such result will be applied to Theorem A2.

Theorem B2. *For $z \geq z_0$, $h \leq \frac{1}{2}z$, $k \geq 1$, P the product of any k primes, all of which are in the interval $(z - h, z]$, we have*

$$(3.8) \quad (-1)^j r_P(y) \doteq (-1)^j \left(\sum_{\substack{n \leq y \\ (n, P)=1}} 1 - \frac{\phi(P)}{P} y \right) \geq \frac{1}{4} y \binom{k}{j} z^{-j}$$

for every integer j with $1 \leq j \leq \frac{k}{3}$ and every real y with $(z - h)^j \geq y \geq \frac{4j}{k-j+1} z^j$.

We shall deduce Theorems B1 and B3 from a more fundamental result, Theorem C below, and two auxiliary propositions. Theorem C gives an asymptotic formula for a "smoothed" version of $r(y, z)$ in terms of the generating Dirichlet series

$$F(s) = F(z; s) = \prod_{p < z} \left(1 - \frac{1}{p^s}\right) \zeta(s).$$

To state this result, we require some notation. With $F(s)$ defined above, we let

$$\phi(s) = \phi(z; s) = \log F(z; s),$$

where the branch of the logarithm is chosen such that $\phi(\sigma)$ is real for $0 < \sigma < 1$. Such a branch exists and is an analytic function in the region $0 < \sigma < 1$, $|t| \leq 1$, since $\zeta(s)$, and hence $F(s)$, is analytic and non-zero in this region. Next, given $\varepsilon > 0$ and real numbers y, z in the range (R_{ε}) , we define $s_0 = s_0(y, z)$ as the unique solution to the equation

$$(3.9) \quad \phi'(s_0) = -\log y = -u\mathcal{L}$$

satisfying

$$(3.10) \quad s_0 = \sigma_0 + it_0, \quad \frac{\varepsilon}{2} < \sigma_0 < 1 - \frac{3}{\mathcal{L}}, \quad 0 < t_0 < \frac{2\pi}{\mathcal{L}},$$

and we set

$$\rho(y, z) = \frac{e^{\gamma} F(z; s_0) y^{s_0-1}}{s_0 \sqrt{\pi u/2}},$$

where u and \mathcal{L} are defined by (3.1). (The existence and uniqueness of s_0 under the stated conditions will be shown in Proposition 3.1 below.) Finally, we introduce a "smoothed" form of $r(y, z)$ by setting

$$r(y, z; \lambda) = \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r(ye^v, z) e^{-\lambda v^2/2} dv,$$

where λ is a positive real parameter. We then have

Theorem C . *Let $\varepsilon > 0$ be fixed. For (y, z) in the range (R_ε) and $\log u \leq \lambda \leq \exp(\mathcal{L}^{3/2-\varepsilon})$ we have*

$$(3.11) \quad r(y, z; \lambda) = \operatorname{Re} \rho(y, z) + O\left(\frac{|\rho(y, z)|}{\log u}\right),$$

where u and \mathcal{L} are defined by (3.1) and the implied constant depends at most on ε .

For the proof of this result we shall use the saddle point method. The basic idea of this method is to write $r(y, z)$ as a Perron integral over $F(s)y^{s-1}/s$ along a vertical line segment, whose abscissa is $\operatorname{Re} s_0$. The main contribution to the integral will then come from small neighborhoods of the points s_0 and \bar{s}_0 which are saddle points of the function $F(s)y^{s-1}$, and integration over these neighborhoods will produce the main term $\operatorname{Re} \rho(y, z)$ in (3.11).

This approach is very similar to the one used by Hildebrand and Tenenbaum [H-T] in the "dual" problem of estimating the function

$$\psi(y, z) = \#\{n \leq y : p|n \Rightarrow p \leq z\}.$$

In fact, the main term in (3.7) is formally almost identical to the main term in their asymptotic formula for $\psi(y, z)$, except that different Dirichlet series are involved. However, there are two important aspects in which the estimation of $r(y, z)$ differs from that of $\psi(y, z)$. The first is that, while in the case of $\psi(y, z)$ there is a single dominating saddle point on the real line, the function $F(s)y^{s-1}$ does not have a real saddle point, but instead two saddle points s_0 and \bar{s}_0 off the real line whose contributions to the integral are complex conjugates. (The function has other saddle points as well, but their contribution turns out to be negligible.) This explains the appearance of the real part function in (3.11).

The second difference between the estimations of $r(y, z)$ and $\psi(y, z)$ is that in the case of $\psi(y, z)$, the generating Dirichlet series can be bounded trivially on any vertical line by its value at the abscissa. This is not possible in the case of $r(y, z)$, and as a result the estimation of the integrand in the Perron integral at large heights becomes a non-trivial problem which prevents us from obtaining a formula of the type (3.11) for $r(y, z)$ if u is large, say a fixed power of z . In order to obtain a better convergence in the Perron integral, we have to work with the smoother functions $r(y, z; \lambda)$ rather than $r(y, z)$ itself.

To complement Theorem C, we shall prove the following results which give estimates for the implicitly defined quantities s_0 and $\rho(y, z)$ in the main term of (3.11). Here ε is an arbitrary given positive number, and the implied constants depend at most on ε .

Proposition 3.1. *If (y, z) is in the range (R_ε) , then the equation (3.9) has a unique solution $s_0 = s_0(y, z)$ satisfying (3.10), and we have*

$$(3.12) \quad s_0(y, z) = 1 - \frac{\xi_u}{\mathcal{L}} + i\frac{\pi}{\mathcal{L}}\left(1 + \frac{1}{\xi_u}\right) + O\left(\frac{1}{\mathcal{L} \log^2 u}\right),$$

$$(3.13) \quad \frac{\partial}{\partial u} s_0(z^u, z) = -\frac{1}{\mathcal{L}u}\left(1 + O\left(\frac{1}{\log u}\right)\right).$$

Proposition 3.2. *Suppose (y, z) is in the range (R_ε) . Then we have*

$$(3.14) \quad \rho(y, z) = \exp\{-u\xi_u + O(u)\},$$

$$(3.15) \quad \rho(z^{u-v}, z) = \rho(z^u, z) \exp\left\{v(\xi_u - i\pi) + O\left(\frac{v}{\log u}\right) + O\left(\frac{v^2}{u}\right)\right\} \quad (0 \leq v \leq (1 - \varepsilon)u),$$

$$(3.16) \quad \rho(\tilde{z}^u, \tilde{z}) = \rho(z^u, z) \exp \left\{ O \left(\frac{u}{z} \left(|E(\tilde{z}, z)| + \frac{\tilde{z} - z}{\mathcal{L}^2} + z^\varepsilon \right) \right) \right\} \quad (z \leq \tilde{z} \leq 2z),$$

$$(3.17) \quad \rho(\tilde{z}^u, \tilde{z}) = \rho(z^u, z) \exp \left\{ O \left(\frac{u}{\mathcal{L}} \right) \right\} \quad (\tilde{z} \geq z),$$

where

$$E(\tilde{z}, z) = \sum_{z < p \leq \tilde{z}} \log p - (\tilde{z} - z).$$

Theorem B2 will be proved in Section 4. Propositions 3.1 and 3.2 and Theorem C will be proved in Sections 5-8. In Sections 9 and 10 we shall deduce Theorems B1 and B3 from Theorem C and the two propositions. We conclude this section by deriving two simple corollaries of Theorem C and then reducing Theorems B1 and B3 to the range $u \geq u_0$, where u_0 is arbitrary. We first show that the estimates of Theorem C hold with $r(y, z)$ in place of $r(y, z; \lambda)$, provided $u = \log y / \log z$ is not too large.

Corollary 3.1. *Let $\varepsilon > 0$ and (y, z) be in the range (R_ε) , and suppose that $u \leq \mathcal{L}^{3/2-\varepsilon}$. Then we have*

$$r(y, z) = \operatorname{Re} \rho(y, z) + O \left(\frac{|\rho(y, z)|}{\log u} \right).$$

To deduce this result, we apply Theorem C with ε replaced by $\varepsilon/2$ and $\lambda = \exp(\mathcal{L}^{3-\varepsilon}/2)$. Without loss of generality we may assume that $\varepsilon < 1/2$. Using the estimates

$$r(ye^v, z) = r(y, z) + O \left((|v| + \frac{1}{y}) \mathcal{L} \right) \quad (|v| \leq 1)$$

and

$$r(y', z) = O(\mathcal{L}) \quad (y' > 0),$$

which easily follow from the definition of $r(y, z)$, we get

$$\begin{aligned} r(y, z; \lambda) &= \frac{r(y, z) \sqrt{\lambda}}{\sqrt{2\pi}} \int_{-1}^1 e^{-\lambda v^2/2} dv + O \left(\mathcal{L} \sqrt{\lambda} \int_{-1}^{-1} (|v| + \frac{1}{y}) e^{-\lambda v^2/2} dv \right) \\ &\quad + O \left(\mathcal{L} \sqrt{\lambda} \int_1^{\infty} e^{-\lambda v^2/2} dv \right). \end{aligned}$$

The error terms here can be bounded by

$$\ll \frac{\mathcal{L}}{\sqrt{\lambda}} + \frac{\mathcal{L}}{y} + \mathcal{L} e^{-\lambda/2} \ll \exp\{-u^{1+\varepsilon/3}\} \ll \frac{\rho(y, z)}{\log u},$$

if we use the definition of λ , the bound $u \leq \mathcal{L}^{3/2-\varepsilon}$ and the estimates (3.14) and (3.2) for $\rho(y, z)$ and ξ_u . Moreover, the integral in the main term equals

$$\int_{-1}^1 e^{-\lambda v^2/2} dv = \frac{\sqrt{2\pi}}{\sqrt{\lambda}} (1 + O(e^{-\lambda/2})) = \frac{\sqrt{2\pi}}{\lambda} (1 + O(\frac{1}{\mathcal{L}})).$$

Thus,

$$r(y, z; \lambda) = r(y, z) \left(1 + O \left(\frac{1}{\log u} \right) \right) + O \left(\frac{|\rho(y, z)|}{\log u} \right)$$

and the estimate of the corollary follows from (3.11).

We note that under the Riemann hypothesis λ can be taken as large as $\exp(z^{1/2-\varepsilon})$ in Theorem C, and the estimate of the corollary can be shown to hold in the range $(R_{1/2+\varepsilon})$ for any $\varepsilon > 0$. By appealing to Corollary 3.1 instead of Theorem C in the proof of Theorems B1 and B3, one can then show that, under the Riemann Hypothesis, one can take $y_{\pm} = y$ in (3.7), as we had remarked earlier.

Our second corollary concerns the asymptotic behavior of the Buchstab function $\omega(u)$, defined as the continuous solution of the system

$$\begin{aligned}\omega(u) &= \frac{1}{u} & (1 \leq u \leq 2), \\ (u\omega(u))' &= \omega(u-1) & (u > 2).\end{aligned}$$

This function was introduced by Buchstab [Bu] who showed that

$$(3.18) \quad \lim_{z \rightarrow \infty} r(z^u, z) = e^{\gamma}(\omega(u) - e^{-\gamma})$$

holds for every $u > 1$, where γ is Euler's constant. Buchstab [Bu] also obtained the estimate

$$\omega(u) = e^{-\gamma} + O(\exp\{-u\xi_u + O(u)\}).$$

In view of this relation it is natural to consider the function

$$W(u) = \omega(u) - e^{-\gamma}.$$

Maier [Ma] showed that $W(u)$ changes sign in every interval of length 1. Cheer and Goldston [C-G] proved that $W(u)$ has at most two sign changes in any interval of length 1. Furthermore they showed that

$$\max_{v \geq u} W(v) = \max_{u \leq v \leq u+2} W(v), \quad \min_{v \geq u} W(v) = \min_{u \leq v \leq u+2} W(v).$$

Theorem C and Proposition 3.2 enable us to describe the behavior of $W(u)$ more precisely as follows.

Corollary 3.2. *There exists a decreasing function $W_0(u)$ satisfying*

$$(3.19) \quad W_0(u) = \exp\{-u\xi_u + O(u)\} \quad (u \geq u_0)$$

and a function $\theta(u)$ satisfying

$$\theta(u+t) - \theta(u) = t\pi(1 + O(\frac{1}{\log u})) \quad (u \geq u_0, 0 \leq t \leq 1)$$

such that, as $u \rightarrow \infty$,

$$W(u) = W_0(u)\{\cos\theta(u) + O(\frac{1}{\log u})\}.$$

This result clearly exhibits the oscillatory behavior of $W(u)$. Aside from a "decay factor" $W_0(u)$, the function $W(u)$ behaves like a trigonometric function with period 2. An immediate consequence of this behavior and the estimate (3.19) is that

$$\max_{u \leq v \leq u+2} W(v) = \exp\{-u\xi_u + O(u)\}$$

and similarly

$$\min_{u \leq v \leq u+2} W(v) = -\exp\{-u\xi_u + O(u)\}.$$

To prove the corollary, we note that, by Buchstab's result (3.18) and Theorem C

$$\begin{aligned} W(u) &= e^{-\gamma} \lim_{z \rightarrow \infty} r(z^u, z) \\ &= e^{-\gamma} \lim_{z \rightarrow \infty} \left\{ \operatorname{Re} \rho(z^u, z) + O\left(\frac{|\rho(z^u, z)|}{\log u}\right) \right\}, \end{aligned}$$

provided u is sufficiently large. Also, the estimates (3.17), (3.14) and (3.15) of Proposition 3.2 imply that the limit

$$\ell(u) = \lim_{z \rightarrow \infty} \rho(z^u, z)$$

exists for every sufficiently large u and satisfies

$$|\ell(u)| = \exp\{-u\xi_u + O(u)\}$$

and

$$\arg \ell(u+t) - \arg \ell(u) = t\pi\left(1 + O\left(\frac{1}{\log u}\right)\right), \quad (u \geq u_0, 0 \leq t \leq 1)$$

where $\arg \ell(u)$ denotes a continuous branch of the argument of $\ell(u)$. Taking $W_0(u) = e^{-\gamma}|\ell(u)|$ and $\theta(u) = \arg \ell(u)$ then yields the asserted estimates.

As already mentioned, the results of Theorems B1 and B3 follow fairly easily in the range $1 + \varepsilon \leq u \leq u_0$ for fixed ε and u_0 . In this range the formula (3.18) of Buchstab still holds with r replaced by r_k (subject to (3.5)). In the case $u \geq 2$ this is an immediate consequence of [F-G, Lemma 5]; in the case $1 + \varepsilon \leq u \leq 2$ that proof needs only a slight modification. An easy variation on Buchstab's original arguments shows that (3.18) also holds with r replaced by $\bar{r}^{(p)}$. Theorems B1 and B3 then follow for $1 + \varepsilon \leq u \leq u_0$ from these facts in conjunction with Maier's result that $W(u)$ changes sign in every interval of length 1 [Ma, Lemma 4].

Note: Professor Warlimont has kindly pointed out to us an omission from the identity in the proof of Lemma 5 in [F-G]: specifically that the contribution $\sum'_{x/y < n \leq x} 1$ was overlooked.

Here, as there, \sum' denotes a sum over integers all of whose prime factors divide q , and q is assumed to have fewer than y^f distinct prime factors where $f = f(y)$ tends to zero as $y \rightarrow \infty$ with $f(y) > 3 \log_2 y / \log y$.

In order to complete the proof of Lemma 5 it suffices to establish, for $x \geq y \geq y_0$, the estimate

$$(3.20) \quad \sum'_{n \leq x} 1 \ll x / \log^3 y + \frac{q}{\phi(q)} \frac{xf}{\log x}.$$

By considering first the contribution to the sum of those $n \leq x^{1/2}$, then those n divisible by some prime $> y^{2f}$, and finally the remaining n , we find that (compare Exercise 01 of [Ha-T]) for $\sigma = 1/f \log y$,

$$\begin{aligned} (3.21) \quad \sum'_{n \leq x} 1 &\ll x^{\frac{1}{2}} + \nu(q)x/y^{2f} + \sum_{x^{1/2} < n \leq x}^* (n/x^{1/2})^\sigma \\ &\ll x / \log^3 y + x^{-\sigma/2} \sum_{n \leq x}^* n^\sigma \end{aligned}$$

where \sum^* denotes the restriction to those n divisible only by prime divisors of q which are $\leq y^{2f}$. By a Theorem of Halberstam and Richert [Ha-T, Theorem 00], we have

$$\begin{aligned}
(3.22) \quad \sum_{n \leq x}^* n^\sigma &\ll \frac{x}{\log x} \sum_{n \geq 1}^* \frac{1}{n^{1-\sigma}} \ll \frac{q}{\phi(q)} \frac{x}{\log x} \exp\left\{ \sum_{p \leq y^{2f}} \left(\frac{1}{p^{1-\sigma}} - \frac{1}{p} \right) \right\} \\
&\ll \frac{q}{\phi(q)} \frac{x}{\log x}.
\end{aligned}$$

Therefore (3.21) and (3.22) yield (3.20), since $x^{-\sigma/2} \ll f \log y / \log x$, and Lemma 5 follows as stated.

4. Proof of Theorem B2

For m a positive integer let

$$s(m) = \sum_{p|m} \frac{1}{p}, \quad \nu(m) = \sum_{p|m} 1.$$

We require the following lemma.

Lemma 4.1 . *For any positive integers m, j with m squarefree and $s(m) \leq j/2$ we have*

$$(4.1) \quad \sum_{\substack{d|m \\ \nu(d) \geq j+1}} \frac{1}{d} \leq \frac{2s(m)}{j} \sum_{\substack{d|m \\ \nu(d)=j}} \frac{1}{d}$$

Proof: We first show that, for every integer $i \geq 0$,

$$(4.2) \quad \sum_{\substack{d|m \\ \nu(d)=j+i}} \frac{1}{d} \leq \frac{j!}{(j+i)!} s^i(m) \sum_{\substack{d|m \\ \nu(d)=j}} \frac{1}{d}.$$

We observe that any divisor d of m with $\nu(d) = j + i$ can be written in the form $d = \delta p_1 \dots p_i$ (where $\nu(\delta) = j$ and the p_μ are prime) in $\binom{j+i}{j} i! = (j+i)!/j!$ distinct ways, if we distinguish amongst different orderings of the p_μ .

By multiplying out and collecting terms on the right side we get (4.2). To prove (4.1) we note that the left side is obtained by summing over i the left side of (4.2). The application of (4.2) thus yields

$$\begin{aligned}
\sum_{\nu(d) \geq j+1} \frac{1}{d} &\leq \sum_{\nu(d)=j} \frac{1}{d} \sum_{i \geq 1} \frac{j!}{(j+i)!} s^i(m) \\
&\leq \sum_{\nu(d)=j} \frac{1}{d} \sum_{i \geq 1} \left(\frac{s(m)}{j} \right)^i,
\end{aligned}$$

and, since $s(m) \leq j/2$, this gives (4.1).

Proof of Theorem B2. If $d|P$ with $\nu(d) \geq j$, then $d > (z-h)^j \geq y$ so $[y/d] = 0$. Now, since

$$r_P(y) = \sum_{\substack{n \leq y \\ (n, P)=1}} 1 - \frac{\phi(P)}{P}y = \sum_{d|P} \mu(d) \left(\left\lfloor \frac{y}{d} \right\rfloor - \frac{y}{d} \right),$$

we have

$$(4.3) \quad \left| r_P(y) - (-1)^j y \sum_{\substack{d|P \\ \nu(d)=j}} \frac{1}{d} \right| \leq y \sum_{\substack{d|P \\ \nu(d) \geq j+1}} \frac{1}{d} + \sum_{\substack{d|P \\ \nu(d) \leq j-1}} 1.$$

Since $s(P) \leq \pi(z)/(z-h) \leq 1/8$ for $z > z_0$ we may use Lemma 4.1 to estimate the first term on the right hand side of (4.3). For the second term we have, since $\binom{k}{i-1} \leq \frac{1}{2} \binom{k}{i}$ for $i \leq k/3$,

$$\begin{aligned} \sum_{\substack{d|P \\ \nu(d) \leq j-1}} &= \sum_{i=0}^{j-1} \binom{k}{i} \leq 2 \binom{k}{j-1} \\ &\leq \frac{y}{2z^j} \binom{k}{j} \leq \frac{y}{2} \sum_{\substack{d|P \\ \nu(d)=j}} \frac{1}{d}. \end{aligned}$$

Substituting these in (4.3) we complete the proof of Theorem B2.

5. Proof of Proposition 3.1

We begin with a lemma which will be needed again in later sections.

Lemma 5.1. *Let $\varepsilon > 0$ and $k \geq 1$. Uniformly for $z \geq 2$ and $s = \sigma + it$ in the range*

$$(5.1) \quad \varepsilon \leq \sigma < 1 - \frac{1}{\mathcal{L}}, \quad |t| \leq 1$$

we have

$$(5.2) \quad \phi(z; s) = - \int_1^{(1-s)\mathcal{L}} \frac{e^{wz}}{w} dw + O\left(\frac{z^{1-\sigma}}{\mathcal{L}^k}\right) + O(1),$$

$$(5.3) \quad \phi'(z; s) = \left(1 + O\left(\frac{1}{\mathcal{L}^k}\right)\right) \frac{z^{1-s}}{1-s} + O(1),$$

$$(5.4) \quad \phi''(z; s) = \left(1 + O\left(\frac{1}{\mathcal{L}^k}\right)\right) \frac{z^{1-s}(1 - (1-s)\mathcal{L})}{(1-s)^2} + O(1),$$

$$(5.5) \quad |\phi'''(z; s)| \leq \left(1 + O\left(\frac{1}{(1-\sigma)\mathcal{L}}\right)\right) \frac{z^{1-\sigma}}{1-\sigma} \mathcal{L}^2,$$

where $\mathcal{L} = \log z$ and the implied constants depend at most on ε and k .

Proof. Differentiating $\phi(z; s) = \log(\prod_{p < z} (1 - \frac{1}{p^s}) \zeta(s))$, we obtain

$$(5.6) \quad \phi'(s) = \sum_{p < z} \frac{\log p}{p^s - 1} + \frac{\zeta'(s)}{\zeta(s)}.$$

In the range (5.1) we have

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{1-s} + O(1)$$

and

$$\begin{aligned} \sum_{p < z} \frac{\log p}{p^s - 1} &= \sum_{p < z} \frac{\log p}{p^s} + O\left(\sum_{p < z} \frac{\log p}{p^{2\sigma}}\right) \\ &= (1 + O(\frac{1}{\mathcal{L}^k})) \int_1^z \frac{du}{u^s} + O(1) + O(z^{1-2\sigma} \log z) \\ &= (1 + O(\frac{1}{\mathcal{L}^k})) \frac{z^{1-s} - 1}{1-s} + O(1), \end{aligned}$$

using partial summation and the prime number theorem with logarithmic error term. Substituting these estimates into (5.6) yields (5.3). The estimates (5.4) and (5.5) can be proved in the same way.

To obtain (5.2), we note that by (5.3)

$$\begin{aligned} \phi(z; s) - \phi(z; 1 - \frac{1}{\mathcal{L}}) &= \int_{1-1/\mathcal{L}}^s \phi'(z; s') ds' = -\frac{1}{\mathcal{L}} \int_1^{(1-s)\mathcal{L}} \phi'(z; 1 - \frac{w}{\mathcal{L}}) dw \\ &= -\int_1^{(1-s)\mathcal{L}} \frac{e^w}{w} dw + O\left(\frac{1}{\mathcal{L}^{k+1}} \int_1^{(1-s)\mathcal{L}} \left|\frac{e^w}{w}\right| |dw|\right) + O(1) \\ &= -\int_1^{(1-s)\mathcal{L}} \frac{e^w}{w} dw + O\left(\frac{z^{1-\sigma}}{\mathcal{L}^k}\right) + O(1). \end{aligned}$$

Since

$$\begin{aligned} \phi(z; 1 - \frac{1}{\mathcal{L}}) &= \sum_{p < z} \log\left(1 - \frac{1}{p^{1-1/\mathcal{L}}}\right) + \log \zeta\left(1 - \frac{1}{\mathcal{L}}\right) \\ &= -\sum_{p < z} \frac{1}{p^{1-1/\mathcal{L}}} + \log \mathcal{L} + O(1) \\ &= -\sum_{p < z} \frac{1}{p} + \log_2 z + O(1) = O(1), \end{aligned}$$

this implies the desired estimate (5.2).

We now proceed to prove Proposition 3.1. We assume that (y, z) lies in the range (R_ε) with a given parameter $\varepsilon > 0$ and sufficiently large constants z_0 and u_0 . We first establish the existence and uniqueness of $s_0(y, z)$ as solution to (3.9) in the range (3.10). To this end it suffices to show that (i) for every $\sigma \in [\frac{\varepsilon}{2}, 1 - \frac{3}{\mathcal{L}}]$ there exists a unique number $t = t(\sigma) \in (0, \frac{2\pi}{\mathcal{L}})$ such that $\phi'(\sigma + it)$ is real and negative, and (ii) the function $f(\sigma) = \phi'(\sigma + it(\sigma))$ is strictly increasing in the interval $[\frac{\varepsilon}{2}, 1 - \frac{3}{\mathcal{L}}]$ with $f(\varepsilon/2) < -z^{1-\varepsilon} \mathcal{L}$ and $f(1 - \frac{3}{\mathcal{L}}) = O(\mathcal{L})$.

For the proof of (i) we observe first that, by (5.6), $\phi(\sigma)$ is positive for $0 < \sigma < 1$, since $\zeta'(\sigma)/\zeta(\sigma) > 0$ in this interval. The assertion (i) will then follow if we can show that

$$(5.7) \quad \frac{d}{dt} \arg \phi'(\sigma + it) \in \left(\frac{-3}{2}\mathcal{L}, \frac{-1}{2}\mathcal{L} \right)$$

holds for $\frac{\varepsilon}{2} \leq \sigma \leq 1 - \frac{3}{\mathcal{L}}$ and $0 \leq t \leq \frac{2\pi}{\mathcal{L}}$. Using Lemma 5.1 with $\varepsilon/2$ in place of ε and $k = 1$ we see that in this range

$$\begin{aligned} \frac{d}{dt} \arg \phi'(\sigma + it) &= \frac{d}{dt} \operatorname{Im} \log \phi'(\sigma + it) = \operatorname{Re} \frac{\phi''(\sigma + it)}{\phi'(\sigma + it)} \\ &= \operatorname{Re} \left(1 + O\left(\frac{1}{\mathcal{L}}\right) \right) \left(\frac{1}{1-s} - \mathcal{L} \right) \\ &= \left(1 + O\left(\frac{1}{\mathcal{L}}\right) \right) \left(\frac{1-\sigma}{(1-\sigma)^2 + t^2} - \mathcal{L} \right). \end{aligned}$$

Since

$$0 < \frac{1-\sigma}{(1-\sigma)^2 + t^2} \leq \frac{1}{1-\sigma} \leq \frac{\mathcal{L}}{3}$$

for $0 < \sigma < 1 - 3/\mathcal{L}$, (5.7) follows if z (and hence \mathcal{L}) is sufficiently large, as we may assume. We remark that since $\arg \phi'(\sigma + it)$ is a continuously differentiable function of (σ, t) in the range under consideration, the function $t(\sigma)$ defined by (i) is continuously differentiable as well.

To prove (ii) we note that the function

$$f'(\sigma) = \phi''(\sigma + it(\sigma)) (1 + it'(\sigma))$$

is real-valued, continuous, and has no zeros in the interval $[\frac{\varepsilon}{2}, 1 - \frac{3}{\mathcal{L}}]$, since by (5.4), $\phi''(\sigma + it(\sigma)) \neq 0$ in this interval. Thus $f'(\sigma)$ is either monotonically decreasing or monotonically increasing, as σ increases from $\varepsilon/2$ to $1 - 3/\mathcal{L}$. By Lemma 5.1 and the definition of $t(\sigma)$ we have

$$\begin{aligned} f\left(\frac{\varepsilon}{2}\right) &= \phi'\left(\frac{\varepsilon}{2} + it\left(\frac{\varepsilon}{2}\right)\right) \\ &= -|\phi'\left(\frac{\varepsilon}{2} + it\left(\frac{\varepsilon}{2}\right)\right)| \\ &= \frac{-z^{1-\varepsilon/2}}{|1-\varepsilon/2 - it(\varepsilon/2)|} \left(1 + O\left(\frac{1}{\mathcal{L}}\right)\right), \end{aligned}$$

which is $< -z^{1-\varepsilon}\mathcal{L}$, if z is sufficiently large. Similarly,

$$f\left(1 - \frac{3}{\mathcal{L}}\right) = \frac{-z^{3/\mathcal{L}}}{|(3/\mathcal{L}) - it(1 - 3/\mathcal{L})|} \left(1 + O\left(\frac{1}{\mathcal{L}}\right)\right),$$

which is of order $O(\mathcal{L})$. The assertion (ii) now follows.

It remains to prove the estimates (3.12) and (3.13). Setting $s_0 = 1 - s_1/\mathcal{L}$, $s_1 = \sigma_1 + it_1$, we can restate (3.12) as

$$(5.8) \quad \sigma_1 = \xi_u + O\left(\frac{1}{\log^2 u}\right), \quad t_1 = -\pi\left(1 + \frac{1}{\xi_u}\right) + O\left(\frac{1}{\log^2 u}\right).$$

From (3.9) and (5.3) we get

$$-u\mathcal{L} = \phi'(z; s_0) = \left(1 + O\left(\frac{1}{\mathcal{L}^2}\right)\right) \frac{e^{s_1}}{s_1} \mathcal{L} + O(1),$$

so that

$$(5.9) \quad \frac{e^{s_1}}{s_1} = -u \left(1 + O\left(\frac{1}{\mathcal{L}^2}\right) + O\left(\frac{1}{u\mathcal{L}}\right) \right).$$

Since $|e^{s_1}| = e^{\sigma_1}$ and

$$|s_1| = |\sigma_1 + it_1| = \sigma_1 \left(1 + O\left(\frac{t_1^2}{\sigma_1^2}\right) \right) = \sigma_1 \left(1 + O\left(\frac{1}{\sigma_1^2}\right) \right),$$

we deduce from (5.9) first

$$\frac{e^{\sigma_1}}{\sigma_1} \asymp u,$$

which implies $\sigma_1 \gg \log u$, and then

$$(5.10) \quad \begin{aligned} \frac{e^{\sigma_1}}{\sigma_1} &= u \left(1 + O\left(\frac{1}{\mathcal{L}^2}\right) + O\left(\frac{1}{u\mathcal{L}}\right) + O\left(\frac{1}{\sigma_1^2}\right) \right) \\ &= u \left(1 + O\left(\frac{1}{\log^2 u}\right) \right). \end{aligned}$$

Now, note that the function e^x/x is monotonically increasing in $x \geq 1$ and that, by the definition of ξ_u ,

$$(5.11) \quad \begin{aligned} \frac{e^{\xi_u + \delta}}{\xi_u + \delta} &= \frac{(u\xi_u + 1)(1 + \delta + O(\delta^2))}{\xi_u(1 + \frac{\delta}{\xi_u})} \\ &= u \left(1 + \left(1 - \frac{1}{\xi_u}\right)\delta + O(\delta^2) + O\left(\frac{1}{u\xi_u}\right) \right) \quad (|\delta| \leq 1). \end{aligned}$$

Comparison of (5.10) and (5.11) shows that $\sigma_1 = \xi_u + \delta$ with $\delta = O(1/\log^2 u)$, as asserted in (5.8).

To prove the second estimate of (5.8), we take the argument on both sides of (5.9), keeping in mind that by (3.10), $t_1 = -t_0\mathcal{L}$ lies in the interval $(-2\pi, 0)$. This yields the equation

$$-\pi + O\left(\frac{1}{\mathcal{L}^2}\right) + O\left(\frac{1}{u\mathcal{L}}\right) = t_1 - \arctan \frac{t_1}{\sigma_1} = t_1 - \frac{t_1}{\sigma_1} + O\left(\frac{1}{\sigma_1^2}\right),$$

which implies the desired estimate

$$t_1 = -\pi \left(1 + \frac{1}{\sigma_1} \right) + O\left(\frac{1}{\log^2 u}\right) = -\pi \left(1 + \frac{1}{\xi_u} \right) + O\left(\frac{1}{\log^2 u}\right).$$

Finally, to obtain the estimate (3.13) for the derivative of s_0 , we differentiate both sides of the equation $\phi'(z; s_0) = -u\mathcal{L}$ with respect to u . We get in this way

$$\phi''(z; s_0) \frac{\partial}{\partial u} s_0(z^u, z) = -\mathcal{L},$$

so that

$$\begin{aligned} \frac{\partial}{\partial u} s_0(z^u, z) &= -\frac{\mathcal{L}}{\phi''(z; s_0)} \\ &= \frac{(1 - s_0)^2 \mathcal{L}}{z^{1-s_0} ((1 - s_0)\mathcal{L} - 1)} \left(1 + O\left(\frac{1}{\mathcal{L}}\right) \right) \\ &= \frac{1}{\phi'(z; s_0)} \left(1 + O\left(\frac{1}{(1 - \sigma_0)\mathcal{L}}\right) \right) = -\frac{1}{u\mathcal{L}} \left(1 + O\left(\frac{1}{\log u}\right) \right) \end{aligned}$$

by the estimates of Lemma 5.1 and (3.12). This proves (3.13).

6. Proof of Proposition 3.2

We consider first the behaviour of $\rho(y, z)$ when z is fixed and $y = z^u$ varies. Setting

$$(6.1) \quad \begin{aligned} f(u) &= \log \rho(z^u, z) = \log \frac{e^\gamma y^{s_0-1} F(s_0)}{s_0 \sqrt{\pi u/2}} \\ &= (s_0 - 1)u\mathcal{L} + \phi(s_0) - \log s_0 - \frac{1}{2} \log u + \log \frac{e^\gamma}{\sqrt{\pi/2}}, \end{aligned}$$

we can rewrite the first two estimates of Proposition 3.2 as

$$(6.2) \quad f(u) = -u\xi_u + O(u)$$

and

$$(6.3) \quad f(u-v) = f(u) + v(\xi_u - i\pi) + O\left(\frac{v}{\log u}\right) + O\left(\frac{v^2}{u}\right) \quad (0 \leq v \leq u(1-\varepsilon))$$

respectively, and it suffices to show that these latter two estimates hold for $u_0 \leq u \leq z^{1-\varepsilon}$.

Taking the derivative with respect to u in (6.1) we obtain

$$f'(u) = s'_0 u \mathcal{L} + (s_0 - 1)\mathcal{L} + \phi'(s_0)s'_0 - \frac{s'_0}{s_0} - \frac{1}{2u},$$

where for brevity we have set $s'_0 = (\partial/\partial u)s_0(z^u, z)$. Since $\phi'(s_0) = -u\mathcal{L}$ by (3.9) and

$$\begin{aligned} (s_0 - 1)\mathcal{L} &= -\xi_u + i\pi + O\left(\frac{1}{\log u}\right), \\ \frac{s'_0}{s_0} &= \frac{-1}{u\mathcal{L}s_0} \left(1 + O\left(\frac{1}{\log u}\right)\right) = O\left(\frac{1}{\log u}\right) \end{aligned}$$

by Proposition 3.1, it follows that in the range (R_ε)

$$(6.4) \quad f'(u) = -\xi_u + i\pi + O\left(\frac{1}{\log u}\right).$$

This implies, for $0 \leq v \leq (1-\varepsilon)u$,

$$\begin{aligned} f(u-v) &= f(u) - \int_{u-v}^u f'(w)dw \\ &= f(u) + \int_{u-v}^u (\xi_w - i\pi + O\left(\frac{1}{\log w}\right))dw \\ &= f(u) + v(\xi_u - i\pi) + O\left(\frac{v^2}{u}\right) + O\left(\frac{v}{\log u}\right), \end{aligned}$$

and hence (6.3), after adjusting the constant u_0 in (R_ε) , if necessary. Here we have used the estimate

$$\xi_w = \xi_u + O(|u-w|/u) \quad (|u-w| \leq (1-\delta)u),$$

which follows from the bound (3.3) for $(d/du)\xi_u$.

To prove (6.2), we apply (6.4) again to obtain

$$\begin{aligned} f(u) &= f(u_0) + \int_{u_0}^u f'(w)dw \\ &= f(u_0) - \int_{u_0}^u (\xi_w + O(1))dw \\ &= f(u_0) - u\xi_u + O(u); \end{aligned}$$

(6.2) will then follow, if we can show that $f(u_0) = O(1)$. Since by Proposition 3.1

$$(s_0 - 1)\mathcal{L} = -\xi_u + O(1),$$

we see from (6.1) that $f(u_0) = \phi(s_0) + O(1)$, where $s_0 = s_0(z^{u_0}, z)$. Now, by Lemma 5.1 and Proposition 3.1 we have

$$\begin{aligned} \phi(s_0) &= - \int_1^{(1-s_0)\mathcal{L}} \frac{e^w}{w} dw + O\left(\frac{z^{1-\sigma_0}}{\mathcal{L}^2}\right) + O(1) \\ &= O(e^{(1-s_0)\mathcal{L}}) = O(e^{\xi_{u_0}}) = O(1). \end{aligned}$$

Hence we obtain $f(u_0) = O(1)$, as desired.

It remains to prove the estimates (3.16) and (3.17). To this end we consider $\rho(z^u, z)$ as a function of z with u fixed, and define, in analogy to (6.1),

$$(6.5) \quad g(z) = \log \rho(z^u, z) = (s_0 - 1)u \log z + \phi(z; s_0) - \log s_0 + \log \frac{e^\gamma}{\sqrt{\pi u/2}}.$$

Setting $s_0 = s_0(z^u, z)$ and $\tilde{s}_0 = s_0(\tilde{z}^u, \tilde{z})$, the asserted estimates can be restated as

$$(6.6) \quad |g(\tilde{z}) - g(z)| \ll \frac{u}{z} |E(\tilde{z}, z)| + \frac{\Delta u}{\mathcal{L}^2} + \frac{u}{z^{1-\varepsilon}} \quad (z \leq \tilde{z} \leq 2z),$$

and

$$(6.7) \quad |g(\tilde{z}) - g(z)| \ll \frac{u}{\mathcal{L}} \quad (\tilde{z} \geq z),$$

where $\Delta = (\tilde{z} - z)/z$. We shall prove that (6.6) holds in the case

$$(6.8) \quad \Delta \leq \frac{1}{\mathcal{L}^3};$$

the relation (6.6) for $\mathcal{L}^{-3} \leq \Delta \leq 1$ and the relation (6.7) can be deduced from this by splitting the interval $(z, \tilde{z}]$ into subintervals of the form $(z_1, z_1(1 + \mathcal{L}^{-5})]$ (plus possibly one interval of shorter length) and applying (6.6) (with ε replaced by $\varepsilon/2$) along with the prime number theorem in the form $E(z_1(1 + \mathcal{L}^{-5}), z_1) \ll z_1 \mathcal{L}^{-7}$ to these intervals.

From (6.5) we obtain

$$g(z) - g(\tilde{z}) = R_1 + R_2 + R_3,$$

where

$$\begin{aligned} R_1 &= u(s_0 - \tilde{s}_0) \log z + \phi(z; s_0) - \phi(z; \tilde{s}_0), \\ R_2 &= \phi(z; \tilde{s}_0) - \phi(\tilde{z}; \tilde{s}_0) + u(1 - \tilde{s}_0) \log \frac{\tilde{z}}{z}, \\ R_3 &= \log(\tilde{s}_0/s_0). \end{aligned}$$

To prove (6.6), we shall show that each of the terms R_i is bounded by the right-hand side of (6.6).

We first derive an estimate for $\tilde{s}_0 - s_0$. By the definition of s_0 and \tilde{s}_0 we have

$$-u \log \tilde{z} + u \log z = \phi'(\tilde{z}; \tilde{s}_0) - \phi'(z; s_0) = R_4 + R_5$$

with

$$R_4 = \phi'(\tilde{z}; \tilde{s}_0) - \phi'(z; \tilde{s}_0) = \sum_{z \leq p < \tilde{z}} \frac{\log p}{p^{\tilde{s}_0} - 1},$$

$$R_5 = \phi'(z; \tilde{s}_0) - \phi'(z; s_0) = \int_{s_0}^{\tilde{s}_0} \phi''(z; s) ds.$$

Since

$$\begin{aligned} |R_4| &\ll \tilde{z}^{-\tilde{\sigma}_0} \sum_{z \leq p < \tilde{z}} \log p \\ &\ll \tilde{z}^{1-\tilde{\sigma}_0} \left(\Delta + \frac{1}{z^{1-\varepsilon}} \right) \\ &\ll e^{\xi u} \left(\Delta + \frac{1}{z^{1-\varepsilon}} \right) \ll u(\log u) \left(\Delta + \frac{1}{z^{1-\varepsilon}} \right) \end{aligned}$$

by a standard upper bound for primes in short intervals and Proposition 3.1, it follows that

$$\begin{aligned} (6.9) \quad |R_5| &\ll |R_4| + u \log \frac{\tilde{z}}{z} \\ &\ll u(\log u) \left(\Delta + \frac{1}{z^{1-\varepsilon}} \right). \end{aligned}$$

On the other hand, since by Proposition 3.1 and our assumption (6.8)

$$\begin{aligned} \tilde{s}_0 - s_0 &= \frac{\xi_u - i\pi}{\log z} - \frac{\xi_u - i\pi}{\log \tilde{z}} + O\left(\frac{1}{\log z \log u}\right) \\ &= O\left(\frac{\log(\tilde{z}/z)}{\mathcal{L}^2} \log u\right) + O\left(\frac{1}{\mathcal{L} \log u}\right) \\ &= O\left(\frac{\Delta \log u}{\mathcal{L}^2}\right) + O\left(\frac{1}{\mathcal{L} \log u}\right) = O\left(\frac{1}{\mathcal{L} \log u}\right), \end{aligned}$$

we obtain from Lemma 5.1 for $s \in [s_0, \tilde{s}_0]$

$$\begin{aligned} (6.10) \quad \phi''(z; s) &= -(1 + O\left(\frac{1}{\mathcal{L}}\right)) \frac{z^{1-s}((1-s)\mathcal{L} - 1)}{(1-s)^2} + O(1) \\ &= -(1 + O\left(\frac{1}{\log u}\right)) \frac{z^{1-s_0}\mathcal{L}}{1-s_0} \\ &= (1 + O\left(\frac{1}{\log u}\right)) u \mathcal{L}^2, \end{aligned}$$

so that

$$|R_5| = \left| \int_{s_0}^{\tilde{s}_0} \phi''(z; s) ds \right| \gg u \mathcal{L}^2 |\tilde{s}_0 - s_0|.$$

Comparing this bound with (6.9), we deduce

$$(6.11) \quad |\tilde{s}_0 - s_0| \ll \frac{\log u}{\mathcal{L}^2} \left(\Delta + \frac{1}{z^{1-\varepsilon}} \right).$$

Since $R_3 = \log(\tilde{s}_0/s_0) = O(|\tilde{s}_0 - s_0|)$, this shows that R_3 is bounded by the right-hand side of (6.6).

To estimate R_1 , we write

$$\begin{aligned}\phi(z; s_0) - \phi(z; \tilde{s}_0) &= - \int_{s_0}^{\tilde{s}_0} \phi'(z; s) ds \\ &= -\phi'(z; s_0)(\tilde{s}_0 - s_0) - \int_{s_0}^{\tilde{s}_0} \phi''(z; s)(\tilde{s}_0 - s) ds\end{aligned}$$

and note that, by (3.9), the first term on the right-hand side is equal to $u\mathcal{L}(\tilde{s}_0 - s_0)$. It follows that

$$\begin{aligned}|R_1| &= \left| \int_{s_0}^{\tilde{s}_0} \phi''(z; s)(\tilde{s}_0 - s) ds \right| \\ &\leq |\tilde{s}_0 - s_0|^2 \max_{s \in [s_0, \tilde{s}_0]} |\phi''(z; s)| \\ &\ll \frac{\log^2 u}{\mathcal{L}^4} (\Delta + \frac{1}{z^{1-\varepsilon}})^2 u\mathcal{L}^2 \ll \frac{u \log^2 u}{\mathcal{L}^5} (\Delta + \frac{1}{z^{1-\varepsilon}}) \ll \frac{\Delta u}{\mathcal{L}^3} + \frac{u}{z^{1-\varepsilon}}\end{aligned}$$

by (6.11), (6.10), and (6.8). This proves the desired bound for R_1 .

It remains to estimate R_2 . We have

$$\begin{aligned}(6.12) \quad R_2 &= u(1 - \tilde{s}_0) \log \frac{\tilde{z}}{z} - \sum_{z \leq p < \tilde{z}} \log(1 - \frac{1}{p^{\tilde{s}_0}}) \\ &= u(1 - \tilde{s}_0) \log \frac{\tilde{z}}{z} + \frac{1}{\tilde{z}^{\tilde{s}_0} \log \tilde{z}} \sum_{z \leq p < \tilde{z}} \log p \\ &\quad + O\left(\sum_{z \leq p < \tilde{z}} \left| \frac{\log p}{\tilde{z}^{\tilde{s}_0} \log \tilde{z}} - \frac{1}{p^{\tilde{s}_0}} \right| \right) + O(|\tilde{z} - z + 1| \tilde{z}^{-2\tilde{\sigma}_0}).\end{aligned}$$

The second error term here is of order

$$\begin{aligned}&\ll \tilde{z}^{2(1-\tilde{\sigma}_0)} \left(\frac{\Delta}{\tilde{z}} + \frac{1}{\tilde{z}^2} \right) \\ &\ll (u \log u)^2 \left(\frac{\Delta}{z} + \frac{1}{z^2} \right) \ll \frac{u\mathcal{L}^2}{z^\varepsilon} (\Delta + \frac{1}{z})\end{aligned}$$

by Proposition 3.1 and the hypothesis $u \leq z^{1-\varepsilon}$, and the first can be bounded, using (6.8), by

$$\begin{aligned}&\ll \tilde{z}^{-\tilde{\sigma}_0} \sum_{z \leq p < \tilde{z}} \log(\tilde{z}/z) \ll \tilde{z}^{1-\tilde{\sigma}_0} \frac{\tilde{z} - z + 1}{\tilde{z}} \log(\tilde{z}/z) \\ &\ll (u \log u) (\Delta + \frac{1}{z}) \Delta \ll \frac{\Delta u \log u}{\mathcal{L}^3} + \frac{\Delta u \log u}{z} \ll \frac{\Delta u}{\mathcal{L}^2}.\end{aligned}$$

Moreover, since by Lemma 5.1 and (3.9)

$$\begin{aligned}\tilde{z}^{1-\tilde{s}_0} &= \frac{\tilde{z}^{1-\tilde{s}_0}}{1-\tilde{s}_0} (1-\tilde{s}_0) \\ &= (1 + O(\frac{1}{\mathcal{L}^2})) \phi'(\tilde{z}; \tilde{s}_0) (1-\tilde{s}_0) + O(|1-\tilde{s}_0|) \\ &= - \left(1 + O(\frac{1}{\mathcal{L}^2}) + O(\frac{1}{u\mathcal{L}}) \right) u(1-\tilde{s}_0) (\log \tilde{z}),\end{aligned}$$

we have

$$\begin{aligned}
\frac{1}{\tilde{z}^{\tilde{s}_0} \log \tilde{z}} \sum_{z \leq p < \tilde{z}} \log p &= \frac{\tilde{z} - z + E(\tilde{z}, z)}{\tilde{z}^{\tilde{s}_0} \log \tilde{z}} \\
&= - \left(1 + O\left(\frac{1}{\mathcal{L}^2}\right) + O\left(\frac{1}{u\mathcal{L}}\right) \right) u(1 - \tilde{s}_0) \left(\frac{\tilde{z} - z}{\tilde{z}} + \frac{E(\tilde{z}, z)}{\tilde{z}} \right) \\
&= -u(1 - \tilde{s}_0) \left(\log \frac{\tilde{z}}{z} + \frac{E(\tilde{z}, z)}{\tilde{z}} \right) + O\left(\frac{u\Delta}{\mathcal{L}^2}\right) + O(u\Delta^2) + O\left(\frac{|1 - \tilde{s}_0|\Delta}{\mathcal{L}}\right) + O\left(\frac{u}{z^{1-\varepsilon}}\right).
\end{aligned}$$

Inserting these estimates into (6.12), we obtain

$$|R_2| \ll \frac{u}{z} |E(\tilde{z}, z)| + \frac{u\Delta}{\mathcal{L}^2} + u\Delta^2 + \frac{|1 - \tilde{s}_0|\Delta}{\mathcal{L}} + \frac{u}{z^{1-\varepsilon}},$$

which proves the desired bound (6.6) for R_2 in view of (6.8) and since $|1 - \tilde{s}_0| \ll \xi_u/\mathcal{L} \ll \log u/\mathcal{L}$ by Proposition 3.1 and (3.2).

The proof of Proposition 3.2 is thus complete.

7. Proof of Theorem C: Preliminaries

In this section we shall derive estimates for the functions $F(z; s)$ and $\phi(z; s)$ that will be needed in the proof of Theorem C. We first state two auxiliary results from prime number theory.

Lemma 7.1. *Let $0 < \varepsilon \leq 1$ be fixed. Then we have, uniformly for $z \geq 2$, $0 \leq \sigma < 1$, and $|t| \leq \exp((\log z)^{3/2-\varepsilon})$,*

$$\sum_{n < z} \frac{\Lambda(n)}{n^s} = \frac{z^{1-s} - 1}{1-s} + O\left(\frac{z^{1-\sigma}}{1-\sigma} \exp(-(\log z)^{\varepsilon/2})\right) + O(1).$$

Proof. This result is stated, in a slightly weaker form, in [H-T, Lemma 6]. It can be established in a standard manner by writing the left-hand side as a Perron integral over the function ζ'/ζ and then shifting the line of integration into the critical strip, using Vinogradov's zero-free region. We omit the details.

Lemma 7.2. *For any fixed $\varepsilon > 0$ and uniformly for $\sigma \geq \varepsilon$, $|t| \geq 1$, we have*

$$|\zeta(\sigma + it)| \ll \max(1, |t|^{1-\sigma}) \log(1 + |t|),$$

$$|\zeta'(\sigma + it)| \ll \max(1, |t|^{1-\sigma}) \log^2(1 + |t|).$$

Proof. The first estimate is well-known; see, e. g., [Iv, Theorem 1.9]. The second estimate follows from the first by Cauchy's integral formula.

For the remainder of this section we let $z \geq 2$ be fixed and write $F(s) = F(z; s)$, $\phi(s) = \phi(z; s)$, and $\mathcal{L} = \log z$. The next three lemmas give upper bounds for these functions for various ranges of values s .

Lemma 7.3. *Let $0 < \varepsilon \leq 1$ be fixed. Then we have, uniformly for $\sigma \geq \varepsilon$ and $|t| \geq 1$,*

$$|F(s)| \ll \begin{cases} \mathcal{L}|t|^{1/\mathcal{L}} \log(1+|t|) & \text{if } \sigma \geq 1 - \frac{1}{\mathcal{L}}, \\ \mathcal{L}|t|^{1-\sigma} \log(1+|t|) \exp\left(\frac{2z^{1-\sigma}}{(1-\sigma)\mathcal{L}}\right) & \text{if } \varepsilon \leq \sigma < 1 - \frac{1}{\mathcal{L}}. \end{cases}$$

Proof. Lemma 7.2 yields

$$\begin{aligned} |F(s)| &= |\zeta(s) \prod_{p < z} (1 - \frac{1}{p^s})| \\ &\leq |\zeta(s)| \prod_{p < z} (1 + \frac{1}{p^\sigma}) \\ &\ll \max(1, |t|^{1-\sigma}) \log(1+|t|) \prod_{p < z} (1 + \frac{1}{p^\sigma}). \end{aligned}$$

In the case $\sigma \geq 1 - 1/\mathcal{L}$ the desired bound follows from the estimate

$$\begin{aligned} \prod_{p < z} (1 + \frac{1}{p^\sigma}) &\leq \exp\left\{\sum_{p < z} \frac{1}{p^{1-1/\mathcal{L}}}\right\} \\ &= \exp\left\{\sum_{p < z} \frac{1}{p} + O(1)\right\} \ll \mathcal{L}. \end{aligned}$$

In the remaining case $\varepsilon \leq \sigma \leq 1 - 1/\mathcal{L}$ we obtain the result from the estimates

$$\prod_{p < z} (1 + \frac{1}{p^\sigma}) \ll \left| \zeta(\sigma)(1-\sigma) \prod_{p < z} (1 - \frac{1}{p^\sigma}) \right|^{-1} = \frac{\exp(-\operatorname{Re}\phi(\sigma))}{1-\sigma} \leq \mathcal{L} \exp(|\phi(\sigma)|)$$

and (cf. Lemma 5.1)

$$\begin{aligned} |\phi(\sigma)| &= (1 + O(\frac{1}{\mathcal{L}})) \int_1^{(1-\sigma)\mathcal{L}} \frac{e^w}{w} dw + O(1) \\ &\leq (1 + O(\frac{1}{\mathcal{L}})) \frac{z^{1-\sigma}}{(1-\sigma)\mathcal{L}} + O\left(\frac{z^{1-\sigma}}{(1-\sigma)^2\mathcal{L}^2}\right) + O(1) \\ &\leq \frac{2z^{1-\sigma}}{(1-\sigma)\mathcal{L}} + O(1). \end{aligned}$$

Lemma 7.4. *Let $0 < \varepsilon \leq 1$ be fixed. Uniformly in the range*

$$(7.1) \quad \varepsilon \leq \sigma \leq 1 - \frac{1}{\mathcal{L}}, \quad 1 \leq |t| \leq \exp(\mathcal{L}^{3/2-\varepsilon})$$

we have

$$(7.2) \quad |F(s)| \ll |t|^{1-\sigma} \log^2(1+|t|) \exp\left((1+\varepsilon)\frac{z^{1-\sigma}}{\mathcal{L}}\right),$$

$$(7.3) \quad |F'(s)| \ll |t|^{1-\sigma} \log^3(1+|t|) z^{1-\sigma} \exp\left((1+\varepsilon)\frac{z^{1-\sigma}}{\mathcal{L}}\right).$$

Proof. Setting

$$\psi(s) = \sum_{p < z} \log\left(1 - \frac{1}{p^s}\right)$$

we have by Lemma 7.2 in the range (7.1)

$$|F(s)| = |\zeta(s)| \exp(\operatorname{Re}\psi(s)) \ll |t|^{1-\sigma} \log(1 + |t|) \exp(|\psi(s)|).$$

Thus, to prove (7.2) it suffices to show that

$$(7.4) \quad |\psi(\sigma + it)| \leq (1 + \varepsilon) \frac{z^{1-\sigma}}{\mathcal{L}} + \log_2(1 + |t|) + O(1)$$

holds in this range. Similarly, since

$$\begin{aligned} |F'(s)| &= |\zeta'(s)| \prod_{p < z} \left(1 - \frac{1}{p^s}\right) + \zeta(s) \prod_{p < z} \left(1 - \frac{1}{p^s}\right) \sum_{p < z} \frac{\log p}{p^s - 1} \\ &\ll |t|^{1-\sigma} \log^2(1 + |t|) \exp(|\psi(s)|) \left(1 + \left| \sum_{p < z} \frac{\log p}{p^s - 1} \right| \right) \end{aligned}$$

by Lemma 7.2, (7.3) is seen to be a consequence of (7.4) and the bound

$$(7.5) \quad \left| \sum_{p < z} \frac{\log p}{p^s - 1} \right| \ll z^{1-\sigma}.$$

Lemma 7.1 yields for the range (7.1)

$$\begin{aligned} \sum_{p < z} \frac{\log p}{p^s - 1} &= \sum_{n < z} \frac{\Lambda(n)}{n^s} + O\left(\sum_{p < z} \frac{\log p}{p^{2\sigma}}\right) + O(1) \\ &= \frac{z^{1-s} - 1}{1-s} + O\left(\frac{z^{1-\sigma}}{1-\sigma} \exp(-\mathcal{L}^{\varepsilon/2})\right) + O(1). \end{aligned}$$

Since $|1-s| \geq |t| \geq 1$ and $1-\sigma \geq 1/\mathcal{L}$ by (7.1), this gives (7.5). The same estimate shows that for $\sigma \leq \sigma' \leq 1 - 1/\mathcal{L}$ we have

$$|\psi'(\sigma' + it)| = \left| \sum_{p < z} \frac{\log p}{p^{\sigma'+it} - 1} \right| \leq \left(1 + O\left(\frac{1}{\mathcal{L}}\right)\right) z^{1-\sigma'} + O(1).$$

This implies

$$\begin{aligned} |\psi(\sigma + it) - \psi\left(1 - \frac{1}{\mathcal{L}} + it\right)| &\leq \left(1 + O\left(\frac{1}{\mathcal{L}}\right)\right) \int_{\sigma}^1 z^{1-\sigma'} d\sigma' + O(1) \\ &\leq \left(1 + O\left(\frac{1}{\mathcal{L}}\right)\right) \frac{z^{1-\sigma}}{\mathcal{L}} + O(1) \leq (1 + \varepsilon) \frac{z^{1-\sigma}}{\mathcal{L}} + O(1), \end{aligned}$$

and since

$$\begin{aligned} \left| \psi\left(1 - \frac{1}{\mathcal{L}} + it\right) \right| &= \left| \sum_{p < z} \frac{1}{p^{1-1/\mathcal{L}+it}} \right| + O(1) = \left| \sum_p \frac{1}{p^{1+1/\mathcal{L}+it}} \right| + O(1) \\ &= \left| \log \zeta\left(1 + \frac{1}{\mathcal{L}} + it\right) \right| + O(1) \leq \log_2(1 + |t|) + O(1) \end{aligned}$$

by Lemma 7.2, (7.4) follows.

For the last lemma of this section we let ε be a given positive number, y and z fixed numbers in the range (R_ε) and $u = \log y / \log z$. We further let $s_0 = s_0(y, z) = \sigma_0 + it_0$ be defined as in the theorem by (3.9) and (3.10). In what follows, the implied constants are allowed to depend at most on ε .

Lemma 7.5. For $s = \sigma_0 + it$, $|t| \leq 1$, $||t| - t_0| > 1/\mathcal{L}$, we have

$$(7.6) \quad |F(s)| \ll |F(s_0)| \exp\left(-\frac{u}{3 \log^2 u}\right).$$

Moreover, we have

$$(7.7) \quad |F(s_0)| = \exp\left\{u\left(1 + O\left(\frac{1}{\log u}\right)\right)\right\}.$$

Proof. Lemma 5.1 yields, for $s = \sigma + it$ as in the lemma,

$$F(s) = \exp(\phi(s)) = \exp\left\{-\text{Ei}((1-s)\mathcal{L}) + O\left(\frac{z^{1-\sigma}}{\mathcal{L}^4}\right) + O(1)\right\},$$

where

$$\text{Ei}(w) = \int_1^w \frac{e^v}{v} dv$$

is a variant of the exponential integral. By Proposition 3.1 and (3.2) the first error term can be bounded by

$$\frac{z^{1-\sigma_0}}{\mathcal{L}^4} \ll \frac{e^{\xi_u}}{\mathcal{L}^4} \ll \frac{u \log u}{\mathcal{L}^4} \ll \frac{u}{\log^3 u}.$$

Thus, the estimates (7.6) and (7.7) are seen to be equivalent to

$$(7.8) \quad \text{Re}\{\text{Ei}((1-s_0)\mathcal{L}) - \text{Ei}((1-s)\mathcal{L})\} \leq -\frac{u}{3 \log^2 u} + O(1)$$

and

$$(7.9) \quad \text{Re}\{\text{Ei}(1-s_0)\mathcal{L}\} = -u\left(1 + O\left(\frac{1}{\log u}\right)\right),$$

respectively, and it suffices to prove the last two estimates.

We shall use the easily derived expansion

$$(7.10) \quad \text{Ei}(w) = e^w \left(\frac{1}{w} + \frac{1}{w^2} + \frac{2}{w^3} + O\left(\frac{1}{|\text{Re} w|^4}\right)\right),$$

which holds uniformly in $\text{Re} w \geq 1$. Applying (7.10) first with $w = (1-s_0)\mathcal{L}$ gives

$$(7.11) \quad \text{Ei}((1-s_0)\mathcal{L}) = \frac{z^{1-s_0}}{(1-s_0)\mathcal{L}} \left(1 + \frac{1}{(1-s_0)\mathcal{L}} + \frac{2}{(1-s_0)^2 \mathcal{L}^2} + O\left(\frac{|(1-s_0)\mathcal{L}|}{(1-s_0)^4 \mathcal{L}^4}\right)\right).$$

Now,

$$(7.12) \quad \begin{aligned} \frac{z^{1-s_0}}{(1-s_0)\mathcal{L}} &= (1 + O\left(\frac{1}{\mathcal{L}^3}\right)) \frac{\phi'(s_0)}{\mathcal{L}} + O\left(\frac{1}{\mathcal{L}}\right) \\ &= -(1 + O\left(\frac{1}{\mathcal{L}^3}\right))u + O\left(\frac{1}{\mathcal{L}}\right) = -(1 + O\left(\frac{1}{\log^3 u}\right))u \end{aligned}$$

by Lemma 5.1 and (3.9), and since

$$|(1-s_0)\mathcal{L}| \asymp (1-\sigma_0)\mathcal{L} \asymp \xi_u \asymp \log u$$

by Proposition 3.1, we see that (7.11) implies (7.9). A further application of Proposition 3.1 shows that

$$\begin{aligned}\operatorname{Re}\frac{1}{(1-s_0)\mathcal{L}} &= \operatorname{Re}\left\{\frac{1}{(1-\sigma_0)\mathcal{L}}\left(1+i\frac{t_0\mathcal{L}}{(1-\sigma_0)\mathcal{L}}\right)+O\left(\frac{(t_0\mathcal{L})^2}{(1-\sigma_0)^3\mathcal{L}^3}\right)\right\} \\ &= \frac{1}{(1-\sigma_0)\mathcal{L}}+O\left(\frac{1}{\log^3 u}\right)=\frac{1}{\xi_u}+O\left(\frac{1}{\log^3 u}\right)\end{aligned}$$

and

$$\operatorname{Re}\frac{1}{(1-s_0)^2\mathcal{L}^2}=\frac{1}{(1-\sigma_0)^2\mathcal{L}^2}+O\left(\frac{1}{\log^3 u}\right)=\frac{1}{\xi_u^2}+O\left(\frac{1}{\log^3 u}\right).$$

Therefore we obtain from (7.11) and (7.12)

$$(7.13) \quad \operatorname{Re}(\operatorname{Ei}((1-s_0)\mathcal{L}))=-u\left(1+\frac{1}{\xi_u}+\frac{2}{\xi_u^2}+O\left(\frac{1}{\log^3 u}\right)\right).$$

Next we apply (7.10) with $w=(1-\sigma_0-it)\mathcal{L}$, where $|t|\leq 1$. Estimating each term on the right-hand side by its modulus we obtain in the same way as before

$$(7.14) \quad \begin{aligned}|\operatorname{Ei}((1-s)\mathcal{L})| &\leq \frac{z^{1-\sigma_0}}{|1-s|\mathcal{L}}\left(1+\frac{1}{\xi_u}+\frac{2}{\xi_u^2}\right)+O\left(\frac{z^{1-\sigma_0}}{\log^4 u}\right) \\ &= \frac{|1-s_0|u}{|1-s|}\left(1+\frac{1}{\xi_u}+\frac{2}{\xi_u^2}\right)+O\left(\frac{u}{\log^3 u}\right).\end{aligned}$$

Now, if $1\geq |t|\geq t_0+1/\mathcal{L}$, then

$$\begin{aligned}\frac{|1-s_0|}{|1-s|} &= \left(\frac{(1-\sigma_0)^2+t_0^2}{(1-\sigma_0)^2+t^2}\right)^{1/2}\leq\left(1+\frac{1}{((1-\sigma_0)^2+t_0^2)\mathcal{L}^2}\right)^{-1/2} \\ &= 1-\frac{1}{2((1-\sigma_0)^2+t_0^2)\mathcal{L}^2}+O\left(\frac{1}{\mathcal{L}^4(1-\sigma_0)^4}\right) \\ &= 1-\frac{1}{2\xi_u^2}+O\left(\frac{1}{\log^3 u}\right).\end{aligned}$$

Thus, comparison of (7.13) with (7.14) shows that

$$\operatorname{Re}(\operatorname{Ei}((1-s_0)\mathcal{L}))+|\operatorname{Ei}((1-s)\mathcal{L})|\leq-\frac{u}{2\xi_u^2}+O\left(\frac{u}{\log^3 u}\right),$$

and hence (7.8), holds in the range $t_0+1/\mathcal{L}\leq |t|\leq 1$.

Finally, to prove (7.8) for $|t|\leq t_0-1/\mathcal{L}$ we apply (7.10) with $w=(1-s)\mathcal{L}$ to obtain

$$\begin{aligned}\operatorname{Re}(\operatorname{Ei}((1-s)\mathcal{L})) &= \operatorname{Re}\frac{z^{1-s}}{(1-s)\mathcal{L}}+O\left(\frac{z^{1-\sigma_0}}{(1-\sigma_0)^2\mathcal{L}^2}\right) \\ &= \frac{z^{1-\sigma_0}}{(1-\sigma_0)\mathcal{L}}\cos(t\mathcal{L})+O\left(\frac{u}{\log u}\right) \\ &= u\left(\cos(t\mathcal{L})+O\left(\frac{1}{\log u}\right)\right).\end{aligned}$$

Since $-\cos(t\mathcal{L})\leq 0.9+O(1/\log u)$ for $|t|\leq t_0-1/\mathcal{L}=(\pi-1+O(1/\log u))/\mathcal{L}$, this implies

$$-\operatorname{Re}(\operatorname{Ei}((1-s)\mathcal{L}))\leq\frac{9}{10}u\left(1+O\left(\frac{1}{\log u}\right)\right),$$

which, in conjunction with (7.13), proves (7.8) for the range $|t| \leq t_0 - 1/\mathcal{L}$.

8. Proof of Theorem C: Completion

We begin by expressing $r(y, z)$ and $r(y, z; \lambda)$ as Perron integrals over the functions $F(s) = F(z; s)$.

Lemma 8.1. *For any fixed $\varepsilon > 0$ and uniformly for $y > 0$, $z \geq z_0$, $\varepsilon < \sigma < 1$, and $T \geq 1$ we have*

$$(8.1) \quad \begin{aligned} r(y, z) V(z) &= \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \frac{F(s)y^{s-1}}{s} ds + O\left(\frac{\log \bar{y}}{T}\right) \\ &+ O\left(\frac{\log \bar{y}}{y}\right) + O\left(\left(1 + \frac{1}{y}\right) \frac{\mathcal{L}}{T^{\varepsilon/2}} \exp\left(\frac{2z^{1-\sigma}}{(1-\sigma)\mathcal{L}}\right)\right), \end{aligned}$$

where $V(z) = \prod_{p < z} (1 - 1/p)$, $\bar{y} = \max(y, 3)$, $\mathcal{L} = \log z$, and the implied constants depend at most on ε .

Proof. Since the right-hand side of (8.1) is a continuous function of y , and since

$$r(y, z) V(z) = \frac{1}{y} \Phi(y, z) - V(z),$$

as a function of y , has jumps of order at most $O(1/y)$, it suffices to prove (8.1) in the case y is not an integer. Perron's formula then gives, for any $\sigma_1 > 1$,

$$\Phi(y, z) = \frac{1}{2\pi i} \int_{\sigma_1-iT}^{\sigma_1+iT} \frac{F(s)y^s}{s} ds + O\left(\sum_{n=1}^{\infty} \left(\frac{y}{n}\right)^{\sigma_1} \min\left(1, \frac{1}{T|\log(y/n)|}\right)\right).$$

Taking $\sigma_1 = 1 + 1/\log \bar{y}$, the error term becomes

$$\begin{aligned} &\ll \frac{y+1}{T} \sum_{n=1}^{\infty} \frac{1}{n^{\sigma_1}} + \sum_{y/2 < n < 2y} \min\left(1, \frac{y}{T|y-n|}\right) \\ &\ll \frac{y+1}{T(\sigma_1-1)} + \frac{y \log \bar{y}}{T} + 1 \ll \frac{y \log \bar{y}}{T} + \log \bar{y}. \end{aligned}$$

Replacing the line of integration by the line $[\sigma - iT, \sigma + iT]$ introduces a residue $yV(z)$ from the pole of $F(s) = \zeta(s) \prod_{p < z} (1 - 1/p^s)$ at $s=1$, and an error term which is of order

$$\begin{aligned} &\ll \left(\int_{\sigma-iT}^{\sigma_1-iT} + \int_{\sigma+iT}^{\sigma_1+iT} \right) \left| \frac{F(s)y^s}{s} \right| |ds| \\ &\ll \frac{y+1}{T} \max_{\sigma' \geq \sigma} (|F(\sigma' + iT)| + |F(\sigma' - iT)|) \\ &\ll \frac{y+1}{T} T^{1-\sigma+1/\mathcal{L}} \log(1+T) \mathcal{L} \exp\left(-\frac{2z^{1-\sigma}}{(1-\sigma)\mathcal{L}}\right) \\ &\ll \frac{y+1}{T^{\varepsilon/2}} \mathcal{L} \exp\left(\frac{2z^{1-\sigma}}{(1-\sigma)\mathcal{L}}\right) \end{aligned}$$

by Lemma 7.3. Thus we have

$$\begin{aligned} yr(y, z) V(z) &= \phi(y, z) - y V(z) \\ &= \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \frac{F(s)y^s}{s} ds + O\left(\frac{y \log \bar{y}}{T}\right) + O(\log \bar{y}) \\ &\quad + O\left(\frac{y+1}{T^{\varepsilon/2}} \mathcal{L} \exp\left(\frac{2z^{1-\sigma}}{(1-\sigma)\mathcal{L}}\right)\right), \end{aligned}$$

from which (8.1) follows.

For the remainder of this section, we fix a positive number ε , which we can assume to be sufficiently small, and real numbers y and z in the range R_ε , and we let $s_0 = \sigma_0 + it_0$ be defined by (3.9) and (3.10). Thus, in particular, we may assume that y, z , and $u = \log y / \log z$, are larger than any given constant. We also fix a real number λ in the range

$$(8.2) \quad 2 \leq \lambda \leq \exp(\mathcal{L}^{3/2-\varepsilon}),$$

and define the function

$$(8.3) \quad k(s) = k_\lambda(s) = \exp\left(\frac{s^2}{2\lambda}\right).$$

In the sequel, the constants implied in the symbols " O " and " \ll " are allowed to depend on ε , but not on y, z , or λ .

Lemma 8.2. *With $T_0 = y^{3/\varepsilon}$ we have*

$$(8.4) \quad r(y, z; \lambda) V(z) = \frac{1}{2\pi i} \int_{\sigma_0-iT_0}^{\sigma_0+iT_0} \frac{F(s)y^{s-1}}{s} k(s-1) ds + O\left(\frac{\log y}{y}\right).$$

Proof. Applying (8.1) with $\sigma = \sigma_0$, $T = T_0$ and ye^v in place of y , we obtain

$$\begin{aligned} r(y, z; \lambda) V(z) &= \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r(ye^v, z) V(z) e^{-\lambda v^2/2} dv \\ &= \frac{1}{2\pi i} \int_{\sigma_0-iT_0}^{\sigma_0+iT_0} \frac{F(s)y^{s-1}}{s} k(s-1) ds + O\left(\sum_{i=1}^3 R_i\right), \end{aligned}$$

where

$$k(s-1) = \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{v(s-1)-\lambda v^2/2} dv = e^{(s-1)^2/2\lambda}$$

is defined as in the lemma by (8.3),

$$\begin{aligned} R_1 &= \sqrt{\lambda} \int_{-\infty}^{\infty} \frac{\log(\overline{ye^v})}{T_0} e^{-\lambda v^2/2} dv \\ &\ll \sqrt{\lambda} \int_{-\infty}^{\infty} \frac{\log y + e^v}{T_0} e^{-\lambda v^2/2} dv \ll \frac{\log y}{y^{3/\varepsilon}}, \\ R_2 &= \sqrt{\lambda} \int_{-\infty}^{\infty} \frac{\log(\overline{ye^v})}{ye^v} e^{-\lambda v^2/2} dv \\ &\ll \sqrt{\lambda} \frac{\log y}{y} \int_{-\infty}^{\infty} (1 + e^{|v|}) e^{-\lambda v^2/2} dv \ll \frac{\log y}{y}, \end{aligned}$$

and

$$\begin{aligned} R_3 &= \frac{\sqrt{\lambda}}{T_0^{\varepsilon/2}} \mathcal{L} \exp\left(\frac{2z^{1-\sigma_0}}{(1-\sigma_0)\mathcal{L}}\right) \int_{-\infty}^{\infty} \left(1 + \frac{1}{ye^v}\right) e^{-\lambda v^2/2} dv \\ &\ll \frac{\mathcal{L}}{y^{3/2}} \exp\left(\frac{2z^{1-\sigma_0}}{(1-\sigma_0)\mathcal{L}}\right). \end{aligned}$$

The error terms R_1 and R_2 are clearly of order $O(\log y/y)$ (assuming, as we may, that $\varepsilon \leq 2$), and the same holds for the term R_3 , since $\mathcal{L} = \log z \ll \log y$, and by Proposition 3.1

$$\frac{z^{1-\sigma_0}}{(1-\sigma_0)\mathcal{L}} \ll u = \frac{\log y}{\mathcal{L}}.$$

This proves (8.4).

Lemma 8.3. *We have*

$$(8.5) \quad \begin{aligned} r(y, z; \lambda)V(z) &= \frac{1}{2\pi i} \int_{\sigma_0-i}^{\sigma_0+i} \frac{F(s)y^{s-1}}{s} k(s-1) ds \\ &\quad + O\left(\frac{y^{\sigma_0-1} e^{(1-\frac{\varepsilon}{3})u} \lambda^{1-\sigma_0} (\log \lambda)^4}{\mathcal{L}}\right). \end{aligned}$$

Proof. In view of the previous lemma and since

$$\frac{\log y}{y} \ll \frac{y^{\varepsilon/2-1}}{\mathcal{L}} \leq \frac{y^{\sigma_0-1}}{\mathcal{L}}$$

it suffices to show that the contribution of the range $1 \leq |t| \leq T_0$ to the integral in (8.4) is bounded by the error term in (8.5). We split this range into the two parts $1 \leq |t| \leq T_1$ and $T_1 < |t| \leq T_0$, where

$$T_1 = \min(T_0, \exp(\mathcal{L}^{3/2-\varepsilon})).$$

Since

$$(8.6) \quad |k(\sigma_0 - 1 + it)| = \exp\left(\frac{(\sigma_0 - 1)^2 - t^2}{2\lambda}\right) \ll e^{-t^2/2\lambda}$$

and

$$(8.7) \quad \begin{aligned} \max_{|t| \leq T_0} |F(\sigma_0 + it)| &\ll T_0^{1-\sigma_0} \log(1 + T_0) \mathcal{L} \exp\left(\frac{2z^{1-\sigma_0}}{(1-\sigma_0)\mathcal{L}}\right) \\ &\ll T_0 \mathcal{L} \exp(O(u)) \ll y^{4/\varepsilon} \end{aligned}$$

by Lemma 7.3 and Proposition 3.1, the integral over the range $T_1 < |t| \leq T_0$ is bounded by

$$\begin{aligned} &\ll \max_{|t| \leq T_0} |F(\sigma_0 + it)| \int_{T_1}^{T_0} |k(\sigma_0 - 1 + it)| dt \\ &\ll y^{4/\varepsilon} \int_{T_1}^{\infty} e^{-t^2/2\lambda} dt \ll y^{4/\varepsilon} \sqrt{\lambda} e^{-T_1^2/2\lambda}. \end{aligned}$$

If $\lambda \leq T_0$, then we have $T_1 \geq \lambda$ by (8.2) and the definition of T_1 , and the above bound becomes

$$\ll y^{4/\varepsilon} \sqrt{\lambda} e^{-T_1/2} \ll y^{6/\varepsilon} e^{-z/2} \ll \frac{1}{y},$$

since $T_1 \geq \min(y, \exp(\mathcal{L})) = z$ and $z \gg (\log y)^{1/(1-\varepsilon/2)}$ in the range (R_ε) . On the other hand, if $\lambda > T_0$, then (8.2) implies $\exp(\mathcal{L}^{3/2-\varepsilon}) > T_0$, in which case $T_1 = T_0$ and the range $T_1 < |t| \leq T_0$ is empty. In either case the contribution of this range is of the desired order, and to prove (8.5) it remains to show that the same is true for the range $1 \leq |t| \leq T_1$.

Since $F(\bar{s}) = \overline{F(s)}$ and $k(\bar{s}) = \overline{k(s)}$, it suffices to consider the integral

$$I = \int_{\sigma_0+i}^{\sigma_0+iT_1} \frac{F(s)y^{s-1}}{s} k(s-1) ds.$$

Now,

$$I = iy^{\sigma_0-1} I_1,$$

where

$$I_1 = \int_1^{T_1} G(t)y^{it} dt$$

with

$$(8.8) \quad G(t) = \frac{F(\sigma_0 + it)}{\sigma_0 + it} k(\sigma_0 - 1 + it).$$

Hence, in order to prove the desired estimate, it suffices to show that

$$(8.9) \quad |I_1| \ll \frac{1}{\mathcal{L}} \lambda^{1-\sigma_0} (\log \lambda)^4 \exp\left\{\left(1 - \frac{\varepsilon}{3}\right)u\right\}.$$

At this point a trivial bound of I_1 by the integral over the modulus of $G(t)$ would not be sufficient to yield (8.9). Therefore, we have to proceed more carefully, and we first apply an integration by parts to get

$$I_1 = \frac{y^{it}}{i \log y} G(t) \Big|_1^{T_1} - \frac{1}{i \log y} \int_1^{T_1} G'(t)y^{it} dt,$$

which implies

$$|I_1| \ll \frac{1}{\mathcal{L}} \left\{ \max_{t=1, T_1} |G(t)| + \int_1^{T_1} |G'(t)| dt \right\}.$$

We shall estimate the terms on the right-hand side using Lemmas 7.4 and 7.5.

By Lemma 7.4 (with $\varepsilon/3$ in place of ε) and Proposition 3.1 we have, for $1 \leq t \leq T_1$,

$$\begin{aligned} |F(\sigma_0 + it)| &\ll t^{1-\sigma_0} \log^2(1+t) \exp\left\{\left(1 + \frac{\varepsilon}{3}\right) \frac{z^{1-\sigma_0}}{\mathcal{L}}\right\} \\ &\ll t^{1-\sigma_0} \log^2(1+t) \exp\left\{\left(1 + \frac{\varepsilon}{3}\right) \frac{e^{\xi_u}}{\mathcal{L}} \left(1 + O\left(\frac{1}{\log u}\right)\right)\right\} \\ &\ll t^{1-\sigma_0} \log^2(1+t) \exp\left\{\left(1 - \frac{\varepsilon}{3}\right)u\right\}, \end{aligned}$$

where in the last estimate we have used the identity $e^{\xi_u} = u\xi_u + 1$, the estimate (3.2) for ξ_u , and the bound $\log u \leq (1 - \varepsilon)\mathcal{L}$. It follows that

$$\begin{aligned} \max_{t=1, T_1} |G(t)| &\ll \max_{t=1, T_1} \frac{|F(\sigma_0 + it)|}{t} e^{-t^2/2\lambda} \\ &\ll \exp\left\{\left(1 - \frac{\varepsilon}{3}\right)u\right\}. \end{aligned}$$

A similar argument shows that, for $1 \leq t \leq T_1$,

$$|F'(\sigma_0 + it)| \ll t^{1-\sigma_0} \log^3(1+t) \exp\left\{\left(1 - \frac{\varepsilon}{3}\right)u\right\}$$

and hence

$$\begin{aligned} |G'(t)| &\ll \frac{|F'(\sigma_0 + it)|e^{-t^2/2\lambda}}{t} + \frac{|F(\sigma_0 + it)|e^{-t^2/2\lambda}}{t} \left(\frac{1}{t} + \frac{t}{\lambda}\right) \\ &\ll t^{-\sigma_0} \left(1 + \frac{t}{\lambda}\right) e^{-t^2/2\lambda} \log^3(1+t) \exp\left\{\left(1 - \frac{\varepsilon}{3}\right)u\right\}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_1^{T_1} |G'(t)| dt &\ll \left\{ \int_1^\lambda \frac{dt}{t^{\sigma_0}} + \frac{1}{\lambda} \int_\lambda^\infty t e^{-t^2/2\lambda} dt \right\} (\log \lambda)^3 \exp\left\{\left(1 - \frac{\varepsilon}{3}\right)u\right\} \\ &\ll \lambda^{1-\sigma_0} (\log \lambda)^4 \exp\left\{\left(1 - \frac{\varepsilon}{3}\right)u\right\}. \end{aligned}$$

Combining these estimates we obtain (8.9).

Lemma 8.4. *With $\delta = 1/\mathcal{L}$ we have*

$$\begin{aligned} (8.10) \quad r(y, z; \lambda)V(z) &= \frac{1}{\pi} \operatorname{Re} \int_{t_0-\delta}^{t_0+\delta} \frac{F(\sigma_0 + it)y^{\sigma_0-1+it}}{\sigma_0 + it} k(\sigma_0 - 1 + it) dt \\ &+ O\left(\frac{y^{\sigma_0-1} e^{(1-\frac{\varepsilon}{3})u} \lambda^{1-\sigma_0} (\log \lambda)^4}{\mathcal{L}}\right) + O\left(\frac{y^{\sigma_0-1} |F(s_0)|}{\mathcal{L}} \exp\left(-\frac{u}{4 \log^2 u}\right)\right). \end{aligned}$$

Proof. Since $F(\sigma_0 - it) = \overline{F(\sigma_0 + it)}$ and $k(\sigma_0 - 1 - it) = \overline{k(\sigma_0 - 1 + it)}$, the integrand in (8.10) changes into its complex conjugate when t is replaced by $-t$. Thus, the main term on the right-hand side of (8.10) can be written as

$$\frac{1}{2\pi i} \left(\int_{\sigma_0 - i(t_0 + \delta)}^{\sigma_0 - i(t_0 - \delta)} + \int_{\sigma_0 + i(t_0 - \delta)}^{\sigma_0 + i(t_0 + \delta)} \right) \frac{F(s)y^{s-1}}{s} k(s-1) ds.$$

By Lemma 8.3, it suffices then to show that the contribution of the ranges $|t| \leq t_0 - \delta$ and $t_0 + \delta \leq |t| \leq 1$ to the integral in (8.5) is of the order of the second error term in (8.10).

For the range $|t| \leq t_0 - \delta$ the trivial estimate

$$\left| \int_{\sigma_0 - i(t_0 - \delta)}^{\sigma_0 + i(t_0 - \delta)} \frac{F(s)y^{s-1}}{s} k(s-1) \right| \ll y^{\sigma_0-1} t_0 \max_{|t| \leq t_0 - \delta} |F(\sigma_0 + it)|$$

is sufficient, since $t_0 \ll 1/\mathcal{L}$ by Proposition 3.1 and

$$\max_{|t| \leq t_0 - \delta} |F(\sigma_0 + it)| \ll |F(s_0)| \exp\left(-\frac{u}{3 \log^2 u}\right)$$

by Lemma 7.5. The second range $t_0 + \delta \leq |t| \leq 1$ requires a more delicate argument. It suffices to consider the integral

$$\begin{aligned} I &= \int_{t_0+\delta}^1 \frac{F(\sigma_0 + it)y^{\sigma_0-1+it}}{\sigma_0 + it} k(\sigma_0 - 1 + it) dt \\ &= y^{\sigma_0-1} \int_{t_0+\delta}^1 G(t)y^{it} dt = y^{\sigma_0-1} I_1, \end{aligned}$$

say, where $G(t)$ is defined by (8.8). We have to show that

$$(8.11) \quad |I_1| \ll \frac{|F(s_0)|}{\mathcal{L}} \exp\left(-\frac{u}{4\log^2 u}\right).$$

Partial integration gives

$$I_1 \ll \frac{1}{\log y} \left\{ \max_{t_0+\delta \leq t \leq 1} |G(t)| + \left| \int_{t_0+\delta}^1 G'(t) y^{it} dt \right| \right\}.$$

By (8.8), (8.6) and Lemma 7.5 we have

$$(8.12) \quad \max_{t_0+\delta \leq t \leq 1} |G(t)| \ll \max_{t_0+\delta \leq t \leq 1} |F(\sigma_0 + it)| \ll |F(s_0)| \exp\left(-\frac{u}{3\log^2 u}\right).$$

Thus, in order to prove (8.11), it remains to show that the integral

$$I_2 = \int_{t_0+\delta}^1 G'(t) y^{it} dt$$

satisfies

$$(8.13) \quad |I_2| \ll |F(s_0)| \exp\left(-\frac{u}{4\log^2 u}\right).$$

By (8.8), (8.6), Lemma 5.1, and Proposition 3.1 we have, for $|t| \leq 1$,

$$(8.14) \quad \begin{aligned} G'(t) &= \frac{iF(\sigma_0 + it)k(\sigma_0 - 1 + it)}{\sigma_0 + it} \left\{ \frac{F'(\sigma_0 + it)}{F(\sigma_0 + it)} - \frac{1}{\sigma_0 + it} + \frac{k'(\sigma_0 - 1 + it)}{k(\sigma_0 - 1 + it)} \right\} \\ &= iG(t) \{ \phi'(\sigma_0 + it) + O(1) \} \\ &= iG(t) \left\{ \frac{z^{1-\sigma_0-it}}{1-\sigma_0-it} + O\left(\frac{z^{1-\sigma_0}}{(1-\sigma_0)\mathcal{L}}\right) + O(1) \right\} \\ &= G_1(t) z^{-it} + O(u|G(t)|), \end{aligned}$$

where

$$G_1(t) = \frac{iz^{1-\sigma_0}}{1-\sigma_0-it} G(t).$$

In view of the bound (8.12), the contribution of the O-term to the integral I_2 is of order

$$\ll u \max_{t_0+\delta \leq t \leq 1} |G(t)| \ll u |F(s_0)| \exp\left(-\frac{u}{3\log^2 u}\right),$$

which is sufficient. Moreover, a further integration by parts yields

$$(8.15) \quad \left| \int_{t_0+\delta}^1 G_1(t) z^{-it} y^{it} dt \right| \ll \frac{1}{\log(y/z)} \left\{ \max_{t_0+\delta \leq t \leq 1} |G_1(t)| + \int_{t_0+\delta}^1 |G_1'(t)| dt \right\}.$$

Since

$$\log(y/z) = (u-1)\mathcal{L} \gg u\mathcal{L}$$

if $u \geq 2$, as we may assume, and since

$$\begin{aligned}
|G_1(t)| &\ll \frac{z^{1-\sigma_0}}{|1-\sigma_0-it|} |G(t)| \ll \frac{u \log u}{|t|} |G(t)|, \\
|G'_1(t)| &\ll \frac{z^{1-\sigma_0}}{|1-\sigma_0-it|^2} |G(t)| + \frac{z^{1-\sigma_0}}{|1-\sigma_0-it|} |G'(t)| \\
&\ll \frac{z^{1-\sigma_0}}{t^2} |G(t)| + \frac{z^{1-\sigma_0}}{t} (|G_1(t)| + u|G(t)|) \\
&\ll \frac{u^2 \log^2 u}{t^2} |G(t)|
\end{aligned}$$

by (8.14) and Proposition 3.1, the right-hand side of (8.15) is bounded by

$$\begin{aligned}
&\ll \frac{1}{\mathcal{L}} \max_{t_0+\delta \leq t \leq 1} |G(t)| \left\{ \frac{u \log u}{t_0} + \int_{t_0+\delta}^1 \frac{u^2 \log^2 u}{t^2} dt \right\} \\
&\ll \frac{u^2 \log^2 u}{t_0 \mathcal{L}} \max_{t_0+\delta \leq t \leq 1} |G(t)| \\
&\ll u^2 (\log u)^2 |F(s_0)| \exp\left(-\frac{u}{3 \log^2 u}\right),
\end{aligned}$$

where the last estimate follows from (8.12) and the bound $t_0 \gg 1/\mathcal{L}$ (cf. (3.12)). This implies the desired bound (8.13) and completes the proof of Lemma 8.4.

Lemma 8.5. *With $\delta = 1/\mathcal{L}$ we have*

$$(8.16) \quad \int_{t_0-\delta}^{t_0+\delta} \frac{F(\sigma_0+it)y^{\sigma_0-1+it}}{\sigma_0+it} k(\sigma_0-1+it) dt = (1 + O(\frac{1}{\log u})) \frac{y^{s_0-1} F(s_0) k(s_0-1)}{s_0 \mathcal{L} \sqrt{u/2\pi}}.$$

Proof. The integrand in (8.16) can be written as

$$(8.17) \quad (1 + O(\frac{1}{\mathcal{L}})) \frac{y^{s_0-1} k(s_0-1)}{s_0} \exp\{i(t-t_0) \log y + \phi(\sigma_0+it)\},$$

since $t = t_0 + O(1/\mathcal{L}) = O(1/\mathcal{L})$ and $k(\sigma_0-1+it) = k(s_0-1)(1 + O(1/\mathcal{L}))$ in the range of integration. Expanding $\phi(\sigma_0+it)$ in a Taylor series about $t = t_0$, we obtain

$$\begin{aligned}
i(t-t_0) \log y + \phi(\sigma_0+it) &= \phi(s_0) + i(t-t_0)(\log y + \phi'(s_0)) \\
&\quad - \frac{1}{2}(t-t_0)^2 \phi''(s_0) - \frac{i}{3!}(\tau-t_0)^3 \phi'''(\sigma_0+i\tau)
\end{aligned}$$

with a suitable $\tau = \tau(t)$, $|\tau-t_0| \leq |t-t_0|$. By the definition of s_0 , the linear term here vanishes. Moreover, from Lemma 5.1 and Proposition 3.1 we see that

$$\begin{aligned}
\phi''(s_0) &= \frac{-z^{1-s_0} \mathcal{L}}{1-s_0} (1 + O(\frac{1}{(1-\sigma_0)\mathcal{L}})) \\
&= u \mathcal{L}^2 (1 + O(\frac{1}{\log u}))
\end{aligned}$$

and

$$|\phi'''(\sigma_0+i\tau)| \leq \frac{z^{1-\sigma_0} \mathcal{L}^2}{1-\sigma_0} (1 + O(\frac{1}{(1-\sigma_0)\mathcal{L}})) = u \mathcal{L}^3 (1 + O(\frac{1}{\log u})).$$

Thus we can write (8.17) as

$$(8.18) \quad \left(1 + O\left(\frac{1}{\mathcal{L}}\right)\right)M \exp\left\{-\frac{1}{2}(t-t_0)^2 u \mathcal{L}^2 + R(t-t_0)\right\},$$

where

$$M = \frac{y^{s_0-1} e^{\phi(s_0)} k(s_0-1)}{s_0} = \frac{y^{s_0-1} F(s_0) k(s_0-1)}{s_0}$$

and the function $R(t)$ satisfies

$$|R(t)| \leq \frac{1}{3!} t^3 \mathcal{L}^3 u + O\left(t^2 \mathcal{L}^2 \frac{u}{\log u}\right) + O\left(t^3 \mathcal{L}^3 \frac{u}{\log u}\right).$$

Setting

$$\delta_1 = \delta \left(\frac{\log u}{u}\right)^{1/2} = \frac{1}{\mathcal{L}} \left(\frac{\log u}{u}\right)^{1/2},$$

we have, for $|t-t_0| \leq \delta_1$, $R(t-t_0) = O(1)$ and

$$\exp(R(t-t_0)) = 1 + O(R(t-t_0)) = 1 + O\left((t-t_0)^2 \mathcal{L}^2 \frac{u}{\log u}\right),$$

and for $|t-t_0| \leq \delta = 1/\mathcal{L}$

$$\begin{aligned} & \exp\left\{-\frac{1}{2}(t-t_0)^2 u \mathcal{L}^2 + R(t-t_0)\right\} \\ & \leq \exp\left\{-\frac{1}{2}(t-t_0)^2 u \mathcal{L}^2 + \frac{1}{3!}(t-t_0)^2 u \mathcal{L}^2 \left(1 + O\left(\frac{1}{\log u}\right)\right)\right\} \\ & \ll \exp\left\{-\frac{1}{4}(t-t_0)^2 u \mathcal{L}^2\right\}. \end{aligned}$$

Thus, the integral of (8.18) over $|t-t_0| \leq \delta$ equals

$$\begin{aligned} & M \int_{t_0-\delta_1}^{t_0+\delta_1} \left(1 + O\left(\frac{1}{\mathcal{L}}\right) + O\left((t-t_0)^2 \mathcal{L}^2 \frac{u}{\log u}\right)\right) e^{-(t-t_0)^2 u \mathcal{L}^2 / 2} dt \\ & \quad + O\left(|M| \int_{t_0+\delta_1}^{t_0+\delta} e^{-(t-t_0)^2 u \mathcal{L}^2 / 4} dt\right) \\ & = \frac{M}{\sqrt{u} \mathcal{L}} \int_{-\delta_1 \sqrt{u} \mathcal{L}}^{\delta_1 \sqrt{u} \mathcal{L}} \left(1 + O\left(\frac{1}{\mathcal{L}}\right) + O\left(\frac{\tau^2}{\log u}\right)\right) e^{-\tau^2 / 2} d\tau \\ & \quad + O\left(\frac{|M|}{\sqrt{u} \mathcal{L}} \int_{\delta_1 \sqrt{u} \mathcal{L}}^{\infty} e^{-\tau^2 / 4} d\tau\right) \\ & = \frac{M \sqrt{2\pi}}{\sqrt{u} \mathcal{L}} \left(1 + O\left(\frac{1}{\log u}\right)\right), \end{aligned}$$

where the last estimate follows since

$$\int_{-\infty}^{\infty} e^{-\tau^2 / 2} d\tau = \sqrt{2\pi}$$

and

$$\int_{\delta_1 \sqrt{u} \mathcal{L}}^{\infty} e^{-\tau^2/4} d\tau = \int_{\sqrt{\log u}}^{\infty} e^{-\tau^2/4} d\tau \ll e^{-(\log u)/4} \ll \frac{1}{\log u}.$$

This proves (8.16).

Proof of Theorem C. We now suppose that λ satisfies the hypothesis $\log u \leq \lambda \leq \exp(\mathcal{L}^{3/2-\varepsilon})$ of the theorem. Combining Lemmas 8.4 and 8.5 we obtain

$$(8.19) \quad \begin{aligned} r(y, z; \lambda) = & \operatorname{Re} \frac{y^{s_0-1} F(s_0) k(s_0-1)}{\pi V(z) s_0 \mathcal{L} \sqrt{u/2\pi}} + O\left(\frac{y^{\sigma_0-1} |F(s_0)|}{V(z) \mathcal{L} \sqrt{u} \log u}\right) \\ & + O\left(\frac{y^{\sigma_0-1} e^{(1-\frac{\varepsilon}{3})u} \lambda^{1-\sigma_0} (\log \lambda)^4}{V(z) \mathcal{L}}\right) + O\left(\frac{y^{\sigma_0-1} |F(s_0)|}{V(z) \mathcal{L}} \exp\left(-\frac{u}{4 \log^2 u}\right)\right). \end{aligned}$$

Since

$$k(s_0-1) = e^{(s_0-1)^2/2\lambda} = 1 + O\left(\frac{1}{\lambda}\right) = 1 + O\left(\frac{1}{\log u}\right)$$

by the assumption $\lambda \geq \log u$ of Theorem C, and since by Mertens' formula

$$V(z) \mathcal{L} = \prod_{p < z} \left(1 - \frac{1}{p}\right) \log z = e^{-\gamma} \left(1 + O\left(\frac{1}{\mathcal{L}}\right)\right),$$

the main term in (8.19) is equal to

$$\begin{aligned} & \operatorname{Re} \frac{y^{s_0-1} F(s_0)}{\pi e^{-\gamma} s_0 \sqrt{u/2\pi}} + O\left(\frac{y^{\sigma_0-1} |F(s_0)|}{\sqrt{u} \log u}\right) \\ & = \operatorname{Re} \rho(y, z) + O\left(\frac{|\rho(y, z)|}{\log u}\right). \end{aligned}$$

The same argument shows that the first and third error terms are bounded by $O(|\rho(y, z)|/\log u)$. Finally, using the second part of Lemma 7.5, the second error term is seen to be of order

$$\begin{aligned} & \ll y^{\sigma_0-1} e^{(1-\frac{\varepsilon}{3})u} \lambda^{1-\sigma_0} (\log \lambda)^4 \\ & = y^{\sigma_0-1} |F(s_0)| \exp\left\{-\frac{\varepsilon}{3}u \left(1 + O\left(\frac{1}{\log u}\right)\right)\right\} \lambda^{1-\sigma_0} (\log \lambda)^4 \\ & \ll |\rho(y, z)| e^{-\varepsilon u/4} \lambda^{1-\sigma_0} (\log \lambda)^4. \end{aligned}$$

Altogether, we obtain

$$r(y, z; \lambda) = \operatorname{Re} \rho(y, z) + O\left(\frac{|\rho(y, z)|}{\log u}\right) + O(|\rho(y, z)| e^{-\varepsilon u/4} \lambda^{1-\sigma_0} (\log \lambda)^4).$$

This implies the desired estimate (3.11) in the case λ satisfies, in addition to the hypotheses of the theorem, the inequalities $\lambda^{1-\sigma_0} \leq e^{\varepsilon u/8}$ and $\log \lambda \leq u^2$. It remains therefore to show that the same estimate holds for values λ in the range

$$(8.20) \quad \min\left(u^2, \frac{\varepsilon u}{8(1-\sigma_0)}\right) \leq \log \lambda \leq \mathcal{L}^{3/2-\varepsilon}.$$

Since $1 - \sigma_0 = (\xi_u/\mathcal{L})(1 + O(1/\log^2 u)) \ll \log u/\mathcal{L}$ by Lemma 5.1, the range (8.20) is only non-empty in the case $u \ll \mathcal{L}$, which we may henceforth assume. We shall presently show that

$$(8.21) \quad r(y, z; \lambda) = r(y, z) + O\left(\frac{1}{\sqrt{\lambda}}\right) + O\left(\frac{z}{y}\right)$$

holds for any $\lambda \geq 1$. If now λ is in the range (8.20), then the error terms in (8.21) are bounded by

$$\begin{aligned} \frac{1}{\sqrt{\lambda}} + \frac{z}{y} &\ll \exp\left\{-\frac{\varepsilon u}{16(1-\sigma_0)}\right\} + e^{-u^2/2} + e^{-(u-1)\mathcal{L}} \\ &\ll \exp\left\{-\frac{\varepsilon u\mathcal{L}}{16\xi_u}\left(1 + O\left(\frac{1}{\log u}\right)\right)\right\} + e^{-u^2/2} + e^{-(u-1)\mathcal{L}} \\ &\ll \exp(-u^{3/2}) \ll \frac{|\rho(y, z)|}{\log u} \end{aligned}$$

in view of Propositions 3.1 and 3.2, and the bound $u \ll \mathcal{L}$. Thus, setting

$$\lambda_1 = \exp\left\{\min\left(u^2, \frac{\varepsilon u}{8(1-\sigma_0)}\right)\right\}$$

we obtain

$$\begin{aligned} r(y, z; \lambda) &= r(y, z; \lambda_1) + O\left(\frac{|\rho(y, z)|}{\log u}\right) \\ &= \operatorname{Re}\rho(y, z) + O\left(\frac{|\rho(y, z)|}{\log u}\right) \end{aligned}$$

by applying the above proved case of the theorem with $\lambda = \lambda_1$. This shows that the desired estimate (3.11) remains valid in the range (8.20).

To prove (8.21), we note that by sieve methods we have, for $y_2 \geq y_1 > 0$,

$$\Phi(y_2, z) - \Phi(y_1, z) \ll (y_2 - y_1 + z)V(z).$$

It follows that for $|v| \leq 1$

$$\begin{aligned} |r(ye^v, z) - r(y, z)| &= \left| \frac{\Phi(ye^v, z)}{ye^v V(z)} - \frac{\Phi(y, z)}{yV(z)} \right| \\ &\leq \frac{\Phi(ye^v, z)}{ye^v V(z)} |1 - e^v| + \frac{1}{yV(z)} |\Phi(ye^v, z) - \Phi(y, z)| \\ &\ll \frac{(ye^v + z)|v|}{ye^v} + \frac{y|e^v - 1| + z}{y} \\ &\ll |v| + \frac{z}{y}. \end{aligned}$$

Since for any $y' > 0$

$$|r(y', z)| \leq 1 + \frac{\Phi(y', z)}{y'V(z)} \ll 1 + \frac{z}{y'},$$

we obtain

$$\begin{aligned}
r(y, z; \lambda) &= \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r(ye^v, z) e^{-\lambda v^2/2} dv \\
&= \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \int_{-1}^1 \left(r(y, z) + O(|v| + \frac{z}{y}) \right) e^{-\lambda v^2/2} dv \\
&\quad + O\left(\sqrt{\lambda} \left(\int_1^{\infty} + \int_{-\infty}^{-1} \right) \left(1 + \frac{z}{ye^v} \right) e^{-\lambda v^2/2} dv \right) \\
&= r(y, z) + O\left(\sqrt{\lambda} \int_1^{\infty} e^{-\lambda v^2/2} dv \right) + O\left(\sqrt{\lambda} \int_{-\infty}^{\infty} |v| e^{-\lambda v^2/2} dv \right) \\
&\quad + O\left(\sqrt{\lambda} \int_{-\infty}^{\infty} \frac{z}{ye^v} e^{-\lambda v^2/2} dv \right) \\
&= r(y, z) + O(e^{-\lambda/2}) + O\left(\frac{1}{\sqrt{\lambda}} \right) + O\left(\frac{z}{y} \right).
\end{aligned}$$

This proves (8.21) and completes the proof of Theorem C.

9. Proof of Theorem B1

We fix a (small) positive number ε , real numbers y and z in the range R_ε , and a positive integer q . We define as before $u = \log y / \log z$ and $\mathcal{L} = \log z$. Throughout this and the next section the implied constants will depend at most on ε , and thus, in particular, will be independent of k .

Since the function r_k and the hypothesis (3.5) of Theorem B1 depend only on the prime divisors of k that are $< z$, we may assume that k has no prime factors $\geq z$. Thus (3.5) reduces to

$$(9.1) \quad \sum_{p|k} 1 \leq 2 \log \mathcal{L}.$$

We begin with the following simple lemma.

Lemma 9.1. *Let \sum' denote summation over integers all of whose prime factors divide k .*

We have, for any $\alpha \geq \varepsilon/2$ and $D \geq 1$,

$$\sum'_{d \geq D} \frac{1}{d^\alpha} \ll \frac{\mathcal{L}^{1/2}}{D^{\alpha(1-\varepsilon/4)}}.$$

Proof. We have

$$\begin{aligned}
\sum'_{d \geq D} \frac{1}{d^\alpha} &\leq \sum'_{d \geq 1} \frac{1}{d^\alpha} \left(\frac{d}{D} \right)^{\alpha(1-\varepsilon/4)} \\
&= D^{-\alpha(1-\varepsilon/4)} \sum'_{d \geq 1} \frac{1}{d^{\alpha\varepsilon/4}} \\
&= D^{-\alpha(1-\varepsilon/4)} \prod_{p|k} \left(1 - \frac{1}{p^{\alpha\varepsilon/4}} \right)^{-1}.
\end{aligned}$$

Now, if ν is the number of prime factors of k and p_ν denotes the ν th prime, then

$$\begin{aligned}
(9.2) \quad \prod_{p|k} \left(1 - \frac{1}{p^{\alpha\varepsilon/4}}\right)^{-1} &\leq \prod_{p \leq p_\nu} \left(1 - \frac{1}{p^{\alpha\varepsilon/4}}\right)^{-1} \leq \prod_{p \leq p_\nu} \left(1 - \frac{1}{p^{\varepsilon^2/8}}\right)^{-1} \\
&= \exp \left\{ O \left(\sum_{p \leq p_\nu} \frac{1}{p^{\varepsilon^2/8}} \right) \right\} = \exp \left\{ O \left(p_\nu^{1-\varepsilon^2/8} \right) \right\} \\
&\ll \exp(\nu^{1-\varepsilon^2/10}) \ll \mathcal{L}^{1/2},
\end{aligned}$$

since $p_\nu \ll \nu \log \nu$ and $\nu \ll \log \mathcal{L}$ by (9.1). This proves the asserted estimate.

Next, we define a smoothed form $r_k(y, z; \lambda)$ of r_k , analogous to the function $r(y, z; \lambda)$ in Theorem C, by

$$r_k(y, z; \lambda) = \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r_k(ye^v, z) e^{-\lambda v^2/2} dv.$$

The following lemma relates the functions $r_k(y, z; \lambda)$ and $r(y, z; \lambda)$. Here, and in the sequel, \sum' has the same meaning as in Lemma 9.1.

Lemma 9.2. *For any $\lambda \geq 1$ we have*

$$(9.3) \quad r_k(y, z; \lambda) = \prod_{p|k} \left(1 - \frac{1}{p}\right) \left\{ \sum'_{d \leq y^{1-\varepsilon/2}} \frac{1}{d} r\left(\frac{y}{d}, z; \lambda\right) + O\left(\frac{1}{y^{1-4\varepsilon/5}}\right) \right\}.$$

Proof. We start out with the identity

$$\Phi_k(y', z) = \sum'_{d \geq 1} \Phi\left(\frac{y'}{d}, z\right) \quad (y' > 0),$$

which follows easily from the definition of $\Phi_k(y, z)$. By Lemma 9.1 the contribution of the range $d > y^{1-\varepsilon/2}$ to the sum is bounded by

$$\leq \sum'_{d > y^{1-\varepsilon/2}} \frac{y'}{d} \ll y' \frac{\mathcal{L}^{1/2}}{y^{(1-\varepsilon/2)(1-\varepsilon/4)}} \ll y' \frac{\mathcal{L}^{1/2}}{y^{1-3\varepsilon/4}}.$$

Thus we obtain

$$\begin{aligned}
\Phi_k(y', z) &= \sum'_{d \leq y^{1-\varepsilon/2}} \Phi\left(\frac{y'}{d}, z\right) \\
&= \sum'_{d \leq y^{1-\varepsilon/2}} \frac{y'}{d} = y' V(z) \prod_{p|k} \left(1 - \frac{1}{p}\right)^{-1} + y' V(z) \sum'_{d \leq y^{1-\varepsilon/2}} \frac{1}{d} r\left(\frac{y'}{d}, z\right) + O\left(y' \frac{\mathcal{L}^{1/2}}{y^{1-3\varepsilon/4}}\right),
\end{aligned}$$

where $V(z) = \prod_{p < z} \left(1 - \frac{1}{p}\right)$ and where in the last estimate we have used the bound

$$\sum'_{d \leq y^{1-\varepsilon/2}} \frac{y'}{d} = y' \sum'_{d \geq 1} \frac{1}{d} + O\left(\sum'_{d > y^{1-\varepsilon/2}} \frac{y'}{d}\right) = y' \prod_{p|k} \left(1 - \frac{1}{p}\right)^{-1} + O\left(y' \frac{\mathcal{L}^{1/2}}{y^{1-3\varepsilon/4}}\right).$$

It follows that

$$\begin{aligned} r_k(y', z) &= \prod_{p|k} \left(1 - \frac{1}{p}\right) \frac{\Phi_k(y', z)}{y'V(z)} - 1 \\ &= \prod_{p|k} \left(1 - \frac{1}{p}\right) \left\{ \sum'_{d \leq y^{1-\varepsilon/2}} \frac{1}{d} r\left(\frac{y'}{d}, z\right) + O\left(\frac{\mathcal{L}^{1/2}}{V(z)y^{1-3\varepsilon/4}}\right) \right\}. \end{aligned}$$

Letting $y' = ye^v$ and integrating both sides of this identity against the weight function $\sqrt{\lambda/2\pi} \exp(-\lambda v^2/2)$ we obtain

$$r_k(y, z; \lambda) = \prod_{p|k} \left(1 - \frac{1}{p}\right) \left\{ \sum'_{d < y^{1-\varepsilon/2}} \frac{1}{d} r\left(\frac{y}{d}, z; \lambda\right) + O\left(\frac{\mathcal{L}^{1/2}}{V(z)y^{1-3\varepsilon/4}}\right) \right\},$$

which in view of the bound

$$\mathcal{L}^{1/2}V(z)^{-1} \ll (\log z)^{3/2} \ll (\log y)^{3/2}$$

implies the asserted estimate (9.3).

Lemma 9.3. *We have*

$$(9.4) \quad \sum'_{d \leq y^{1-\varepsilon/2}} \frac{1}{d} \rho\left(\frac{y}{d}, z\right) = c_k(y, z) \prod_{p|k} \left(1 - \frac{1}{p}\right)^{-1} \rho(y, z) \left(1 + O\left(\frac{1}{\mathcal{L}^{1/3}}\right)\right)$$

and

$$(9.5) \quad \sum'_{d \leq y^{1-\varepsilon/2}} \frac{1}{d} |\rho\left(\frac{y}{d}, z\right)| \ll c_k(y, z) \prod_{p|k} \left(1 - \frac{1}{p}\right)^{-1} |\rho(y, z)|,$$

where

$$(9.6) \quad c_k(y, z) = \prod_{p|k} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^{1-\xi_u/\mathcal{L}}}\right)^{-1}.$$

Proof. By the estimate (3.15) of Proposition 3.2 (with $\varepsilon/2$ in place of ε) we have for $d \leq y^{1-\varepsilon/2}$

$$\rho\left(\frac{y}{d}, z\right) = \rho(y, z) \exp\left\{\frac{\log d}{\mathcal{L}} (\xi_u + O(1))\right\} = d^{\xi_u/\mathcal{L} + O(1/\mathcal{L})} \rho(y, z).$$

Hence the left-hand side of (9.4) equals

$$(9.7) \quad \begin{aligned} &\rho(y, z) \sum'_{d \leq y^{1-\varepsilon/2}} \frac{1}{d^{1-\xi_u/\mathcal{L} + O(1/\mathcal{L})}} \\ &= \rho(y, z) \left\{ \sum'_{d \geq 1} \frac{1}{d^{1-\xi_u/\mathcal{L}}} + O\left(\sum'_{d \geq z} \frac{1}{d^{1-\xi_u/\mathcal{L} + O(1/\mathcal{L})}}\right) + O\left(\frac{1}{\mathcal{L}} \sum'_{d < z} \frac{\log d}{d^{1-\xi_u/\mathcal{L}}}\right) \right\}. \end{aligned}$$

The main term inside the braces can be written as

$$\sum'_{d \geq 1} \frac{1}{d^{1-\xi_u/\mathcal{L}}} = \prod_{p|k} \left(1 - \frac{1}{p^{1-\xi_u/\mathcal{L}}}\right)^{-1} = c_k(y, z) \prod_{p|k} \left(1 - \frac{1}{p}\right)^{-1}$$

by the definition of $c_k(y, z)$. Further, since $1 - \xi_u/\mathcal{L} \geq 2\varepsilon/3$ in the range R_ε (assuming that the constants u_0 and z_0 in the definition of R_ε have been chosen sufficiently large), we get from Lemma 9.1

$$\sum'_{d \geq z} \frac{1}{d^{1-\xi_u/\mathcal{L}+O(1/\mathcal{L})}} \ll \sum'_{d \geq z} \frac{1}{d^{\varepsilon/2}} \ll \frac{\mathcal{L}^{1/2}}{z^{(\varepsilon/2)(1-\varepsilon/4)}} \ll \frac{1}{\mathcal{L}^{1/2}}$$

and

$$\frac{1}{\mathcal{L}} \sum'_{d < z} \frac{\log d}{d^{1-\xi_u/\mathcal{L}}} \ll \frac{1}{\mathcal{L}} \sum_{p|k} \frac{\log p}{p^{\varepsilon/2}} \sum'_{d \geq 1} \frac{1}{d^{\varepsilon/2}} \ll \frac{1}{\mathcal{L}} \left(\sum_{p|k} 1\right) \prod_{p|k} \left(1 - \frac{1}{p^{\varepsilon/2}}\right)^{-1} \ll \frac{1}{\mathcal{L}^{1/3}}$$

in view of (9.1) and (9.2). The error terms in (9.7) are therefore both bounded by $O(\mathcal{L}^{-1/3})$, and the right-hand side of (9.7) becomes

$$\rho(y, z) c_k(y, z) \prod_{p|k} \left(1 - \frac{1}{p}\right)^{-1} \left(1 + O\left(\frac{1}{\mathcal{L}^{1/3}}\right)\right).$$

This proves (9.4); the second estimate (9.5) can be established in the same way.

For the remainder of this section we restrict (y, z) as in the statement of Theorem B1 to the range $(R_{1/6+\varepsilon})$. This restriction would not be necessary if the Riemann Hypothesis were assumed.

Lemma 9.4. *There exist numbers z_\pm in the range*

$$(9.8) \quad \min\left(z^{1-\frac{2}{u+2}}, \frac{1}{2}z\right) \leq z_\pm \leq z$$

such that

$$(9.9) \quad \begin{cases} \operatorname{Re} \rho(y, z_+) \geq \exp\{-u\xi_u - c_1 u\}, \\ \operatorname{Re} \rho(y, z_-) \leq -\exp\{-u\xi_u - c_1 u\}, \end{cases}$$

holds, where c_1 is a suitable positive constant depending at most on ε .

Proof. By Proposition 3.2 and (3.2) we have, for any z' in the range (9.8),

$$|\rho(y, z')| = \exp\{-u'\xi_{u'} + O(u')\},$$

where $u' = \log y / \log z'$. Since

$$u' = u \frac{\log z}{\log z'} = u \left(1 + O\left(\frac{1}{u} + \frac{1}{\log z}\right)\right)$$

for $\min(z/2, z^{1-2/(u+2)}) \leq z' \leq z$, we may replace u' by u in this estimate. To prove (9.9) it therefore suffices to show that there exist values z_\pm satisfying (9.8) such that

$$(9.10) \quad |\operatorname{Arg} \rho(y, z_+)| \leq \frac{\pi}{3}, \quad |\operatorname{Arg} \rho(y, z_-) - \pi| \leq \frac{\pi}{3},$$

where Arg denotes the principal value of the argument.

We consider first the case when $u \leq \sqrt{\mathcal{L}}$. By the estimate (3.15) of Proposition 3.2 we have, for $u \leq u' \leq u + 2$,

$$\arg \rho(z^{u'}, z) = \arg \rho(z^u, z) + \pi(u' - u) + O\left(\frac{1}{\log u}\right).$$

Hence there exist numbers u_{\pm} in $[u, u + 2]$ such that

$$(9.11) \quad |\text{Arg } \rho(z^{u_{\pm}}, z)| \leq \frac{\pi}{4}, \quad |\text{Arg } (z^{u_{\pm}}, z) - \pi| \leq \frac{\pi}{4},$$

provided $u \geq u_0$ with a sufficiently large constant u_0 , as we may assume. Setting $z_{\pm} = z^{u/u_{\pm}}$, we then have

$$z^{u/(u+2)} \leq z_{\pm} \leq z,$$

so that (9.8) is satisfied, and

$$\arg \rho(y, z_{\pm}) = \arg \rho(z_{\pm}^{u_{\pm}}, z_{\pm}) = \arg \rho(z^{u_{\pm}}, z) + O\left(\frac{u}{\mathcal{L}}\right)$$

by the last estimate of Proposition 3.2. In view of (9.11) and our assumption $u \leq \sqrt{\mathcal{L}}$ this implies (9.10).

The above argument would go through for $u \leq z^{5/12}$, if we would use Huxley's prime number theorem and the estimate (3.16) of Proposition 3.1, but it would break down for larger values of u . For the range $\sqrt{\mathcal{L}} < u \leq z^{5/6-\varepsilon}$ we shall therefore use a somewhat different approach that depends on an "almost all" type estimate for primes in short intervals. Let

$$h = \frac{z\sqrt{\mathcal{L}}}{u}, \quad z_n = z - nh \quad (1 \leq n \leq n_0 = \lfloor \frac{z}{2h} \rfloor),$$

so that, for $1 \leq n \leq n_0$, $z/2 \leq z_n \leq z$ and $z_n - z_{n+1} = h \gg z^{1/6+\varepsilon}$ by our assumption that (y, z) lies in the range $(R_{1/6+\varepsilon})$. We shall show that (9.9) holds with $z_{\pm} = z_{n_{\pm}}$ for suitable indices $n_{\pm} \leq n_0$.

We note first that the error term function $E(\tilde{z}, z)$ defined in Proposition 3.2 satisfies

$$\int_{z/2}^z |E(t, t+h)| dt \ll_A \frac{zh}{(\log z)^A}$$

for any fixed constant A . This follows from Selberg's work [Se] and Huxley's zero-density estimate [Hu] in view of the bound $h \gg z^{1/6+\varepsilon}$; for the argument see Lemma 2 of Nicolas [Ni]. A simple consequence of this estimate is that

$$(9.12) \quad |E(z_n, z_{n+1})| \ll \frac{h}{\mathcal{L}^2}$$

holds for all $n \leq n_0$ with at most $O(n_0/\mathcal{L}^2)$ exceptions. If z is large enough, this implies that there exist at least \mathcal{L} consecutive indices $n \leq n_0$ for which (9.12) holds, say $n = n_1 + 1, \dots, n_1 + J$, where $J = \lfloor \mathcal{L} \rfloor$ and $1 \leq n_1 \leq n_0 - J$.

Let

$$u_j = \frac{\log y}{\log z_{n_1+j}}.$$

By the definition of z_n and h we have, for $j = 1, \dots, J$,

$$\begin{aligned} u_{j+1} - u_j &= \frac{\log y}{\log(z_{n_1} - (j+1)h)} - \frac{\log y}{\log(z_{n_1} - jh)} \\ &= \frac{u\mathcal{L} \log((z_{n_1} - jh)/(z_{n_1} - (j+1)h))}{(\log z_{n_1})^2 + O(\mathcal{L})} \asymp \frac{hu}{z\mathcal{L}} = \frac{1}{\sqrt{\mathcal{L}}}. \end{aligned}$$

Thus, in view of (3.15), the argument of $\rho(z_{n_1}^{u_j}, z_{n_1})$ increases in steps of size $\asymp 1/\sqrt{\mathcal{L}}$, as j ranges over the integers in the interval $[1, \mathcal{L}]$. It follows that, for suitable indices $j_{\pm} \leq J$ we have (9.11) with z_{n_1} in place of z and $u_{\pm} = u_{j_{\pm}}$. Using the estimate (9.12) for $n = n_1 + j$, $1 \leq j \leq J$, Proposition 3.2, and the definitions of h and J , we see that with $z_{\pm} = z_{n_1+j_{\pm}} = y^{1/u_{\pm}}$ we have

$$\begin{aligned} \arg \rho(y, z_{\pm}) &= \arg \rho(z_{n_1+j_{\pm}}^{u_{\pm}}, z_{n_1+j_{\pm}}) \\ &= \arg \rho(z_{n_1}^{u_{\pm}}, z_{n_1}) + O\left(\frac{u}{z} \sum_{n=n_1}^{n_1+J} \left(|E(z_n, z_{n+1})| + \frac{h}{\mathcal{L}^2} + z^{\varepsilon}\right)\right) \\ &= \arg \rho(z_{n_1}^{u_{\pm}}, z_{n_1}) + O\left(J \frac{u}{z} \left(\frac{h}{\mathcal{L}^2} + z^{\varepsilon}\right)\right) \\ &= \arg \rho(z_{n_1}^{u_{\pm}}, z_{n_1}) + O\left(\frac{1}{\sqrt{\mathcal{L}}}\right). \end{aligned}$$

This estimate, in conjunction with (9.11) (with z_{n_1} in place of z), implies (9.10), and thus completes the proof of Lemma 9.4.

Proof of Theorem B1. From Lemma 9.2, Theorem C and Lemma 9.3 we obtain, for $\log u \leq \lambda \leq \exp(\mathcal{L}^{3/2-\varepsilon})$,

$$r_k(y, z; \lambda) = c_k(y, z) \left\{ \operatorname{Re} \rho(y, z) + O\left(\frac{|\rho(y, z)|}{\log u}\right) + O\left(\frac{|\rho(y, z)|}{\mathcal{L}^{1/3}}\right) \right\} + O\left(\frac{1}{y^{1-4\varepsilon/5}}\right).$$

The last error term here can be estimated, using the bound $u \leq z^{5/6-\varepsilon}$ and Proposition 3.2, by

$$\begin{aligned} y^{-1+4\varepsilon/5} &\ll \exp\left\{-\left(1 - \frac{4}{5}\varepsilon\right)u\mathcal{L}\right\} \ll \exp\left\{-\left(1 + \frac{\varepsilon}{10}\right)u\xi_u\right\} \\ &\ll \frac{|\rho(y, z)|}{\log u} \ll c_k(y, z) \frac{|\rho(y, z)|}{\log u}, \end{aligned}$$

since $c_k(y, z) \geq 1$. It follows that

$$r_k(y, z; \lambda) = c_k(y, z) \left\{ \operatorname{Re} \rho(y, z) + O\left(\frac{|\rho(y, z)|}{\log u}\right) + O\left(\frac{|\rho(y, z)|}{\mathcal{L}^{1/3}}\right) \right\}.$$

We apply this estimate with $y_1 = y - y/2 \log y$ in place of y , and z_{\pm} in place of z , where the numbers z_{\pm} are chosen according to Lemma 9.4 so that (9.9) (with y replaced by y_1) is satisfied. Adjusting the constant c_1 if necessary, we then obtain

$$(9.13) \quad \pm r_k(y_1, z_{\pm}; \lambda) \geq \exp\{-u\xi_u - c_1 u\}.$$

Note that the numbers z_{\pm} here depend only on y and z , but not on k or λ .

We now set $\delta = 1/2 \log y$, $\Delta = \log(1 + \delta)$, so that

$$(9.14) \quad \begin{aligned} y &\geq y_1(1 + \delta) = y_1 e^\Delta, \\ y(1 - \frac{1}{\log y}) &\leq y_1(1 - \delta) \leq y_1 e^{-\Delta}, \end{aligned}$$

and choose $\lambda = (\log y)^4 = (2\delta)^{-4}$. With this choice of parameters we have

$$\begin{aligned} r_k(y_1, z_+; \lambda) &= \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \int_{-\Delta}^{\Delta} r_k(y_1 e^v, z_+) e^{-\lambda v^2/2} dv + O(\sqrt{\lambda} \mathcal{L} \int_{\Delta}^{\infty} e^{-\lambda v^2/2} dv) \\ &= \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \int_{-\Delta}^{\Delta} r_k(y_1 e^v, z_+) e^{-\lambda v^2/2} dv + O(\mathcal{L} e^{-\lambda \Delta^2/2}) \\ &\leq \max_{|v| \leq \Delta} r_k(y_1 e^v, z_+) + O(\frac{1}{y}) \\ &\leq \max_{y(1-1/\log y) \leq y' \leq y} r_k(y', z_+) + O(\frac{1}{y}), \end{aligned}$$

and a similar estimate holds with z_- in place of z_+ . Combining these estimates with (9.14) yields

$$\max_{y(1-1/\log y) \leq y' \leq y} \pm r_k(y', z_{\pm}) \geq \exp\{-u\xi_u - c_1 u\} + O(\frac{1}{y}).$$

In view of the hypothesis $u \leq z^{5/6-\varepsilon}$ the error term here is of smaller order than the main term. Therefore we obtain the desired estimates (3.7) with suitable numbers y_{\pm} satisfying (3.6), after adjusting the constant c_1 if necessary.

10. Proof of Theorem B3

Since the proof is to a large degree analogous to that of Theorem B1, we shall keep the exposition shorter. We fix $\varepsilon > 0$, real numbers y, z in the range R_{ε} , and a prime $p \geq z$. We will concentrate on the case when $p \geq z$; the argument for $p < z$ is similar and simpler.

We write

$$\Phi^{(p)}(y, z) = \sum_{n \leq x}^{(p)} \frac{n}{\phi(n)},$$

where $\sum^{(p)}$ denotes summation over integers n that are not divisible by p or by any prime $< z$,

$$V^{(p)}(z) = \prod_{p' < z} (1 - \frac{1}{p'})(1 - \frac{1}{p}),$$

and

$$C = C^{(p)}(z) = \prod_{\substack{p' \geq z \\ p' \neq p}} (1 + \frac{1}{p'(p'-1)}).$$

We recall that $r^{(p)}$ and $\bar{r}^{(p)}(y, z)$ are defined by

$$\Phi^{(p)}(y, z) = y C V^{(p)}(z) (1 + r^{(p)}(y, z))$$

and

$$\bar{r}^{(p)}(y, z) = \int_{y/2}^y r^{(p)}(t, z) \frac{dt}{t}.$$

For $\lambda > 0$ we set

$$r^{(p)}(y, z; \lambda) = \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r^{(p)}(ye^v, z) e^{-\lambda v^2/2} dv,$$

$$\bar{r}^{(p)}(y, z; \lambda) = \int_{y/2}^y r^{(p)}(t, z; \lambda) \frac{dt}{t}.$$

The following lemma is analogous to Lemma 9.2.

Lemma 10.1. *For any $\lambda \geq 1$ we have*

$$(10.1) \quad r^{(p)}(y, z; \lambda) = \frac{1}{C(1 - \frac{1}{p})} \left\{ \sum_{d \leq y^{1-\varepsilon/2}}^{(p)} \frac{\mu^2(d)}{d\phi(d)} r\left(\frac{y}{d}, z; \lambda\right) - \frac{1}{p} \sum_{d \leq y^{1-\varepsilon/2}}^{(p)} \frac{\mu^2(d)}{d\phi(d)} r\left(\frac{y}{dp}, z; \lambda\right) \right\} + O\left(\frac{\mathcal{L}}{y^{1-\varepsilon/2}}\right),$$

where the summation in $\sum^{(p)}$ is restricted to integers d that are not divisible by p or by any prime $< z$.

Proof. Using the identity

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)} \quad (n \geq 1),$$

we obtain, for any $y' > 0$,

$$\begin{aligned} \Phi^{(p)}(y', z) &= \sum_{n \leq y'}^{(p)} \sum_{d|n} \frac{\mu^2(d)}{\phi(d)} = \sum_{d \geq 1}^{(p)} \frac{\mu^2(d)}{\phi(d)} \sum_{n \leq y'/d}^{(p)} 1 \\ &= \sum_{d \geq 1}^{(p)} \frac{\mu^2(d)}{\phi(d)} \left(\Phi\left(\frac{y'}{d}, z\right) - \Phi\left(\frac{y'}{dp}, z\right) \right) \\ &= \sum_{d \leq y^{1-\varepsilon/2}}^{(p)} \frac{\mu^2(d)}{\phi(d)} \frac{y'}{d} V(z) \left\{ 1 - \frac{1}{p} + r\left(\frac{y'}{d}, z\right) - \frac{1}{p} r\left(\frac{y'}{dp}, z\right) \right\} + O\left(\frac{y'}{y^{1-\varepsilon/2}}\right) \\ &= Cy'V^{(p)}(z) \left\{ 1 + \frac{1}{C(1 - \frac{1}{p})} \sum_{d \leq y^{1-\varepsilon/2}}^{(p)} \frac{\mu^2(d)}{d\phi(d)} r\left(\frac{y'}{d}, z\right) - \frac{1}{C(1 - \frac{1}{p})p} \sum_{d \leq y^{1-\varepsilon/2}}^{(p)} \frac{\mu^2(d)}{d\phi(d)} r\left(\frac{y'}{dp}, z\right) \right\} + O\left(\frac{y'}{y^{1-\varepsilon/2}}\right), \end{aligned}$$

where

$$C = \sum_{d \geq 1}^{(p)} \frac{\mu^2(d)}{\phi(d)d} = \prod_{\substack{p' \geq z \\ p' \neq p}} \left(1 + \frac{1}{p'(p'-1)} \right)$$

is defined as above and the error term absorbs the contribution of the range $d > y^{1-\varepsilon/2}$ to the sum over $\Phi(y'/d, z)$ and that of the range $d > y^{1-\varepsilon/2}$ to the sum over $\Phi(y'/dp, z)$. It follows that

$$r^{(p)}(y', z) = \frac{1}{C(1 - \frac{1}{p})} \left\{ \sum_{d \leq y^{1-\varepsilon/2}}^{(p)} \frac{\mu^2(d)}{d\phi(d)} r\left(\frac{y'}{d}, z\right) - \frac{1}{p} \sum_{d \leq y^{1-\varepsilon/2}}^{(p)} \frac{\mu^2(d)}{d\phi(d)} r\left(\frac{y'}{dp}, z\right) \right\} + O\left(\frac{\mathcal{L}}{y^{1-\varepsilon/2}}\right).$$

Integrating both sides with $y' = ye^v$ against the weight function $\sqrt{\lambda/2\pi} \exp(-\lambda v^2/2)$, we obtain (10.1).

Lemma 10.2. *We have*

$$\sum_{d \leq y^{1-\varepsilon/2}}^{(p)} \frac{\mu^2(d)}{d\phi(d)} \rho\left(\frac{y}{d}, z\right) = (1 + O(\frac{1}{\mathcal{L}})) \rho(y, z)$$

and

$$\sum_{d \leq y^{1-\varepsilon/2}}^{(p)} \frac{\mu^2(d)}{d\phi(d)} |\rho\left(\frac{y}{d}, z\right)| \ll |\rho(y, z)|.$$

Proof. This result can be proved in the same way as Lemma 9.2, using the estimates

$$\begin{aligned} \sum_{d \geq 1}^{(p)} \frac{\mu^2(d)}{d^\sigma \phi(d)} &= \exp \left\{ O\left(\sum_{p' \geq z} \frac{1}{p'^\sigma (p' - 1)}\right) \right\} \\ &= 1 + O\left(\frac{1}{\sqrt{z}}\right) \quad (\sigma \geq \varepsilon/2) \end{aligned}$$

and

$$\sum_{d \geq z}^{(p)} \frac{1}{d^{\varepsilon/2} \phi(d)} \ll \frac{1}{z^{\varepsilon/4}} \ll \frac{1}{\mathcal{L}}.$$

We omit the details.

Proof of Theorem B3. From Lemma 10.1, Theorem C, Lemma 10.2, and Proposition 3.1 we deduce that, for $\log u \leq \lambda \leq \exp(\mathcal{L}^{3/2-\varepsilon})$,

$$r^{(p)}(y, z; \lambda) = \operatorname{Re} \rho(y, z) + O\left(\frac{|\rho(y, z)|}{\log u}\right).$$

It follows that

$$\begin{aligned} \bar{r}^{(p)}(y, z; \lambda) &= \int_{y/2}^y r^{(p)}(y', z; \lambda) \frac{dy'}{y'} \\ &= \operatorname{Re} \int_{y/2}^y \rho(y', z) \frac{dy'}{y'} + O\left(\frac{1}{\log u} \int_{y/2}^y |\rho(y', z)| \frac{dy'}{y'}\right). \end{aligned}$$

Since by Proposition 3.2

$$\rho(y', z) = \rho(y, z) \left(\frac{y'}{y}\right)^{-\xi_u/\mathcal{L}} \left(1 + O\left(\frac{1}{\log u}\right)\right) \quad (y/2 \leq y' \leq y),$$

we have

$$\operatorname{Re} \int_{y/2}^y \rho(y', z) \frac{dy'}{y'} = \frac{2\xi_u/\mathcal{L} - 1}{\xi_u/\mathcal{L}} \operatorname{Re} \rho(y, z) + O\left(\frac{|\rho(y, z)|}{\log u}\right).$$

Replacing here y by $y_1 = y - y/2 \log y$ and applying Lemma 9.4, we deduce that

$$\pm \overline{\tau}^{(p)}(y_1, z_{\pm}; \lambda) \geq \exp\{-u\xi_u - c_1 u\}$$

holds with suitable numbers z_{\pm} satisfying (3.4) and an appropriate constant c_1 . Taking $\lambda = (\log y)^4$, we complete the proof of Theorem B3 by following the argument at the end of the preceding section.

III. Oscillation Theorems for Primes

11. Proof of Theorem A1

For the proof of Theorem A1 (as well as that of Theorem A2) we require the following result which refines Proposition 2 of [H-M]. It is here that we appeal to the Linnik-Gallagher prime number theorem.

Proposition 11.1. *Fix $\varepsilon > 0$. For any squarefree integer $n > 1$, all of whose prime factors are $\leq n^{1-\varepsilon}$, there exists a divisor P of n , with n/P prime, such that if $(a, P)=1$, $x \geq P^2$, and $x \geq h \geq x/\exp(\sqrt{\log x})$, then*

$$(11.1) \quad \theta(x+h; P, a) - \theta(x; P, a) = \frac{h}{\phi(P)} \left\{ 1 + O\left(e^{-c \log x / \log P} + e^{-c \sqrt{\log x}}\right) \right\},$$

where c is a constant depending only on ε .

Proof. For a given value $c > 0$, a character $\chi \pmod{q}$ is called " c -exceptional" if $L(\beta, \chi) = 0$ for some real $\beta > 1 - c/\log q$. It is well known (cf. [Da, p. 94]) that there exists a constant $c_0 > 0$ for which there is at most one c_0 -exceptional character $\chi \pmod{q}$ for each $q \geq 2$. We first show that there is a divisor P of n , where n/P is prime, for which there are no εc_0 -exceptional characters $\chi \pmod{P}$.

Suppose that this is false and let q_1 be a divisor of n with n/q_1 prime. Let χ_1 be the εc_0 -exceptional character $\pmod{q_1}$ and let p be a prime divisor of the conductor of χ_1 . Set $q_2 = n/p$; note that p is a divisor of q_1 and so of n . Let χ_2 be the εc_0 -exceptional character $\pmod{q_2}$, and denote by χ_3 and χ_4 the characters of modulus n induced by χ_1 and χ_2 , respectively. Since p divides the conductor of χ_3 but not that of χ_4 , χ_3 and χ_4 cannot be the same character. However, since by our assumption $L(\beta_i, \chi_i) = 0$ for some numbers $\beta_i > 1 - \varepsilon c_0 / \log q_i$ and $i = 1, 2$, we have $L(\beta_i, \chi_{i+2}) = 0$ where $\beta_i > 1 - \varepsilon c_0 / \log q_i \geq 1 - c_0 / \log n$ as each $q_i \geq n^\varepsilon$. Thus χ_3 and χ_4 are distinct c_0 -exceptional characters \pmod{n} which, as already pointed out, cannot happen.

Choosing P as above we have, under the hypotheses of the proposition,

$$\theta(x+h; P, a) - \theta(x; P, a) = \frac{1}{\phi(P)} \sum_{\chi \pmod{P}} \overline{\chi}(a) \sum_{x < p \leq x+h} \chi(p) \log p$$

and therefore

$$\begin{aligned} & \left| \theta(x+h; P, a) - \theta(x; P, a) - \frac{h}{\phi(P)} \right| \\ & \leq \frac{1}{\phi(P)} \left\{ \left| \sum_{x < p \leq x+h} \log p - h \right| + \sum_{\substack{k|P \\ k > 1}} \sum_{\chi \pmod{k}}^* \left| \sum_{x < p \leq x+h} \chi(p) \log p \right| \right\}, \end{aligned}$$

where in \sum^* the summation is restricted to primitive characters. By the choice of P , the characters appearing in the last sum all satisfy $L(\beta, \chi) \neq 0$ for $\beta > 1 - \varepsilon c_0 / \log P$. Thus, estimating the sum over p by a sharp form of the prime number theorem and the character sum by Theorem 7 of Gallagher [Ga] (with $Q = \max\{P, \exp(\sqrt{\log x})\}$), we obtain

$$\left| \theta(x+h; P, a) - \theta(x; P, a) - \frac{h}{\phi(P)} \right| \ll \frac{h}{\phi(P)} \left\{ e^{-c \log x / \log P} + e^{-c \sqrt{\log x}} \right\}.$$

This proves (11.1).

Proof of Theorem A1. We give the proof of (2.3), the proof of (2.4) being almost identical. Given q and x satisfying (2.1) and (2.2), we let $y = x/q$ and define $z = \log x / \log y$ so that (y, z) lies in the range $(R_{1/6+\varepsilon})$ of Theorem B1 (with $u_0 = 1 + \varepsilon$) provided that ε is sufficiently small and $q_0(\varepsilon)$ sufficiently large. We next take z_+ as in Theorem B1, define $n = \prod_{p < z_+, p/q} p$ and choose P as in Proposition 11.1. Finally we set $k = qn/P$, so that

$$(11.2) \quad P = \prod_{\substack{p < z_+ \\ p/k}} p.$$

The hypothesis (2.1) on q ensures that k satisfies the condition (3.5) of Theorem B1 for $q \geq q_0(\varepsilon)$, since $k = qn/P$ has only one prime divisor more than q and we have

$$z = \frac{\log x}{\log(x/q)} \ll \frac{\log q}{\log_2 q}$$

in the range (2.2). Thus, by Theorem B1 there exists a number y_+ in the range (3.6) for which the first inequality in (3.7) holds. Since

$$\Phi_k(y_+, z_+) = y_+ \prod_{\substack{p < z_+ \\ p/k}} \left(1 - \frac{1}{p}\right) (1 + r_k(y_+, z_+)) = y_+ \frac{\phi(P)}{P} (1 + r_k(y_+, z_+))$$

by (11.2), this inequality can be restated as

$$(11.3) \quad \Phi_k(y_+, z_+) \geq y_+ \frac{\phi(P)}{P} (1 + \exp(-u\xi_u - c_1 u)),$$

where $u = \log y / \log z$.

Consider now the matrix $\mathcal{M} = (a_{rs})$, where

$$a_{rs} = \begin{cases} \log(rP + qs) & \text{if } rP + qs \text{ prime,} \\ 0 & \text{else,} \end{cases}$$

and where r and s run over the values $r \sim R$ (that is $R < r \leq 2R$) and $1 \leq s \leq y_+$ with

$$(11.4) \quad R = \frac{x}{2P} \exp(-\sqrt{\log x}).$$

Let $|M|$ denote the sum of the entries of \mathcal{M} . For given s , the sum of entries in the s -th column is

$$\theta(2RP + qs; P, qs) - \theta(RP + qs; P, qs).$$

This vanishes if $(qs, P) > 1$, and for those s with $(qs, P) = 1$ it may be estimated by applying (11.1) with x , h , and a replaced by PR , PR , and qs , respectively. By (11.2) and the definition of Φ_k there are precisely $\Phi_k(y_+, z_+)$ values of s satisfying $(s, P) = 1$, or equivalently (since $(q, P) = 1$), $(qs, P) = 1$. Thus, we obtain

$$(11.5) \quad |M| = \frac{RP}{\phi(P)} \left(1 + O(e^{-c \log RP / \log P}) + O(e^{-c \sqrt{\log RP}}) \right) \Phi_k(y_+, z_+) \\ \geq R y_+ \{1 + \exp(-u\xi_u - c'_1 u)\}$$

with a suitable constant c'_1 , since

$$\frac{\log RP}{\log P} \asymp \frac{\log x}{z_+} \geq \frac{\log x}{z} = \log y = u \log z$$

and

$$\sqrt{\log RP} \gg \sqrt{\log x} \geq \log y = u \log z$$

by (11.4) and the definitions of z_+ and z .

On the other hand, the sum of the entries in row r is

$$\theta(rP + qy_+; q, rP) - \theta(rP; q, rP) = \theta(rP + qy_+; q, rP),$$

since $rP \leq 2RP < q$ by (11.4) and (2.2), and rP is not prime. This vanishes unless $(r, q) = 1$. Thus,

$$(11.6) \quad |M| = \sum_{\substack{r \sim R \\ (r, q) = 1}} \theta(rP + qy_+; q, rP).$$

The number of r satisfying $r \sim R$ and $(r, q) = 1$ is just $R\phi(q)/q + O(\tau(q))$, and, since $R > x^{1/2}$, the error is negligible. Thus, by (11.3), (11.5), and (11.6), there is at least one r for which

$$\theta(rP + qy_+; q, rP) \geq \frac{qy_+}{\phi(q)} \{1 + \exp(-u\xi_u - c''_1 u)\},$$

where c''_1 is an appropriate constant depending at most on ε . Choosing this r and taking $x_+ = rP + qy_+$ we obtain

$$\Delta(x_+; q, rP) \geq \exp(-u\xi_u - c''_1 u) - \frac{2rP}{x_+},$$

which yields the desired estimate, since by (11.4) and (3.6),

$$x_+ = O(x \exp(-\sqrt{\log x})) + qy(1 + O(\frac{1}{\log y})) = x(1 + O(\frac{1}{\log_2 x})),$$

$$\frac{rP}{x_+} \ll \exp(-\sqrt{\log x}) \ll \exp(-u \log z),$$

and by (3.2) and the definition $\delta_1(x, y)$

$$\begin{aligned} u\xi_u &= \left(1 + O\left(\frac{\log_2 u}{\log u}\right)\right) u \log u \\ &= \left(1 + O\left(\frac{\log_2 u}{\log u}\right)\right) \log y \frac{\log(\log y / \log z)}{\log z} = \left(1 + O\left(\frac{\log_2 u}{\log u}\right)\right) \delta_1(x, y) \log y. \end{aligned}$$

Remarks: Under the Generalized Riemann Hypothesis, (11.1) holds with the error replaced by $O(Px^{\varepsilon-1/2})$ and, in the above proof, we may choose $z = (1/2) \log x$ leading to the conclusion of Theorem A1 with $\delta_1(x, y)$ replaced by

$$\delta_3(x, y) = \frac{\log(\log y / \log_2 x)}{\log_2 x}$$

and the larger range $q(\log q)^{1+\varepsilon} \leq x \leq q \exp((\log q)^{1/2-\varepsilon})$ in place of (2.2).

As remarked in §2 we can use a modification of the above argument to delete from (2.12) and (2.13) of Theorem A3 the factor $\log_2 x$, at least for sums of the form $\sum_{q \sim Q} \max_{(a,q)=1} \sum_{q \sim Q} \min_{(a,q)=1}$. Indeed, since we are able to assume $\log y \ll \log_2 x \log_3 x$, the estimate of Corollary 3.1, in conjunction with (2.14), is sufficient for the proof. In this case we must choose y_+ above so that qy_+ is independent of q and, for y this small, Corollary 3.1 allows us to do that.

12. Proof of Theorem A2

Let $y = x/q$ and $u = \log y / \log(\log x \log y)$ so that

$$(12.1) \quad N(\varepsilon) \ll u \leq \log^{1/3} x.$$

Define v to be the positive real solution of

$$(12.2) \quad \left[2 + \frac{10v}{\log_2 x}\right] \log x^v = y,$$

and let $w = v / \log_2 x$. We pick $j = [v] - 1$ or $[v]$ so that j is even when we wish to prove (2.6) and odd when we wish to prove (2.7). Clearly, if $N(\varepsilon)$ has been chosen sufficiently large then $|j - u| < \varepsilon u / 2$.

Now take $\ell = y^{1/j} / \log x$, $h = 2 \log x$ and $z = (\ell + 2) \log x$, so that $y = (z - h)^j$. As

$$\ell = \frac{y^{1/v}}{\log x} y^{1/j-1/v} = (2 + 10w) \exp\left(\frac{v-j}{jv} \log y\right)$$

we have

$$(12.3) \quad 2 + 10 \frac{j}{\log_2 x} \leq \ell \ll w \exp(4/w) \ll \log^{1/2} x$$

provided that $N(\varepsilon)$ is sufficiently large. Therefore

$$(12.4) \quad j^2(1 + 2/\ell)^j \leq j^2 e^{2j/\ell} \leq (\log x)^{2/3} (\log x)^{1/5} = (\log x)^{13/15}$$

as $2j/\ell \leq \frac{1}{5} \log_2 x$ by the first inequality in (12.3). Let

$$k = 1 + [c \log x / 10j \log_2 x \log(\ell \log_2 x)]$$

with c as in Proposition 11.1 with $\varepsilon = 1/2$. Then, by the last inequality in (12.3), and by (12.4),

$$\begin{aligned} k - j + 1 &\geq \frac{c}{20} \frac{\log x}{j(\log_2 x)^2} - j \\ &\geq \frac{c}{30} \frac{(\log x)^{2/15}}{(\log_2 x)^2} j \left(1 + \frac{2}{\ell}\right)^j \\ &\geq 4j z^j / y \end{aligned}$$

so that

$$(12.5) \quad (z - h)^j = y \geq \frac{4j}{(k - j + 1)} z^j.$$

Let n be the product of any $k + 1$ primes in $(z - h, z]$ that do not divide q . Note that, as $z^{5/8} \leq h \leq z$, Huxley's theorem [Hu] gives $\pi(z) - \pi(z - h) \sim h / \log z$ and so there are at least $\nu(q) + k + 1$ primes in $(z - h, z]$. Choose P as in Proposition 11.1, with $\varepsilon = 1/2$.

Define the matrix \mathcal{M} exactly as in the proof of Theorem A1, except with y_+ replaced by y . By counting down each column, we see that the sum of the elements of \mathcal{M} is

$$\sum_{\substack{n \leq y \\ (n, P) = 1}} \frac{RP}{\phi(P)} \{1 + O(e^{-c \log x / \log P} + e^{-c \sqrt{\log x}})\}$$

by Proposition 11.1. Now

$$\begin{aligned} c \frac{\log x}{\log P} &\geq \frac{c \log x}{k \log z} \geq 9j \log(\ell \log_2 x) \frac{\log_2 x}{\log(\ell \log x)} \\ &\geq 2j \log(j^2 \ell \log_2 x) \end{aligned}$$

by (12.3), and $c \sqrt{\log x} \geq 2j \log(j^2 \ell \log_2 x)$. Therefore the sum of the elements of \mathcal{M} is

$$(12.6) \quad R \left(y + \frac{P}{\phi(P)} r_P(y) \right) \{1 + O(1/(j^2 \ell \log_2 x)^{2j})\}.$$

There are

$$R \frac{\phi(q)}{q} + O(\tau(q)) = \frac{R \phi(q)}{q} \{1 + O(1/(j^2 \ell \log_2 x)^{2j})\}$$

row numbers r , with $r \sim R$ and $(q, rP) = 1$; therefore we may choose some such row (say row r_0) which contains more than (respectively less than)

$$\frac{q}{\phi(q)} \left(y + \frac{P}{\phi(P)} r_P(y) \right) \{1 + O(1/(j^2 \ell \log_2 x)^{2j})\}$$

primes, in order to prove (2.6) (respectively (2.7)).

Let $x_0 = x_{\pm} = r_0P + qy$ and $a = a_{\pm} = r_0P$, so that $(a, q) = 1$. Now, as in the proof of Theorem A1, row r_0 contains exactly $\theta(x_0; q, a)$ primes so that

$$(12.7) \quad (-1)^j \Delta(x_0; q, a) \geq (-1)^j \frac{P}{\phi(P)} \frac{r_P(y)}{y} + O\left(\frac{1}{(j^2 \ell \log_2 x)^{2j}}\right).$$

By Theorem B2 and (12.5) we have that

$$\begin{aligned} (-1)^j \frac{r_P(y)}{y} &\geq \frac{1}{4} \binom{k}{j} \frac{1}{z^j} \gg \frac{1}{\sqrt{j}} \left(\frac{ek}{jz}\right)^j \\ &\gg \left(\frac{c'}{j^2 \ell \log_2 x \log(\ell \log_2 x)}\right)^j \end{aligned}$$

where $c' = ce/20$, and so, by (12.7),

$$(12.8) \quad \begin{aligned} (-1)^j \Delta(x_0; q, a) &\geq 1/(u^2 \ell \log_2 x)^{(1+\varepsilon/3)u} \\ &\geq 1/(u^{3(1+\varepsilon/2)u} \log^5 x) \end{aligned}$$

by the second inequality in (12.3).

Finally, as $\log u = \{1 + o(1)\} \log(\log y / \log_2 x)$, we see that

$$3(1 + \varepsilon/2)u \log u \leq (1 + \varepsilon)\delta_2(x, y) \log y,$$

and so (2.6) and (2.7) both follow from (12.8).

Remarks: In general this is the best possible result, using this method, up to the power of $\log x$ in the denominator. If $\log y \leq (\log_2 x)^2$ then $u \leq \log_2 x$ and so the factor $\exp(4/w)$ in the upper bound in (12.3) becomes relevant. It does not seem possible to remove this factor because of the ℓ^u in the denominator of the right-hand side of (12.8). More precisely, in order to use Theorem B2 we need $\ell \geq 2$ and so

$$\ell^j \log^j x \geq y \geq \frac{(\ell + 2)^j \log^j x}{k} \geq \log^{j-1} x.$$

Now in the proof above we need two values of j , one odd and one even, so if $y = \log^J x - 1$ for some given integer J , we can take $j = J$ and $j = J - 1$ only if $\ell \geq (\log x)^{1/(J-1)}$. But then we are guaranteed the factor $\ell^u \approx (\log x)^{1+o(1)}$ in the denominator.

If we only need to get a good estimate on one side (i.e. for $|\Delta(x; q, a)|$) when $\log y$ is small compared to $(\log_2 x)^2$ then we do this with $j = 1 + \lceil \log y / \log(2 \log x) \rceil$ and $\ell = 2$. Select h, k and z as in the proof of Theorem A2, and we can then get the lower bound $1/(u^2 \log_2 x)^{u(1+\varepsilon)}$.

If we were to suppose that the error term in Proposition 11.1 could be taken to be $O(Px^{\varepsilon-\frac{1}{2}})$ (which would follow from the Generalized Riemann Hypothesis) then, in the proof of Theorem A2, we could take $k = 1 + \lceil \log x / 1000 \log_2 x \rceil$. This would imply that, in the definition of $\delta_2(x, y)$, we could replace the "3" by a "2" and (2.6) and (2.7) would still hold.

As remarked in §2 we can also use Theorem B2 to prove a slightly weaker version of Theorem A3 and its Corollary. By just considering those integers q in $[Q, 2Q]$ containing no prime factor in $[2 \log y, 4 \log y]$ and by taking $\ell = 1$ in the proof of Theorem A2 one can, for example, deduce (2.15) in the range

$$Q \geq x \exp(-(1 - \varepsilon) B (\log_2 x)^2 / \log_3 x)$$

where $A = 3B + 5$.

13. Proofs of Theorem A3 and Proposition 2.1

We shall require the following simple result.

Lemma 13.1. For any non-zero integer a and all $x \geq 2$ we have

$$(13.1) \quad \sum_{\substack{m \leq x \\ (m,a)=1}} \frac{m}{\phi(m)} = xc_1(a) + O(\tau(a) \log x),$$

$$(13.2) \quad \sum_{\substack{m \leq x \\ (m,a)=1}} \frac{1}{\phi(m)} = c_1(a) \log x + c_2(a) + O(\tau(a)x^{-1} \log x),$$

where

$$c_1(a) = \frac{\phi(a)}{a} \prod_{p|a} \left(1 + \frac{1}{p(p-1)}\right)$$

and $c_2(a) \ll \tau(a)$.

Proof. It suffices to prove (13.1) since (13.2) then follows by partial summation. Writing $m/\phi(m) = \sum_{d|m} \mu^2(d)/\phi(d)$ and inverting the order of summation, we have

$$\sum_{\substack{m \leq x \\ (m,a)=1}} \frac{m}{\phi(m)} = \sum_{\substack{d \leq x \\ (d,a)=1}} \frac{\mu^2(d)}{\phi(d)} \sum_{\substack{r \leq x/d \\ (r,a)=1}} 1.$$

Since

$$\sum_{\substack{r \leq x/d \\ (r,d)=1}} 1 = \frac{x}{d} \frac{\phi(a)}{a} + O(\tau(a)),$$

it follows that

$$\sum_{\substack{m \leq x \\ (m,a)=1}} \frac{m}{\phi(m)} = x \frac{\phi(a)}{a} \sum_{\substack{d \leq x \\ (d,a)=1}} \frac{\mu^2(d)}{d\phi(d)} + O(\tau(a) \log x).$$

The sum over d may be extended to a sum over all d prime to a , the error in doing so being absorbed by the error $O(\tau(a) \log x)$. Once this is done, the main term is just $xc_1(a)$ and we have (13.1).

Proof of Theorem A3. It is well-known (cf. [Da, p. 124]) that, with a suitable constant c , $0 < c < 1$, the estimate

$$\theta(x; r, a) = \frac{x}{\phi(r)} + O(x \exp(-2c\sqrt{\log x}))$$

holds uniformly for all $r \leq \exp(2\sqrt{\log x})$ and all integers a with $(a, r) = 1$, except possibly for those moduli r that are multiples of a certain exceptional modulus $r_1 > 1$. It follows that if a is a non-zero integer satisfying $(a, r_1) > 1$ (if r_1 exists), then uniformly for $R_1 \leq R_2 \leq \exp(c\sqrt{\log x})$, $x^{1/2} \leq x_1 \leq x_2 \leq x$ we have

$$(13.3) \quad \sum_{\substack{R_1 < r \leq R_2 \\ (r,a)=1}} \{\theta(x_2; r, a) - \theta(x_1; r, a)\} = \sum_{\substack{R_1 < r \leq R_2 \\ (r,a)=1}} \frac{x_2 - x_1}{\phi(r)} + O(x \exp(-c'\sqrt{\log x}))$$

for any $c' < 2c$. Now let $x \exp(-c\sqrt{\log x}) \leq Q_1 \leq x/2$ and $0 < |a| < Q_1$. Then

$$\begin{aligned} \sum_{\substack{q \sim Q_1 \\ (q,a)=1}} \theta(x; q, a) &= \sum_{\substack{q \sim Q_1 \\ (q,a)=1}} \sum_{\substack{a < p \leq x \\ p-a=qr}} \log p + O(Q_1 \log |a|) \\ &= \sum_{\substack{r \leq x/2Q_1 \\ (r,a)=1}} \{\theta(a + 2rQ_1; r, a) - \theta(a + rQ_1; r, a)\} \\ &\quad + \sum_{\substack{r \sim x/2Q_1 \\ (r,a)=1}} \{\theta(x; r, a) - \theta(a + rQ_1; r, a)\} \\ &\quad + O(Q_1 \log |a|) + O(Q_1 \log_2(2x/Q_1)), \end{aligned}$$

where the first error term absorbs the contribution of the primes $\leq a$ to $\theta(x; q, a)$ and thus, in view of the bound $|a| < Q_1 < q$, may be dropped unless a itself is a prime. The latter error term accounts for the replacement of $(x - a)$ by x in the ranges of summation. By (13.3) this implies

$$\sum_{\substack{q \sim Q_1 \\ (q,a)=1}} \theta(x; q, a) = \sum_{\substack{r \leq x/2Q_1 \\ (r,a)=1}} \frac{rQ_1}{\phi(r)} - \sum_{\substack{r \sim x/2Q_1 \\ (r,a)=1}} \frac{rQ_1}{\phi(r)} + \sum_{\substack{r \sim x/2Q_1 \\ (r,a)=1}} \frac{x}{\phi(r)} + O(Q_1 \log(|a|x/Q_1)).$$

provided that $(a, r_1) > 1$ in the case the exceptional modulus r_1 exists. Setting

$$S_a(t) = \sum_{\substack{r \leq t \\ (r,a)=1}} r/\phi(r),$$

we can write the right-hand side as

$$Q_1 S_a\left(\frac{x}{2Q_1}\right) - Q_1 \left(S_a\left(\frac{x}{Q_1}\right) - S_a\left(\frac{x}{2Q_1}\right) \right) + x \sum_{\substack{r \sim x/2Q_1 \\ (r,a)=1}} \frac{1}{\phi(r)} + O(Q_1 \log(|a|x/Q_1)),$$

which by partial summation reduces to

$$x \int_{x/2Q_1}^{x/Q_1} S_a(t) \frac{dt}{t^2} + O(Q_1 \log(|a|x/Q_1)).$$

Since

$$\begin{aligned} \sum_{\substack{q \sim Q_1 \\ (q,a)=1}} \theta(x; q, a) &= x \left\{ \sum_{\substack{q \sim Q_1 \\ (q,a)=1}} \frac{1}{\phi(q)} + \sum_{\substack{q \sim Q_1 \\ (q,a)=1}} \frac{1}{\phi(q)} \Delta(x; q, a) \right\} \\ &= x \left\{ c_1(a) \log 2 + \sum_{\substack{q \sim Q_1 \\ (q,a)=1}} \frac{1}{\phi(q)} \Delta(x; q, a) \right\} + O(x^{1/2}) \end{aligned}$$

by Lemma 13.1 and the bound $\tau(a) \ll |a|^{1/3} \leq Q_1^{1/3} \leq x^{1/3}$, it follows that

$$(13.4) \quad \sum_{\substack{q \sim Q_1 \\ (q,a)=1}} \frac{1}{\phi(q)} \Delta(x; q, a) = \int_{x/2Q_1}^{x/Q_1} (S_a(t) - tc_1(a)) \frac{dt}{t^2} + O\left(\frac{Q_1 \log(|a|x/Q_1)}{x}\right).$$

Now, let x and Q be given subject to (2.11) and let $y = x/Q$. We shall give the proof of (2.13), the proof of (2.12) being almost identical. Set $z = (1/2) \log x$. In case there is an exceptional modulus $r_1 \leq \exp(\sqrt{\log x})$, let p be a prime divisor of r_1 , otherwise let $p = 2$. By Theorem B3 there exist z_- in the range (3.4) and y_- in the range (3.6) so that $\bar{r}^p(y_-, z_-)$ satisfies (3.7). Let $Q_- = x/y_-$ and $a_- = p \prod_{p' < z_-} p'$. By (3.6) Q_- satisfies $Q \leq Q_- \leq Q(1 - 1/\log y)^{-1}$, and thus, in view of (2.11), lies in the range $x \exp(-c\sqrt{\log x}) \leq Q_- \leq x/2$. Moreover we have $a_- \neq 0$, $Q_- > |a_-|$ and, if r_1 exists, $(a_-, r_1) \geq (a_-, p) = p > 1$. Hence the above argument applies with $Q_1 = Q_-$, $a = a_-$, and (13.4) holds. Now it is easily seen that, with the above choice of a_- ,

$$\int_{x/2Q_-}^{x/Q_-} (S_{a_-}(t) - tc_1(a_-)) \frac{dt}{t^2} = c_1(a_-)(\log 2)\bar{r}^p(y_-, z_-).$$

Thus, (2.13) follows from (13.4), (3.7), and the estimate

$$c_1(a_-) \gg \frac{\phi(a_-)}{a_-} \gg \prod_{p' < z_-} \left(1 - \frac{1}{p'}\right) \gg \frac{1}{\log z_-} \gg \frac{1}{\log_2 x}.$$

Proof of Proposition 2.1. In the case $Q \leq x^{2/3}$ the result follows at once from Theorem 9 of [B-F-I] (for fixed a) or from Fouvry [Fö]; in fact their results apply in a much larger range.

In the case $Q > x^{2/3}$ we follow the above argument except that instead of using the estimate (13.3) we bound $\theta(x; r, a)$ on average by means of the Bombieri-Vinogradov theorem. We obtain in this way (13.4) with $Q_1 = Q$ and an additional error term $O_A((\log x)^{-A})$ on the right-hand side. Since, by (13.1),

$$\int_{x/2Q}^{x/Q} (S_a(t) - tc_1(a)) \frac{dt}{t^2} = O\left(\int_{x/2Q}^{x/Q} \tau(a) \log t \frac{dt}{t^2}\right) = O\left(\tau(a) \frac{Q \log(x/Q)}{x}\right),$$

this proves the desired estimate (2.16).

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