CORRIGENDUM FOR

PRIME DIVISORS ARE POISSON DISTRIBUTED

ANDREW GRANVILLE

Kevin Ford pointed out that the proof of Theorem 3 of [1] contains several significant mistakes (perhaps I did not check it over carefully enough because it is the most easily proved result in the paper!). We give here a correct proof of a slightly modified version of that Theorem.

Almost all integers up to x have $\sim \log\log x$ distinct prime factors. In our article [1] we examined the distribution of the sizes of those prime factors, and Theorem 3 looked at the integers with far fewer prime factors than is typical. If $S_k(x)$ denotes the set of integers $n \le x$ which have exactly k distinct prime factors then it is well-known that

$$|S_k(x)| = \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!} \ \left\{1 + O\left(\frac{k}{\log\log x}\right)\right\} \tag{1}$$

for $k = o(\log\log x)$. We will denote by $p_1(n) < p_2(n) < \cdots < p_k(n)$ the distinct prime factors of $n \in S_k(x)$. Theorem 3 of [1] shows that the numbers

$$\{\log\log p_i(n)/\log\log n : \quad 1 \le i \le k-1\}$$

are distributed on $(0,1)$ like $k-1$ random numbers, as we vary over $n \in S_k(x)$, for $k = o(\log\log x)$. More precisely we prove the following:

Theorem 3 (of [1]). *Suppose that* $2 \le k = o(\log\log x)$. *Let* $\epsilon \in [1/\log\log x, 1/k)$ *and* $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{k-1} \le \alpha_k = 1$ *where* $\alpha_{j+1} - \alpha_j \ge \epsilon$ *for each* j *and* $\alpha_k - \alpha_{k-1} \ge \epsilon + \log\log\log x/\log\log x$. *Then there are*

$$(k-1)! \ \epsilon^{k-1} \ |S_k(x)| \ \left\{1 + O\left(\frac{1}{\epsilon \log\log x}\right)\right\}$$

integers $n \in S_k(x)$ *for which* $\log\log p_i(n)/\log\log x \in [\alpha_i, \alpha_i + \epsilon)$ *for every* i *in the range* $1 \le i \le k-1$.

L'auteur est partiellement soutenu par une bourse de la CRSNG du Canada. Thanks are due to Kevin Ford for finding my mistake and for reading over this corrigendum.

Key Words: Prime divisors, Local distribution, Permutation, Poisson, Hardy-Ramanujan, factorization.

MSC 2000 subject classifications: Primary 11A51; Secondary 11N60, 11N05, 62E17, 62E20, 11Y05.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Proof. The number of $n \in S_k(x)$ for which $\log\log p_i(n)/\log\log x \in [\alpha_i, \alpha_i+\epsilon)$ for each $1 \le i \le k-1$, equals

$$\sum_{p_1\in I_1, p_2\in I_2,\dots,p_{k-1}\in I_{k-1}} \ \sum_{a_1,a_2,\dots,a_{k-1}\ge 1} \pi^*\left(p_{k-1}; \frac{x}{p_1^{a_1}\dots p_{k-1}^{a_{k-1}}}\right) \tag{2}$$

where $I_j = [\exp((\log x)^{\alpha_j}), \exp((\log x)^{\alpha_j+\epsilon}))$, and $\pi^*(b;T)$ counts the number of prime powers $p^a \le T$ with prime $p > b$. By the prime number theorem we have $\pi^*(b;T) = T/\log T(1+O(1/\log T))$ if $b \le T/\log T$, and we always have $\pi^*(b;T) \le T$.

Since $(\log x)^{\alpha_{j+1}} \ge (\log x)^{\alpha_j+\epsilon} \ge e(\log x)^{\alpha_j}$, we deduce that

$$\log(p_1\dots p_{k-1}) < \sum_{j=1}^{k-1} (\log x)^{\alpha_j+\epsilon} \le (1-1/e)^{-1}(\log x)^{\alpha_{k-1}+\epsilon} < 2\ \frac{\log x}{\log\log x}\ .$$

This implies that in each term of the outer sum in (2) we have $x/(p_1\dots p_{k-1}) > x^{1-o(1)}$ and $p_{k-1} \le x^{o(1)}$. In order to estimate the inner sum in (2) we treat the terms differently depending on whether $p_1^{a_1}\dots p_{k-1}^{a_{k-1}}$ is smaller or larger than x^δ, where $\delta = 100/\log\log x$. The terms with $p_1^{a_1}\dots p_{k-1}^{a_{k-1}} \le x^\delta$ each contribute

$$\frac{x}{\log x}\ \frac{1}{p_1^{a_1}\dots p_{k-1}^{a_{k-1}}}\ \left\{1+O\left(\frac{1}{\log\log x}\right)\right\}. \tag{3}$$

The terms with $p_1^{a_1}\dots p_{k-1}^{a_{k-1}} > x^\delta$ each contribute $\le x^{1-\delta}$ and there are $\ll (\log x)^k/k!$ such terms (since we must have $p_1^{a_1}\dots p_{k-1}^{a_{k-1}} \le x$). Thus their total contribution is $\ll x^{1-\delta/2}$. Therefore summing up (3) over all values of the a_i we find that the inner sum in (2) equals

$$\frac{x}{\log x}\ \frac{1}{(p_1-1)\dots(p_{k-1}-1)}\ \left\{1+O\left(\frac{1}{\log\log x}\right)\right\}. \tag{4}$$

Therefore the sum in (2) becomes, since the intervals I_J are disjoint,

$$\left\{1+O\left(\frac{1}{\log\log x}\right)\right\} \frac{x}{\log x} \prod_{j=1}^{k-1} \sum_{p_j\in I_j} \frac{1}{p_j-1} \tag{5}$$

Now $\sum_{a<p<b} 1/(p-1) = \log\log b - \log\log a + O(1/\log a)$, so that

$$\sum_{p\in I_j} \frac{1}{p-1} = \epsilon\log\log x + O\left(\frac{1}{(\log x)^{\alpha_j}}\right) = \epsilon\log\log x + O\left(\frac{1}{e^j}\right),$$

and hence (5) equals

$$\frac{x}{\log x}\ (\epsilon\log\log x)^{k-1}\left\{1+O\left(\frac{1}{\epsilon\log\log x}\right)\right\}.$$

The result follows by comparing this to (1).

References

1. Granville, A. (2007), *Prime divisors are Poisson distributed*, Int. J. Number Theory **3**, 1–18.

Départment de Mathématiques et Statistique, Université de Montréal, CP 6128 succ Centre-Ville, Montréal, Québec H3C 3J7, Canada.
andrew@dms.umontreal.ca http://www.dms.umontreal.ca/∼andrew/