Subdesigns in Steiner Quadruple Systems

by

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Abstract

A *Steiner quadruple system of order* v, denoted $SQS(v)$, is a pair $(X, \mathcal{B})$ where X is a set of cardinality v, and $\mathcal{B}$ is a set of 4-subsets of X (called *blocks*), with the property that any 3-subset of X is contained in a unique block. If $(X, \mathcal{B})$ is an $SQS(v)$ and (Y, C) is an $SQS(w)$ with $Y \subseteq X$ and $C \subseteq \mathcal{B}$, we say that (Y, C) is a *subdesign* of $(X, \mathcal{B})$. Hanani has shown that an $SQS(v)$ exists for all $v \equiv 2$ or $4 \pmod 6$ and when $v \in \{0, 1\}$; such integers v are said to be *admissible*. A necessary condition for the existence of an $SQS(v)$ with a subdesign of order w is that $v = w$ or $v \geq 2w$. In this paper we show the existence of an explicitly computable constant k (independent of w) such that for all admissible v and all admissible w with $v \geq kw$ there exists an $SQS(v)$ containing a subdesign of order w. We also show that for any sufficiently large w we can take $k = 12.54$. To establish these results we introduce several new constructions for SQS, and we also consider the subdesign problem for related classes of designs.

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1. Introduction

This paper is concerned with the existence problem for $SQS(v)$ with a subdesign of order w. Let $A_S = \{0,1\} \cup \{v \geq 2 : \ v \equiv 2 \ or \ 4 (\mathrm{mod}\, 6)\}$ denote the set of admissible integers. For $w \in A_S$, we define S_w to be the set of all orders v for which there exists an $SQS(v)$ with a subdesign of order w. Define s_w to be the least admissible integer v_0, such that for all $v \in A_S, \ v \geq v_0$ we have $v \in S_w$, if such a v_0 exists; otherwise $s_w = \infty$. In section 3 we shall show that s_w is finite for all $w \in A_S$, and then in section 4 we shall show that $s_w \leq kw$ for some absolute constant k. In section 5 we improve the constant k. To do this we need constructions for $SQS(v)$ with subdesigns. These constructions are described in section 2, including a review of existing constructions as applied to the subsystem problem. Several new constructions are also described in section 2, and the details of these new results are given in section 6. In section 7 we give some generalizations of our results and, in section 8, pose some problems.

To put our question into a more general context, we define a $t-(v,K,\lambda)$ design, for integers $t \geq 0, \ v \geq 0, \ \lambda \geq 1$, and K a set of non-negative integers, to be a pair $(X, \mathcal{B})$ where X is a set of cardinality v, and $\mathcal{B}$ is a set of subsets of X called blocks with the properties that

(i) $B \in \mathcal{B} \ \Rightarrow \ |B| \in K$. i.e. K is the set of block sizes,

(ii) every t-subset of X is contained in precisely λ blocks.

An $SQS(v)$ is just a $3-(v,\{4\},1)$ design, and a Steiner triple system ($STS(v)$) is a $2-(v,\{3\},1)$ design.

A three-wise balanced design is a $3-(v,K,1)$design. In his second paper on $3-(v,\{4\},\lambda)$ designs Hanani [7] showed the importance of $3-(v,\{4,6\},1)$ designs for the construction of SQS, and we denote such a design by $T(v)$ (T for three-wise balanced). These systems have been studied by many authors (e.g. [2] and [3]), usually under the name of generalized Steiner systems. Hanani showed that a $T(v)$ exists if and only if $v \equiv 0 (\mathrm{mod}\, 2)$ or $v = 1$, and this set of integers is denoted by A_T. Our bounds on s_w follow directly from our study of the subdesign problem for $T(v)$ designs. Accordingly we define T_w to be the set of all orders v for which there exist a $T(v)$ with a subdesign of order w; and t_w to be the smallest integer v_0 such that for all $v \in A_T, \ v \geq v_0$ we have $v \in T_w$, if such an integer exists; otherwise $t_w = \infty$.

A special kind of design $T(v)$ can be constructed when $v \equiv 0 (\mathrm{mod}\, 6)$ - these are $3-(v,\{4,6\},1)$ designs with precisely $\frac{v}{6}$ blocks of size 6 (which form a parallel class), and all the remaining blocks of size 4. Such designs will be denoted by $G(v)$. The existence of designs $G(v)$ for all $v \equiv 0 (\bmod 6)$ was first shown by Mills [17] in his studies of packing and covering problems.

As the cases of $SQS(v)$ and $T(v)$, we define A_G, G_w and g_w for designs $G(v)$ with subdesigns $G(w)$. The subdesign problem for $G(v)$ has been studied in the special case $w = 498$, by Hartman, Mills and Mullin [12]. Their results easily generalize to give a quadratic upper bound on g_w - and this is outlined in section 3.

Our main result is that $s_v \leq kv$ for some fixed constant k. We also give linear bounds on t_v and g_v. It has been conjectured by many authors that $s_v = 2v$, and this conjecture has been proved for $v \leq 8$. Finite bounds on s_v for $v \leq 40$ were given by Hartman [9], but to this date no proof of the finiteness of s_v has appeared. We believe the old conjecture to be true, and further conjecture that $t_v = 2v$ and $g_v = 2v$; these conjectures are true for $v \leq 6$. (Note that a very easy counting argument gives $s_v \geq 2v$, and similarly for t_v and g_v.)

The study of the subdesign problem for $SQS(v)$ is a natural generalization of the Doyen-Wilson theorem [5] for Steiner triple systems. Doyen and Wilson showed that for all $v \equiv 1$ or $3 \, (\bmod 6)$ and $w \equiv 1$ or $3 \, (\bmod 6)$ with $v = w$ or $v \geq 2w + 1$, there exists an $STS(v)$ with a subdesign of order w.

2. Constructions with subdesigns

In this section we state the constructive results which enable us to show the existence of designs with subdesigns. Several of these constructions have appeared in the literature before, and we restate these results in a form which highlights the subdesign properties. The new constructions are described here and a detailed account of them appears in Section 6.

When the subdesign is trivial (has 0 or 1 blocks) then Hanani's and Mills' existence theorems have the following corollaries:

Theorem 2.0

(a)
$$\begin{aligned} S_1 &= A_S \setminus \{0\} \\ S_2 &= A_S \setminus \{0,1\} \\ S_4 &= A_S \setminus \{0,1,2\} \\ S_8 &= A_S \setminus \{0,1,2,4,10,14\} = \{8\} \cup \{v \in A_S :\ v \geq 16\} \text{ [Hartman, 10]} \end{aligned}$$

(b) $T_1 = A_T \setminus \{0\}$
$T_2 = A_T \setminus \{0,1\}$
$T_4 = A_T \setminus \{0,1,2,6\}$

(c) $G_6 = A_G \setminus \{0\}$ [Mills, 17]

A basic result in the construction of designs is the replacement property for subdesigns. That is, given a design $(X, \mathcal{B})$ with a subdesign (Y, C) and another design $(Y, \mathcal{P})$ on the same points, then $(X, (\mathcal{B} - C) \cup \mathcal{P})$ is a design which contains the second subdesign. More formally

Theorem 2.1

(a) If $v \in S_w$ and $w \in S_x$ then $v \in S_x$. ($w \in S_x$ implies $S_w \subseteq S_x$)

(b) If $v \in T_w$ and $w \in T_x$ then $v \in T_x$.

(c) If $v \in G_w$ and $w \in G_x$ then $v \in G_x$.

Other constructions for designs are recursive in nature and this naturally induces subsystems in the constructed design. We discuss five families of recursive constructions below.

2.1 Doubling Constructions

The well known doubling construction for SQS described in Hanani [6], Lindner and Rosa [16] and Hartman [8] has the following implications for the subdesign problems.

Theorem 2.2

(a) For all $v \in A_T$, $2v \in T_v$

(b) For all $v \in A_S$, $2v \in S_v$

(c) For all $v \in A_G$, $2v \in G_v$

(d) If $v \in T_w$ then $2v \in T_{2w}$

(e) If $v \in S_w$ then $2v \in S_{2w}$

(f) If $v \in G_w$ then $2v \in G_{2w}$

There is also another doubling construction for $T(v)$ due to Hanani [7]. His construction uses a $T(v)$ to construct a $T(2v-2)$ for any even v. Close inspection of this construction gives

Theorem 2.3 If $v \in T_w$ then $(2v-2) \in T_{2w-2}$.

2.2 Tripling Constructions

The constructions described in this section are for designs of order $3v - 2w$, given an input design of order v with a subsystem of order w. The cases where $w = 0, 1$ are rather special, and we discuss these cases first.

Theorem 2.4

(a) If $v \in T_1$ then $3v - 2 \in T_v$ [Proved in §6]

(b) If $v \in S_1$ then $3v - 2 \in S_v$ [Hanani]

Corollary 2.5

For all even v we have $3v \in T_v$.

Proof By induction on the power of 2 dividing v: If $v \equiv 2 (\bmod 4)$ then let $x = \frac{v}{2} + 1 \equiv 0 (\bmod 2)$. Therefore $3x - 2 \in T_x$ by Theorem 2.4(a), and so $3v = 2(3x-2) - 2 \in T_{2x-2=v}$, by Theorem 2.3.

Otherwise let $x = \frac{v}{2} \equiv 0 (\bmod 2)$. Then $3x \in T_x$ by the induction hypothesis, and so $3v = 6x \in T_{2x=v}$ by Theorem 2.2(d).

The next result is the key to our use of $T(v)$ for the study of $SQS(v)$.

Theorem 2.6 (Hanani [7]) If $v \in T_w$ then $3v - 2 \in S_{3w-2} \subset T_{3w-2}$.

When w is even then the tripling constructions due to Hanani [6] and Hartman [8] [11] (see also Lenz [14]) have the following form

Theorem 2.7 For all even $w \geq 2$,

(a) If $v \in T_w$ and $v \equiv \pm w (\bmod 3)$ then $3v - 2w \in T_v$.

(b) If $v \in S_w$ then $3v - 2w \in S_v$.

(c) If $v \in G_w$ then $3v - 2w \in G_v$.

2.3 Quadrupling Constructions

In his original paper, Hanani [6] proved that $4v-6 \in S_v$ for all $v \in S_2$. His construction also gives $4v-6 \in T_v$ for all $v \in T_2$. In section 5 we give a generalization of his construction which proves the following result.

Theorem 2.8 For all even w we have

(a) If $v \in T_w$ then $4v - 3w \in T_v$

(b) If $v \in S_w$ then $4v - 3w \in S_v$

(c) If $v \in G_w$ then $4v - 3w \in G_v$

Our construction also shows that if there exists a design $(X, \mathcal{B})$ of order v with two subdesigns (Y_1, C) and $(Y_2, \mathcal{P})$ with $Y_1 \cap Y_2 = \emptyset$, $|Y_1| = w$, $|Y_2| = x$ and $x = v - w$ or $x \leq \frac{v-w}{2}$, then there exists a design of order $4v - 3w$ containing a subdesign of order $2x$. In particular, when $w = 2$, we have the following Corollary to Hanani's construction.

Theorem 2.9 If v, w are even, with $v > w$ then

(a) If $v \in T_w$ then $4v - 6 \in T_{2w}$

(b) If $v \in S_w$ then $4v - 6 \in S_{2w}$

Note that $w \leq \frac{v-2}{2}$ is no real condition since if $w = \frac{v}{2}$ then $4v - 6 \in T_v = T_{2w}$ (by Theorem 2.8).

2.4 Hextupling Constructions

The constructions described here generalize the constructions of Hanani [7] and Hartman, Mills and Mullin [12]. The result stated below is proved in section 5.

Theorem 2.10 For all $w \geq 2$, if $v \in S_w \cup G_w$ then

(a) $6(v-2) + k \in S_{6(w-2)+k}$ for $k = 2, 4, 8$ or 10 and

(b) $6(v-2) + k \in G_{6(w-2)+k}$ for $k = 6$ or 12.

This clearly implies

Corollary 2.11 For $w \geq 2$, if $v \in S_w \cup G_w$ then

$$6(v-2) + k \in T_{6(w-2)+k} \qquad \text{for} \qquad k = 2, 4, 6, 8, 10 \text{ or } 12.$$

2.5 The Singular Direct Product

This contruction, due to Hanani [6], [7], has been rediscovered many times; see for example Aliev [1], Lindner and Rosa [16], and others. The result is a consequence of Theorems 2.4 and 2.7, although it appears to contain these results as a special case.

Theorem 2.12 If $n \in S_k$ then

(a) If $v \in T_w$ and $v \equiv \pm w \pmod 3$ then $(n-1)(v-w) + w \in T_{(k-1)(v-w)+w}$.

(b) If $v \in S_w$ then $(n-1)(v-w)+w \in S_{(k-1)(v-w)+w}$.

(c) If $v \in G_w$ then $(n-1)(v-w)+w \in G_{(k-1)(v-w)+w}$.

Setting $n = 4,\ k = 2$ almost gives a restatement of Theorems 2.4 and 2.7; however Theorem 2.4 (a) is stronger, since the restriction $v \equiv \pm 1 (\bmod 3)$ is not needed in that result.

We have a number of results that combine the above theorems. We shall concentrate just on S:

Theorem 2.13.

(a) If $n \in S_k$ and $v \in S_w$ then $(3n-2)(v-w)+w \in S_{(3k-2)(v-w)+w}$.

(b) Suppose A is a positive integer with $A \equiv 0$ or $2 \pmod 6$ or $A \equiv 3$ or $9 \pmod{18}$. For any integer B and $w \in A_S$, if $v = Aw + B \in S_w$ then $Av + B \in S_v$.

Proof:

(a) We have $4v - 3w \in S_v$ by 2.8(b) and so the result follows from substituting this into 2.12(b).

(b) For $A \equiv 0$ or $2 \pmod 6$ take $n = A + 2, k = 2$ in 2.12(b) to get $Av + B = Av + (v - Aw) = (A+1)(v-w)+w \in S_v$. For $A \equiv 3$ or $9 \pmod{18}$ take $n = A/3 + 1, k = 1$ in (a) to get $Av + B = (A+1)(v-w)+w \in S_v$.

Theorem 2.14

(a) If $v \equiv 4 \pmod 6$ then $2v + 2 \in S_v$

(b) If $v \equiv 2 \pmod 6$ then $4v + 2 \in S_v$

Proof:

(a) Let $w = (v+2)/3$ so that $w \in A_T$. Then $2w \in T_w$ by 2.2(a) and so $2v + 2 = 6w - 2 \in S_{3w-2=v}$ by 2.6.

(b) By (a), $4v + 2 \in S_{2v}$ and as $S_{2v} \subset S_v$ by 2.2(b) and 2.1(a), we get the result.

3. Finite embedding bounds

In this section we use the constructions given in section 2 to show that s_w and t_w are bounded above by a polynomial in w. We shall also refer to the techniques of Hartman, Mills and Mullin [12] to give a quadratic bound on g_w. We begin by showing the existence of certain classes of designs in T_w for $w = 6, 8, 10, 18, 20$ and 30.

Lemma 3.1 If $v \equiv 2 \pmod 8$ and $v \geq 26$ then $v \in T_8$.

Proof: As $w = (v+6)/4 \in T_4$ by 2.0(b), so $v = 4w - 6 \in T_8$ by 2.9.

Lemma 3.2 If $v \equiv 0 \pmod 2$ and $v \geq 12$ then $v \in T_6$.

Proof: By induction on the power of 2 dividing v: If $v \equiv 2 \pmod 4$ then $w = (v+2)/2 \in T_4$ by 2.0 (b); and so $v = 2w - 2 \in T_6$ by 2.3. Now $12 \in T_6$ by 2.2(a). Chouinard et al [4] have shown that $16 \in T_6$. Kreher [13] has shown that $20 \in T_6$. For any other $v \equiv 0 \pmod 4$ we have $v \in T_{v/2} \subset T_6$ by the induction hypothesis.

Lemma 3.3 If $v \equiv 2 \pmod 6$ and $v \geq 20$ then $v \in T_{10}$, except possibly $v = 32$.

Proof: For a given value of v let m be the largest odd number dividing v. If $m = 1$ let $w = 64$; if $m = 7$ let $w = 28$; otherwise let $w = 2m$. By definition, if w divides v then $v = w2^k$ for some value of k, so that $v \in T_w$ by k applications of 2.2(a).

Now as $10, 22 \in T_4$ by 2.0(a) we have $28 = 3.10 - 2$ and $64 = 3.22 - 2 \in T_{10}$ by 2.6. The other values of w satisfy $w \equiv 2 \pmod 4$ and $w = 10$ or $w \geq 22$. Let $z = w/2 + 1$, so that $z \equiv 0 \pmod 2$ and $z = 6$ or $z \geq 12$. Then $z \in T_6$ by 3.2 and so $w = 2z - 2 \in T_{10}$ by 2.3. Therefore $v \in T_w \subset T_{10}$ by 2.1(b).

Lemma 3.4 If $v \equiv 2 \pmod{24}$ and $v \geq 50$ then $v \in T_{20}$.

Proof: For $v \geq 74$, $v \neq 122$ we let $w = (v+6)/4$. Then $w \in T_{10}$ by 3.3 and so $v = 4w - 6 \in T_{20}$ by 2.9(a). As $20 \in T_{10}$ by 2.2(a) thus $50 = 4.20 - 3.10 \in T_{20}$ by 2.8(a). Similarly $50 = 4.14 - 3.2 \in T_{14}$ by 2.8(a) and so $122 = 3.50 - 2.14 \in T_{50} \subset T_{20}$ by 2.7(a).

Lemma 3.5 If $v \equiv 12 \ (\bmod\ 24)$ and $v \geq 36$ then $v \in T_{18}$.

Proof: If $v \geq 84$ then let $w = \frac{v}{2}$ so that $w \equiv 6 (\bmod 12)$ and $w \geq 42$. Now let $x = \frac{w+2}{2}$, so that $x \equiv 4 (\bmod 6)$ and $x \geq 22$; finally let $z = \frac{x+2}{3}$, so that $z \equiv 0 (\bmod 2)$ and $z \geq 8$.

Now $z \in T_4$ and so $x = 3z - 2 \in T_{10}$ (by Theorem 2.6), $w = 2x - 2 \in T_{18}$ (by Theorem 2.3), and $v = 2w \in T_w \subseteq T_{18}$ (by Theorem 2.2).

Finally $36 \in T_{18}$ by 2.2(a) and $60 = 4 \cdot 18 - 3 \cdot 4 \in T_{18}$ by 2.8, as $18 \in T_4$.

Lemma 3.6 If $v \equiv 12 \ (\bmod\ 24)$ and $v \geq 60$ then $v \in T_{30}$.

Proof: If $v \geq 132$ then let $w = \frac{v}{2}$ so that $w \equiv 6 \pmod{12}$ and $w \geq 66$. Let $x = \frac{w+2}{2}$, so that $x \equiv 4 \pmod 6$ and $x \geq 34$; finally let $z = \frac{x+2}{3}$, so that $z \equiv 0 \pmod 2$ and $z \geq 12$.

Now $z \in T_6$ (by Lemma 3.2) and so $3z-2 = x \in T_{16}$ (by Theorem 2.6), $2x-2 = w \in T_{30}$ (by Theorem 2.3), and $2w = v \in T_w \subseteq T_{30}$ (by Theorem 2.2).

For $v \in \{60, 84, 108\}$ we note that $60 = 2 \cdot 30 \in T_{30}$ and $84 = 4 \cdot 30 - 3 \cdot 12 \in T_{30}$ since $30 = 4 \cdot 12 - 3 \cdot 6 \in T_{12}$ as $12 \in T_6$. Also $108 = 4 \cdot 30 - 3 \cdot 4 \in T_{30}$ as $30 \in T_4$.

We can now establish the finiteness of t_w for all even w.

Theorem 3.7 For all even $w \geq 2$ we have

$$t_w \leq \begin{array}{ll} \max\{12t_{\frac{w}{2}} - 18, 110\} & \text{if } w \equiv 0 \pmod 4 \\ \max\{12t_{\frac{w+2}{2}} - 16, 120\} & \text{if } w \equiv 2 \pmod 4. \end{array}$$

Proof: The proof is by induction; the case $w = 2$ is trivial. Assume that the theorem holds for all $x < w$. (Throughout we shall use Theorem 2.1 without explicitly mentioning it.)

Case 1 $w \equiv 0 \pmod 4$: Let $x = \frac{w}{2}$ and $v_0 = t_x$.

(a) If $v \equiv 0 \pmod 4$ and $v \geq 2v_0$ then $\frac{v}{2} \in T_x$ and so $v \in T_w$ by 2.2(d).

(b) If $v \equiv 2 \pmod 8$ and $v \geq 4v_0 - 6$ then $\frac{v+6}{4} \in T_x$ and so $v \in T_w$ by 2.9(a).

(c) If $v \equiv 10 \pmod{12}$ and $v \geq 6v_0 - 2$ then $y = \frac{v+2}{3} \in T_w$ by (a), and so $v \in T_y \subseteq T_w$ by 2.4(a).

(d) If $v \equiv 6 \pmod{24}$ and $v \geq 12v_0 - 18$ then $y = \frac{v}{3} \in T_w$ by (b), and so $v \in T_y \subseteq T_w$ by 2.5.

(e) If $v \equiv 14$ or $62 \pmod{72}$ and $v \geq \max\{12v_0 - 34, 62\}$ then let $y = \frac{v+16}{3}$ so that $y \equiv 2$ or $10 \pmod{24}$ and $y \geq \max\{4v_0 - 6, 26\}$. Therefore $y \in T_w \cap T_8$ by (b) and 3.1 and so $v = 3y - 2 \cdot 8 \in T_y \subset T_w$ by 2.7(a).

(f) If $v \equiv 38 \pmod{72}$ and $v \geq \max\{12v_0 - 58, 110\}$ then let $y = \frac{v+40}{3}$ so that $y \equiv 2 \pmod{24}$ and $y \geq \max\{4v_0 - 6, 50\}$. Therefore $y \in T_w \cap T_{20}$ by (b) and 3.4, and so $v = 3y - 2.20 \in T_y \subset T_w$ by 2.7(a).

This completes Case 1, since all values of $v \geq \max\{12t_{\frac{w}{2}} - 18, 110\}$ are covered by one of the subcases (a) to (f).

Case 2 $w \equiv 2 \pmod 4$, $w > 2$: Let $x = \frac{w+2}{2}$ and $v_0 = t_x$.

(a) If $v \equiv 2 \pmod 4$ and $v \geq 2v_0 - 2$ then $\frac{v+2}{2} \in T_x$ and so $v \in T_w$ by 2.3.

(b) If $v \equiv 4 \pmod 8$ and $v \geq 4v_0 - 4$ then $\frac{v}{2} \in T_w$ by (a), and so $v \in T_w$ by 2.2(a).

(c) If $v \equiv 4 \pmod{12}$ and $v \geq 6v_0 - 8$ then $\frac{v+2}{3} \in T_w$ by (a), and so $v \in T_w$ by 2.4(a).

(d) If $v \equiv 8 \pmod{24}$ and $v \geq 12v_0 - 16$ then $\frac{v}{2} \in T_w$ by (c), and so $v \in T_w$ by 2.2(a).

(e) If $v \equiv 24 \pmod{72}$ and $v \geq \max\{12v_0 - 24\}$ then $\frac{v+12}{3} \in T_w \cap T_6$ by (b) and 3.2, and so $v \in T_w$ by 2.7(a).

(f) If $v \equiv 0 (\mathrm{mod}\, 72)$ and $v \geq \max\{12v_0 - 48, 72\}$ then $\frac{v+36}{3} \in T_w \cap T_{18}$ by (b) and 3.5, and so $v \in T_w$ by 2.7(a).

(g) If $v \equiv 48 (\mathrm{mod}\, 72)$ and $v \geq \max\{12v_0 - 72, 120\}$ then $\frac{v+60}{3} \in T_w \cap T_{30}$ by (b) and 3.6, and so $v \in T_w$ by 2.7(a).

This completes the proof.

Theorem 3.7 gives an explicit bound on t_w, by iterating from $t_2 = 2$:

Theorem 3.8 For all even $w \geq 2$ we have

$$t_w \leq \frac{4}{3} w^4.$$

We now prove that s_w is finite, using a short argument based on Theorem 3.8. In section 4 we give more intricate arguments and derive linear bounds on both t_x and s_x.

Lemma 3.9 If $v \equiv 0 \pmod 4$ and $v \geq 24$ then $v \in T_{12}$.

Proof: Apply the doubling construction (2.2(d)) to the results of Lemma 3.2.

Lemma 3.10 If $v \equiv 10 \pmod{12}$ and $v \geq 34$ then $v \in S_{14}$, except possibly $v = 46$.

Proof: For $v = 34 = 3 \cdot 14 - 2 \cdot 4$ we have $34 \in S_{14}$ by 2.7(b). As $20 \in T_{10}$ and $28 \in S_{14}$ by 2.2, we have $58 = 3.20 - 2 \in S_{28=3.10-2} \subset S_{14}$. For $v \geq 70$ let $y = \frac{v+2}{3}$ so that $y \equiv 0 (\mathrm{mod}\, 4)$ and $y \geq 24$. Then $y \in T_{12}$ by 3.9 and so $v = 3y - 2 \in S_{34=3.12-2} \subseteq S_{14}$ by 2.6.

Theorem 3.11 For all $w \equiv 4 (\mathrm{mod}\, 6)$ we have $s_w \leq \max\{9t_{\frac{w+2}{3}} - 10, 146\}$.

Proof: Let $x = \frac{w+2}{3} \equiv 0 (\mathrm{mod}\, 2)$, and $v_0 = t_x$.

(a) If $v \equiv 4 (\mathrm{mod}\, 6)$ and $v \geq 3v_0 - 2$ then $y = \frac{v+2}{3} \in T_x$ and so $v = 3y - 2 \in S_w$ by 2.6.

(b) If $v \equiv 8 (\mathrm{mod}\, 12)$ and $v \geq 6v_0 - 4$ then $\frac{v}{2} \in S_w$ by (a), and so $v \in S_{2w} \subseteq S_w$ by 2.2(e).

(c) If $v \equiv 8 (\mathrm{mod}\, 18)$ and $v \geq 9v_0 - 10$, then $y = \frac{v+4}{3} \in S_w \cap S_2$ by (a), and so $v = 3y - 2 \cdot 2 \in S_y \subseteq S_w$ by 2.7(b).

(d) If $v \equiv 14 (\mathrm{mod}\, 18)$ and $v \geq \max\{9v_0 - 22, 32\}$ then $\frac{v+16}{3} \in S_w \cap S_8$ by (a) and 2.0(i), and so $v \in S_w$ by 2.7(b).

(e) If $v \equiv 2 (\mathrm{mod}\, 36)$ and $v \geq \max\{9v_0 - 34, 146\}$ then let $y = \frac{v+28}{3}$, so that $y \equiv 10 (\mathrm{mod}\, 12)$ and $y \geq \max\{3v_0 - 2, 58\}$. Therefore $y \in S_w \cap S_{14}$ by (a) and 3.10, and so $v \in S_w$ by 2.7(b).

This completes the proof since all $v \in A_S$ are covered by one of the cases (a) to (e).

The previous theorem, together with the doubling construction, now gives us a bound on s_w for $w \equiv 2 (\mathrm{mod}\, 6)$:

Corollary 3.12 For all $w \equiv 2 (\bmod 6)$ we have $S_{2w} \subseteq S_w$ and hence

$$s_w \leq s_{2w} \leq \max\{9t_{\frac{2w+2}{3}} - 10, 146\}.$$

The last two results together with Theorem 3.8 then yield.

Corollary 3.13 For all $w \in A_S$ we have

$$s_w \leq \frac{64}{27}(w+1)^4 \ .$$

We turn now to the problem of subdesigns in G-designs, the results here are due to the methods of Hartman, Mills and Mullin [12] and their results are quoted without proof.

We also need the notion of an H-design $H(v)$, which is a triple $(X, \mathcal{B}, \mathcal{G})$ where X is a set of points of cardinality $v \equiv 0 (\bmod 6)$. The group set $\mathcal{G} = \{G_1, G_2, \ldots, G_{\frac{v}{6}}\}$ is a partition of X into groups of size 6. The block set $\mathcal{B}$ consists of 4-subsets of X with the properties that

(1) $B \in \mathcal{B} \ \Rightarrow \ |B \cap G_i| \leq 1 \quad$ for all i;

(2) Every 3-subset T of X with $|T \cap G_i| \leq 1$ for all i is contained in a unique block.

Hartman, Mills and Mullin showed that there exists an $H(v)$ for all $v \equiv 0 (\bmod 6)$ except $v = 18$, and possibly $v = 9 \cdot 6, \ v = 27 \cdot 6$, or $v = 81 \cdot 6$.

The existence of a design $H(v)$ implies the existence of a design $G(v)$ as follows. Let $F_1^i | F_2^i | \cdots | F_5^i$ be a one-factorization of K_6 with vertex set G_i. Form the block set $\mathcal{B}_2 = \{[x, y, z, t] : [x, y] \in F_j^i, \ [z, t] \in F_j^k, \ 1 \leq i < k \leq \frac{v}{6}, \ 1 \leq j \leq 5\}$, then $(X, \mathcal{G} \cup \mathcal{B} \cup \mathcal{B}_2)$ is a $G(v)$. Furthermore, if the H-design contains a subdesign on w points (or is missing a non-existent subdesign on $w = 6 \cdot 3^i, \ i = 1, 2, 3, 4$ points) then the $G(v)$ can be constructed, as above, to contain a subdesign $G(w)$. Let H_w be the set of integers v such that there exists an $H(v)$ with a (possibly non-existent) $H(w)$ subdesign, so that $g_w \leq h_w$. In the paper [12], several results on the structure of H_w were obtained, the most powerful of these results is given below.

Theorem 3.14 Suppose that w is a non-negative integer and q is a prime power with $q \geq \max\{w - 1, 88\}$. Then $6v \in H_{6w}$ for all integers v such that

$$q^2 - q + w - (q-3)\lfloor \frac{q-83}{2} \rfloor \leq v \leq q^2 - q + w.$$

This theorem, together with a quantitative form of the prime number theorem gives us a bound on h_w, and thus on g_w.

Corollary 3.15 For all $x \equiv 0 (\bmod 6)$ we have

$$g_x \leq h_x \leq \max \{8654 + \frac{x}{6}, \frac{1.21}{2}(\frac{x}{6})^2 + 46(\frac{x}{6})\}$$

Proof: Let $w = \frac{x}{6}$ and q_1 be the smallest prime power $\geq \max\{w-1, 97\}$. Using the formulae in [18] it is easy to show that there is a prime in the interval $[n, 1.1n]$ for all $n \geq 117$. This result, together with some hand calculations for $97 \leq q_1 \leq 121$, shows that for all $v \geq q_1^2 - q_1 + w - (q_1 - 3)\lfloor \frac{q_1 - 83}{2} \rfloor$ we have $6v \in H_x$. The first term of the statement of the result is given by setting $q_1 = 97$, the second by setting $q_1 = 1.1(w-1)$.

To finish this section we will construct certain subdesigns that we we will need in section 6. First define for $i = 0, 1, 2, \ldots,$ the following table of values of n_i and σ_i

i	0	1	2	3	4	5	6	7	8
σ_i	34	38	16	4	32	448	8	58	16
n_i	158	172	68	14	130	1808	28	218	56

i	9	10	11	12	13	14	15	16	17	18
σ_i	14	56	2	38	20	14	28	4	26	8
n_i	44	176	4	106	52	34	68	8	58	16

Lemma 3.16 For $i = 0, 1, \ldots, 18$ we have $n_i \in S_{\sigma_i}$.

Proof: The result for $i = 3, 6, 11, 16, 18$ comes from 2.0. We know that $28, 34 \in S_8$ by 2.0 and so $56, 68 \in S_{16}$ by 2.2(e), giving $i = 2$ and 8. As $10 \in S_2$ we have $26 = 2.10 + 6 \in S_{10}$ and $58 = 2.26 + 6 \in S_{26}$ by 2.13(b); therefore $52 = 2.26 \in S_{20=2.10}$ by 2.2(e), giving $i = 13, 17$. As $14 \in S_4$ we have $34 = 2.14 + 6 \in S_{14}$ and $74 = 2.34 + 6 \in S_{34}$ by 2.13(b), giving $i = 14$. Also $68 = 2.34 \in S_{28}$ by 2.2(e), giving $i = 15$. Now $28 \in S_{14}$ by 2.2(b) and so $158 = 6(28-2) + 2 \in S_{6(14-2)+2=74} \subset S_{34}$ by 2.10(a), giving $i = 0$. As $38 \in S_4$ we have $106 = 3.38 - 2.4 \in S_{38}$ by 2.7(b), giving $i = 12$. Furthermore $130 \in S_{32}$ and $226 \in S_{56}$ by 2.14(b), giving $i = 4$. Applying 2.2(e) to this 3 times gives $1808 \in S_{448}$, giving $i = 5$. Now we've seen $(i = 6)$ that $28 \in S_8$ and $10 \in S_2$ and so $218 = (28-1)(10-2) + 2 \in S_{(8-1)(10-2)+2=58}$ by 2.12(b), giving $i = 7$. As $14 \in S_2$ we have $38 = 3.14 - 2.2 \in S_{14}$ and $86 = 3.38 - 2.14 \in S_{38}$ by 2.7(b). Then $172 \in S_{86} \subset S_{38}$

by 2.2(b), giving $i = 1$. As $14 \in S_4$ we have $44 = 4.14 - 3.4 \in S_{14}$ by 2.8(b), and $176 = 2^2$. $44 \in S_{56=2^2.14}$ by 2.2(e), thus giving $i = 9$ and 10.

Lemma 3.17 If $v \equiv 0 \pmod 2$ and $v \geq 24$ then $v \in T_{12}$, except possibly $v = 26, 38, 46$.

Proof: By imitating the proof of Case I in Theorem 3.7, we see that, as $T_6 = \{v \equiv 0 \pmod 2 : v = 6 \text{ or } v \geq 12\}$, thus $\{v \equiv 0 \pmod 2 : v = 12 \text{ or } v \geq 24\} \backslash A \subset T_{12}$ where $A = \{26, 30, 38, 46, 54, 62, 78, 102\}$. For $v \in G_6$ with $v \geq 12$, we have a design $T(v)$ with two disjoint blocks of size 6; hence by the remarks following Theorem 2.8 we have $12, 18, 24, 30 \in G_6$ which implies $30, 54, 78, 102 \in T_{12}$. Finally, $32 \in T_{16}$ by 2.2(a), and so $62 = 2.32 - 2 \in T_{30=2.16-2} \subset T_{12}$ by 2.3.

Theorem 3.18.

(a) If $v \equiv 4 \pmod 6$ and $v \geq 68$ then $v \in S_{34}$, except possibly $v \in A = \{76\}$.

(b) If $v \equiv 2 \pmod 6$ and $v \geq 68$ then $v \in S_{34}$, except possibly $v \in B = \{80, 92, 104, 110, 116, 122, 128, 146, 152\}$.

Proof:

(a) By applying Theorem 2.6 to 3.17, we see that all such v lie in S_{34}, with the possible exceptions of 112 and 136. But $136 \in S_{68} \subset S_{34}$ by 2.2(b) and, as $34 \in S_8$, we have $112 = 4.34 - 3.8 \in S_{34}$ by 2.8(b).

(b) We now apply to the values in (a), the steps (b) to (e) of the proof of 3.11. Carefully using this algorithm one gets that for all the values of v in the hypothesis, except perhaps for 134, 158, 170, 212 and 224 we have $v \in S_{34}$. But $212 \in S_{106} \subset S_{34}$ and $224 \in S_{112} \subset S_{34}$ by 2.2(b). $158 \in S_{34}$ was shown ($i = 0$) in 3.16. As $68 \in S_{34}$ we have $170 = 4.68 - 3.34 \in S_{68} \subset S_{34}$ by 2.8(b). The fact that $134 \in S_{34}$ is proved in Section 6, see 6.3.

4. Linear bounds

In this section we shall find linear upper bounds for s_x, t_x and g_x. The values of the constants involved will be improved considerably in section 5, where we shall only be concerned with "large" values of x.

Theorem 4.1. There exist constants c_s, c_t and c_g such that

(a) If $x \in A_S$ then $s_x \leq c_s x$

(b) If $x \in A_G$ then $g_x \leq c_g x$

(c) If $x \in A_T$ then $t_x \leq c_t x$.

In particular we may take $c_s = 10^{20}$, $c_g = 10^{15}$ and $c_t = 3.10^{20}$.

Actually (c) is derived from (a) and (b) by the following:

Lemma 4.2. Suppose that there exists constants $c > 0$ and x_0 such that if $x \geq x_0$ and (a) if $x \in A_S$ then $s_x \leq cx$; (b) if $x \in A_G$ then $g_x \leq cx$. Then $t_x \leq 3cx$ for all $x \in A_T$ with $x \geq x_0$.

Proof: If $x \in A_S$ then $S_x \subset T_x$ and, as $3x \in T_x$, by 2.5, we have $G_{3x} \subset T_{3x} \subset T_x$. Thus $S_x \cup G_{3x} \subset T_x$ and so $t_x \leq \max\{s_x, g_{3x}\} \leq 3cx$.

If $x \in A_G$ then $G_x \subset T_x$ and, as $3x-2 \in T_x$, by 2.4(a), we have $S_{3x-2} \subset T_{3x-2} \subset T_x$. Thus $G_x \cup S_{3x-2} \subset T_x$ and so $t_x \leq \max\{g_x, s_{3x-2}\} \leq 3cx$.

If we take $x_0 = 0$ and $c = \max\{c_g, c_s\}$ in 4.1, then, by 4.2, we see that $c_t \leq 3c$.

In what follows we shall let R stand for G or S and similarly r_x, A_R, etc. We define $b_G = 12$ and $b_S = 9$ and we fix $\sigma \in A_S$ so that $\sigma \geq 6b_r + 2$.

In order to prove Theorem 4.1 we will need a series of technical lemmas: (Actually 4.3, 4.4 and 4.6 are really two lemmas each, one for $R = S$ the other for $R = G$).

Let $N = N_\sigma^R = \max\{(\sigma-1)r_x - (\sigma-2)x : \ x \in A_R,\ 0 \leq x \leq 6\sigma - 6\}$.

Lemma 4.3 For any $x \geq N$ there exist integers u and y with $u \equiv 0 \pmod 6$, $u \leq x/(\sigma-1)$ and $y \equiv x \pmod{2u}$ such that

$$\{z \geq \left(\frac{s_\sigma - 1}{\sigma - 1}\right) x : \ z \equiv y \quad \text{or} \quad y + 2u \pmod{6u}\} \subset R_x.$$

Proof: Let w be the least non-negative residue of $x \pmod{6\sigma-6}$ and $u = \frac{x-w}{\sigma-1}$ which is divisible by 6. Let $y = u + w = \frac{x+(\sigma-2)w}{\sigma-1}$ so that, by the definition of N, we have $y \geq r_w$.

Therefore whenever $n \in S_\sigma$ we get $(n-1)u + w \in R_x$ by 2.12. So by taking those $n \geq s_\sigma$ with $n \equiv 2$ or $4 \pmod 6$ we get

$$\{z \geq \left(\frac{s_\sigma - 1}{\sigma - 1}\right) x - \left(\frac{s_\sigma - \sigma}{\sigma - 1}\right) w : \quad z \equiv u + w \text{ or } 3u + w \pmod{6u}\} \subset R_x.$$

Now $s_\sigma \geq 2\sigma$ so that $\frac{s_\sigma - \sigma}{\sigma - 1} > 0$ and

$$\begin{aligned} x = (\sigma-1)u + w &\equiv u + w \text{ or } 3u + w \pmod{6u} \\ &\equiv u + w = y \qquad \pmod{2u} \end{aligned}$$

Lemma 4.4 Suppose that $x, y \in A_R$ and n and d are integers with d divisible by 6, such that $\{v \geq n : \ v \equiv y \ \text{ or } \ y + 2d \pmod{6d}\} \subset R_x$. If $y \in A_R$ and $r_w \leq n$ for each $w \in A_R$ with $w \leq 6d$ then $\{z \geq 3n : \ z \in A_R \text{ and } z \equiv y \pmod{m_r}\} \subset R_x$, where $m_S = 4$ and $m_G = 12$.

Proof: Note that $3y \equiv y \pmod{m_r}$. It is a matter of elementary number theory to prove that any integer $z \geq 3n$ with $z \equiv y \pmod{m_r}$ and $z \in A_R$ can be written in the form $z = 3v - 2w$ where $v \geq n,\ 0 \leq w \leq 6d,\ w \in A_R$ and $v \equiv y$ or $y + 2d \pmod{6d}$. But then $v \geq n \geq r_w$ by hypothesis, so that $z = 3v - 2w \in R_v (\subset R_x)$ by 2.7.

Lemma 4.5 If $x \in A_S$ then $x, 3x - 2 \in S_x$ and are congruent to 0 and 2 $\pmod 4$.

If $x \in A_G$ then $4x$ and $4x - 18 \in G_x$, and are congruent to 0 and 6 $\pmod{12}$.

Proof: The result follows from 2.4(b), 2.2(c) and 2.8(c) with $w = 6$.

Lemma 4.6 For any $x \in A_R$ with $x \geq N$ let

$$\lambda = \frac{1}{x} \max\{r_w : w \leq \frac{2b_r x}{\sigma - 1}\} \quad \text{and} \quad \ell = \max\{3\lambda, b_r \left(\frac{s_\sigma - 1}{\sigma - 1}\right)\}.$$

Then $r_x \leq \ell x$.

Proof: By Lemma 4.3 there exist integers u and y, with $u \leq \frac{x}{\sigma - 1}$, $\quad u \equiv 0 \pmod 6$ and $y \equiv x \pmod{2u}$, for which

$$\{z \geq \frac{\ell}{b_r} x : \ z \equiv y \text{ or } y + 2u \pmod{6u}\} \subset R_x.$$

By Lemma 4.5 we see that

$$B_i = \{z \geq \frac{\ell}{3} x : \ z \equiv y_i \text{ or } y_i + 2u_i \pmod{6u_i}\} \subset R_x$$

where $y_1 = x,\ y_2 = 3x - 2,\ u_1 = u,\ u_2 = 3u$, for $x \in A_S$,
and $y_3 = 4x, y_4 = 4x - 18,\ u_3 = u_4 = 4u$ for $x \in A_G$.

If $w \leq 6u_i$ then $w \leq \frac{2b_r x}{\sigma - 1}$ so that $r_w \leq \lambda x$. By applying Lemma 4.4 with $d = u_i$ and $n = \frac{\ell x}{3}$ we get

$$C_i = \{z \geq \ell x : \ z \in A_R \quad \text{and} \quad z \equiv y_i \pmod{m_r}\} \subset R_x.$$

Taking the union of the C_i in each case gives the result by Lemma 4.5.

We now can proceed to the

Proof of Theorem 4.1:

For $R = S$ or G let $\lambda_0 = \max\{\frac{r_w}{w} :\ w \le N,\ w \in A_R\}$ and $c_r = \max\{3\lambda_0,\ b_r\left(\frac{s_\sigma - 1}{\sigma - 1}\right)\}$. We shall prove that $r_x \le c_r x$ for all $x \in A_R$, by induction on x. For $x \le N$ we have $r_x \le \lambda_0 x < c_r x$. Suppose that $x > N$ and that the result holds for each $y < x$. Then, in 4.6 we have

$$\lambda \le \frac{1}{x}\max\{c_r w :\ w < \frac{x}{3}\} < \frac{c_r}{3} \quad \text{and} \quad \ell \le \max\{c_r, b_r\left(\frac{s_\sigma - 1}{\sigma - 1}\right)\} \le c_r$$

and so $r_x \le c_r x$.

By taking $\sigma = 56$ and 74 for $R = S$ and G respectively, and by using the bounds in 3.13 and 3.15, we get the upper bounds on c_s and c_g. The bound for c_t follows from 4.2.

5. Better Linear bounds

Throughout this section let $\kappa = 4 + \sqrt{10} \approx 7.1623$. Our main result is

Theorem 5.1. For all $\delta > 0$ there exists a constant x_δ such that if $x \ge x_\delta$ then

(a) If $v \in A_S$ and $v \ge (\kappa + \delta)x$ then $v \in S_x$, except possibly when $v \in [12x - 52, \left(\frac{7}{4}\kappa + \delta\right)x]$, with $v \not\equiv x \pmod 6$.

(b) If $v \in A_G$ and $v \ge (\kappa + \delta)x$ then $v \in G_x$, except possibly when $v \in [12x - 144, \left(\frac{7}{4}\kappa + \delta\right)x]$, with $v \not\equiv x \pmod{18}$.

We thus have

Corollary 5.2. For all $\delta > 0$ there exists a constant x_δ such that if $x \ge x_\delta$ then

(a) If $x \in A_S$ then $s_x \le \left(\frac{7}{4}\kappa + \delta\right)x$.

(b) If $x \in A_G$ then $g_x \le \left(\frac{7}{4}\kappa + \delta\right)x$.

(c) If $x \in A_T$ then $t_x \le \left(\frac{21}{4}\kappa + \delta\right)x$.

(N.B. (c) follows from (a) and (b) by Lemma 4.2).

We define $\bar{S}_x \equiv \{v \in S_x :\ v \equiv x \pmod 6\}$ and $\bar{G}_x = \{v \in G_x :\ v \equiv x \pmod{18}\}$. Define $\bar{s}_x$ to be the least integer v_0 such that for all $v \ge v_0$ with $v \equiv x \pmod 6$ we have $v \in S_x$; similarly define $\bar{g}_x$. We again take $R = S$ or G throughout the section.

Corollary 5.3. For all $\delta > 0$ there exists x_δ such that $\bar{r}_x \le (\kappa + \delta)x$ for each $x \in A_R$ with $x \ge x_\delta$.

The proofs of these results come from the following technical result:

Proposition 5.4. Fix $\epsilon > 0$. Suppose that (a) $\bar{r}_x \le kx$ or (b) $r_x \le kx$, for all $x \in A_R$ with $x \ge x_0$, and $n \in S_\sigma$ where (a) $3 \not| n - \sigma$ or (b) $9 \not| n - \sigma$. There exists x_1 such that for all $x \in A_R$ with $x \ge x_1$, and for all $v \in \Big[(4 - \frac{3}{k})\big(\frac{n-1}{\sigma-1}\big)x, \Big(4\big(\frac{n-1}{\sigma-1}\big) - \varepsilon\Big)x\Big]$ with (a) $v \in A_R$ or (b) $v \equiv x \pmod 6$ (if $R = S$), $v \equiv x \pmod{18}$ (if $R = G$), we have $v \in R_x$.

We postpone the proof until the end of the section. From Proposition 5.4(a) we immediately derive

Corollary 5.5. Suppose that $\bar{r}_x \le kx$ for all $x \in A_R$ with $x \ge x_0$, and $n \in S_\sigma$ where $3 \not| n - \sigma$ and $k < 4\big(\frac{n-1}{\sigma-1}\big)$. Then $r_x \le k'x$ for all $x \in A_R$ with $x \ge x_1$, where $k' = \big(4 - \frac{3}{k}\big)\big(\frac{n-1}{\sigma-1}\big) \Big(\le 4\big(\frac{n-1}{\sigma-1}\big) - \frac{3}{4}\Big)$.

¿From 5.5 together with 4.1 and 3.18(b) we have

Lemma 5.6. We have $r_x \le kx$ for all $x \in A_R$ with $x \ge x_0$, where

$$k = 4\big(\tfrac{158-1}{34-1}\big) - \tfrac{3}{4} < 18.29.$$

Proof: Let m be the smallest element of S_{34} with $m \equiv 2 \pmod 6$ for which there exists x_0 such that $r_x \le \big(4\big(\frac{m-1}{33}\big) - \frac{3}{4}\big)x$ for all $x \ge x_0$ with $x \in A_R$. By 4.1 we know that such a value of m exists (just choose m so that $c_R \le 4\big(\frac{m-1}{33}\big) - \frac{3}{4}$) and we claim that $m \le 158$, which would establish the result. If $m > 158$ then let $n = m - 6$ so that $n \in S_{34}$ by 3.18. Therefore, as $k\Big(= 4\big(\frac{m-1}{\sigma-1}\big) - \frac{3}{4}\Big)$ is less than $4\big(\frac{n-1}{\sigma-1}\big)$, we have $r_x \le k'x$ where $k' = 4\big(\frac{n-1}{\sigma-1}\big) - \frac{3}{4}$, for x sufficiently large, by 5.5, giving a contradiction.

Lemma 5.7. We have $r_x \le 13.21x$ for all $x \in A_R$ with $x \ge x_0$.

Proof: Define n_i and σ_i for $i = 0, 1, 2, \ldots, 8$ as in Lemma 3.16, and let $n_{8+j} = 2^j.56$, $\sigma_{8+j} = 2^j.16$ for each $j \ge 0$. Note that each $n_i \in S_{\sigma_i}$ by 3.16 and by 2.2(d). Define $k_0 = 4\big(\frac{n_0-1}{\sigma_0-1}\big) - \frac{3}{4}$ and $k_i = \big(4 - \frac{3}{k_{i-1}}\big)\big(\frac{n_i-1}{\sigma_i-1}\big)$ for each $i \ge 1$. We shall prove by induction on $i \ge 0$ that, for some x_i, we have $r_x \le k_i x$ for all $x \in A_R$ with $x \ge x_i$. For $i = 0$ this comes from 5.6. If the result is true for i then it follows for $i + 1$ by an immediate application of 5.5, with $k = k_i$, $k' = k_{i+1}$, $n = n_{i+1}$, $\sigma = \sigma_{i+1}$.

Now $\lim_{i\to\infty} k_i$ is seen to be the solution τ of the equation $\big(4 - \frac{3}{\tau}\big)\big(\frac{14}{4}\big) = \tau$ as $\lim_{i\to\infty} \frac{n_i-1}{\sigma_i-1}$ exists and equals $\frac{14}{4}$. Thus $\tau^2 - 14\tau + \frac{42}{4} = 0$ so that $\tau = \frac{14+\sqrt{154}}{2}$, and the result follows as $\frac{14+\sqrt{154}}{2} < 13.21$.

With this preparation we can now give

The Proof of Theorem 5.1. For $i = 9, \ldots, 18$ define n_i and σ_i as in 3.16. For all $j \ge 0$ let $n_{18+j} = 2^{4+j}$ and $\sigma_{18+j} = 2^{3+j}$ and let $k_8 = 13.21$, $k_i = \big(4 - \frac{3}{k_{i-1}}\big)\big(\frac{n_i-1}{\sigma_i-1}\big)$ for all $i \ge 9$. We claim that for each $i \ge 8$, $\bar{r}_x \le k_i x$ for sufficiently large x, and that

$A_R \cap [k_i x, \Big(4\big(\frac{n_i-1}{\sigma_i-1}\big) - \varepsilon\Big)x] \subset R_x$, for each $i \geq 11$, which we shall prove by induction on i. For $i = 8$ this statement is weaker than 5.7 and so follows immediately. Now fix $k = k_0$ in Proposition 5.4 (b) and take $n = n_i$, $\sigma = \sigma_i$. This gives the result for $i = 9$ and 10. Taking $k = k_{i-1}$, $n = n_i$, $\sigma = \sigma_i$ in Proposition 5.4(a), for $i = 11, 12, \ldots$ gives the result for $i + 1$ immediately. Now $\kappa = \lim_{i\to\infty} k_i$ exists and is the solution of $\big(4 - \frac{3}{\kappa}\big)2 = \kappa$, i.e. $\kappa^2 - 8k + 6 = 0$, i.e. $\kappa = 4 + \sqrt{10}$. Thus Corollary 5.3 follows immediately.

If we take the union of the intervals above for $i = 11, 12, \ldots$ we see that we have proved that $\Big[(\kappa+\delta)x, (12-\varepsilon)x\Big] \cap A_R \subset R_x$ for x sufficiently large.

Now we define $n_i = 56.2^i$, $\sigma_i = 16.2^i$ for each $i \geq 0$. Taking $k = \kappa + \delta/20$ in 5.4(a) gives us that $r_x \leq k'x$ where $k' = \lim_{i\to\infty}\big(4 - \frac{3}{k}\big)\big(\frac{n_i-1}{\sigma_i-1}\big) = \big(4 - \frac{3}{k}\big)\big(\frac{14}{4}\big) > \frac{7}{4}\kappa + \delta/2$. This completes the proof of 5.2.

To complete the proof of 5.1 we need only show that for some fixed $\varepsilon > 0$, $[(12 - \varepsilon)x, 12x - t_r] \cap A_R \subset R_x$, for all sufficiently large x, where $t_S = 53$ and $t_G = 145$. The proof here will give the "flavour" of the difficult proof for 5.4. If $R = S$ let $B = \{3x-2, 3x-4, 3x-8, 3x-16\}$ and if $R = G$ let $B = \{3x-12, 3x-24, 3x-36\}$. By 2.4(b) and 2.7 we have $B \subset R_x$, for sufficiently large x. By 5.7 we know that if $z \in A_R$ and $z \leq x/14$ then $x \in R_z$, and so $B \subset R_z$. Thus by 2.8, $4b - 3z \in R_b \subset R_z$ for each such $b \in B$ and z, and by elementary number theory it is easy to show that this gives all the numbers in the interval required.

The Proof of Proposition 5.4. We choose $x \geq x_1$ (i.e. sufficiently large, which we shall specify as we go along).

Let w_0 be the least residue of x (mod $6\sigma - 6$). Define

$$\begin{aligned} w_i &= w_0 + 2i(\sigma-1) \\ u_i &= \tfrac{x-w_i}{\sigma-1} = u_0 - 2i \\ v_i &= u_i + w_i = v_0 + 2i(\sigma-2) \qquad \text{and} \\ a_i &= (n-1)u_i + w_i = a_0 - 2i(n-\sigma), \end{aligned}$$

for $i = 0, 1, 2, \ldots, 24$.

Now, as each $w_i \leq 54(\sigma-1)$, we have $v_i \in R_{w_i}$ (as v_i increases with x) for x sufficiently large (i.e. $x \geq x_1$) by 4.1, provided that $v_i, w_i \in A_R$. Then $a_i \in R_x$ by 2.12.

We define the the set B as follows:

If $R = S$ and $x \equiv \sigma$ (mod 6) then (a) $B = \{a_0, a_3, a_6, a_1, a_4, a_7\}$, (b) $B = \{a_0, a_1\}$.

If $R = S$ and $x \not\equiv \sigma$ (mod 6) then (a) $B = \{a_0, a_3, a_6, a_2, a_5, a_8\}$, (b) $B = \{a_0, a_2\}$.

If $R = G$ then (a) $B = \{a_0, a_3, a_6, \ldots, a_{24}\}$, (b) $B = \{a_0\}$.

Now suppose, for a given $b \in B$, we have $z \le b/k$ with $z \in A_R$ (and, in (a), $z \equiv b \pmod 6$) if $R = S$; $z \equiv b \pmod{18}$ if $R = G$). We choose x_1 sufficiently large so that for any such z, the hypothesis tells us that (a) $b \in \bar{R}_z$, (b) $b \in R_z$. Then, by 2.8, $4b - 3z \in R_b \subset R_x$.

Now, just as in the proof of 4.7, it is simply a matter of elementary number theory to show that any v in the range $\left[\left(4 - \frac{3}{k}\right)a_0, 4a_{24}\right]$ with $v \in A_R$ (and, in (b) $v \equiv x \pmod 6$ if $R = S$; $v \equiv x \pmod{18}$ if $R = G$) can be written in the form $4b - 3z$ for some $b \in B$ and such a value of z.

Finally note that as,

$$a_0 = \frac{(n-1)}{(\sigma-1)}x - \frac{(n-\sigma)}{\sigma-1}w_0 \ \le \ \frac{(n-1)}{(\sigma-1)}x$$

and

$$\begin{aligned} a_{24} \ &= \ \frac{(n-1)}{\sigma-1}x - \left(\frac{n-\sigma}{\sigma-1}\right)w_0 - 48(n-\sigma) \ \ge \ \left(\frac{n-1}{\sigma-1}\right)x - 54(n-\sigma) \\ &\ge \ \left(\frac{n-1}{\sigma-1} - \frac{\varepsilon}{4}\right)x \qquad \text{for} \qquad x \ge x_1, \quad \text{the result follows.} \end{aligned}$$

We finish this section by noting the following

Lemma 5.8. If $n \in S_\sigma$ where $3 \nmid n - \sigma$ and

$$16 - 4\sqrt{10}(\approx 3.35) > \frac{n-1}{\sigma-1} > \frac{7}{16}(4+\sqrt{10})(\approx 3.13349)$$

then $s_x \le (\kappa+\delta)x, g_x \le (\kappa+\delta)x$ and $t_x \le (3\kappa+\delta)x$ for each sufficiently large $x \in A_S, A_G, A_T$, respectively.

Proof: By 5.1 and 4.2 we see that it is sufficient to show that if $v \in [12x - 144, \left(\frac{7}{4}\kappa + \delta\right)x] \cap A_R$ (where $R = G$ or S) then $v \in R_x$. By 5.4(a) this follows immediately for n and σ in the range above, as we may take $k = \kappa + \delta$ by 5.3.

A good example of such an n and σ would be $n = 64, \sigma = 20$.

6. Constructions in Detail

This section consists of detailed proofs of the results quoted in Section 2, which have not appeared elsewhere. In particular we give proofs of Theorem 2.4(a), 2.8 and 2.10.

6.1 A tripling construction

We wish to show that if there exists a $T(v)$ then there exists a $T(3v-2)$ containing a subdesign on v points. Let $(X \cup \{A\}, \mathcal{B})$ be a $T(v)$. We shall construct a $T(3v-2)$ with point set $(X \times Z_3) \cup \{\infty\}$, containing a subdesign with point set $(X \times \{0\}) \cup \{\infty\}$, isomorphic to the input design.

For each block of size 4, $[A, x, y, z] \in \mathcal{B}$ which contains A we construct an $SQS(10)$ with point set $(\{x, y, z\} \times Z_3) \cup \{\infty\}$, in such a way that $[x_0, x_1, x_2, \infty][y_0, y_1, y_2, \infty]$ $[z_0, z_1, z_2, \infty]$ and $[x_0, y_0, z_0, \infty]$ are all blocks of the new design. This is easily achieved, since the unique $SQS(10)$ has this configuration through any of its points.

For each block of size 6, $[A, x, y, z, w, t] \in \mathcal{B}$ which contains A we construct the $T(16)$ due to Chouinard et al. [4] on the point set $(\{x, y, z, t, w\} \times Z_3) \cup \{\infty\}$, in such a way that

$$[\infty, x_0, x_1, x_2][\infty, y_0, y_1, y_2] \cdots [\infty, w_0, w_1, w_2] \text{ and } [\infty, x_0, y_0, z_0, t_0, w_0]$$

are all blocks of the new design. It is easily verified that the design in question has such a subconfiguration. (Note that blocks of the form $[\infty, x_0, x_1, x_2]$ are not to be repeated in the final design).

For each block of size 4 $[x, y, z, t] \in \mathcal{B}$ not containing A, we construct the blocks $[x_i, y_j, z_k, t_m]: \ i + j + k + m \equiv 0 (\bmod 3) \ i, j, k, m \in Z_3$. (This clearly contains the block $[x_0, y_0, z_0, t_0]$.)

For each block of size 6, $[x, y, z, u, t, w] \in \mathcal{B}$, not containing A, we construct the 18 point configuration on $\{x, y, z, u, t, w\} \times Z_3$ given below. A concise description of the configuration is obtained by using the point set $Z_{15} \cup \{\infty_0, \infty_1, \infty_2\}$, then identifying x_i with ∞_i, y_i with $5i$, z_i with $5i+3, \ldots, w_i$ with $5i+12$. Let G be the cyclic group generated by the permutation $\alpha(j) = j + 1 (\bmod 15)$, $\alpha(\infty_i) = \infty_{i+1 (\bmod 3)}$ and consider the G-orbits of the following blocks:

$$[\infty_0, 0, 3, 6, 9, 12]$$

$$[\infty_0, 0, 1, 2, 4, 8]$$

$$[\infty_0, 0, 7, 13]$$

$$[\infty_0, 0, 11, 14]$$

$$[7, 11, 13, 14]$$

After the identification we see that $[x_0, y_0, z_0, u_0, t_0, w_0]$ is indeed in the block set, and the subdesign on $X \times \{0\}$ has been preserved. To satisfy oneself that the entire configuration is in fact a $T(3v-2)$ requires the reader to verify that every 3-subset has been included in precisely one block. This construction is a variant of Hanani's Proposition 8 from [7].

6.2 The quadrupling construction

In this section we shall prove that if there exists a three-wise balanced design on an even number of points, v, with a block (or subdesign) of even size w, then there exists a design on $4v - 3w$ points with blocks of size 4, and the other blocksizes from the original design. The new design contains four copies of the old design which intersect in a common block (subdesign).

Let $v - w = 2f$ and let $w = 2s$. Let $X = Z_{2f} \cup \{\infty_0, \infty_1, \ldots, \infty_{2s-1}\} \ = Z_{2f} \cup B_\infty$ and let $\mathcal{B}$ be the block set of a 3-wise balanced design on X containing a subdesign on B_∞. We shall construct a new design with point set $X' = (Z_{2f} \times \{0,1,2,3\}) \cup B_\infty$. Define the embeddings $\lambda_i :\ X \to X'\ ,\ i = 0,1,2,3$ by

$$\lambda_i(x) \ = \ \begin{cases} (x_i) & \text{if } \ x \in Z_{2f} \\ x & \text{if } \ x \in B_\infty. \end{cases}$$

The new block set $\mathcal{B}'$ will contain all the embedded blocks $\lambda_i(B)$ for each $i \in \{0,1,2,3\}$ and each $B \in \mathcal{B}$. The other blocks in $\mathcal{B}$ are all of size 4, and contain each triple from X', except those in $\lambda_i(X)$, precisely once. Any triple from $\lambda_i(X)$ is contained precisely once in an embedded bock $\lambda_i(B)$, by the 3-wise balance of $(X, \mathcal{B})$. Before constructing the remaining blocks we define a key ingredient of the construction which we have called a Hanani factorization.

For integers $f \geq s \geq 1$, we define a Hanani factorization $HF(2f, 2s)$ to be a four-tuple $(D, E, \mathcal{G}, \mathcal{H})$ with the following properties

(1) $D \subseteq \{1,3,5,\ldots,2f-1\}\ ,\ |D| = s$

(2) $E \subseteq \{0,2,4,\ldots,2f-2\}\ ,\ |E| = s$

(3) $\mathcal{G} = \{G_0, G_1, \ldots, G_{f-1}\}$ is a set of partial one-factors of the complete graph with vertex set Z_{2f}. The number of edges in each factor G_i is $f - s$, and the set of vertices covered by G_i is precisely $Z_{2f} \setminus ((D \cup E) + 2i)$, for each $i \in \{0,1,2,\ldots,f-1\}$.

(4) $\mathcal{H} = \{H_0, H_1, \ldots, H_{f-1}, H_f, \ldots, H_{f+s-2}\}$ is a set of one-factors of the complete graph with vertex set Z_{2f}.

(5) $\mathcal{G} \cup \mathcal{H}$ is a partition of the edge set of the complete graph with vertex set Z_{2f}.

In his paper [6], Hanani constructed $HF(2f, 2)$ for all integers $f \geq 1$. Hanani's construction had $D = \{1\},\ E = \{0\}$. We now give the existence theorem for Hanani factorizations.

Theorem 6.1. There exists an $HF(2f, 2s)$ for all integers $f \geq s \geq 1$.

Proof: The proof is by direct construction.

If f is odd, consider the one-factor

$$G = \{[f - i,\ f + 1 + i] :\ i = 0, 1, 2, \cdots, f - 1\}$$

The partial one-factor G_0 is constructed from G as follows.

If s is odd then omit the edge with $i = (f-1)/2$, and the edges with $i = \frac{(f-1)}{2} \pm j$ for $j = 1, 2, \ldots, \frac{s-1}{2}$. If s is even omit the edges with $i = \frac{(f-1)}{2} \pm j$ for $j = 1, 2, \ldots, \frac{s}{2}$.

Now let $G_k = G_0 + 2k$ for $k = 0, 1, 2, \ldots, f-1$ The graph Γ, covered by edges in $\mathcal{G}$ is cyclic, and it is not difficult to verify that its complement satisfies the conditions of Stern and Lenz's theorem on cyclic graphs [19]. This theorem implies the existence of a one-factorization $\mathcal{H}$ of Γ's complement. The sets D and E are the odd and even vertices, respectively, of edges omitted from G.

If f is even, consider the partial one-factor

$$G = \{[f - i,\ f + 1 + i] :\ i = 0, 1, 2, \ldots, \frac{f}{2} - 2\} \cup \{[j, -1 - j] :\ j = 0, 1, 2, \ldots, \frac{f}{2} - 1\}$$

The partial one-factor G_0 is formed from G as follows. If $s = 1,\ G_0 = G$. If $s > 1$ and s is odd remove pairs of edges from G with $i = j = 0, 1, 2, \ldots, \frac{s-3}{2}$.

If s is even remove the edge with $j = \frac{f}{2} - 1$, and pairs of edges with $i = j = 0, 1, \ldots, \frac{s-2}{2}$. As in the previous case, let $G_k = G_0 + 2k$ for $k = 0, 1, 2, \ldots, f - 1$. When s is even, the one-factorization $\mathcal{H}$ exists by Stern and Lenz's theorem. When s is odd construct

$$H_0 = \{[\frac{f}{2} + 2i,\ 1 - \frac{f}{2} + 2i] :\ i = 0, 1, 2, \ldots, f - 1\}.$$

The remaining edges (i.e. those not in $\mathcal{G}$ or H_0) form a cyclic graph which satisfies the conditions of Stern and Lenz's theorem, and thus the remaining one-factors in $\mathcal{H}$ may be constructed.

The sets D and E are the odd and even vertices omitted from G together with the vertices $\frac{f}{2}$ and $\frac{f}{2} + 1$.

This completes the construction of Hanani factorizations.

We return now to the quadrupling construction. Let $(D, E, \mathcal{G}, \mathcal{H})$ be a Hanani factorization $HF(2f, 2s)$, and let $D = \{d_0, d_1, d_2, \ldots, d_{s-1}\}$ and $E = \{e_0, e_1, \ldots, e_{s-1}\}$.

For each $i = 0, 1, 2, \ldots, s-1$, we construct the following 16 sets of blocks in $\mathcal{B}'$.

$$\{[\infty_{2i}, a_0, b_1, c_2] : \; a+b+c \equiv e_i (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (0,0,0) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i}, a_0, b_1, c_2] : \; a+b+c \equiv d_i (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (1,1,1) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i}, a_0, b_1, c_3] : \; a+b+c \equiv d_i + 1 (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (1,0,1) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i}, a_0, b_1, c_3] : \; a+b+c \equiv e_i + 1 (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (0,1,0) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i}, a_0, b_2, c_3] : \; a+b+c \equiv e_i (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (0,1,1) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i}, a_0, b_2, c_3] : \; a+b+c \equiv d_i (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (1,0,0) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i}, a_1, b_2, c_3] : \; a+b+c \equiv d_i + 1 (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (1,0,1) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i}, a_1, b_2, c_3] : \; a+b+c \equiv e_i + 1 (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (0,1,0) (\mathrm{mod}\, 2)\}$$

$$\{[\infty_{2i+1}, a_0, b_1, c_2] : \; a+b+c \equiv e_i (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (1,1,0) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i+1}, a_0, b_1, c_2] : \; a+b+c \equiv d_i (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (0,0,1) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i+1}, a_0, b_1, c_3] : \; a+b+c \equiv d_i + 1 (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (0,1,1) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i+1}, a_0, b_1, c_3] : \; a+b+c \equiv e_i + 1 (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (1,0,0) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i+1}, a_0, b_2, c_3] : \; a+b+c \equiv e_i (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (0,0,0) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i+1}, a_0, b_2, c_3] : \; a+b+c \equiv d_i (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (1,1,1) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i+1}, a_1, b_2, c_3] : \; a+b+c \equiv d_i + 1 (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (1,1,0) (\mathrm{mod}\, 2)\}$$
$$\{[\infty_{2i+1}, a_1, b_2, c_3] : \; a+b+c \equiv e_i + 1 (\mathrm{mod}\, 2f), \; (a,b,c) \equiv (0,0,1) (\mathrm{mod}\, 2)\}$$

Note that each of these sets of blocks contains f^2 distinct blocks. Every 3-subset of X' of the form $\infty_j x_k y_m$ with $k \neq m$, is contained in precisely one block. This can be checked methodically by considering the six possibilities for $\{k, m\}$ and the eight possible parities of (j, x, y).

The other 3-subsets of X' contained in the above blocks are those of the forms:

$$\begin{array}{lll} a_0 b_1 c_2 & \text{with} & a+b+c \in D \cup E, \text{ and } a \equiv b (\mathrm{mod}\, 2) \\ a_0 b_1 c_3 & \text{with} & a+b+c \in (D \cup E) + 1 \text{ and } a \not\equiv b (\mathrm{mod}\, 2) \\ a_0 b_2 c_3 & \text{with} & a+b+c \in D \cup E, \text{ and } b \equiv c (\mathrm{mod}\, 2) \\ a_1 b_2 c_3 & \text{with} & a+b+c \in (D \cup E) + 1 \text{ and } b \not\equiv c (\mathrm{mod}\, 2) \end{array}$$

We now form the following sets of blocks in $\mathcal{B}'$.

$$\{[a_0, b_1, g_2, \overline{g}_2] : \ a \equiv b (\bmod 2), \ \{g, \overline{g}\} \in G_c, \ a+b+2c \equiv 0 (\bmod 2f)\}$$
$$\{[a_0, b_1, g_3, \overline{g}_3] : \ a \not\equiv b (\bmod 2), \ \{g, \overline{g}\} \in G_c, \ a+b+2c \equiv 1 (\bmod 2f)\}$$
$$\{[g_0, \overline{g}_0, b_2, c_3] : \ b \equiv c (\bmod 2), \ \{g, \overline{g}\} \in G_a, \ 2a+b+c \equiv 0 (\bmod 2f)\}$$
$$\{[g_1, \overline{g}_1, b_2, c_3] : \ b \not\equiv c (\bmod 2), \ \{g, \overline{g}\} \in G_a, \ 2a+b+c \equiv 1 (\bmod 2f)\}$$

Each of these sets of blocks contains $2f^2(f-s)$ distinct blocks.

The 3-subsets of X' contained in these blocks include those of the forms

$$a_0 b_1 g_2 \quad \text{with} \quad a+b+g \notin D \cup E, \text{ and } a \equiv b (\bmod 2).$$

Since G_c is a partition of $Z_{2f} \setminus ((D \cup E) + 2c)$, we see that $g \notin (D \cup E) + 2c$

$$\text{and therefore} \quad a+b+g \notin \ (D \cup E) + 2c + a + b = D \cup E.$$

$$a_0 b_1 g_3 \quad \text{with} \quad a+b+g \notin (D \cup E) + 1, \text{ and } a \not\equiv b (\bmod 2)$$
$$g_0 b_2 c_3 \quad \text{with} \quad g+b+c \notin (D \cup E), \text{ and } b \equiv c (\bmod 2)$$
$$g_1 b_2 c_3 \quad \text{with} \quad g+b+c \notin (D \cup E) + 1, \text{ and } b \not\equiv c (\bmod 2)$$

The other 3-subsets of X' contained in these blocks are those of the form $g_i \overline{g}_i a_j$ with $\{i, j\} \in \{\{0,2\}\{0,3\}\{1,2\}\{1,3\}\}$, and $[g, \overline{g}]$ is an edge in one of the partial one factors $G_x \in \mathcal{G}$. The next sets of blocks are the following:

$$\{[a_0, b_1, h_2, \overline{h}_2] : \ a \not\equiv b (\bmod 2), [h, \overline{h}] \in H_c, \ a+b+2c \equiv 1 (\bmod 2f), \ 0 \le c < f\}$$
$$\{[a_0, b_1, h_3, \overline{h}_3] : \ a \equiv b (\bmod 2), [h, \overline{h}] \in H_c, \ a+b+2c \equiv 0 (\bmod 2f), \ 0 \le c < f\}$$
$$\{[h_0, \overline{h}_0, b_2, c_3] : \ b \not\equiv c (\bmod 2), [h, \overline{h}] \in H_a, \ 2a+b+c \equiv 1 (\bmod 2f), \ 0 \le a < f\}$$
$$\{[h_1, \overline{h}_1, b_2, c_3] : \ b \equiv c (\bmod 2), [h, \overline{h}] \in H_a, \ 2a+b+c \equiv 0 (\bmod 2f), \ 0 \le a < f\}$$

These blocks cover all 3-subsets of X' of the forms

$$a_0 b_1 c_2 \qquad \text{with} \qquad a \not\equiv b (\bmod 2)$$
$$a_0 b_1 c_3 \qquad \text{with} \qquad a \equiv b (\bmod 2)$$
$$a_0 b_2 c_3 \qquad \text{with} \qquad b \not\equiv c (\bmod 2)$$
$$a_1 b_2 c_3 \qquad \text{with} \qquad b \equiv c (\bmod 2),$$

and also those 3-subsets of the form $h_i, \overline{h}_i, a_j$ with $\{i, j\} \in \{\{0,2\}\{0,3\}\{1,2\}\{1,3\}\}$ and $[h, \overline{h}]$ an edge in one of the one factors $H_x \in \mathcal{H}$ with $0 \le x < f$.

The penultimate group of blocks has the form

$$\{[h_i,\overline{h}_i,a_j,\overline{a}_j] : \{i,j\} \in \{\{0,2\}\{0,3\}\{1,2\}\{1,3\}\},$$
$$[h,\overline{h}] \in H_x, [a,\overline{a}] \in H_x,\ f \leq x < f+s-1\}.$$

To construct the final group of blocks, let $J_0, J_1, \ldots, J_{2f-2}$, be a one-factorization of the complete graph with vertex set Z_{2f}, and the final group of blocks is

$$\Big\{[h_i,\overline{h}_i,a_j,\overline{a}_j] : \{i,j\} \in \big\{\{0,1\}\{2,3\}\big\},\ [h,\overline{h}] \in J_x\ ,\ [a,\overline{a}] \in J_x\ ,\ 0 \leq x < 2f-1\Big\}.$$

These last two sets of blocks cover all remaining 3-subsets of X', and the number of blocks in each set is $4f^2(s-1)$ and $2f^2(2f-1)$. The total number of blocks constructed is thus $16f^2s + 8f^2(f-s) + 4\cdot 2f^3 + 4f^2(s-1) + 2f^2(2f-1)$, and an easy counting argument shows that each 3-subset of X' is covered precisely once.

Note that if the one-factorization $J_0, J_1, \ldots, J_{2f-2}$ contains a sub one-factorization on $2r$ points, and the original design contained a block of size $2r$ on $R \subseteq Z_{2f}$, then the new design contains two subdesigns of size $4r$ on $R\times\{0,1\}$ and $R\times\{2,3\}$. Lindner et al [15] have shown that a one-factorization on $2f$ vertices with a sub one-factorization on $2r$ vertices exists if and only if $r = f$ or $r \leq \frac{f}{2}$. This observation justifies the remarks preceding Theorem 2.9.

6.3 The Hextupling Constructions.

The constructions described here are all special cases of the general construction given by Hanani [7, Proposition 9]. The basic idea behind the constructions is to begin with a threewise balanced design, delete two of its points, and "inflate" each of the remaining points x into a group of points of size w_x, and add n new "infinite" points. The blocks of the initial design then determine the structure of the new design in a regular fashion, using a small list of design fragments. Our main contribution here is the construction of some of these fragments. We have called the constructions hextupling, since in each case we use the weight function $w_x = 6$ for all points x of the base design, however we also give an example with $w_x = 10$. Finding a general construction for the fragments remains an open problem, however the astute reader with some familiarity with the tripling constructions of [8] and [11] will be able to see how these fragments are constructed in most cases where w_x is even, n is even, and either

$$w_x \equiv 0 (\mathrm{mod}\, 6), \qquad 0 \le n \le 2w_x$$
$$\text{or} \qquad w_x \equiv 2 (\mathrm{mod}\, 6) \qquad 4 \le n \equiv 4 (\mathrm{mod}\, 6) \le 2w_x$$
$$\text{or} \qquad w_x \equiv 4 (\mathrm{mod}\, 6) \qquad 8 \le n \equiv 2 (\mathrm{mod}\, 6) \le 2w_x.$$

In the cases where $w_x = 6$, the design fragments are sometimes G-designs or H-designs, and in general they are somewhere between these two extremes.

We proceed to construct design fragments of types A and B denoted $DFA(n)$ and $DFB(n)$, for $n \in \{2, 4, 6, 8, 10, 12\}$ with $w_x = 6$. The point set of each fragment is $(Z_6 \times Z_3) \cup \{\infty_0, \infty_1, \ldots, \infty_{n-1}\}$, and the blocks of size 4 are constructed below. To simplify the construction we give below a list of some useful one-factors of the graph with vertex set Z_6.

$$F_1 : [0,1][2,3][4,5]$$
$$F_2 : [0,5][1,2][3,4]$$
$$F_3 : [0,3][2,4][1,5]$$
$$F_4 : [1,4][0,2][3,5]$$
$$F_5 : [2,5][0,4][1,3]$$
$$F_6 : [0,3][1,4][2,5]$$

DFA(2)(=Design 39 of [7])

$[a_i, (a+2)_i, (a+3b+1)_{i+1}, (a+3b+1)_{i+2}] : \ a \in Z_6, i \in Z_3, \ b \in \{0,1\}$

$[a_i, (a+2)_i, (a+3)_{i+k}, (a+5)_{i-k}] : \ a \in Z_6, i \in Z_3, \ k \in \{1,2\}$

$[a_i, (a+2)_i, a_{i+1}, (a+2)_{i+1}] : \ a \in Z_6, i \in Z_3$

$[\infty_j, a_0, b_1, c_2] : \ a+b+c \equiv 3j (\mathrm{mod}\, 6), a, b, c \in Z_6, j \in \{0,1\}$

$[a_i, b_i, c_{i+1}, d_{i+1}] : i \in Z_3, [a,b] \in F_6, [c,d] \in F_6.$

DFB(2) (=Design 40 of [7]) = DFB(4)

$[a_i, (a+1)_i, b_{i+1}, c_{i+2}] : \ a+b+c \equiv 2i (\mathrm{mod}\, 6), i \in Z_3, a, b, c \in Z_6.$

DFA(4) (= Design 1.B of [12])

$[a_i, (a+3b)_{i+1}, (1-2a-3b)_{i+2}, (5-2a-3b)_{i+2}] : a \in Z_6, i \in Z_3, b \in \{0,1\}$

$[a_i, (a+2)_i, (a+3b)_{i+1}, (a+3b+2)_{i+1}] : a \in Z_6, i \in Z_3, b \in \{0,1\}$

$[\infty_j, a_0, b_1, c_2] \ : \ a+b+c \equiv k (\mathrm{mod}\, 6), a, b, c \in Z_6, (j,k) \in \{(0,0)(1,2)(2,3)(3,4)\}$

$[a_i, b_i, c_{i+1}, d_{i+1}] : i \in Z_3, [a,b] \in F_6, [c,d] \in F_6$

DFA(6) ($\cong$ H(24)) = DFA(8) = DFA(10) = DFA(12)

$[\infty_j, a_0, b_1, c_2] : a+b+c \equiv j (\mathrm{mod}\, 6), a, b, c, j \in Z_6$

DFB(6) ($\cong$ G(18))

$[a_i, (a+1)_i, b_{i+1}, c_{i+2}] : \ a+b+c \equiv 2i (\mathrm{mod}\, 6), i \in Z_3, a, b, c \in Z_6$

$[a_i, b_i, c_{i+1}, d_{i+1}] : i \in Z_3, [a,b] \in F_k, [c,d] \in F_k, k \in \{3,4,5\}$.

DFB(8)

First three classes of blocks in DFA(2), together with

$[\infty_j, a_0, b_1, c_2] : a+b+c \equiv 3j (\bmod 6), a,b,c \in Z_6, j \in \{6,7\}$

$[a_i, b_i, c_{i+1}, d_{i+1}] : i \in Z_3, [a,b] \in F_k, [c,d] \in F_k, k \in \{1,2,6\}$.

DFB(10)

First two classes of blocks in DFA(4), together with

$[\infty_j, a_0, b_1, c_2] : a+b+c \equiv k (\bmod 6), a,b,c \in Z_6, (j,k) \in \{(6,0)(7,2)(8,3)(9,4)\}$

$[a_i, b_i, c_{i+1}, d_{i+1}] : i \in Z_3, [a,b] \in F_k, [c,d] \in F_k, k \in \{1,2,6\}$.

DFB(12)

$[\infty_j, a_0, b_1, c_2] : a+b+c \equiv j (\bmod 6), j \in \{6,7,\ldots,11\}$

$[a_i, b_i, c_{i+1}, d_{i+1}] : i \in Z_3, [a,b] \in F_k, [c,d] \in F_k, k \in \{1,2,3,4,5\}$.

The essential observation to make about these design fragments are that:

(1) All triples of the form a_0, b_1, c_2 occur precisely once in both the A fragment and the B fragment

(2) All triples of the form $\infty_i a_j b_k$ with $j \neq k$ occur precisely once in either the A fragment or the B fragment, but not both. Specifically A contains all those with $i < n$, B contains all those with $i \geq n$.

(3) All triples of the form $a_i(a+d)_i b_j$ with $i \neq j,\ d \in \{1,2,3\}$ occur precisely once in either the A or the B fragment, but not both. Specifically

$$n = 2,4\ :\ A \text{ contains } d = 2,3\ ;\ B \text{ contains } d = 1.$$

$$n = 6,8,10,12\ :\ A \text{ contains none}\ ;\ B \text{ contains all.}$$

The other ingredients in the hextupling constructions are H(24), H(36) (see Hanani [7] or Mills [17]), and the following designs with subdesigns.

Lemma 6.2. Designs which validate the following assertions exist.

$$\{12+n,\ 24+n\} \subseteq S_n \quad \text{for} \quad n \in \{2,4,8,10\}$$

$$\{12+n,\ 24+n\} \subseteq G_n \quad \text{for} \quad n \in \{6,12\}$$

Proof. The twelve designs are easily constructed using the doubling and tripling constructions.

We now describe the hextupling construction. Let $(X \cup \{A,B\}, \mathcal{B})$ be an $S(v)$ if $v \equiv 2$ or $4 (\bmod 6)$, and let it be a $G(v)$ when $v \equiv 0 (\bmod 6)$, with both A and B in the same block of size 6.

We shall construct an $S(6(v-2)+n)$, respectively $G(6(v-2)+n)$, when $n = 2, 4, 8, 10$, respectively $n = 6, 12$. Let $I_n = \{\infty_0, \infty_1, \ldots, \infty_{n-1}\}$. The point set of the new design will be $(Z_6 \times X) \cup I_n$. The blocks are constructed as follows. For each block in $\mathcal{B}$ containing both A and B, say $\{A, B, x, y\}$ or $\{A, B, x, y, z, t\}$ construct an $S(12+n)$ or $S(24+n)$ or $G(12+n)$ or $G(24+n)$ with a subdesign on I_n and point set $(Z_6 \times \{x, y\}) \cup I_n$ or $(Z_6 \times \{x, y, z, t\}) \cup I_n$.

For each block in $\mathcal{B}$ containing A but not B, say $\{A, x, y, z\}$, construct DFA(n) on the point set $(Z_6 \times \{x, y, z\}) \cup I_n$, and similarly use DFB(n) for blocks $\{B, x, y, z\}$ containing B but not A. (Here we use the fact that none of these blocks has size 6). Finally for each block Y in $\mathcal{B}$ containing neither A nor B construct an H(24) or H(36) on $Z_6 \times Y$, depending on whether $|Y| = 4$ or 6.

Note that a subdesign of order w in the input design, will give rise to a subdesign of order $6(w-2)+n$ in the output design. This completes the proof of Theorem 2.10.

Before closing this section we give a further application of Hanani's construction, using a weighting factor $w_x = 10$.

Let $F_1, F_2, \ldots, F_7$ be a one-factorization of the cyclic graph with vertex set Z_{10} and edges between vertices of distances 2,3,4 and 5 (such a factorization exists by Stern and Lenz's theorem).

Consider now the following design fragments with point set $(Z_{10} \times Z_3) \cup I_{14}$

DFA(10,14)

$[\infty_i, a_0, b_1, c_2]: \ a+b+c \equiv i (\mathrm{mod}\, 10), \ a, b, c, i \in Z_{10}$

DFB(10,14)

$[a_i, (a+1)_i, b_{i+1}, c_{i+2}]: a+b+c \equiv 4+2i (\mathrm{mod}\, 10), \ a, b, c \in Z_{10}, \ i \in Z_3$

$[\infty_i, a_0, b_1, c_2]: \ a+b+c \equiv i (\mathrm{mod}\, 10), \ a, b, c \in Z_{10}, \ i \in \{10, 11, 12, 13\}$

$[a_i, b_i, c_{i+1}, d_{i+1}]: i \in Z_3, [a, b] \in F_k, [c, d] \in F_k, k \in \{1, 2, \ldots, 7\}$.

There exists an $S(34)$ with a subsystem of order 14 (since $34 = 3 \cdot 14 - 2 \cdot 4$), and DFA(10,14) is a generalized H-design with four groups of size 10.

So using the construction given above we obtain the following

Theorem 6.3. If $v \in S_w$ then $10(v-2)+14 \in S_{10(w-2)+14}$.

Since $14 \in S_4$, we deduce that $134 \in S_{34}$, and we know of no other method to construct an $S(134)$ with a subdesign of order 34. (N.B. See Theorem 3.18(b))

7. Sets of Subdesigns

Suppose that M is a finite subset of A_S (with some elements perhaps repeated) and let S_M be the set of integers $v \in A_S$ for which there exist a $3-(v, M \cup \{4\}, 1)$ design, which actually contains at least one block of size m, for each $m \in M$. We define G_M (for $M \subset A_G$) and T_M (for $M \subset A_T$) in the obvious analogous way. We define s_M to be the least integer v_0 such that if $v \geq v_0$ with $v \in A_S$ then $v \in S_M$; similarly define g_M and t_M. Note that $s_M(t_M$ or $g_M) \geq 2m_1$ where m_1 is the largest element of M. In fact if $v \in R_M$ (for $M \in A_R$) then we can get the following lower bound:

If $b_1, \ \ldots, \ b_t$ are blocks, the sizes being the elements of M, then, as no two blocks meet in more than two points, we have

$$\begin{aligned} v \ &\geq \ |\cup_1^t \ b_i| \geq \sum_1^t \ |b_i| - \sum_{1 \leq i < j \leq t} |b_i \cap b_j| \\ &\geq \sum_{m \in M} m - 2\binom{t}{2} \ . \end{aligned}$$

In fact we shall be considering constructions where the blocks are disjoint and so we have the trivial lower bound $v \geq \sum_{m \in M} \ m$.

We shall prove

Theorem 7.1. Let $R = G, S$ or T. There exist constants c'_g, c'_s, c'_t such that if M is any finite subset of A_R then

$$r_M \ \leq \ c'_r \sum_{m \in M} m \ .$$

In particular we may take $c'_r \leq 4t^2$ for any $t \in A_R$ such that $t \geq c_r$ where c_r is as in 4.1.

Proof: First note that in the construction in 2.1 we actually have

$$2w \in R_{\{w,w\}} \quad \text{for any} \quad w \in A_R \qquad (*)$$

Select $t \geq c_r, t \in A_R$ with c_r as in 4.1; for each $i \geq 0$ define $M_i = \{m \in M : \ 2^{i-1} < m \leq 2^i\}$ and $\mu_i = |M_i|$. As $2^i t \geq 2^i c_r \geq m c_r$ we see that $2^i t \in R_m$ for any $m \in M_i$ by 4.1.

If m and $m' \in M_0$ then $2t \in R_{\{t,t\}} \subset R_{\{m,m'\}}$ by $(*)$. Thus the blocks with size in M_0 can be fit into $\rho_0 = \left[\frac{\mu_0+1}{2}\right]$ blocks of size $2t$. Now, together with the elements of M_1, these can be fit into $\rho_0 + \mu_1$ blocks of size $2t$; and so, by $(*)$, we fit these into $\rho_1 = \left[\frac{\rho_0+\mu_1+1}{2}\right]$ blocks of size $2^2 t$. We continue this process, with each successive M_j, until we have that $2^n t \in R_M$ where $2^n \geq \ \sum_{i \geq 0} \ \mu_i 2^i$.

Now, if $x \geq 2^n tc_r$, where $x \in A_R$ then $x \in R_{2^n t} \subset R_M$ by 4.1; and so, as we can choose n so that

$$2^n < 2 \sum_{i \geq 0} \mu_i 2^i \; = \; 4 \sum_{i \geq 0} \sum_{m \in M_i} 2^{i-1} < 4 \sum_{m \in M} m \;,$$

we therefore have $2^n tc_r \leq 4t^2 \; \sum_{m \in M} \; m$.

8. Open Problems

We complete the paper with a set of open problems:

(1) Prove that $r_x = 2x$ for all $x \in A_R$ where $R = G$, S or T!!

Evidently (1) is the object of the paper and we list below a number of questions, the answers to which will lead us nearer to proving (1) for $R = S$ (the case that we are most interested in).

(2) Show that $2x + 6 \in S_x$ for all $x \in A_S$.

Note that if (2) holds and $[2x, 4x+4] \cap A_S \subset S_x$ then $s_x = 2x$, which is easily proved by induction on $v \geq 4x + 6$: If $v \equiv 0 \pmod 4$ then $v/2 \in S_x$ and so $v \in S_{v/2} \subset S_x$ by 2.2(b). If $v \equiv 2 \pmod 4$ then $(v-6)/2 \in S_x$ and so $v \in S_{(v-6)/2} \subset S_x$ by (2).

Note that if $x \in S_w$ for all $w \in A_S$ with $w \leq x/2$ then $4x - 3w \in S_x$ by 2.8(b) and $3x - 2w \in S_x$ by 2.7(b). Thus $\{v \equiv x \pmod 6 : \; 4x \geq v \geq 5x/2\} \subset \; S_x$ and $\{v \equiv x \pmod 4 : \; v \not\equiv 0 \pmod 6 \text{ and } 3x \geq v \geq 2x\} \subset S_x$; so that we don't yet know whether $[2x, 4x+4] \cap A_s \subset S_x$; but we do have a significant subset.

One could also get a similar, strong result if one had $2x + k \in S_x$ for all $x \in A_S$ for any $k \equiv 6 \pmod{12}$.

(3) More generally one needs, for each $v \in A_S$, a subdesign of order w where w is within a constant distance from $v/2$, to get a good result. For $v \equiv 0 \pmod 4$ we have, from 2.2(b), $v \in S_{v/2}$; and for $v \equiv 10 \pmod{12}$ we have, from 2.14(a), $v \in S_{v/2-1}$. Thus we need only concentrate on $v \in A_S$ with $v \equiv 2 \pmod{12}$.

(4) In Lemma 5.8 we showed that a strong consequence would follow from finding $n \in S_\sigma$ with $3 \not| n - \sigma$ and $16 - 4\sqrt{10} (\approx 3.35) > \frac{n-1}{\sigma - 1} > \frac{7}{16}(4 + \sqrt{10}) (\approx 3.13)$. Good examples of such n and σ are $50 \in S_{16}$, $64 \in S_{20}$, $68 \in S_{22}$, $82 \in S_{26}$ etc. Despite all our constructions, we have been unable to find values of n and σ with $n \in S_\sigma$, $3 \not| n - \sigma$ and $3\frac{1}{2} > \frac{n}{\sigma} > 3$.

In a sense this highlights the problem with all our constructions: In the resulting design we always seem to have a damaging congruence condition. For example, look at 2.12: If $x = (n-1)(v-w)+w$ and $z = (k-1)(v-w)+w$ we have $x - z = (n-k)(v-w)$ and so, unless k or $w = 1$, we get a design that has $x \equiv z \pmod 4$. Similarly in 2.2(d), (e), (f), 2.3, 2.5, 2.7. In 2.6 and 2.8 we have $x \equiv z \pmod 6$, in 2.10 and 2.11 $x \equiv z$ (mod 12) in the resulting designs. In 2.2 and 2.4 the constructions only have one parameter which certainly limits their applicability. 2.9 is really the only construction that avoids this problem.

(5) From the comments above we should like to have more constructions that give $v \in R_m$ with $v \equiv w + 2 \pmod 4$. A good example of this is $3v - 6 \in T_v$.

(6) Other "small" subdesign problems that we are unable to determine include $18 \in T_8; 32, 38 \in S_{10}$; and the exceptional values in the lemmata of section 3.

(7) We should like to "cross out" the condition $v \equiv \pm w \pmod 3$ in the statement of 2.7(a) (and consequently 2.12(a)). This shall be done for 2 of the 4 cases in a future paper.

Moving on to the more general question considered in the previous section we ask:

(8) Can one state a similar theorem to 7.1, with arbitrary block sizes, rather than just with elements of A_R? (Obviously satisfying certain trivial necessary conditions).

(9) For an arbitrary subset M of A_R, give a better lower bound for r_M than $\max\{2m_1, \Sigma m - 2\binom{t}{2}\}$.

Two interesting examples in this context are when M_1 consists of 7 sixes, where we are able to show that $14 \in T_{M_1}$, and we conjecture that $14 \le t_{M_1} \le 28$; and when M_2 consists of 30 tens where we can show that $50 \in S_{M_2}$, and we conjecture that $50 \le s_{M_2} \le 100$. It is easy to show that $t_{M_1} > 12$, but we have no proof that $s_{M_2} > 48$.

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