LARGE CHARACTER SUMS: PRETENTIOUS CHARACTERS AND THE PÓLYA-VINOGRADOV THEOREM

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1. Introduction

The best bound known for character sums was given independently by G. Pólya and I. M. Vinogradov in 1918 (see [4], pp. 135-137). For any nonprincipal Dirichlet character $\chi \pmod{q}$ we let

$$M(\chi) := \max_x \Big| \sum_{n \leq x} \chi(n) \Big|,$$

and then the Pólya-Vinogradov inequality reads

$$(1.1) \qquad\qquad M(\chi) \ll \sqrt{q} \log q.$$

There has been no subsequent improvement in this inequality other than in the implicit constant. Moreover it is believed that (1.1) will be difficult to improve since it is possible (though highly unlikely) that there is an infinite sequence of primes $q \equiv 3 \pmod 4$ for which $(\frac{p}{q}) = 1$ for all $p < q^\epsilon$, in which case $M((\frac{\cdot}{q})) \gg_\epsilon \sqrt{q} \log q$.

The unlikely possibility described above involves a quadratic character, and one might imagine that there are similar possibilities preventing one from improving (1.1) for higher-order characters. Surprisingly, one of our main results shows that we can improve (1.1) for characters of odd, bounded order.

Theorem 1. *If $\chi \pmod q$ is a primitive character of odd order g, then*

$$M(\chi) \ll_g \sqrt{q}(\log q)^{1-\frac{\delta_g}{2}+o(1)},$$

where $\delta_g = 1 - \frac{g}{\pi} \sin \frac{\pi}{g}$.

Our proof of Theorem 1 is based on some technical results (described in the next section) which allow us to characterize characters χ for which $M(\chi)$ is large. Our characterization reveals that there is a hidden structure among the characters having large $M(\chi)$. One example of this structure is the following:

Theorem 2. *For $1 \leq j \leq g$, let $\chi_j \pmod{q_j}$ be primitive characters (not necessarily distinct) with $q_j \leq q$ for all j. We suppose that the product $\chi_1 \cdots \chi_g$ gives*

the principal character. If g is odd, then we have that

$$\prod_{j=1}^{g} \frac{M(\chi_j)}{\sqrt{q_j}} \ll_g (\log q)^{g-\frac{1}{2g}} + (\log q)^{g-\frac{1}{7}}.$$

If g is even, then

$$\frac{M(\chi_g)}{\sqrt{q_g} \log q_g} + (\log q)^{-\frac{2(g-1)}{7}} \gg_g \prod_{j=1}^{g-1} \Big(\frac{M(\chi_j)}{\sqrt{q_j} \log q} \Big)^{2(g-1)}.$$

Roughly speaking, the first part of Theorem 2 tells us that if g is odd and $\chi_1 \cdots \chi_g = 1$, then at least one of the $M(\chi_j)$ is small. In particular, taking $\chi_1 = \cdots = \chi_k = \chi$ with $k = g-1$, if $M(\chi)$ is large, then $M(\chi^k)$ is small for small even integers k. The second part of Theorem 2 tells us that if g is even and $M(\chi_1)$, ..., $M(\chi_{g-1})$ are all large, then so is $M(\chi_1 \cdots \chi_{g-1})$ (since $\overline{\chi}_g = \chi_1 \cdots \chi_{g-1}$ and $M(\chi_g) = M(\overline{\chi}_g)$, by definition). In particular, taking $\chi_1 = \cdots = \chi_k = \chi$ with $k = g-1$, if $M(\chi)$ is large, then $M(\chi^k)$ is also large for small odd numbers k.

Another consequence of Theorems 1 and 2 is that if q is prime $\equiv 3 \pmod 4$, and $M(\chi) \gg \sqrt{q} \log q$ for some character $\chi \pmod q$ of bounded order, then $M((\frac{\cdot}{q})) \gg \sqrt{q} \log q$ for the quadratic character $(\frac{\cdot}{q})$. One can deduce further results like this from Theorem 2.

We give yet a third consequence. Suppose that q_1, q_2, q_3 are pairwise coprime, odd, squarefree integers in the interval $[Q, 2Q]$, such that each $M((\frac{\cdot}{q_i})) \gg \sqrt{q_i} \log q_i$. Then we have $M((\frac{\cdot}{q_1 q_2})) \ll \sqrt{q_1 q_2}(\log(q_1 q_2))^{6/7}$, whereas $M((\frac{\cdot}{q_1 q_2 q_3})) \gg \sqrt{q_1 q_2 q_3} \log(q_1 q_2 q_3)$. Similar results can be proved for products of four or more characters.

These bounds are larger than the expected maximal order of character sums. In 1977, H. L. Montgomery and R. C. Vaughan [12] showed if the Generalized Riemann Hypothesis[1] (GRH) is true, then

$$(1.2) \qquad\qquad M(\chi) \ll \sqrt{q} \log \log q.$$

This bound is best possible, up to the evaluation of the constant, in view of R. E. A. C. Paley's 1932 result [13] that there are infinitely many positive integers q such that

$$(1.3) \qquad\qquad M((\tfrac{\cdot}{q})) \geq \Big(\frac{e^\gamma}{\pi} + o(1) \Big) \sqrt{q} \log \log q,$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant.[2] Paley's result gives large character sums for a thin class of carefully constructed quadratic characters, and one may ask if for each large prime q there are characters $\chi \pmod q$ with similarly large $M(\chi)$. Our next result shows that there are indeed many such characters χ, and moreover we can point these character sums in any given direction.

Theorem 3. *Let q be a large prime and let $\theta \in (-\pi, \pi]$ be given. There is an absolute constant C_0 such that for at least $q^{1 - C_0/(\log \log q)^2}$ characters $\chi \pmod q$*

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[1]In their results and in ours (when indicated), the Riemann hypothesis for all Dirichlet L-functions is needed, not merely the Riemann hypothesis for the particular $L(s, \chi)$.

[2]Actually Paley's method gives the constant "1/2" not "e^γ", but such an improvement appeared subsequently in several places, for example [1].

with $\chi(-1) = -1$ we have

$$\sum_{n \leq x} \chi(n) = e^{i\theta} \frac{e^\gamma}{\pi} \sqrt{q} \Big(\log\log q + O((\log\log q)^{1/2}) \Big)$$

for all but $o(q)$ natural numbers $x \leq q$.

In view of Theorem 3 it may be surprising that there are analogues of Theorems 1 and 2 which give a sharper upper bound than (1.2) for characters of small odd order.

Theorem 4. *Assume GRH. If $\chi \pmod q$ is a primitive character of odd order g, then*

$$M(\chi) \ll_g \sqrt{q}(\log\log q)^{1 - \frac{\delta_g}{2} + o(1)}$$

(where $\delta_g = 1 - \frac{g}{\pi} \sin\frac{\pi}{g}$).

On GRH, we can show that there exist arbitrarily large q and primitive characters $\chi \pmod q$ of odd order g such that

(1.4) $$M(\chi) \gg_g \sqrt{q}(\log\log q)^{1 - \delta_g - o(1)}.$$

We believe that $M(\chi) \ll_g \sqrt{q}(\log\log q)^{1 - \delta_g + o(1)}$ for all primitive characters $\chi \pmod q$ of odd order g, so that the exponent $1 - \delta_g$ is "best possible". Perhaps this can be achieved by further developing Lemma 4.3. It would also be interesting to obtain the lower bound (1.4) unconditionally.

Assuming GRH, we will show (just after Theorem 2.5) that for any fixed even $g \geq 2$ there exist arbitrarily large q and primitive characters $\chi \pmod q$ of order g such that

(1.5) $$M(\chi) \gg_g \sqrt{q}\log\log q.$$

Therefore (1.2) is best possible (assuming GRH) for all fixed even g, up to the evaluation of the implicit constant. By suitably modifying the argument given below, one should be able to obtain (1.5) unconditionally.

Theorem 5. *Assume GRH. For $1 \leq j \leq g$, let $\chi_j \pmod{q_j}$ be primitive characters with $q_j \leq q$ for all j. We suppose that the product $\chi_1 \cdots \chi_g$ gives the principal character. If g is odd, then we have that*

$$\prod_{j=1}^{g} \frac{M(\chi_j)}{\sqrt{q_j}} \ll_g (\log\log q)^{g - \frac{1}{2g}} + (\log\log q)^{g - \frac{1}{7}}.$$

If g is even, then

$$\frac{M(\chi_g)}{\sqrt{q_g}\log\log q_g} + (\log\log q)^{-\frac{2(g-1)}{7}} \gg_g \prod_{j=1}^{g-1} \Big(\frac{M(\chi_j)}{\sqrt{q_j}\log\log q} \Big)^{2(g-1)}.$$

One can make deductions from Theorems 4 and 5 analogous to those consequences we gave after Theorems 1 and 2.

The known estimates for character sums strongly resemble bounds for $L(1,\chi)$. Unconditionally it is easy to show that $|L(1,\chi)| \ll \log q$. Assuming GRH, J. E. Littlewood [11] proved that

(1.6) $$L(1,\chi) \sim \prod_{p \leq \log^2 q} \Big(1 - \frac{\chi(p)}{p} \Big)^{-1},$$

from which it follows easily that

(1.7) $|L(1, \chi)| \leq (1 + o(1))2e^{\gamma} \log \log q.$

Apart from a factor of 2, the bound (1.7) is best possible, since S. D. Chowla [3] showed that there exist arbitrarily large q and characters χ (mod q) such that

$$|L(1, \chi)| \geq (1 + o(1))e^{\gamma} \log \log q.$$

Theorem 6. *Assume GRH. If χ is a primitive character* (mod q), *then*

$$\Big| \sum_{n \leq x} \chi(n) \Big| \leq \Big(\frac{2e^{\gamma}}{\pi} + o(1) \Big) \sqrt{q} \log \log q.$$

Furthermore,

$$\Big| \sum_{x \leq n \leq x+y} \chi(n) \Big| \leq \Big(\frac{4e^{\gamma}}{\pi\sqrt{3}} + o(1) \Big) \sqrt{q} \log \log q.$$

Regarding the second part of Theorem 6 we record that with minor modifications to the proof of Theorem 3 we may prove that for any angle $\theta \in (-\pi, \pi]$ and any large prime q there are at least $q^{1-2/(\log \log q)^2}$ characters χ (mod q) with $\chi(-1) = 1$ such that

(1.8) $$\sum_{q/3 \leq n \leq 2q/3} \chi(n) \sim e^{i\theta} \frac{2e^{\gamma}}{\pi\sqrt{3}} \sqrt{q} \log \log q.$$

If $q \equiv 3$ (mod 4) is a prime, then the condition $\chi(-1) = 1$ is equivalent to χ having odd order. Thus (1.8) shows that there are characters of odd order for which $M(\chi) \gg \sqrt{q} \log \log q$, and so there is no improvement of (1.2) which holds for all odd order characters.

Theorem 6 places the situation for large character sums on the same footing as bounds for $L(1, \chi)$: the conditional O-results for character sums differ from Paley's Ω result by only a factor of 2. Moreover the maximal size of character sums in an interval $[x, x + y]$ is also determined up to a factor of 2. It is believed that the Ω result represents the true extreme values of $L(1, \chi)$ (see [8] for arguments in the case when χ is quadratic). Similarly we believe that (1.3) and (1.8) give the largest possible character sums.

Conjecture 1. *If χ is a primitive character* (mod q), *then*

$$\Big| \sum_{n \leq x} \chi(n) \Big| \leq \Big(\frac{e^{\gamma}}{\pi} + o(1) \Big) \sqrt{q} \; \log \log q$$

and

$$\Big| \sum_{x \leq n \leq x+y} \chi(n) \Big| \leq \Big(\frac{2e^{\gamma}}{\pi\sqrt{3}} + o(1) \Big) \sqrt{q} \log \log q.$$

In the final section of this paper we use the improved upper bounds for $L(1, \chi)$ given in [9] to obtain a modest improvement over Hildebrand's results [10] on the constant in the Pólya-Vinogradov inequality.

Theorem 7. *Let χ be a primitive character* (mod q), *and set $c = 1/4$ if q is cubefree, and $c = 1/3$ otherwise. If $\chi(-1) = 1$, then*

$$M(\chi) \leq \frac{69}{70} \frac{c + o(1)}{\pi\sqrt{3}} \sqrt{q} \log q.$$

If $\chi(-1) = -1$, then

$$M(\chi) \le \frac{c + o(1)}{\pi} \sqrt{q} \log q.$$

In the case $\chi(-1) = 1$ our result improves Hildebrand's estimate by a factor of $\frac{69}{70}$. Hildebrand gives an estimate for a slightly different quantity than $M(\chi)$ when $\chi(-1) = -1$.

2. Detailed statement of results

If χ is a primitive character (mod q), then the sum $\sum_{n \le x} \chi(n)$ has a Fourier expansion which is given quantitatively as (see Pólya [14])

$$\sum_{n \le x} \chi(n) = \frac{\tau(\chi)}{2\pi i} \sum_{\substack{n \in \mathbb{Z} \\ 1 \le |n| \le N}} \frac{\overline{\chi}(n)}{n} \left(1 - e(-\tfrac{nx}{q})\right) + O\left(1 + \frac{q \log q}{N}\right).$$

Here $\tau(\chi)$ is the usual Gauss sum (see section 4), and $e(t) = \mathrm{e}^{2\pi i t}$. Choosing $N = q$ above and noting that $L(1, \overline{\chi}) = \sum_{n \le q} \overline{\chi}(n)/n + O(1)$ we obtain that

(2.1)
$$\sum_{n \le x} \chi(n) = \frac{\tau(\chi)}{2\pi i}(1 - \overline{\chi}(-1)) L(1, \overline{\chi}) - \frac{\tau(\chi)}{2\pi i} \sum_{n \le q} \frac{\overline{\chi}(n)}{n} \left(e(-\tfrac{nx}{q}) - \chi(-1) e(\tfrac{nx}{q})\right) + O(\sqrt{q}).$$

All of our work here proceeds from the Fourier expansion (2.1). We wish to understand when the terms appearing in (2.1) can be large. Littlewood's result (1.6) indicates that $L(1, \overline{\chi})$ is large only when $\overline{\chi}(p) \approx 1$ for many small primes p. We will find that the other terms appearing in (2.1) can be large only when $\chi(p) \approx \xi(p)$ for many small primes p, where ξ is a character of small conductor. A. Hildebrand [10] first realized the possibility of such a result.

To formulate our results precisely we define for two characters χ and ψ,

(2.2)
$$\mathbb{D}(\chi, \psi; y) := \left(\sum_{p \le y} \frac{1 - \mathrm{Re}\ \chi(p)\overline{\psi}(p)}{p}\right)^{\frac{1}{2}}.$$

We think of $\mathbb{D}(\chi, \psi; y)$ as measuring the distance between the characters χ and ψ (up to some point y). As we will see below (Lemma 3.1) the triangle inequality holds:

(2.3)
$$\mathbb{D}(\chi_1, \psi_1; y) + \mathbb{D}(\chi_2, \psi_2; y) \ge \mathbb{D}(\chi_1 \chi_2, \psi_1 \psi_2; y).$$

Note that $0 \le \mathbb{D}(\chi, \psi; y) \le (1 + o(1))\sqrt{2 \log \log y}$.

Definition. Let χ and ψ be two characters and let $\delta > 0$. We say that a character χ is (ψ, y, δ)-pretentious if

$$\mathbb{D}(\chi, \psi; y)^2 = \sum_{p \le y} \frac{1 - \mathrm{Re}\ \chi(p)\overline{\psi}(p)}{p} \le \delta \log \log y.$$

Our main results, from which Theorems 1 and 2 will follow, are the following two theorems.

Theorem 2.1. *Of all primitive characters with conductor below* $(\log q)^{\frac{1}{3}}$ *let* ξ (mod m) *denote that character for which* $\mathbb{D}(\chi, \xi; q)$ *is a minimum.*[3] *Then*

$$M(\chi) \ll (1 - \chi(-1)\xi(-1))\frac{\sqrt{qm}}{\phi(m)}\log q \exp\left(-\frac{1}{2}\mathbb{D}(\chi, \xi; q)^2\right) + \sqrt{q}(\log q)^{\frac{6}{7}}.$$

Thus $M(\chi) \ll \sqrt{q}(\log q)^{\frac{6}{7}}$ *unless* $\xi(-1) = -\chi(-1)$ *and* χ *is* $(\xi, q, \frac{2}{7})$-*pretentious.*

In the opposite direction we will show that if χ is close to a character with small conductor (and opposite parity to χ), then $M(\chi)$ is large.

Theorem 2.2. *Let* ψ (mod ℓ) *be a primitive character with* $\psi(-1) = -\chi(-1)$. *Then*

$$M(\chi) + \frac{\sqrt{q\ell}}{\phi(\ell)}\log\log q \gg \frac{\sqrt{q\ell}}{\phi(\ell)}\log q \exp(-\mathbb{D}(\chi, \psi; q)^2).$$

We next turn to results conditional on GRH. Given a real number $y \geq 1$ we let $\mathcal{S}(y)$ denote the set of integers all of whose prime factors are below y. We are motivated by Littlewood's conditional result (1.6) which shows that $L(1, \chi)$ is well approximated by $\sum_{n \in \mathcal{S}(\log^2 q)} \chi(n)/n$. We will show that the terms in (2.1) involving $e(\pm nx/q)$ may also be replaced by sums involving only smooth numbers n.

Proposition 2.3. *Assume GRH. Let* χ *be a primitive character* (mod q) *and let* α *be a real number. Then*

$$\sum_{n \leq q} \frac{\overline{\chi}(n)}{n}e(n\alpha) = \sum_{\substack{n \leq q \\ n \in \mathcal{S}((\log q)^{12})}} \frac{\overline{\chi}(n)}{n}e(n\alpha) + O(1).$$

It follows at once from (1.7), Proposition 2.3 and (2.1) that

$$\left|\sum_{n \leq q\alpha}\chi(n)\right| \leq \frac{\sqrt{q}}{\pi}|L(1, \overline{\chi})| + \frac{\sqrt{q}}{\pi}\prod_{p \leq (\log q)^{12}}\left(1 - \frac{1}{p}\right)^{-1} + O(\sqrt{q})$$

$$\leq (14e^{\gamma} + o(1))\frac{\sqrt{q}}{\pi}\log\log q.$$

Thus Proposition 2.3 already gives a refinement of (1.2), and its proof (given in §5) is simpler than the original proof of (1.2). With Proposition 2.3 in place we can argue as in Theorems 2.1 and 2.2 and arrive at the following conditional analogues from which we will deduce Theorems 4 and 5.

Theorem 2.4. *Assume GRH. Of all primitive characters with conductor below* $(\log\log q)^{\frac{1}{3}}$ *let* ξ (mod m) *denote that character for which* $\mathbb{D}(\chi, \xi; \log q)$ *is a minimum. Then*

$$M(\chi) \ll (1 - \chi(-1)\xi(-1))\frac{\sqrt{qm}}{\phi(m)}\log\log q \exp\left(-\frac{1}{2}\mathbb{D}(\chi, \xi; \log q)^2\right) + \sqrt{q}(\log\log q)^{\frac{6}{7}}.$$

Thus $M(\chi) \ll \sqrt{q}(\log\log q)^{\frac{6}{7}}$ *unless* $\xi(-1) = -\chi(-1)$ *and* χ *is* $(\xi, \log q, \frac{2}{7})$-*pretentious.*

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[3] If there are several characters attaining this minimum, then one can pick ξ to be any one of those characters.

Theorem 2.5. *Assume GRH. Let ψ (mod ℓ) be a primitive character with $\psi(-1) = -\chi(-1)$. Then*

$$M(\chi) + \frac{\sqrt{q\ell}}{\phi(\ell)} \log\log\log q \gg \frac{\sqrt{q\ell}}{\phi(\ell)} \log\log q \exp(-\mathbb{D}(\chi,\psi;\log q)^2).$$

Deduction of (1.5). For a given even integer $g \geq 4$ we fix a character ψ_g of order g. By Paley's result (1.3) we may find a quadratic character χ (mod q) such that $M(\chi) \gg \sqrt{q} \log\log q$. If we assume GRH, then Theorem 2.4 tells us that there must be a primitive character ξ (mod m) with $\chi(-1)\xi(-1) = -1$, $m \ll 1$, and $\mathbb{D}(\chi,\xi;\log q) \ll 1$. Now let us define χ_g to be the primitive character inducing $\chi\psi_g$, and note that χ_g will have order g and its conductor will be of size $\asymp_g q$. Correspondingly let us define ξ_g to be the primitive character inducing $\xi\psi_g$, and note that ξ_g will have conductor $\ll_g 1$, and moreover $\xi_g(-1)\chi_g(-1) = \xi(-1)\chi(-1)\psi(-1)^2 = -1$, and $\mathbb{D}(\chi_g,\xi_g;\log q) \ll 1 + \mathbb{D}(\chi,\xi;\log q) \ll 1$. Thus an application of Theorem 2.5 yields $M(\chi_g) \gg_g \sqrt{q} \log\log q$, which establishes (1.5). $\qquad\square$

Our work also allows us to make the following refined version of Conjecture 1.

Conjecture 2.6. *Let χ (mod q) be a primitive character. If $1 \leq x \leq q/2$, then*

$$\left| \sum_{n \leq x} \chi(n) \right| \leq \left(\frac{e^\gamma}{\pi} + o(1) \right) \sqrt{q} \log\log q,$$

and equality holds here if and only if $\chi(-1) = -1$, $x \geq q/(\log q)^{o(1)}$, and

$$\sum_{p \leq \log q} \frac{1 - \text{Re } \chi(p)}{p} = o(1).$$

Furthermore, if $1 \leq x \leq x + y \leq q$, then

$$\left| \sum_{x \leq n \leq x+y} \chi(n) \right| \leq \left(\frac{2e^\gamma}{\pi\sqrt{3}} + o(1) \right) \sqrt{q} \, \log\log q,$$

and equality holds here if and only if $\chi(-1) = 1$, both $|x - q/3|$ and $|x + y - 2q/3|$ are $\leq q/(\log q)^{h(q)}$ where $h(q) \to \infty$ as $q \to \infty$, and

$$\sum_{\substack{p \leq \log q \\ p \neq 3}} \frac{1 - \text{Re}(\chi(p)(\frac{p}{3}))}{p} = o(1).$$

The main purpose of this paper was to improve (1.1) and (1.2) in as many situations as possible. Of course we believe in the Generalized Riemann Hypothesis so that (1.2) should hold, and beyond that results such as Theorem 4. In the discussion after Theorem 4 we explained that for primitive characters χ (mod q) of odd order g we expect that the "best possible" result is $M(\chi) \ll_g \sqrt{q}(\log\log q)^{1-\delta_g+o(1)}$, while for even order g the estimate (1.2) cannot be improved in general. Analogously we may expect our methods to give unconditionally that $M(\chi) \ll_g \sqrt{q}(\log q)^{1-\delta_g+o(1)}$ when g is odd, and we do not expect an improvement over (1.1) for even g. Just after (1.1) we gave an unlikely but possible scenario in which (1.1) could not be improved for characters of order 2. If we had such examples then, arguing as in the deduction of (1.5) (after Theorem 2.5), we could create characters of any fixed even order g for which (1.1) could not be improved.

The exponents in Theorems 1, 2, 4, and 5 can all be improved by refining the technical Lemmas 3.4 and 4.3. Although we can give some improvements to both

lemmas, we have refrained from doing so in the interests of a simpler exposition. We invite the reader to attain our objectives and reap the improved theorems, by replacing the lower bound given in Lemma 3.4 by the best possible result, and to replace the exponent '1/2' by '1' in the first part of Lemma 4.3 (perhaps at the cost of an extra term of smaller order of magnitude).

3. The distance between characters, and deductions

In this section we gather together information on the distance between characters defined in (2.2) and show how Theorems 1, 2, 4, and 5 may be deduced from Theorems 2.1, 2.2, 2.4 and 2.5. Let $\mathbf{z}$ and $\mathbf{w}$ denote sequences $(z(2), z(3), \ldots)$ and $(w(2), w(3), \ldots)$ indexed by the primes, and such that $|z(p)| \leq 1$ and $|w(p)| \leq 1$ for all p. For two such sequences we define (generalizing (2.2))

$$\mathbb{D}(\mathbf{z}, \mathbf{w}; y) = \Big(\sum_{p \leq y} \frac{1 - \operatorname{Re} z(p)\overline{w(p)}}{p}\Big)^{\frac{1}{2}}.$$

Given two sequences $\mathbf{z}_1$ and $\mathbf{z}_2$ we will denote by $\mathbf{z}_1\mathbf{z}_2$ the sequence obtained by multiplying componentwise: $(z_1(2)z_2(2), z_1(3)z_2(3), \ldots)$.

Lemma 3.1. *With the above notation we have the triangle inequality*

$$\mathbb{D}(\mathbf{z}_1, \mathbf{w}_1; y) + \mathbb{D}(\mathbf{z}_2, \mathbf{w}_2; y) \geq \mathbb{D}(\mathbf{z}_1\mathbf{z}_2, \mathbf{w}_1\mathbf{w}_2; y).$$

Proof. Since $\mathbb{D}(\mathbf{z}, \mathbf{w}; y) = \mathbb{D}(\mathbf{1}, \overline{\mathbf{z}}\mathbf{w}; y)$ we may assume that $\mathbf{z}_1 = \mathbf{1} = \mathbf{z}_2$. Using the Cauchy-Schwarz inequality we see that $(\mathbb{D}(\mathbf{1}, \mathbf{w}_1; y) + \mathbb{D}(\mathbf{1}, \mathbf{w}_2; y))^2$ is

$$= \sum_{p \leq y} \Big(\frac{1 - \operatorname{Re} w_1(p)}{p} + \frac{1 - \operatorname{Re} w_2(p)}{p}\Big) + 2\mathbb{D}(\mathbf{1}, \mathbf{w}_1; y)\mathbb{D}(\mathbf{1}, \mathbf{w}_2; y)$$

$$\geq \sum_{p \leq y} \frac{1}{p}\Big(1 - \operatorname{Re} w_1(p) + 1 - \operatorname{Re} w_2(p) + 2\sqrt{1 - \operatorname{Re} w_1(p)}\sqrt{1 - \operatorname{Re} w_2(p)}\Big)$$

$$\geq \sum_{p \leq y} \frac{1}{p}\Big(1 - \operatorname{Re} w_1(p) + 1 - \operatorname{Re} w_2(p) + |\operatorname{Im} w_1(p)||\operatorname{Im} w_2(p)|\Big)$$

$$\geq \sum_{p \leq y} \frac{1}{p}\Big(1 - \operatorname{Re} w_1(p)w_2(p)\Big),$$

which proves the lemma. $\square$

More generally, given a sequence $(a(2), a(3), \ldots)$ of nonnegative real numbers we could define the distance between $\mathbf{z}$ and $\mathbf{w}$ as $(\sum_{p \leq y} a(p)(1 - \operatorname{Re} z(p)\overline{w(p)}))^{\frac{1}{2}}$. A simple modification of the proof above shows that this also satisfies the triangle inequality.

We now turn to estimates on distances between characters. We first record a consequence of the prime number theorem in arithmetic progressions that we will find useful below. Suppose $a \pmod{\ell}$ is a reduced residue class. Then for any x such that $\ell \leq (\log x)^A$ (A an arbitrary constant) we have that

(3.1) $$\sum_{\substack{\ell \leq p \leq x \\ p \equiv a \pmod{\ell}}} \frac{1}{p} = (1 + o(1))\frac{1}{\phi(\ell)} \log \log x.$$

Lemma 3.2. *Let $\chi \pmod q$ be a primitive character of odd order g. Suppose ξ (mod m) is a primitive character such that $\chi(-1)\xi(-1) = -1$. If $m \le (\log y)^A$, then*

$$\mathbb{D}(\chi, \xi; y)^2 \ge (\delta_g + o(1)) \log \log y.$$

Proof. Since χ has odd order, $\chi(-1) = 1$. Thus $\xi(-1) = -1$ and ξ must have even order $k \ge 2$ say. We have

$$\mathbb{D}(\chi, \xi; y)^2 \ge \sum_{-k/2 < \ell \le k/2} \Big(\sum_{\substack{p \le y \\ \xi(p)=e(\ell/k)}} \frac{1}{p} \Big) \min_{z^g=0,1} (1 - \text{Re } ze(-\ell/k)).$$

If $\|\lambda\|$ denotes the distance of λ from the nearest integer, then we may check that

$$\min_{z^g=0,1} (1 - \text{Re } ze(-\ell/k)) = 1 - \cos(\tfrac{2\pi}{g}\|\ell g/k\|).$$

An application of (3.1) gives that

$$\sum_{\substack{p \le y \\ \xi(p)=e(\ell/k)}} \frac{1}{p} \ge (1 + o(1))\frac{1}{k} \log \log y.$$

Writing $g/k = g^*/k^*$ in lowest terms (note that $k^* \ge 2$ is even) we deduce that

$$\mathbb{D}(\chi, \xi; y)^2 \ge (1 + o(1))\frac{1}{k} \log \log y \cdot \frac{k}{k^*} \sum_{-k^*/2 < \ell \le k^*/2} (1 - \cos \tfrac{2\pi\ell}{gk^*})$$

$$\sim \Big(1 - \frac{\sin(\pi/g)}{k^* \tan(\pi/(gk^*))} \Big) \log \log y.$$

Since $k^* \tan(\pi/(gk^*)) > \pi/g$, the lemma follows.   □

Deducing Theorem 1 from Theorem 2.1. Suppose χ has odd order g and let ξ be the character with conductor below $(\log q)^{\frac{1}{3}}$ with smallest $\mathbb{D}(\chi, \xi; q)$. If $\chi(-1)\xi(-1) = 1$, then Theorem 2.1 gives $M(\chi) \ll \sqrt{q}(\log q)^{\frac{6}{7}}$, which is stronger than Theorem 1. If $\chi(-1)\xi(-1) = -1$, then Lemma 3.2 gives that $\mathbb{D}(\chi, \xi; q)^2 \ge (\delta_g + o(1)) \log \log q$, and Theorem 1 follows at once from Theorem 2.1.   □

Deducing Theorem 4 from Theorem 2.4. This is entirely analogous to the above deduction.   □

Lemma 3.3. *Let $g \ge 2$ be fixed. Suppose that for $1 \le j \le g$, $\chi_j \pmod{q_j}$ is a primitive character. Let y be large, and suppose $\xi_j \pmod{m_j}$ are primitive characters with conductors $m_j \le \log y$. Suppose that $\chi_1 \cdots \chi_g$ is the trivial character, but $\xi_1 \cdots \xi_g$ is not trivial. Then*

$$\sum_{j=1}^{g} \mathbb{D}(\chi_j, \xi_j; y)^2 \ge \Big(\frac{1}{g} + o(1)\Big) \log \log y.$$

Proof. We decompose $\mathbb{D}(\chi_j, \xi_j; y)^2$ as $\mathbb{D}_0(\chi_j, \xi_j; y)^2 + \mathbb{D}_1(\chi_j, \xi_j; y)^2$ where in $\mathbb{D}_0$ we sum over primes $p \le y$ dividing the l.c.m. of q_1, ..., q_g, and in $\mathbb{D}_1$ we sum over all other primes $p \le y$. Then the triangle inequality holds for $\mathbb{D}_1$, and using Cauchy-Schwarz we find that

$$\sum_{j=1}^{g} \mathbb{D}_1(\chi_j, \xi_j; y)^2 \ge \frac{1}{g}\Big(\sum_{j=1}^{g} \mathbb{D}_1(\chi_j, \xi_j; y)\Big)^2 \ge \frac{1}{g} \sum_{\substack{p \le y \\ p \nmid q_1 \cdots q_g}} \frac{1 - \text{Re } \overline{\xi_1 \cdots \xi_g}(p)}{p}.$$

Trivially

$$\sum_{j=1}^{g} \mathbb{D}_0(\chi_j, \xi_j; y)^2 \geq \sum_{\substack{p \leq y \\ p|q_1 \cdots q_g}} \frac{1}{p} \geq \frac{1}{g} \sum_{\substack{p \leq y \\ p|q_1 \cdots q_g}} \frac{1 - \mathrm{Re}\ \overline{\xi_1 \cdots \xi_g}(p)}{p},$$

and so we deduce that

$$\sum_{j=1}^{g} \mathbb{D}(\chi_j, \xi_j; y)^2 \geq \frac{1}{g} \mathbb{D}(1, \xi_1 \cdots \xi_g; y)^2.$$

The lemma now follows from (3.1). $\qquad\qquad\square$

Deducing Theorem 2 from Theorems 2.1 and 2.2. We first consider the case when g is odd. For each $1 \leq j \leq g$ let $\xi_j \pmod{m_j}$ denote that primitive character with conductor below $(\log q_j)^{\frac{1}{3}}$ for which $\mathbb{D}(\chi_j, \xi_j; q_j)$ is a minimum. If for some j we have $\chi_j(-1)\xi_j(-1) = 1$, then Theorem 2.1 gives that $M(\chi_j) \ll \sqrt{q_j}(\log q_j)^{\frac{6}{7}}$ and our claimed bound follows. Suppose now that $\chi_j(-1)\xi_j(-1) = -1$ for all j. By Theorem 2.1, and since $q_j \leq q$, we see that

$$\frac{M(\chi_j)}{\sqrt{q_j}} \ll (\log q_j) \exp\Big(-\frac{1}{2}\mathbb{D}(\chi_j, \xi_j; q_j)^2 \Big) + (\log q_j)^{\frac{6}{7}}$$

$$\ll (\log q) \exp\Big(-\frac{1}{2}\mathbb{D}(\chi_j, \xi_j; q)^2 \Big) + (\log q)^{\frac{6}{7}}.$$

Therefore

$$\prod_{j=1}^{g} \frac{M(\chi_j)}{\sqrt{q_j}} \ll (\log q)^g \exp\Big(-\frac{1}{2} \sum_{j=1}^{g} \mathbb{D}(\chi_j, \xi_j; q)^2 \Big) + (\log q)^{g-\frac{1}{7}}.$$

We know that $\chi_1 \cdots \chi_g$ is the trivial character, and since g is odd, $(\xi_1 \cdots \xi_g)(-1) = (-1)^g = -1$ and so $\xi_1 \cdots \xi_g$ is not trivial. Lemma 3.1 now gives the bound of the theorem.

Now we consider the case when g is even. If $g = 2$, then χ_1 and χ_2 are complex conjugates and there is nothing to prove. Suppose now that $g \geq 4$. If for any $1 \leq j \leq g - 1$ we have $M(\chi_j) \ll \sqrt{q_j}(\log q)^{\frac{6}{7}}$, then the bound of the theorem holds trivially. Suppose now that for each $1 \leq j \leq g - 1$ we have that $M(\chi_j) \gg \sqrt{q_j}(\log q)^{\frac{6}{7}}$. If $\xi_j \pmod{m_j}$ denotes the primitive character with conductor below $(\log q_j)^{\frac{1}{3}}$ with minimum $\mathbb{D}(\chi_j, \xi_j; q_j)^2$, then by Theorem 2.1 we have that $\chi_j(-1)\xi_j(-1) = -1$, and that

$$M(\chi_j) \ll \frac{\sqrt{q_j m_j}}{\phi(m_j)}(\log q_j) \exp\Big(-\frac{1}{2}\mathbb{D}(\chi_j, \xi_j; q_j)^2 \Big)$$

$$\ll \frac{\sqrt{q_j m_j}}{\phi(m_j)}(\log q) \exp\Big(-\frac{1}{2}\mathbb{D}(\chi_j, \xi_j; q)^2 \Big),$$

so that (for $j \leq g - 1$)

$$\mathbb{D}(\chi_j, \xi_j; q)^2 \leq 2 \log\Big(\frac{\sqrt{q_j m_j}\, \log q}{M(\chi_j)\phi(m_j)} \Big) + O(1).$$

Now note that χ_g is the primitive character inducing $\overline{\chi_1 \cdots \chi_{g-1}}$ and so we let ψ denote the primitive character inducing $\overline{\xi_1 \cdots \xi_{g-1}}$. We note (using the triangle and

the Cauchy-Schwarz inequalities, as well as an argument as in Lemma 3.3 to handle the primes dividing $q_1 \cdots q_{g-1} m_1 \cdots m_{g-1}$) that

$$\mathbb{D}(\chi_g, \psi; q)^2 \leq (g-1) \sum_{j=1}^{g-1} \mathbb{D}(\chi_j, \xi_j; q)^2,$$

and that $\chi_g(-1)\psi(-1) = (-1)^{g-1} = -1$. Appealing to Theorem 2.2 we obtain the theorem. $\qquad\square$

Deducing Theorem 5 from Theorems 2.4 and 2.5. This is entirely analogous to our deduction above. $\qquad\square$

Finally we record a lemma which will be useful later.

Lemma 3.4. *Let χ (mod q) be a primitive character. Of all primitive characters with conductor below $\log y$, suppose that ψ_j (mod m_j) ($1 \leq j \leq A$) give the smallest distances $\mathbb{D}(\chi, \psi_j; y)$ arranged in ascending order. Then for each $1 \leq j \leq A$ we have that*

$$\mathbb{D}(\chi, \psi_j; y)^2 \geq \left(1 - \frac{1}{\sqrt{j}} + o(1)\right) \log \log y.$$

Proof. Notice that

$$\mathbb{D}(\chi, \psi_j; y)^2 \geq \frac{1}{j} \sum_{k=1}^{j} \mathbb{D}(\chi, \psi_k; y)^2 = \frac{1}{j} \sum_{p \leq y} \frac{1}{p} \sum_{k=1}^{j} (1 - \mathrm{Re}\ \chi(p)\overline{\psi_k}(p))$$

$$(3.2) \qquad\qquad \geq \frac{1}{j} \sum_{p \leq y} \frac{1}{p} \left(j - \Big| \sum_{k=1}^{j} \psi_k(p) \Big|\right).$$

By Cauchy-Schwarz we have that

$$(3.3) \qquad \left(\sum_{p \leq y} \Big| \sum_{k=1}^{j} \psi_k(p) \Big|\right)^2 \leq \left(\sum_{p \leq y} \frac{1}{p}\right)\left(\sum_{p \leq y} \frac{1}{p} \Big| \sum_{k=1}^{j} \psi_k(p) \Big|^2\right).$$

The first term in the RHS above is $\sim \log \log y$. The second term is

$$\sum_{p \leq y} \frac{1}{p}\left(j + \sum_{\substack{1 \leq k, \ell \leq j \\ k \neq \ell}} \frac{\psi_k(p)\overline{\psi_\ell}(p)}{p}\right) \sim j \log \log y,$$

by appealing to (3.1). Using these estimates to bound the quantity in (3.3), and inserting that bound in (3.2) we obtain the lemma. $\qquad\square$

4. PRELIMINARY LEMMAS

Here we collect together some lemmas used below. For any character χ (mod q) we recall the *Gauss sum*

$$(4.1) \qquad\qquad \tau(\chi) = \sum_{a \ (\mathrm{mod}\ q)} \chi(a)e(a/q).$$

It is immediate that if $(b, q) = 1$, then

$$(4.2) \qquad\qquad \sum_{a \ (\mathrm{mod}\ q)} \chi(a)e(ab/q) = \overline{\chi}(b)\tau(\chi).$$

Lemma 4.1. *Suppose that χ (mod q) is induced by the primitive character χ' (mod q'). Then*

$$\tau(\chi) = \mu(q/q')\chi'(q/q')\tau(\chi').$$

If χ (mod q) is primitive, then $|\tau(\chi)| = \sqrt{q}$ and (4.2) holds for all integers b.

Proof. Note that

$$\tau(\chi) = \sum_{\substack{a \ (\mathrm{mod}\ q) \\ (a,q/q')=1}} \chi'(a)e(a/q) = \sum_{d|(q/q')} \mu(d)\chi'(d) \sum_{a \ (\mathrm{mod}\ q/d)} \chi'(a)e(ad/q).$$

The inner sum vanishes unless $d = q/q'$ and the first result follows. The second statement is well known; see for example [4]. $\square$

Lemma 4.2. *Let f be a completely multiplicative function with $|f(n)| \leq 1$ for all n. Suppose $|\alpha - b/r| \leq 1/r^2$ with $(b,r) = 1$. For any $2 \leq R \leq r$ and any $N \geq Rr$ we have*

$$\sum_{n \leq N} f(n)e(n\alpha) \ll \frac{N}{\log N} + N\frac{(\log R)^{3/2}}{\sqrt{R}}$$

and

$$\sum_{Rr \leq n \leq N} \frac{f(n)}{n}e(n\alpha) \ll \log\log N + \frac{(\log R)^{3/2}}{\sqrt{R}}\log N.$$

Proof. The first bound follows from Corollary 1 of Montgomery and Vaughan [12]. The second estimate follows easily from the first estimate and partial summation. $\square$

Lemma 4.3. *If f is a multiplicative function with $|f(n)| \leq 1$ for all n, then*

$$\sum_{n \leq x} \frac{f(n)}{n} \ll 1 + \log x \exp\left(-\sum_{p \leq x} \frac{2 - |1 + f(p)|}{p}\right) \ll 1 + \log x \exp\left(-\frac{1}{2}\mathbb{D}(1,f;x)^2\right).$$

Furthermore, if $y \geq 2$, then

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} \ll \log y \exp\left(-\frac{1}{2}\mathbb{D}(1,f;y)^2\right).$$

Proof. For the first assertion see the remark after Proposition 8.1 of [7] and note that if $|z| \leq 1$, then $2 - |1 + z| \geq \frac{1}{2}(1 - \mathrm{Re}\ z)$. To see the second assertion note that if $y \leq x$, then

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} \ll 1 + \log x \exp\left(-\sum_{p \leq y} \frac{2 - |1 + f(p)|}{p} - \sum_{y < p \leq x} \frac{1}{p}\right)$$

$$\ll 1 + \log y \exp\left(-\frac{1}{2}\mathbb{D}(1,f;y)^2\right) \ll \log y \exp\left(-\frac{1}{2}\mathbb{D}(1,f;y)^2\right).$$

Moreover, if $y > x$, then $\log x \exp(-\frac{1}{2}\mathbb{D}(1,f;x)^2) \ll \log y \exp(-\frac{1}{2}\mathbb{D}(1,f;y)^2)$, and so the second assertion holds in this case also. $\square$

We remark that the second assertion of Lemma 4.3 cannot in general be improved. To see this, pick any $x < y$ and take $f(p) = 1$ for $p \leq x$ and $f(p) = -1$ for $x < p \leq y$; then it is easy to check that

$$\sum_{n \leq x} \frac{f(n)}{n} = \log x + O(1) \gg \log y \exp\Big(-\frac{1}{2}\mathbb{D}(1, f; y)^2\Big).$$

Lemma 4.4. *Let f be a completely multiplicative function with $|f(n)| \leq 1$ for all n. Then for any integer $\ell \geq 1$ we have*

$$\sum_{n \leq x} \frac{f(n)}{n} = \prod_{p | \ell} \Big(1 - \frac{f(p)}{p}\Big)^{-1} \sum_{\substack{n \leq x \\ (n, \ell) = 1}} \frac{f(n)}{n} + O((\log \log(\ell + 2))^2).$$

Proof. Writing n as uv where u is composed only of primes dividing ℓ and v is coprime to ℓ we see that

$$\sum_{n \leq x} \frac{f(n)}{n} = \sum_{p | u \implies p | \ell} \frac{f(u)}{u} \sum_{\substack{v \leq x/u \\ (v, \ell) = 1}} \frac{f(v)}{v}$$

$$= \prod_{p | \ell} \Big(1 - \frac{f(p)}{p}\Big)^{-1} \sum_{\substack{v \leq x \\ (v, \ell) = 1}} \frac{f(v)}{v} + O\Big(\sum_{p | u \implies p | \ell} \frac{\log u}{u}\Big).$$

The error term is seen to be $\ll (\sum_{p|\ell} \frac{\log p}{p}) \prod_{p | \ell} (1 + 1/p) \ll (\log \log(\ell + 2))^2$. $\qquad \square$

5. Proof of Proposition 2.3

We begin by recalling a consequence of GRH. If $\psi \pmod m$ is a nonprincipal character, then for all $x \geq 2$,

$$\sum_{n \leq x} \psi(n)\Lambda(n) \ll \sqrt{x} \log x \log(mx).$$

This follows from standard arguments: for example, take $T = x^2$ in (13) on page 120 of [4] and use GRH. It follows from the above and partial summation that

$$(5.1) \qquad\qquad \sum_{p \leq x} \psi(p) \ll \sqrt{x} \log(mx).$$

Lemma 5.1. *Assume GRH. If $\chi \pmod q$ is a primitive character and $x < q^{3/2}$, then, uniformly for all θ, we have that*

$$\sum_{p \leq x} \chi(p) e(p\theta) \ll x^{5/6} \log q.$$

Proof. First we show that if $(b, r) = 1$ with $r < q$, then for any $x \geq 2$,

$$(5.2) \qquad\qquad \sum_{p \leq x} \chi(p) e(bp/r) \ll \sqrt{rx} \log(qx).$$

To see this, note that

$$\sum_{p \le x} \chi(p)e(bp/r) = \sum_{\substack{p \le x \\ (p,r)=1}} \chi(p)e(bp/r) + O\Big(\sum_{p|r} 1\Big)$$

$$= \frac{1}{\phi(r)} \sum_{\psi \pmod r} \overline{\psi}(b)\tau(\psi) \sum_{p \le x} \chi(p)\overline{\psi}(p) + O(\log q).$$

Since $r < q$ and χ is primitive we know that $\chi\overline{\psi}$ is a nonprincipal character $\pmod{qr}$. Appealing now to (5.1) and using that $|\tau(\psi)| \le \sqrt{r}$ from Lemma 4.1 we obtain (5.2).

We now turn to the proof of the lemma. Set $R = x^{2/3}$ and find $r \le R$ such that $\theta = b/r + \beta$ where $(b, r) = 1$ and $|\beta| \le 1/(rR)$. If $x < q^{3/2}$, then $r < q$ and by (5.2) we obtain that $\sum_{p \le N} \chi(p)e(bp/r) \ll \sqrt{rN}\log(qN)$ for all $N \ge 2$. By partial summation we see that

$$\sum_{p \le x} \chi(p)e(p\theta) = \sum_{p \le x} \chi(p)e(pb/r)e(p\beta) = \int_2^x e(t\beta)d\Big(\sum_{p \le t} \chi(p)e(pb/r)\Big),$$

and integrating by parts using our bound above, we obtain that

$$\sum_{p \le x} \chi(p)e(p\theta) \ll (1 + |\beta|x)\sqrt{rx}\log q \ll x^{5/6}\log q.$$

$\square$

Lemma 5.2. *Assume GRH. If $\chi \pmod q$ is a primitive character and $x < q^{3/2}$, then*

$$\sum_{n \le x} \chi(n)e(n\alpha) = \sum_{\substack{n \le x \\ n \in \mathcal{S}(y)}} \chi(n)e(n\alpha) + O(xy^{-1/6}\log q).$$

Proof. Write $n \notin \mathcal{S}(y)$ as pm where p is the largest prime divisor of n. Thus $x/m \ge p > y$ and $m \le x/y$ and so (denoting by $P(m)$ the largest prime factor of m)

$$\sum_{\substack{n \le x \\ n \notin \mathcal{S}(y)}} \chi(n)e(n\alpha) = \sum_{m \le x/y} \chi(m) \sum_{\max(P(m)-1,y)<p\le x/m} \chi(p)e(pm\alpha),$$

and by Lemma 5.1 this is

$$\ll \sum_{m \le x/y} (x/m)^{5/6}\log q \ll xy^{-1/6}\log q,$$

as required. $\square$

Lemma 5.2 and partial summation give that

$$\sum_{n \le q} \frac{\overline{\chi}(n)e(n\theta)}{n} = \sum_{\substack{n \le q \\ n \in \mathcal{S}(y)}} \frac{\overline{\chi}(n)e(n\theta)}{n} + O(y^{-1/6}\log^2 q),$$

and Proposition 2.3 follows.

We remark that Lemma 5.2 with $\alpha = 0$ shows how character sums may be approximated by character sums involving only smooth numbers. This question is explored in greater depth in our paper [6].

6. Proof of Theorems 2.1, 2.2, 2.4 and 2.5

The main ideas of our proof work whether or not GRH is assumed, the only difference being that the relevant parameters need to be chosen differently in each case. To present this in a unified manner we adopt the following convention. We set $Q = \log q$ if GRH is assumed, and $Q = q$ if no assumption is being made. Accordingly we warn the reader that the results in this section must all be read keeping this convention in mind.

By (2.1), to understand $M(\chi)$ we must first gain an understanding of $\sum_{n \leq q} \overline{\chi}(n)e(n\alpha)/n$ where $\alpha \in [0,1]$. If we assume GRH, then Proposition 2.3 shows that we may restrict ourselves to $\sum_{n \leq q, n \in \mathcal{S}((\log q)^{12})} \overline{\chi}(n)e(n\alpha)/n$. Thus, with our convention, we seek to understand

$$(6.1) \qquad \sum_{\substack{n \leq q \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha),$$

since in the unconditional case the criterion $n \in \mathcal{S}(Q^{12})$ is vacuous.

We now define $s = (\log Q)^{\frac{1}{3}}$ and $S = \exp((\log Q)^{\frac{5}{6}})$. We say that α lies on a *minor arc* if there is a rational approximation $|\alpha - b/r| \leq 1/(rS)$ with $(b,r) = 1$ and $s < r \leq S$. Otherwise we say that α lies on a *major arc*; in this case there is a rational approximation $|\alpha - b/r| \leq 1/(rS)$ with $(b,r) = 1$ and $r \leq s$.

Lemma 6.1. *With the above conventions, if α lies on a minor arc, then*

$$\sum_{\substack{n \leq q \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha) \ll (\log Q)^{\frac{5}{6} + o(1)}.$$

Proof. Suppose $|\alpha - b/r| \leq 1/(rS)$ where $(b,r) = 1$ and $s < r \leq S$.

First we consider the unconditional case. By Lemma 4.2 with $R = r$ we see that

$$\sum_{n \leq q} \frac{\overline{\chi}(n)e(n\alpha)}{n} = \sum_{n \leq r^2} \frac{\overline{\chi}(n)e(n\alpha)}{n} + \sum_{r^2 \leq n \leq q} \frac{\overline{\chi}(n)e(n\alpha)}{n} \ll (\log q)^{\frac{5}{6} + o(1)},$$

which proves the lemma in this case.

Now we consider the case when we assume GRH. By Lemma 4.2 with $R = r$ we see that

$$\sum_{\substack{r^2 \leq n \leq (\log q)^{\log s} \\ n \in \mathcal{S}((\log q)^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha) \ll \log \log s + \log \log \log q + \frac{(\log s)^{5/2}}{\sqrt{s}} \log \log q.$$

Furthermore,

$$\left| \sum_{\substack{n > (\log q)^{\log s} \\ n \in \mathcal{S}((\log q)^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha) \right| \leq \frac{1}{s} \sum_{n \in \mathcal{S}((\log q)^{12})} \frac{1}{n^{1 - 1/\log \log q}} \ll \frac{1}{s} \log \log q,$$

and, trivially,

$$\sum_{\substack{n \leq r^2 \\ n \in \mathcal{S}((\log q)^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha) \ll \log r \leq \log S.$$

Combining these estimates we get the lemma in this situation. $\square$

We now consider (6.1) when α lies on a major arc. Thus we suppose that $|\alpha - b/r| \leq 1/(rS)$ where $(b, r) = 1$ and $r \leq s$, and no such approximation exists with $s \leq r \leq S$. Define $N = N_{q,\alpha,b/r} = \min(q, 1/|r\alpha - b|)$.

Lemma 6.2. *With the above conventions, we have*

$$\sum_{\substack{n \leq q \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha) = \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(nb/r) + O(\log \log Q).$$

Proof. If $N = q$, then the lemma follows easily from $|e(n\alpha) - e(nb/r)| \ll n|\alpha - b/r| \leq n/N$. Now suppose that $S \leq N = 1/|r\alpha - b| < q$. We find an approximation $|\alpha - b_1/r_1| \leq 1/(r_1 N)$ where $(b_1, r_1) = 1$ and $r_1 \leq N$. Note that $1/(rr_1) \leq |b/r - b_1/r_1| \leq 1/(rN) + 1/(r_1 N)$ and so $r_1 \geq N - r \geq N - s \geq N/2$. We now set $R = (\log Q)^5$ and divide the interval $(N, q]$ into three intervals: I_1 which contains the integers in $(N, q]$ that are in $(N, Rr_1]$, I_2 which contains the integers in $(N, q]$ that are in $(Rr_1, \exp((\log Q)^2)]$, and I_3 which contains the integers in $(N, q]$ that are larger than $\exp((\log Q)^2)$.

Since $N/2 \leq r_1 \leq N$ it follows that

$$\sum_{\substack{n \in I_1 \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha) \ll \log R \ll \log \log Q.$$

An application of Lemma 4.2 shows that

$$\sum_{\substack{n \in I_2 \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha) \ll \log \log Q.$$

Finally, since each element of I_3 is at least $\exp((\log Q)^2)$ we see that

$$\sum_{\substack{n \in I_3 \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha) \ll \frac{1}{Q} \sum_{n \in \mathcal{S}(Q^{12})} \frac{1}{n^{1-1/\log Q}} \ll 1.$$

Combining these estimates we obtain that

$$\sum_{\substack{n \leq q \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha) = \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha) + O(\log \log Q),$$

and since $|e(n\alpha) - e(nb/r)| \ll n|\alpha - b/r| \leq n/N$, the lemma follows. $\square$

6.1. Lower bounds for $M(\chi)$: Proof of Theorems 2.2 and 2.5. We consider the quantity (6.1) for $\alpha_{b,N} = b/\ell + 1/N$ where b runs over reduced residue classes (mod ℓ) and $1 \leq N \leq q$. We multiply this by $\overline{\psi}(b)$ and sum over all reduced residue classes b (mod ℓ). Thus we arrive at

$$\sum_{b \pmod{\ell}} \overline{\psi}(b) \sum_{\substack{n \leq q \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha_{b,N}) = \tau(\overline{\psi}) \sum_{\substack{n \leq q \\ n \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\psi)(n)}{n} e(n/N).$$

Exactly as in the proof of Lemma 6.2, set $R = (\log Q)^5$ and divide the integers in $(N, q]$ into intervals I_1, I_2 and I_3. Then we deduce that

$$\sum_{b \pmod{\ell}} \overline{\psi}(b) \sum_{\substack{n \leq q \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(n\alpha_{b,N}) = \tau(\overline{\psi}) \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\psi)(n)}{n} + O(\sqrt{\ell} \log\log Q).$$

Now consider $\sum_{b=1}^{\ell} \overline{\psi}(b) \sum_{n \leq q\alpha_{b,N}} \chi(n)$ which in magnitude is plainly $\leq \phi(\ell) M(\chi)$. We see by (2.1), Proposition 2.3 (in the conditional case), and the above remarks that if $\ell > 1$,

$$\sum_{b=1}^{\ell} \overline{\psi}(b) \sum_{n \leq q\alpha_{b,N}} \chi(n) = -\frac{\tau(\chi)\tau(\overline{\psi})}{2\pi i}(\psi(-1) - \chi(-1)) \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\psi)(n)}{n}$$

$$\text{(6.2a)} \qquad\qquad\qquad\qquad + O(\sqrt{q\ell} \log\log Q),$$

while if $\ell = 1$ we have (because of the extra $L(1, \overline{\chi})$ term)

$$\text{(6.2b)} \qquad \sum_{n \leq q\alpha_{1,N}} \chi(n) = \frac{\tau(\chi)}{2\pi i}(1 - \chi(-1)) \sum_{\substack{N \leq n \leq q \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} + O(\sqrt{q} \log\log Q).$$

If we assume GRH, then taking $N = q$ in the case $\ell > 1$ and $N = 1$ when $\ell = 1$ we obtain easily that the sum over n in (6.2a,b) is $\gg \log Q \exp(-\mathbb{D}(\chi, \psi; Q)^2)$ and Theorem 2.5 follows. In the unconditional case we show in the next lemma that a similar lower bound holds for some $1 \leq N \leq q$, which proves Theorem 2.2.

Lemma 6.3. *Let* $\eta \pmod{r}$ *be a primitive character. Then there exists* $1 \leq N \leq r$ *such that*

$$\left| \sum_{n \leq N} \frac{\eta(n)}{n} \right| + 1 \gg \log r \exp(-\mathbb{D}(\eta, 1; r)^2).$$

There also exists $1 \leq N \leq r$ *with*

$$\left| \sum_{N \leq n \leq r} \frac{\eta(n)}{n} \right| + 1 \gg \log r \exp(-\mathbb{D}(\eta, 1; r)^2).$$

Proof. Set $\delta = 1/\log r$ and observe that

$$\frac{1}{e} \sum_{n \leq r} \frac{\eta(n)}{n} + \int_1^r \frac{\delta}{t^{1+\delta}} \sum_{n \leq t} \frac{\eta(n)}{n} dt = \sum_{n \leq r} \frac{\eta(n)}{n^{1+\delta}}.$$

It follows that

$$\max_{N \leq r} \left| \sum_{n \leq N} \frac{\eta(n)}{n} \right| \geq \left| \sum_{n \leq r} \frac{\eta(n)}{n^{1+\delta}} \right|.$$

We see easily that $L(1+\delta, \eta) = \sum_{n \leq r} \eta(n)/n^{1+\delta} + O(1)$, and from the Euler product that $L(1 + \delta, \eta) \gg \log r \exp(-\mathbb{D}(\eta, 1; r)^2)$. The first part of the lemma follows. The second part is similar starting from

$$\sum_{n \leq r} \frac{\eta(n)}{n} - \int_1^r \frac{\delta}{t^{1+\delta}} \sum_{t \leq n \leq r} \frac{\eta(n)}{n} dt = \sum_{n \leq r} \frac{\eta(n)}{n^{1+\delta}}.$$

$\square$

6.2. Upper bounds for $M(\chi)$: Proof of Theorems 2.1 and 2.4. We continue from Lemma 6.2 our analysis of (6.1) in the case when α lies on a major arc. Of all characters with conductor below s we let ξ (mod m) denote that character for which $\mathbb{D}(\chi, \psi; Q)$ is a minimum.

Lemma 6.4. *We keep the conventions of this section. Suppose $(b, r) = 1$ with $r \leq s$. Then*

$$\sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(nb/r) = O((\log Q)^{\frac{6}{7}}),$$

unless $m | r$ in which case it equals

$$\frac{\xi(b)\tau(\overline{\xi})}{\phi(r)} \prod_{\substack{p^\alpha \| r/m \\ \alpha \geq 1}} (\overline{\chi}(p^\alpha) - \overline{\xi}(p)\overline{\chi}(p^{\alpha-1})) \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^2)}} \frac{(\overline{\chi}\xi)(n)}{n} + O((\log Q)^{\frac{6}{7}}).$$

Proof. Note that

$$(6.3) \qquad \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n} e(nb/r) = \sum_{d | r} \frac{\overline{\chi}(d)}{d} \sum_{\substack{n \leq N/d \\ n \in \mathcal{S}(Q^{12}) \\ (n, r/d) = 1}} \frac{\overline{\chi}(n)}{n} e\left(\frac{nb}{r/d}\right).$$

Since $(nb, r/d) = 1$ we see that

$$e\left(\frac{nb}{r/d}\right) = \frac{1}{\phi(r/d)} \sum_{a \ (\mathrm{mod} \ r/d)} e\left(\frac{a}{r/d}\right) \sum_{\psi \ (\mathrm{mod} \ r/d)} \overline{\psi}(a)\psi(nb)$$

$$\overline{\psi}(a)\psi(nb) = \frac{1}{\phi(r/d)} \sum_{\psi \ (\mathrm{mod} \ r/d)} \psi(nb)\tau(\overline{\psi}).$$

Therefore

$$(6.4) \qquad \sum_{\substack{n \leq N/d \\ n \in \mathcal{S}(Q^{12}) \\ (n, r/d) = 1}} \frac{\overline{\chi}(n)}{n} e\left(\frac{nb}{r/d}\right) = \frac{1}{\phi(r/d)} \sum_{\psi \ (\mathrm{mod} \ r/d)} \tau(\overline{\psi})\psi(b) \sum_{\substack{n \leq N/d \\ n \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\psi)(n)}{n}.$$

By Lemma 4.3 we see that

$$\left| \sum_{\substack{n \leq N/d \\ n \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\psi)(n)}{n} \right| \ll (\log Q) \exp\left(-\frac{1}{2}\mathbb{D}(\chi, \psi; Q)^2\right).$$

Using Lemma 3.4 we see that if ψ is not induced by ξ, then

$$\mathbb{D}(\chi, \psi; Q)^2 \geq (1 - 1/\sqrt{2} + o(1)) \log\log Q,$$

and further that there are at most 9 characters ψ (mod r/d) for which $\mathbb{D}(\chi, \psi; Q)^2 \leq \frac{2}{3} \log\log Q$. Since $|\tau(\overline{\psi})| \leq \sqrt{r/d}$ we deduce that the contribution of all characters not induced by ξ to (6.4) is

$$\ll \frac{\sqrt{r/d}}{\phi(r/d)} (\log Q)^{\frac{1}{2} + \frac{1}{2\sqrt{2}} + o(1)} + \sqrt{r/d}(\log Q)^{\frac{2}{3}}.$$

The contribution of these terms to (6.3) is $\ll (\log Q)^{\frac{1}{2}+\frac{1}{2\sqrt{2}}+o(1)}+\sqrt{r}(\log Q)^{\frac{2}{3}+o(1)} \ll (\log Q)^{\frac{6}{7}}$.

We must now handle the contribution to (6.3) from characters ψ induced by ξ (mod m). If $m \nmid r$, then there are no such characters ψ, and the lemma follows in this case. If $m|r$, then we have to account for the contribution of the characters ψ (mod r/d) induced by ξ (thus d must be a divisor of r/m). By Lemma 4.1 and (6.4) we see that the contribution of these induced characters to (6.3) is

$$(6.5) \qquad \sum_{d|r/m} \frac{\overline{\chi}(d)}{d} \frac{1}{\phi(r/d)} \xi(b)\tau(\overline{\xi})\mu\left(\frac{r}{dm}\right)\overline{\xi}\left(\frac{r}{dm}\right) \sum_{\substack{n \leq N/d \\ (n,r/d)=1 \\ n \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\xi)(n)}{n}.$$

By Lemma 4.4,

$$\sum_{\substack{n \leq N/d \\ (n,r/d)=1 \\ n \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\xi)(n)}{n} = \sum_{\substack{n \leq N \\ (n,r/d)=1 \\ n \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\xi)(n)}{n} + O(\log d)$$

$$= \prod_{p|r/d} \left(1 - \frac{(\overline{\chi}\xi)(p)}{p}\right) \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\xi)(n)}{n} + O(\log\log Q).$$

Therefore (6.5) equals, up to an error $O(\log\log Q)$,

$$\xi(b)\tau(\overline{\xi}) \sum_{d|r/m} \frac{\overline{\chi}(d)}{d} \frac{1}{\phi(r/d)} \mu\left(\frac{r}{md}\right)\overline{\xi}\left(\frac{r}{md}\right) \prod_{p|r/(md)} \left(1 - \frac{(\overline{\chi}\xi)(p)}{p}\right) \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\xi)(n)}{n},$$

which by a straightforward calculation is

$$(6.6) \qquad \frac{\xi(b)\tau(\overline{\xi})}{\phi(r)} \prod_{\substack{p^\alpha\|r/m \\ \alpha \geq 1}} (\overline{\chi}(p^\alpha) - \overline{\xi}(p)\overline{\chi}(p^{\alpha-1})) \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\xi)(n)}{n}.$$

To complete the proof of the lemma, it remains to show that the terms in the sum in (6.6) may be restricted to $n \in \mathcal{S}(Q^2)$ with an acceptable error, in the case where GRH is assumed. We must therefore estimate the contribution of terms n which lie in $\mathcal{S}(Q^{12})$ but not in $\mathcal{S}(Q^2)$. We may write such n uniquely as $p\ell$ where p, the largest prime factor of n, lies between $\max(P(\ell)-1, Q^2)$ and $\min(Q^{12}, N/\ell)$, and $\ell \leq N/Q^2$ is in $\mathcal{S}(Q^{12})$. Thus the contribution of these terms is

$$(6.7) \qquad \sum_{\substack{\ell \leq N/Q^2 \\ \ell \in \mathcal{S}(Q^{12})}} \frac{(\overline{\chi}\xi)(\ell)}{\ell} \sum_{\substack{\max(P(\ell)-1,Q^2) \leq p \\ p \leq \min(Q^{12}, N/\ell)}} \frac{(\overline{\chi}\xi)(p)}{p}.$$

Using (5.1) and partial summation to handle the primes larger than $Q^2(\log Q)^2$, and estimating the smaller primes trivially, we obtain that the sum over primes in (6.7) above is $\ll (\log\log Q)/\log Q$. Thus the quantity (6.7) is $\ll \log\log Q$, and the lemma follows. $\qquad\square$

Combining Lemmas 6.1, 6.2 and 6.4 we arrive at the following:

Lemma 6.5. *Keep the conventions of this section. Then*

$$\sum_{\substack{n \leq q \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n}(e(-n\alpha) - \chi(-1)e(n\alpha)) \ll (\log Q)^{\frac{6}{7}}$$

unless α lies on a major arc $|\alpha - b/r| \leq 1/(rS)$ with $r \leq s$, $(b,r) = 1$ and $m|r$, in which case it equals, up to an error $O((\log Q)^{\frac{6}{7}})$,

$$\frac{(\xi(-1) - \chi(-1))\xi(b)\tau(\overline{\xi})}{\phi(r)} \prod_{\substack{p^a \| r/m \\ a \geq 1}} (\overline{\chi}(p^a) - \overline{\xi}(p)\overline{\chi}(p^{a-1})) \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^2)}} \frac{(\overline{\chi}\xi)(n)}{n},$$

where $N = \min(q, 1/|r\alpha - b|)$.

Proof of Theorems 2.1 and 2.4. From Lemma 6.5 we see easily that

$$\sum_{\substack{n \leq q \\ n \in \mathcal{S}(Q^{12})}} \frac{\overline{\chi}(n)}{n}(e(-n\alpha) - \chi(-1)e(n\alpha)) \ll (\log Q)^{\frac{6}{7}}$$

$$+ \frac{(1 - \chi(-1)\xi(-1))\sqrt{m}}{\phi(m)} \max_{N \leq q} \Big| \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^2)}} \frac{(\overline{\chi}\xi)(n)}{n} \Big|.$$

Using Lemma 4.3 this is

$$(6.8) \qquad \ll (\log Q)^{\frac{6}{7}} + \frac{(1 - \chi(-1)\xi(-1))\sqrt{m}}{\phi(m)}(\log Q) \exp\Big(-\frac{1}{2}\mathbb{D}(\chi, \xi; Q)^2 \Big).$$

We also note that (using $L(1, \overline{\chi}) = \sum_{n \leq q} \overline{\chi}(n)/n + O(1)$ and Lemma 4.3 in the unconditional case, and Littlewood's (1.6) in the conditional case)

$$(6.9) \qquad L(1, \overline{\chi}) \ll (\log Q) \exp\Big(-\frac{1}{2}\mathbb{D}(\chi, 1; Q)^2 \Big).$$

Using (6.8) and (6.9) in (2.1) (and using Proposition 2.3 in the conditional case) we immediately obtain Theorems 2.1 and 2.4 in the case when $m = 1$ and ξ is the trivial character $\xi(n) = 1$ for all n. In the case when $m > 1$ it follows from Lemma 3.4 that $\mathbb{D}(\chi, 1; Q)^2 \geq (1 - 1/\sqrt{2} + o(1)) \log \log Q$. Using this in (6.9) we obtain the bounds claimed in Theorems 2.1 and 2.4. $\qquad\square$

Our development of results in section 2 is designed to dovetail with the easily understood triangle inequality, and its consequences, as described in section 3. Our analysis above shows that one can obtain a more precise (though less readily applicable) evaluation of $M(\chi)$ in terms of

$$M^*(\psi) := \max_N \Big| \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^2)}} \frac{\psi(n)}{n} \Big|.$$

We describe such a result below, which improves, and was inspired by, Corollary 4 of [10].

Corollary 6.6. *Keep the conventions of this section. Then*

$$M(\chi) = \epsilon_\chi \, \frac{\sqrt{q}}{\pi} \, \frac{\sqrt{m}}{\phi(m)} \, M^*(\chi\overline{\xi}) \, \max\{1, |1 - (\chi\overline{\xi})(2)|\} + O(\sqrt{q}(\log Q)^{\frac{6}{7}}),$$

where

$$\epsilon_\chi = \begin{cases} 0 & \text{if } \chi(-1)\xi(-1) = 1, \\ 1 & \text{if } \chi(-1)\xi(-1) = -1, \text{ and } m > 1, \\ \in [1/2, 2] & \text{if } m = 1 \text{ and } \chi(-1) = -1. \end{cases}$$

Proof. If $m > 1$, then $L(1, \chi) = O((\log Q)^{\frac{6}{7}})$ by Lemma 3.4, so the result follows from (2.1) and Lemma 6.5. If $m = 1$, then (2.1) and Lemma 6.5 yield the result when $\chi(-1) = 1$, and when $\chi(-1) = -1$ we obtain

$$M(\chi) = \max_r \max_{N \leq q} \frac{\sqrt{q}}{\pi} \left| L(1, \overline{\chi}) - \frac{1}{\phi(r)} \prod_{\substack{p^a \| r \\ a \geq 1}} (\overline{\chi}(p^a) - \overline{\chi}(p^{a-1})) \sum_{\substack{n \leq N \\ n \in \mathcal{S}(Q^2)}} \frac{\overline{\chi}(n)}{n} \right|$$

$$+ O(\sqrt{q}(\log Q)^{\frac{6}{7}}).$$

From this we easily obtain the upper bound

$$M(\chi) \leq \frac{\sqrt{q}}{\pi} \left(|L(1, \overline{\chi})| + M^*(\chi) \max\{1, |1 - \chi(2)|\} \right) + O(\sqrt{q}(\log Q)^{\frac{6}{7}}),$$

and the upper bound on ϵ_χ follows as $|L(1, \overline{\chi})| \leq (1 + o(1))M^*(\chi)$.

Note that the case $N = 0$ implies that $M(\chi)/\frac{\sqrt{q}}{\pi} \geq |L(1, \overline{\chi})| + O((\log Q)^{\frac{6}{7}})$. Furthermore, the choice of N which yields $M^*(\chi)$ together with $r = 1$ or 2 (the latter if $|1 - \chi(2)| > 1$) implies that $M(\chi)/\frac{\sqrt{q}}{\pi} \geq M^*(\chi) \max\{1, |1 - \chi(2)|\} - |L(1, \overline{\chi})| + O((\log Q)^{\frac{6}{7}})$. Taking the average of these two lower bounds yields the lower bound on ϵ_χ. $\qquad\square$

7. Proof of Theorem 6

To prove Theorem 6, we assume GRH and continue the analysis of the previous section (note that $Q = \log q$). We distinguish two cases: when the nearest character $\xi \pmod{m}$ is the trivial character ($m = 1$ and $\xi(n) = 1$ for all n), and when $m > 1$.

We start with the easier second case. By Lemma 3.4 we have that $\mathbb{D}(\chi, 1; \log q)^2 \geq (1 - 1/\sqrt{2} + o(1)) \log\log\log q$ and therefore, by (6.9), $L(1, \overline{\chi}) = o(\log\log q)$. From this, (2.1), Proposition 2.3, and Lemma 6.5 we obtain that $M(\chi) + o(\sqrt{q}\log\log q)$ is

$$\leq \frac{\sqrt{q}|\tau(\overline{\xi})|}{\pi} \max_{m|r, N \leq q} \left| \frac{1}{\phi(r)} \prod_{\substack{p^a \| r/m \\ a \geq 1}} (\overline{\chi}(p^a) - \overline{\xi}(p)\overline{\chi}(p^{a-1})) \sum_{\substack{n \leq N \\ n \in \mathcal{S}(\log^2 q)}} \frac{(\overline{\chi}\xi)(n)}{n} \right|.$$

By Lemma 4.4 we see that, up to $o(\sqrt{q}\log\log q)$, the above is

$$\leq \frac{\sqrt{qm}}{\pi} \max_{m|r, N \leq q} \left| \frac{1}{\phi(r)} \prod_{\substack{p^a \| r/m \\ a \geq 1}} \frac{\overline{\chi}(p^a) - \overline{\xi}(p)\overline{\chi}(p^{a-1})}{1 - (\overline{\chi}\xi)(p)/p} \sum_{\substack{v \leq N \\ v \in \mathcal{S}(\log^2 q) \\ (v,r)=1}} \frac{(\overline{\chi}\xi)(v)}{v} \right|.$$

The product above is bounded in magnitude by $\prod_{p|r/m} 2p/(p+1)$, and the sum over v above has size $\leq \prod_{p\leq \log^2 q, p\nmid r}(1-1/p)^{-1} = (2e^\gamma + o(1))(\phi(r)/r)\log\log q$. It follows readily that when $m > 1$,

$$M(\chi) \leq \left(\frac{2e^\gamma}{\pi\sqrt{m}} + o(1)\right)\sqrt{q}\log\log q.$$

Since there is no primitive character $(\mathrm{mod}\,2)$ we have that $m \geq 3$ and so the bounds of Theorem 6 follow.

We now consider the more involved case when $m = 1$ and ξ is the trivial character. We consider $\sum_{\alpha_1 q \leq n \leq \alpha_2 q} \chi(n)$ where $0 \leq \alpha_1 < \alpha_2 \leq 1$ and by (2.1) and Proposition 2.3 this is

(7.1)
$$-\frac{\tau(\chi)}{2\pi i} \sum_{\substack{n\leq q \\ n\in\mathcal{S}((\log q)^{12})}} \frac{\overline{\chi}(n)}{n}(e(-n\alpha_1)-\chi(-1)e(n\alpha_1)-e(-n\alpha_2)+\chi(-1)e(n\alpha_2))+O(\sqrt{q}).$$

There arise three cases: both α_1 and α_2 lie on minor arcs, exactly one of α_1 and α_2 lies on a major arc, and both α_1 and α_2 lie on major arcs. In the first case we obtain from Lemma 6.5 that the above is $\ll \sqrt{q}(\log\log q)^{\frac{6}{7}+o(1)}$. We examine the third case in detail, and omit the second case which is similar and simpler. Suppose (for $j = 1,\,2$) that $|\alpha_j - b_j/r_j| \leq 1/(r_j S)$ where $r_j \leq s$, and $(b_j, r_j) = 1$. Set $N_j = \min(q, 1/|r_j\alpha_j - b_j|)$. Using Lemma 6.5 and Lemma 4.4 we see that (7.1) equals, up to an error $O(\sqrt{q}(\log\log q)^{\frac{6}{7}+o(1)})$,

(7.2)
$$-(1-\chi(-1))\frac{\tau(\chi)}{2\pi i}\left(\lambda_1 \sum_{\substack{v\leq N_1 \\ v\in\mathcal{S}((\log q)^2) \\ (v,r_1 r_2)=1}} \frac{\overline{\chi}(v)}{v} - \lambda_2 \sum_{\substack{v\leq N_2 \\ v\in\mathcal{S}((\log q)^2) \\ (v,r_1 r_2)=1}} \frac{\overline{\chi}(v)}{v}\right),$$

where

$$\lambda_j = \frac{1}{\phi(r_j)}\prod_{\substack{p^a\|r_j \\ a\geq 1}}(\overline{\chi}(p^a)-\overline{\chi}(p^{a-1}))\prod_{p|r_1 r_2}\left(1-\frac{\overline{\chi}(p)}{p}\right)^{-1}.$$

It is easy to see that (7.2) is bounded in magnitude by

$$\frac{\sqrt{q}}{\pi}\max(|\lambda_1|,|\lambda_2|,|\lambda_1-\lambda_2|)\sum_{\substack{v\in\mathcal{S}((\log q)^2) \\ (v,r_1 r_2)=1}}\frac{1}{v}$$

$$=\frac{\sqrt{q}}{\pi}\max(|\lambda_1|,|\lambda_2|,|\lambda_1-\lambda_2|)\prod_{p|r_1 r_2}\left(1-\frac{1}{p}\right)(2e^\gamma+o(1))\log\log q.$$

Thus Theorem 6 would follow if

(7.3)
$$\max(|\lambda_1|,|\lambda_2|,|\lambda_1-\lambda_2|)\phi(r_1 r_2)/(r_1 r_2) \leq 1.$$

A simple optimization gives that

$$|\lambda_j|\frac{\phi(r_1 r_2)}{r_1 r_2} \leq \frac{1}{r_j}\prod_{\substack{p^a\|r_j \\ a\geq 1}}\left|\frac{\overline{\chi}(p^a)-\overline{\chi}(p^a-1)}{1-\overline{\chi}(p)/p}\right| \leq \frac{1}{r_j}\prod_{p|r_j}\frac{2p}{p+1}\ (\leq 1).$$

This estimate immediately gives (7.3) in all but the following two cases: one of r_1 or r_2 equals 1, or one of r_1 or r_2 equals 2 and the other equals 3. In the second case we see that

$$|\lambda_1 - \lambda_2|\frac{\phi(6)}{6} = \left|\frac{2\overline{\chi}(2) - \overline{\chi}(3) - 1}{(2 - \overline{\chi}(2))(3 - \overline{\chi}(3))}\right| \leq \frac{2}{3},$$

since this is maximized at $\overline{\chi}(2) = -1$ and $\overline{\chi}(3) = 1$ and so (7.3) holds. Finally we have the case when one of r_1 or r_2 is 1 and the other equals r say. Here we must show that

(7.4) $$\left|1 - \frac{1}{\phi(r)}\prod_{\substack{p^a\|r \\ a\geq 1}}(\overline{\chi}(p^a) - \overline{\chi}(p^{a-1}))\right|\prod_{p|r}\left|\frac{p-1}{p-\overline{\chi}(p)}\right| \leq 1.$$

If $r = p^a$ is a prime power, then the LHS of (7.4) equals

$$\frac{1}{p^{a-1}}\left|\left(1 - \frac{1}{p}\right)\frac{p^a - \overline{\chi}(p)^a}{p - \overline{\chi}(p)} + \frac{\overline{\chi}(p)^{a-1}}{p}\right| \leq \frac{1}{p^{a-1}}\left(\left(1 - \frac{1}{p}\right)(p^{a-1} + \ldots + 1) + \frac{1}{p}\right) = 1,$$

and so (7.4) holds. Now suppose that r has at least two distinct prime factors. For any nonnegative $a_1, \ldots, a_k$ with $k \geq 2$, we have that

$$(1 + a_1 \ldots a_k)^2 \leq (1 + a_1^2)(1 + (a_2 \ldots a_k)^2) \leq \prod_{i=1}^{k}(1 + a_i^2).$$

Therefore

$$\left|1 - \frac{1}{\phi(r)}\prod_{\substack{p^a\|r \\ a\geq 1}}(\overline{\chi}(p^a) - \overline{\chi}(p^{a-1}))\right|^2 \leq \left(1 + \prod_{\substack{p^a\|r \\ a\geq 1}}\frac{|\overline{\chi}(p^a) - \overline{\chi}(p^{a-1})|}{p^{a-1}(p-1)}\right)^2$$

$$\leq \prod_{p|r}\left(1 + \left|\frac{1 - \overline{\chi}(p)}{p-1}\right|^2\right)$$

$$\leq \prod_{p|r}\left|1 + \frac{1 - \overline{\chi}(p)}{p-1}\right|^2$$

as desired since $z = (1 - \overline{\chi}(p))/(p-1)$ is a complex number with nonnegative real part so that $1 + |z|^2 \leq |1 + z|^2$.

8. PALEY'S BOUND IN ALL DIRECTIONS: PROOF OF THEOREM 3

Bateman and Chowla [1] showed that

$$\frac{1}{q}\sum_{N\leq q}\left|\sum_{n\leq N}\chi(n) - \frac{\tau(\chi)}{i\pi}\frac{(1 - \chi(-1))}{2}L(1,\overline{\chi})\right|^2 = \frac{q}{12}\prod_{p|q}\left(1 - \frac{1}{p^2}\right).$$

If $\chi(-1) = -1$ we deduce that

(8.1) $$\sum_{n\leq N}\chi(n) = \frac{\tau(\chi)}{i\pi}(L(1,\overline{\chi}) + O(\log\log\log q))$$

for "almost all" $N \leq q$. We now show that for most characters χ, $L(1,\overline{\chi})$ may be approximated by a short Euler product. Throughout this section we let $y := \log q / \log\log q$.

Proposition 8.1. *For any large prime q,*

$$L(1,\chi) = \prod_{p \leq y} \left(1 - \frac{\chi(p)}{p}\right)^{-1}\left(1 + O\left(\frac{\log\log\log q}{\log\log q}\right)\right),$$

for all but at most $q^{1-1/(4\log\log q)}$ characters $\chi \pmod q$.

Proof. An immediate consequence of Proposition 2.2 of [8] is that

$$L(1,\chi) = \prod_{p \leq (\log q)^3} \left(1 - \frac{\chi(p)}{p}\right)^{-1}\left(1 + O\left(\frac{1}{\log\log q}\right)\right),$$

for all but at most $q^{3/4}$ characters $\chi \pmod q$. Consider

$$\frac{1}{\phi(q)}\sum_{\chi \pmod q}\Big|\sum_{\log q < p \leq (\log q)^3}\frac{\chi(p)}{p}\Big|^{2k} = \sum_{m \equiv n \pmod q}\frac{a_k(m)a_k(n)}{mn},$$

where $a_k(n)$ is the number of ways of writing $n = p_1 \ldots p_k$ where each p_i is a prime in $(\log q, (\log q)^3]$. We choose $k = [\log q/(4\log\log q)]$ so that $a_k(n) = 0$ for $n > q$ and so the congruence $m \equiv n \pmod q$ implies that $m = n$. Since $a_k(n) \leq k!$ it follows that

$$\sum_n \frac{a_k(n)^2}{n^2} \leq k!\sum_n\frac{a_k(n)}{n^2} = k!\Big(\sum_{\log q < p \leq (\log q)^3}\frac{1}{p^2}\Big)^k \leq \Big(\frac{1}{\log\log q}\Big)^{2k}.$$

We deduce that there are fewer than $\phi(q)e^{-2k}$ characters χ with

$$\Big|\sum_{\log q < p \leq (\log q)^3}\chi(p)/p\Big| \geq e/\log\log q.$$

Since $\sum_{y < p \leq \log q}\frac{\chi(p)}{p} \ll \log\log\log q/\log\log q$ trivially, the proposition follows. □

Proposition 8.2. *Given a prime q and an angle $\theta \in (-\pi, \pi]$, there are at least $q^{1-C_0/(\log\log q)^2}$ characters $\chi \pmod q$ with $\chi(-1) = -1$ such that*

$$\frac{\tau(\chi)}{i\sqrt{q}}\prod_{p \leq y}\left(1 - \frac{\overline{\chi}(p)}{p}\right)^{-1} = e^{i\theta}(e^\gamma \log\log q) + O((\log\log q)^{1/2}).$$

Proof of Theorem 3. Theorem 3 follows upon combining (8.1) with Propositions 8.1 and 8.2. □

To prove Proposition 8.2 we require the following consequence of P. Deligne's celebrated bound on hyper-Kloosterman sums.

Lemma 8.3. *We have*

$$\frac{2}{\phi(q)}\Big|\sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}}\chi(a)\tau(\chi)^n\Big| \leq 2nq^{(n-1)/2}.$$

Proof. Using the definition of $\tau(\chi)$ and the orthogonality relation for characters we see that

$$\frac{2}{\phi(q)}\sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}}\chi(a)\tau(\chi)^n = \mathrm{Kl}_n(\overline{a}, q) - \mathrm{Kl}_n(-\overline{a}, q),$$

where
$$\mathrm{Kl}_n(b,q) = \sum_{\substack{x_1,\ldots,x_n \pmod q \\ x_1\cdots x_n \equiv b \pmod q}} e\Big(\frac{x_1+\ldots+x_n}{q}\Big).$$

In (7.1.3) of [5] Deligne gives the bound (for $(b,q)=1$)
$$|\mathrm{Kl}_n(b,q)| \leq nq^{(n-1)/2},$$

from which the lemma follows. □

Proof of Proposition 8.2. Set $R := \prod_{p\leq y}(1-1/p)^{-1} = e^{\gamma}\log\log q + O(\log\log\log q)$ and consider for a natural number k,

$$(8.2) \qquad \frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}} \Big|\frac{\tau(\chi)}{i\sqrt{q}} \prod_{p\leq y}\Big(1-\frac{\overline{\chi}(p)}{p}\Big)^{-1} + Re^{i\theta}\Big|^{2k}.$$

Expanding using the binomial theorem this equals

$$\sum_{0\leq j,\ell\leq k} \binom{k}{j}\binom{k}{\ell} R^{2k-j-\ell} e^{i\theta(\ell-j)} \sum_{m,n\in\mathcal{S}(y)} \frac{d_j(m)}{m}\frac{d_\ell(n)}{n}$$
$$(8.3) \qquad\qquad\qquad\qquad \times \frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}} \overline{\chi}(m)\chi(n)\Big(\frac{\tau(\chi)}{i\sqrt{q}}\Big)^j \overline{\Big(\frac{\tau(\chi)}{i\sqrt{q}}\Big)}^\ell.$$

Using Lemma 8.3 we see that the terms $j \neq \ell$ above contribute an amount bounded in magnitude by

$$(8.4) \qquad \frac{2k}{\sqrt{q}} \sum_{0\leq j,\ell\leq k} \binom{k}{j}\binom{k}{\ell} R^{2k-j-\ell} R^j R^\ell = \frac{2k}{\sqrt{q}} 2^{2k} R^{2k}.$$

Now we focus on the terms $j=\ell$ in (8.3) which give, by the orthogonality relation for characters,

$$(8.5) \qquad \sum_{0\leq j\leq k} \binom{k}{j}^2 R^{2k-2j} \sum_{\substack{m\equiv\pm n \pmod q \\ m,n\in\mathcal{S}(y)}} (\pm 1)\frac{d_j(m)}{m}\frac{d_j(n)}{n}.$$

If $m\equiv\pm n \pmod q$ but $m\neq n$, then either m or n exceeds $q/2$. Thus such terms contribute to the sum in (8.5) an amount

$$\leq 4 \sum_{m\in\mathcal{S}(y)} \frac{d_j(m)}{m} \sum_{\substack{n\geq q/2 \\ n\in\mathcal{S}(y)}} \frac{d_j(n)}{n} \leq 4R^j\Big(\frac{2}{q}\Big)^{1/\log\log q} \sum_{n\in\mathcal{S}(y)} \frac{d_j(n)}{n^{1-1/\log\log q}}$$
$$\ll C^j R^{2j} q^{-1/\log\log q},$$

for some absolute constant $C > 1$. From this and (8.4) we conclude that our quantity (8.2) is

$$(8.6) \qquad \sum_{0\leq j\leq k} \binom{k}{j}^2 R^{2k-2j} \sum_{n\in\mathcal{S}(y)} \frac{d_j(n)^2}{n^2} + O\Big((4C)^k R^{2k} q^{-1/\log\log q}\Big).$$

Note that

$$\sum_{n\in\mathcal{S}(y)} \frac{d_j(n)^2}{n^2} = \prod_{p\leq y} \sum_{\ell=0}^{\infty} \frac{d_j(p^\ell)^2}{p^{2\ell}} = \prod_{p\leq y} \int_{-1/2}^{1/2} \left|1-\frac{e(\theta)}{p}\right|^{-2j} d\theta.$$

Observe that $\int_{-1/2}^{1/2} |1-\frac{e(\theta)}{p}|^{-2j} d\theta \geq 1$ always, and that if $p \leq j$, then it is

$$\geq \int_{-p/(2j)}^{p/(2j)} \left|1-\frac{e(\theta)}{p}\right|^{-2j} d\theta \geq \frac{cp}{j}\left(1-\frac{1}{p}\right)^{-2j},$$

for a suitable positive constant c. It follows that if $2 \leq j \leq y$, then

$$(8.7) \qquad \sum_{n\in\mathcal{S}(y)} \frac{d_j(n)^2}{n^2} \geq \prod_{p\leq j}\left(1-\frac{1}{p}\right)^{-2j} \exp\left(-\frac{Cj}{\log j}\right),$$

for some positive constant C.

We now take $k = [c'y]$ for a suitably small constant $c' > 0$, and consider only the contribution of $j = [k/2]$ in (8.6). Using (8.7) we deduce easily that the main term in (8.6) exceeds $2^{2k}R^k \prod_{p\leq k}(1-1/p)^{-k} \exp(-Ck/\log k)$ and that the error term there is substantially smaller. We conclude that

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi(-1)=-1}} \left|\frac{\tau(\chi)}{i\sqrt{q}} \prod_{p\leq y}\left(1-\frac{\overline{\chi}(p)}{p}\right)^{-1} + Re^{i\theta}\right|^{2k} \geq (2R)^{2k} \exp\left(-\frac{Ck}{\log k}\right),$$

for some positive absolute constant C. From this estimate we deduce that (for some absolute constant C_0) there are at least $q^{1-C_0/(\log\log q)^2}$ characters $\chi \pmod q$ with $\chi(-1) = -1$ such that

$$\left|\frac{\tau(\chi)}{i\sqrt{q}} \prod_{p\leq y}\left(1-\frac{\overline{\chi}(p)}{p}\right)^{-1} + Re^{i\theta}\right| \geq 2R\left(1-\frac{C'}{\log\log q}\right).$$

Now, if $|z| \leq 1$ and $|1+z| \geq 2-\epsilon$, then we may check easily that $z = 1 + O(\sqrt{\epsilon})$. The proposition follows. $\square$

9. The constant in the Pólya-Vinogradov theorem: Proof of Theorem 7

Let $\xi \pmod m$ denote the primitive character with conductor below $(\log q)^{\frac{1}{3}}$ such that $\mathbb{D}(\chi,\xi;q)$ is a minimum. We distinguish two cases depending on whether $m > 1$ or $m = 1$.

We start with the easier first case. By Lemma 3.4 we know that $\mathbb{D}(\chi,1;q)^2 \geq (1 - 1/\sqrt{2} + o(1))\log\log q$, and so by (6.9) we have that $L(1,\overline{\chi}) = o(\log q)$. Thus by (2.1) and Lemma 6.5 we deduce that $M(\chi) + o(\sqrt{q}\log q)$ is

$$(9.1) \qquad \leq \frac{\sqrt{qm}}{\pi} \max_{m|r,N\leq q} \left|\frac{1}{\phi(r)} \prod_{\substack{p^a\|r/m \\ a\geq 1}} (\overline{\chi}(p^a) - \overline{\xi}(p)\overline{\chi}(p^{a-1})) \sum_{n\leq N} \frac{(\overline{\chi}\xi)(n)}{n}\right|.$$

Observe that $\overline{\chi}\xi$ is a nontrivial character to the modulus $[q,m]$ (this denotes the l.c.m. of q and m). Set $c_0 = 1/4$ if $[q,m]$ is cubefree and $c_0 = 1/3$ otherwise, and note that $c_0 = c$ unless m is divisible by a prime cube, and in any case $c_0 \leq \frac{4}{3}c$.

Burgess's results on character sums (see [2]) show that $\sum_{n \leq x} (\overline{\chi}\xi)(n) = o(x)$ if $x > q^{c_0 + \epsilon}$, from which it follows by partial summation that

$$\sum_{n \leq N} \frac{(\overline{\chi}\xi)(n)}{n} = \sum_{n \leq \min(q^{c_0}, N)} \frac{(\overline{\chi}\xi)(n)}{n} + o(\log q).$$

Using this and Lemma 4.4 in (9.1) we obtain that $M(\chi) + o(\sqrt{q} \log q)$ is

$$(9.2) \qquad \leq \frac{\sqrt{qm}}{\pi} \max_{m | r, N \leq q^{c_0}} \left| \frac{1}{\phi(r)} \prod_{\substack{p^a \| r/m \\ a \geq 1}} \frac{\overline{\chi}(p^a) - \overline{\xi}(p)\overline{\chi}(p^{a-1})}{1 - (\overline{\chi}\xi)(p)/p} \sum_{\substack{n \leq N \\ (n,r)=1}} \frac{(\overline{\chi}\xi)(n)}{n} \right|.$$

The product above is bounded in magnitude by $\prod_{p | r/m} 2p/(p+1)$, while the sum is bounded by $\sim (\phi(r)/r)c_0 \log q$. Thus (9.2) is bounded in magnitude by

$$\frac{\sqrt{q}}{\pi} \frac{\sqrt{m}}{r} \prod_{p | r/m} \frac{2p}{p+1} (c_0 \log q).$$

If $c_0 \neq c$, then m must be at least 8 and the above bound beats the estimates claimed in the theorem. If $1 < m < 8$, then $c_0 = c$ and the bound above suffices in all cases except for $m = r = 3$ (and $\xi = (\frac{\cdot}{3})$). In this final case we have that the quantity in (9.2) is bounded in magnitude by

$$\frac{\sqrt{q}}{\pi} \frac{\sqrt{3}}{2} \max_{N \leq q^c} \left| \sum_{n \leq N} \frac{(\overline{\chi}\xi)(n)}{n} \right|.$$

Applying Theorem 1 of [9] we may see that

$$|L(1, \overline{\chi}\xi)| = \left| \sum_{n \leq q^c} \frac{(\overline{\chi}\xi)(n)}{n} \right| + o(\log q) \leq \left(\frac{34}{35} + o(1) \right) \frac{2}{3}(c \log q),$$

where the $\frac{2}{3}$ accounts for the fact that $(\overline{\chi}\xi)(3) = 0$. It follows that for $N \leq q^c$,

$$\left| \sum_{n \leq N} \frac{(\overline{\chi}\xi)(n)}{n} \right| \leq \left(\frac{2}{3} + o(1) \right) \min \left(\log N, \frac{34}{35}(c \log q) + \log \frac{q^c}{N} \right)$$

$$\leq (1 + o(1)) \frac{69}{70} \cdot \frac{2}{3}(c \log q),$$

which completes the proof of the theorem when $m > 1$.

Now consider the case $m = 1$. Here (2.1), Burgess's estimate, and Lemma 6.5 give that $M(\chi) + o(\sqrt{q} \log q)$ is

$$\leq \frac{\sqrt{q}}{\pi} \max_{r, N \leq q^c} \left| \sum_{n \leq q^c} \frac{\overline{\chi}(n)}{n} - \frac{1}{\phi(r)} \prod_{p^a \| r} (\overline{\chi}(p^a) - \overline{\chi}(p^{a-1})) \sum_{n \leq N} \frac{\overline{\chi}(n)}{n} \right|.$$

The estimate claimed in the theorem now follows from Lemma 4.4 and (7.4).

REFERENCES

1. P.T. Bateman and S. Chowla, *Averages of character sums*, Proc. Amer. Math. Soc. **1** (1950), 781-787. MR0042445 (13:113d)
2. D.A. Burgess, *On character sums and L-series, I*, Proc. London Math. Soc. **12** (1962), 193-206; II, Proc. London Math. Soc. **13** (1963), 524-536. MR0132733 (24:A2570); MR0148626 (26:6133)
3. S.D. Chowla, *On the class number of the corpus $P(\sqrt{-k})$*, Proc. Nat. Inst. Sci. India **13** (1947), 197-200. MR0027303 (10:285e)

4. H. Davenport, *Multiplicative number theory*, Springer-Verlag, New York, 1980. MR0606931 (82m:10001)

5. P. Deligne, *Applications de la formule des traces aux sommes trigonométriques,* In: Cohomologie Etale (SGA $4\frac{1}{2}$), Lecture Notes in Math. **569** (1977). MR0463174 (57:3132)

6. A. Granville and K. Soundararajan, *Large Character Sums*, J. Amer. Math. Soc. **14** (2001), 365–397. MR1815216 (2002h:11074)

7. A. Granville and K. Soundararajan, *The Spectrum of Multiplicative Functions*, Ann. of Math. **153** (2001), 407–470. MR1829755 (2002g:11127)

8. A. Granville and K. Soundararajan, *The distribution of values of* $L(1,\chi_d)$, Geom. Funct. Anal. **13** (2003), 992–1028. MR2024414 (2005d:11129)

9. A. Granville and K. Soundararajan, *Upper bounds for* $|L(1,\chi)|$, Quart. J. of Math. **53** (2002), 265–284. MR1930263 (2003h:11104)

10. A. Hildebrand, *Large values of character sums*, J. Number Theory **29** (1988), 271-296. MR0955953 (89k:11073)

11. J.E. Littlewood, *On the class number of the corpus* $P(\sqrt{-k})$, Proc. London Math. Soc. **27** (1928), 358-372.

12. H.L. Montgomery and R.C. Vaughan, *Exponential sums with multiplicative coefficients*, Invent. Math. **43** (1977), 69-82. MR0457371 (56:15579)

13. R.E.A.C. Paley, *A theorem on characters*, J. London Math. Soc. **7** (1932), 28–32.

14. G. Pólya, *Über die Verteilung der quadratischen Reste und Nichtreste*, Göttingen Nachrichten (1918), 21-29.

15. I.M. Vinogradov, *Über die Verteilung der quadratischen Reste und Nichtreste*, J. Soc. Phys. Math. Univ. Permi **2** (1919), 1-14.

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