UPPER BOUNDS FOR $|L(1,\chi)|$

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1. Introduction

Given a non-principal Dirichlet character $\chi \pmod q$, an important problem in number theory is to obtain good estimates for the size of $L(1,\chi)$. The best bounds known give that $q^{-\epsilon} \ll_\epsilon |L(1,\chi)| \ll \log q$, while assuming the Generalized Riemann Hypothesis, J.E. Littlewood showed that $1/\log\log q \ll |L(1,\chi)| \ll \log\log q$. Littlewood's result reflects the true range of the size of $|L(1,\chi)|$ as it is known that there exist characters $\chi_\pm$ for which $L(1,\chi_+) \asymp \log\log q$ and $L(1,\chi_-) \asymp 1/\log\log q$.

In this paper we focus on sharpening the upper bounds known for $|L(1,\chi)|$; in particular, we wish to determine constants c (as small as possible) for which the bound $|L(1,\chi)| \le (c+o(1))\log q$ holds. To set this in context, observe that if X is such that $\sum_{n\le x}\chi(n) = o(x)$ for all $x > X$ then

$$L(1,\chi) = \sum_{n\le X}\frac{\chi(n)}{n} + o(\log q). \tag{1.1}$$

Trivially $X = q$ is permissible and so $|L(1,\chi)| \le (1+o(1))\log q$. Less trivially the Pólya-Vinogradov inequality gives that $X = q^{\frac12+o(1)}$ is permissible. Finally note that D. Burgess' character sums estimates (and Heath-Brown's improvement [5, Lemma 2.4]) permit one to take $X = q^{\frac14+o(1)}$ if q is cube-free or if χ has order $q^{o(1)}$, and $X = q^{\frac13+o(1)}$ otherwise. In particular we get that $|L(1,\chi)| \le (1/4+o(1))\log p$ when $q = p$ is prime. In [1] Burgess improved on this "trivial" bound, for quadratic characters, obtaining that $L(1,(\frac{\cdot}{p})) \le 0.2456\log p$ for all large primes p. This was subsequently improved by P.J. Stephens [8] to $L(1,(\frac{\cdot}{p})) \le (2-2/\sqrt{e}+o(1))\frac14\log p$. This result is best-possible in the sense that one can construct totally multiplicative functions f, taking only values -1 and 1, such that $\sum_{n\le x} f(n) = o(x)$ for all $x > X$, and $\sum_{n\le X} f(n)/n \sim (2-2/\sqrt{e})\log X$. Stephens' result was extended by Pintz [9] to all quadratic characters. No analogous improvements over the trivial bound were known for complex characters χ. We give such a result below.

Corollary. *Define* $c_2 = 2-2/\sqrt{e} = 0.786938\dots$, $c_3 = 4/3-1/e^{2/3} = 0.819916\dots$, $c_4 = 0.8296539741\dots$, *and* $c_k = c_\infty := 34/35$ *for* $k \ge 5$. *For any primitive Dirichlet character* $\chi \pmod q$ *of order* k, *we have*

$$|L(1,\chi)| \le \begin{cases} \frac14(c_k+o(1))\log q & \text{if } q \text{ is cube-free, or } \chi \text{ has order } q^{o(1)},\\ \frac13(c_k+o(1))\log q & \text{otherwise.}\end{cases}$$

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In particular, it had previously been known that for any primitive Dirichlet character $\chi \pmod q$, if q is cube-free or if χ has order $q^{o(1)}$ then $|L(1,\chi)| \le \{1/4 + +o(1)\}\log q$. We have now improved this upper bound to

$$|L(1,\chi)| \le \left\{\frac{17}{70} + o(1)\right\}\log q.$$

We establish the Corollary by focussing more generally on multiplicative functions satisfying a "Burgess-type" condition. Given a subset S of the unit disc $\mathbb{U}$, we define $\mathcal{F}(S)$ to be the class of all completely multiplicative functions f such that $f(p) \in S$ for all primes p. We denote by S_k the set $\{0\} \cup \{\xi : \xi^k = 1\}$. Our problem is to bound $(1/\log X)|\sum_{n\le X} f(n)/n|$ for $f \in \mathcal{F}(S)$ assuming that

$$\sum_{n\le x} f(n) = o(x), \tag{1.2}$$

for suitable $x \ge X$. More precisely, let $A \ge 1$ be a parameter, and define

$$\gamma(S;A) := \limsup_{X\to\infty} \max_{\substack{f\in\mathcal{F}(S)\\ \text{(1.2) holds for } X\le x\le X^A}} \frac{1}{\log X}\Big|\sum_{n\le X}\frac{f(n)}{n}\Big|.$$

The condition "$X \le x \le X^A$" means that $x \to \infty$ as $X \to \infty$, and so the "$o(x)$" in (1.2) makes sense. Note that for fixed S, $\gamma(S;A)$ is a non-increasing function of A. Further if $S_1 \supset S_2$ then $\gamma(S_1;A) \ge \gamma(S_2;A)$ for all $A \ge 1$. Set $\gamma(S) = \lim_{A\to\infty}\gamma(S;A)$.

By (1.1) and Burgess' estimates we see that if χ is a character $\pmod q$ of order k then

$$|L(1,\chi)| \le \begin{cases} \frac14(\gamma(S_k)+o(1))\log q & \text{if } q \text{ is cube-free, or } \chi \text{ has order } q^{o(1)},\\ \frac13(\gamma(S_k)+o(1))\log q & \text{otherwise.}\end{cases}$$

Thus our corollary above follows from our main Theorem which establishes upper bounds on $\gamma(S;A)$.

Theorem 1. *With the definitions as above, $\gamma(S_k;1) \le c_k$ for $k = 2$, 3 and 4. For $k \ge 5$ we have*

$$\gamma(S_k) \le \gamma(S_k;\sqrt{e}) \le \gamma(\mathbb{U};\sqrt{e}) \le c_\infty.$$

It is possible to show that $\gamma(\mathbb{U};1) \le c$ for an absolute constant $c < 1$. This follows from P.D.T.A. Elliott's groundbreaking result [2] that the magnitude of averages of multiplicative functions varies slowly. Precisely, for any $f \in \mathcal{F}(\mathbb{U})$ and $1 \le w \le x$, we have the following Lipschitz-type estimate

$$\frac1x\Big|\sum_{n\le x} f(n)\Big| - \frac{w}{x}\Big|\sum_{n\le x/w} f(n)\Big| \ll \left(\frac{\log 2w}{\log x}\right)^{\frac{1}{19}}.$$

Thus if $\sum_{n\le X} f(n) = o(X)$, then there is some $\delta < 1$ such that for all $X^\delta \le x \le X$ we have $|\sum_{n\le x} f(n)| \le x/2$. Hence

$$\frac{1}{\log X}\Big|\sum_{n\le X}\frac{f(n)}{n}\Big| = \frac{1}{\log X}\Big|\int_1^X \sum_{n\le t} f(n)\frac{dt}{t^2}\Big| + O\left(\frac{1}{\log X}\right)$$
$$\le \frac{1}{\log X}\left(\int_1^{X^\delta}\frac{dt}{t} + \int_{X^\delta}^X \frac{dt}{2t}\right) + o(1) = c + o(1).$$

where $c = (1+\delta)/2$. Elliott's exponent $1/19$ has recently been improved in [4] to any exponent $< 1 - 2/\pi$, and is probably true for any exponent < 1. However it seems that the value of c given by this method is inevitably much closer to 1 than c_∞.

Although we do not go into this here, one can, via Lipschitz-type estimates, improve the upper bound from Theorem 1 for $k = 3$ to $\gamma(S_3; 1) < c_3 - \delta$ for some tiny $\delta > 0$.

By means of a construction we are also able to give lower bounds for $\gamma(S)$.

Theorem 2a. *We have $\gamma(\mathbb{U}) \ge \gamma(S_{2k}) \ge \gamma(S_2) \ge (2 - 2/\sqrt{e})$. If k is odd then $\gamma(S_k) \ge (1+\delta_k)(1 - e^{-1/(1+\delta_k)})$ where $\delta_k = \cos(\pi/k)$.*

Combining with Theorem 1 we get that $\gamma(S_2) = 2 - 2/\sqrt{e}$. It is tempting to conjecture that $\gamma(\mathbb{U}) = 2 - 2/\sqrt{e}$.

Returning to our application to bounding $L(1,\chi)$, we note that we have not exploited all the information on characters available to us. Namely, if χ is a character of order k then χ^j is a non-principal character for $j = 1, 2, \ldots k-1$, so that the Burgess estimates apply to mean values of χ^j as well. Although we have not been able to take advantage of this fact, we can establish some limits on how much it can imply. The problem is to bound $(1/\log X)|\sum_{n\le X} f(n)/n|$ for a given $f \in \mathcal{F}(S_k)$ satisfying

$$\sum_{n\le x} f(n)^j = o(x) \quad \text{for } 1 \le j \le k-1, \tag{1.3}$$

for suitable $x \ge X$. Precisely, for $A \ge 1$ we wish to determine

$$\gamma_k(A) := \limsup_{X\to\infty} \max_{\substack{f\in\mathcal{F}(S_k)\\ (1.3)\text{ holds for } X\le x\le X^A}} \frac{1}{\log X}\Big|\sum_{n\le X}\frac{f(n)}{n}\Big|.$$

Plainly $\gamma_k(A)$ is a decreasing function of A, and $\gamma_k(A) \le \gamma(S_k; A)$. We set $\gamma_k = \lim_{A\to\infty}\gamma_k(A)$. If $\chi \pmod q$ is a character of order k then

$$|L(1,\chi)| \le \begin{cases} \frac{1}{4}(\gamma_k + o(1))\log q & \text{if } q \text{ is cube-free, or } \chi \text{ has order } q^{o(1)}, \\ \frac{1}{3}(\gamma_k + o(1))\log q & \text{otherwise.}\end{cases}$$

Theorem 2b. *For large k we have*

$$\gamma_k \geq (e^\gamma + o_k(1)) \frac{\log\log k}{\log k}.$$

We prove Theorem 2b in Section 6; indeed there we shall establish a more precise lower bound on γ_k, and give numerical data for small k. We suspect that Theorem 2b gives the correct size of γ_k for large k. At any rate, it seems safe to conjecture that $\gamma_k = o_k(1)$, which would imply that for any fixed $\epsilon > 0$ and if k is sufficiently large then $|L(1,\chi)| \leq \epsilon \log q$ for all characters $\chi \pmod q$ of order k.

Our final result obtains an upper bound for $|L(1,\chi)|$ on "average" over all the characters of order k.

Theorem 3. *Suppose that $f \in S_k$ satisfies (1.3) for all $X \leq x \leq X^{\varphi(k)+1}$. Then*

$$\Big\{ \prod_{\substack{1\leq j\leq k-1\\ (j,k)=1}} \frac{1}{\log X}\Big| \sum_{n\leq X} \frac{f(n)^j}{n}\Big| \Big\}^{1/\varphi(k)} \leq \Big\{ \frac{43}{15} e^\gamma + o_k(1) \Big\} \frac{\log\log k}{\log k}.$$

Consequently

$$\Big\{ \prod_{\substack{\chi \pmod q\\ \chi \text{ of order } k}} |L(1,\chi)| \Big\}^{1/\varphi(k)} \leq \begin{cases} \big\{ \frac{43}{60} e^\gamma + o_k(1) \big\} \frac{\log\log k}{\log k} \ \log q & \text{if } q \text{ is cube-free, or } \chi \text{ has order } q^{o(} \\ \big\{ \frac{43}{45} e^\gamma + o_k(1) \big\} \frac{\log\log k}{\log k} \ \log q & \text{otherwise.} \end{cases}$$

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2. Preliminaries

Define $y := \exp((\log X)^{\frac{1}{4}})$. In proving Theorem 1 it is convenient to restrict attention to completely multiplicative functions f satisfying $f(p) = 1$ for all $p \leq y$. We indicate first why this entails no loss in generality.

Lemma 1. *Let f be a multiplicative function with $|f(n)| \leq 1$ for all n. Then*

$$\frac{1}{\log X}\Big| \sum_{n\leq X} \frac{f(n)}{n}\Big| \ll \exp\Big(-\frac{1}{2} \sum_{p\leq X} \frac{1 - \text{Re } f(p)}{p}\Big).$$

Proof. See Proposition 8.1, and the comments following it, in [3].

In proving Theorem 1 we may thus assume that

$$\sum_{p\leq X} \frac{1-\text{Re } f(p)}{p} \ll 1.$$

Since $|1-f(p)|^2 \ll (1 - \text{Re } f(p))$, we deduce by the Cauchy-Schwarz inequality that

$$\sum_{p\leq X} \frac{|1-f(p)|}{p} \leq \Big(\sum_{p\leq X} \frac{1}{p}\Big)^{\frac{1}{2}} \Big(\sum_{p\leq X} \frac{|1-f(p)|^2}{p}\Big)^{\frac{1}{2}} \ll \sqrt{\log\log X}. \tag{2.1}$$

Lemma 2. *Suppose f is a multiplicative function with $|f(n)| \le 1$ for all n, and that f satisfies* (2.1). *Let $f_s(n)$ be the completely multiplicative function defined by $f_s(p) = f(p)$ if $p > y$ and $f_s(p) = 1$ for $p \le y$. Define*

$$\Theta(f,y) = \prod_{p\le y}\left(1-\frac{1}{p}\right)\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\dots\right).$$

Then for all $X^2 \ge x > X$,

$$\frac{1}{x}\sum_{n\le x} f(n) = \Theta(f,y)\frac{1}{x}\sum_{n\le x} f_s(n) + O((\log x)^{-\frac{1}{2}}),$$

and

$$\frac{1}{\log X}\sum_{n\le X}\frac{f(n)}{n} = \Theta(f,y)\frac{1}{\log X}\sum_{n\le X}\frac{f_s(n)}{n} + O((\log X)^{-\frac{1}{4}}).$$

Proof. The first assertion follows from (2.1) and Proposition 4.5 of [3], while the second assertion follows from Proposition 8.2 of [3].

Note that $|\Theta(f,y)| \le 1$ always. If $|\Theta(f,y)| = o(1)$ (that is, as $X\to\infty$ so that $y\to\infty$) then the bound in Theorem 1 is immediate. If $|\Theta(f,y)| \gg 1$ and f meets the hypothesis of Theorem 1, then f_s meets the hypothesis of Theorem 1, and it suffices to demonstrate the conclusion for f_s. Thus Lemma 2 allows us to restrict attention to completely multiplicative functions f with $f(p) = 1$ for all primes $p \le y$, and we suppose this henceforth.

Lemma 3. *Let f be a completely multiplicative function with $f(p) = 1$ for all $p \le y$, and $|f(p)| \le 1$ otherwise, and let g be the completely multiplicative function defined by $g(p) = |1 + f(p)| - 1$. Put $G(u) = \sum_{n\le u} g(n)$. Then for any $y \le u \le X$*

$$\frac{1}{\log X}\Big|\sum_{n\le X}\frac{f(n)}{n}\Big| \le \frac{1}{\log X}\sum_{n\le X}\frac{g(n)}{n} + o(1)$$
$$\le \frac{1}{\log u}\int_1^u \frac{|G(t)|}{t^2}dt + o(1).$$

Proof. Note that

$$\sum_{n\le X}\frac{f(n)}{n} = \frac{1}{X}\sum_{n\le X}\sum_{d|n} f(d) + O(1).$$

Now $|\sum_{d|n} f(d)| \le \sum_{d|n} g(d)$ unless n is divisible by the square of some prime p with $f(p) \ne 1$, so that $p > y$. The contribution of such n is readily bounded by

$$\frac{1}{X}\sum_{p\ge y}\sum_{\substack{n\le X\\ p^2|n}} d(n) \ll \log X\sum_{p>y}\frac{1}{p^2} \ll \frac{\log X}{y}.$$

It follows that

$$\frac{1}{\log X}\Big|\sum_{n\le X}\frac{f(n)}{n}\Big| \le \frac{1}{\log X}\sum_{n\le X}\frac{g(n)}{n} + o(1) = \frac{1}{\log X}\int_1^X \frac{G(t)}{t^2}dt + o(1).$$

From Lemma 2.1 of [4] we know that

$$\frac{|G(u)|}{u} \le \frac{1}{\log u}\int_1^u \frac{|G(t)|}{t^2}dt + O\left(\frac{1}{\log u}\right).$$

From this it follows that

$$\left(\frac{1}{\log u}\int_1^u \frac{|G(t)|}{t^2}dt\right)' = -\frac{1}{u\log^2 u}\int_1^u \frac{|G(t)|}{t^2}dt + \frac{|G(u)|}{u^2\log u} \le O\left(\frac{1}{u\log^2 u}\right).$$

We deduce that

$$\begin{aligned}\frac{1}{\log X}\int_1^X \frac{|G(t)|}{t^2}dt &\le \frac{1}{\log u}\int_1^u \frac{|G(t)|}{t^2}dt + O\left(\int_u^X \frac{1}{t\log^2 t}dt\right)\\ &= \frac{1}{\log u}\int_1^u \frac{|G(t)|}{t^2}dt + O\left(\frac{1}{\log u}\right),\end{aligned}$$

showing that $(1/\log u)\int_1^u |G(t)|/t^2 dt$ is essentially a non-increasing function, and the Lemma follows.

We record the value of an integral that we will encounter several times. For $C > 0$, we have

$$\frac{1}{\log u}\int_{u^{e^{-C}}}^u \left(C - \log\left(\frac{\log u}{\log t}\right)\right)\frac{dt}{t} = C - 1 + e^{-C}. \tag{2.2}$$

Lemma 4. *Let f and g be as in Lemma 3, and put $I(u) = \sum_{p\le u}(1-g(p))/p$. If $I(u)\le 1$ then*

$$\frac{1}{\log u}\int_1^u \frac{|G(t)|}{t^2}dt \le \frac{1}{2} + \frac{1}{2\log u}\int_1^u (1-I(t))(1-I(u/t))\frac{dt}{t} + o(1), \tag{2.3}$$

Useful bounds on the right hand side of (2.3) *are*

$$1 - \frac{I(\sqrt{u})}{2} + o(1), \qquad 1 - \frac{(1-I(\sqrt{u}))}{\log u}\int_1^u I(t)\frac{dt}{t} + o(1), \qquad 1 - \frac{1}{\log u}\int_{\sqrt{u}}^u I(t)\frac{dt}{t} + o(1),$$

and

$$3 - I(u) - 2e^{-I(u)/2} + o(1).$$

Proof. If $f(p)=1$ for every prime p for which p^2 divides n then, by induction on the number of primes dividing n, we see that

$$g(n)\geq 1-\sum_{p|n}(1-g(p)),\qquad \text{and}\qquad g(n)\leq 1-\sum_{p|n}(1-g(p))+\sum_{pq|n}(1-g(p))(1-g(q)).$$

It follows that

$$G(t)\geq \sum_{n\leq t}\Big(1-\sum_{p|n}(1-g(p))\Big)+O\Big(\sum_{p\geq y}\sum_{\substack{n\leq t\\ p^2|n}}1\Big)$$

$$= t-t\sum_{p\leq t}\frac{1-g(p)}{p}+o(t)=t(1-I(t))+o(t), \tag{2.4}$$

and, similarly, that

$$G(t)\leq t(1-I(t))+\frac{t}{2}\sum_{p\leq t}\sum_{q\leq t/p}\frac{1-g(p)}{p}\frac{1-g(q)}{q}+o(t).$$

Thus if $I(u)\leq 1+o(1)$ then $G(t)\geq o(t)$ for all $t\leq u$, and so

$$\int_1^u\frac{|G(t)|}{t^2}dt\leq\int_1^u\Big(1-I(t)+\frac12\sum_{p\leq t}\sum_{q\leq t/p}\frac{1-g(p)}{p}\frac{1-g(q)}{q}\Big)\frac{dt}{t}+o(\log t).$$

Since

$$\int_1^u I(t)\frac{dt}{t}=\frac12\int_1^u(I(t)+I(u/t))\frac{dt}{t},$$

and

$$\int_1^u I(t)I(u/t)\frac{dt}{t}=\int_1^u\sum_{p\leq t}\sum_{q\leq t/p}\frac{1-g(p)}{p}\frac{1-g(q)}{q}\frac{dt}{t}$$

we obtain the upper bound (2.3). Since both $I(t)$ and $I(u/t)$ are in $[0,1]$ and one of them is at least $I(\sqrt{u})$ we immediately obtain our first alternative bound on the RHS of (2.3). Next

$$\begin{aligned}\text{RHS of (2.3)}&=1-\frac{1}{\log u}\int_1^u I(t)\frac{dt}{t}+\frac{1}{2\log u}\int_1^u I(t)I(u/t)\frac{dt}{t}\\ &\leq 1-\frac{1}{\log u}\int_1^u I(t)\frac{dt}{t}+\frac{1}{\log u}\int_{\sqrt{u}}^u I(t)I(\sqrt{u})\frac{dt}{t},\end{aligned}$$

which proves our second alternative bound. Further

$$\begin{aligned}\frac{1}{2\log u}\int_1^u(1-I(t))(1-I(u/t))\frac{dt}{t}&=\frac{1}{\log u}\int_{\sqrt{u}}^u(1-I(t))(1-I(u/t))\frac{dt}{t}\\ &\leq\frac{1}{\log u}\int_{\sqrt{u}}^u(1-I(t))\frac{dt}{t},\end{aligned}$$

which gives our third alternative bound. Lastly note that $1 - I(t) \ge 0$ and also

$$1 - I(t) = 1 - I(u) + \sum_{t \le p \le u} \frac{1 - g(p)}{p} \le 1 - I(u) + 2\log\left(\frac{\log u}{\log t}\right) + o(1),$$

so that using our third alternative bound we get that the RHS of (2.3) is

$$\le \frac{1}{2} + \frac{1}{\log u}\int_{\sqrt{u}}^{u^{e^{-I(u)/2}}} \frac{dt}{t} + \frac{1}{\log u}\int_{u^{e^{-I(u)/2}}}^{u} \left(1 - I(u) + 2\log\left(\frac{\log u}{\log t}\right)\right)\frac{dt}{t} + o(1),$$

from which our final bound follows by (2.2).

Lemma 5. *Assume that $f \in \mathcal{F}(\mathbb{U})$ satisfies* (1.2) *for $X < x \le Y \le X^2$. Define $h(n) = \sum_{ab=n} f(a)\overline{f(b)}$. If $Y \le X$ then $|\sum_{n \le Y} h(n)| \le Y \log Y + O(Y)$. If $X < Y \le X^2$ then*

$$\Big|\sum_{n \le Y} h(n)\Big| \le Y \log \frac{X^2}{Y} + o(Y \log Y).$$

Proof. Since $|h(n)| \le d(n)$ we see that $|\sum_{n \le Y} h(n)| \le \sum_{n \le Y} d(n) = Y \log Y + O(Y)$ which gives the first assertion. Suppose now that $X < Y \le X^2$ and write

$$\sum_{n \le Y} h(n) = \sum_{ab \le Y} f(a)\overline{f(b)} = \left(\sum_{\substack{ab \le Y \\ a \le Y/X}} + \sum_{\substack{ab \le Y \\ b \le Y/X}} + \sum_{\substack{ab \le Y \\ a,b \ge Y/X}} - \sum_{\substack{ab \le Y \\ a,b \le Y/X}}\right) f(a)\overline{f(b)}.$$

If $a \le Y/X$ then $X^2 \ge Y/a > X$, and so by (1.2) the first sum above is

$$\sum_{a \le Y/X} \sum_{b \le Y/a} f(a)\overline{f(b)} = \sum_{a \le Y/X} o(Y/a) = o(Y \log Y).$$

Similarly the second sum above $\sum_{ab \le Y,\ b \le Y/X}$ is also $o(Y \log Y)$. The third term is (in magnitude)

$$\le \sum_{Y/X \le a \le X} \sum_{Y/X \le b \le Y/a} 1 = Y \log \frac{X^2}{Y} + o(Y \log Y).$$

Finally the last term is $O(Y^2/X^2) = o(Y \log Y)$ since $Y \le X^2$. The Lemma follows.

3. Proof of Theorem 1 for $k = 2, 3$ and 4

Throughout this section we shall only assume that (1.2) holds for $x = X$.

3a. Proof of Theorem 1 for $k = 2$.
Suppose $k = 2$ and $f \in \mathcal{F}(\mathcal{S}_2)$. Note here that $g(p) = f(p)$. From (2.4) it follows that $\sum_{n\le X} f(n) = G(X) \ge X(1-I(X)) + o(X)$, so that by (1.2) we get $I(X) \ge 1 + o(1)$. Let u be the largest integer $\le X$ with $I(u) \le 1$. Plainly $u \ge y$ is large. Note that $I(u) = 1 + o(1)$, and so by Lemma 3 and the final bound in Lemma 4 it follows that

$$\frac{1}{\log X}\Big|\sum_{n\le X}\frac{f(n)}{n}\Big| \le 3 - I(u) - 2e^{-I(u)/2} + o(1) = \left(2 - \frac{2}{\sqrt{e}}\right) + o(1).$$

3b. Proof of Theorem 1 for $k = 3$.
Here note that $g(p) = 1$ if $f(p) = 1$, and $g(p) = 0$ if $f(p) \neq 1$. Also note that

$$\text{Re } f(n) \ge 1 - \sum_{p|n}(1 - \text{Re } f(p)) \ge 1 - \frac{3}{2}\sum_{p|n}(1 - g(p)).$$

Hence

$$o(X) = \sum_{n\le X}\text{Re } f(n) \ge X - \frac{3X}{2}\sum_{p\le X}\frac{1-g(p))}{p} + o(X) = X\left\{1 - \frac{3}{2}I(X) + o(1)\right\},$$

so that $I(X) \ge 2/3 + o(1)$. Let u be the largest integer below X with $I(u) \le 2/3$. Note that $u \ge y$ is large and that $I(u) = 2/3 + o(1)$.

For $t \le u$ note that $0 \le 1 - I(t) \le 1$ and that

$$\begin{aligned}1 - I(t) &= \frac{1}{3} + I(u) - I(t) + o(1) = \frac{1}{3} + \sum_{t\le p\le u}\frac{1-g(p)}{p} + o(1) \le \frac{1}{3} + \sum_{t\le p\le u}\frac{1}{p} + o(1)\\ &= \frac{1}{3} + \log\left(\frac{\log u}{\log t}\right) + o(1).\end{aligned}$$

Hence, using (2.2),

$$\begin{aligned}\int_{\sqrt{u}}^{u}(1 - I(t))\frac{dt}{t} &\le \int_{\sqrt{u}}^{u^{e^{-2/3}}}\frac{dt}{t} + \int_{u^{e^{-2/3}}}^{u}\left(\frac{1}{3} + \log\left(\frac{\log u}{\log t}\right)\right)\frac{dt}{t} + o(\log u)\\ &= \left(\frac{5}{6} - \frac{1}{e^{2/3}} + o(1)\right)\log u.\end{aligned}$$

Applying Lemma 3 and the third bound in Lemma 4 we conclude that

$$\frac{1}{\log X}\Big|\sum_{n\le X}\frac{f(n)}{n}\Big| \le \frac{1}{2} + \frac{1}{\log u}\int_{\sqrt{u}}^{u}(1 - I(t))\frac{dt}{t} + o(1) \le \frac{4}{3} - \frac{1}{e^{2/3}} + o(1).$$

3c. Proof of Theorem 1 for $k=4$.
Note that $g(p)=\sqrt{2}-1$ if $f(p)=\pm i$, and $g(p)=f(p)$ if $f(p)=0,\pm 1$. We may assume that $I(X)\le 1$ else, applying Lemmas 3 and 4 (taking u to be the largest integer with $I(u)\le 1$), we deduce that $|\sum_{n\le X} f(n)/n| \le (2-2/\sqrt{e}+o(1))\log X$.

Let

$$A = \sum_{\substack{p\le X\\ f(p)=0,\pm i}} \frac{1}{p}, \qquad \text{and} \qquad B = \sum_{\substack{p\le X\\ f(p)=-1}} \frac{1}{p}.$$

so that $1\ge I(X)\ge (2-\sqrt{2})A+2B$. For $t\le X$ we have

$$\begin{aligned} I(t) &\ge (2-\sqrt{2}) \sum_{\substack{p\le t\\ f(p)=0,\pm i}} \frac{1}{p} + 2 \sum_{\substack{p\le t\\ f(p)=-1}} \frac{1}{p} \\ &\ge (2-\sqrt{2})\Bigg(A - \sum_{\substack{t<p\le X\\ f(p)=0,\pm i}} \frac{1}{p}\Bigg) + 2\Bigg(B - \sum_{\substack{p\le t\\ f(p)=-1}} \frac{1}{p}\Bigg) \\ &\ge \begin{cases} (2-\sqrt{2})A + 2(B-\log(\log X/\log t)) + o(1) & \text{if } X^{e^{-B}} \le t\le X \\ (2-\sqrt{2})(A+B-\log(\log X/\log t)) + o(1) & \text{if } X^{e^{-A-B}} \le t \le X^{e^{-B}}. \end{cases} \end{aligned}$$

Of course $I(t)\ge 0$ for $t\le X^{e^{-A-B}}$. Using these lower bounds and the third bound in Lemma 4 we deduce that

$$\begin{aligned} \frac{1}{\log X}\Bigg|\sum_{n\le X}\frac{f(n)}{n}\Bigg| &\le 1 - \frac{1}{\log X}\int_{\sqrt{X}}^{X} I(t)\frac{dt}{t} + o(1) \\ &\le 1 - \frac{\sqrt{2}}{\log X}\int_{\max\{X^{e^{-B}},X^{1/2}\}}^{X} \left(B - \log\left(\frac{\log X}{\log t}\right)\right)\frac{dt}{t} \\ &\quad - \frac{(2-\sqrt{2})}{\log X}\int_{\max\{X^{e^{-A-B}},X^{1/2}\}}^{X} \left(A+B-\log\left(\frac{\log X}{\log t}\right)\right)\frac{dt}{t} + o(1) \\ &\le F(A,B) + o(1), \end{aligned}$$

say, where (using (2.2) to compute the integrals)

$$F(A,B) = \begin{cases} 3-2B-(2-\sqrt{2})(A+e^{-A-B}) - \sqrt{2}e^{-B} & \text{if } A+B\le \log 2 \\ 2+1/\sqrt{2}-(1-1/\sqrt{2})(A+\log 2) - \sqrt{2}e^{-B} - (1+1/\sqrt{2})B & \text{if } B\le \log 2\le A+B \\ 2-\log 2 - B - (1-1/\sqrt{2})A & \text{if } B\ge \log 2. \end{cases}$$

By differentiation we find that $F(A,B)$ is a non-increasing function of both A and B, for $A,B\ge 0$.

Since $\text{Re}f(n)\ge 1-\sum_{p|n}(1-\text{Re } f(p))$ we have, by (1.2),

$$o(X) = \sum_{n\le X}\text{Re } f(n) \ge X - \sum_{p\le X}(1-\text{Re } f(p))\frac{X}{p} + o(X),$$

so that $A+2B=\sum_{p\le X}(1-\text{Re } f(p))/p \ge 1+o(1)$. Therefore

$$F(A,B) \le \min_{0\le A\le 1} F(A,(1-A)/2)+o(1) = F(A_0,(1-A_0)/2)+o(1) = 0.8296539741\ldots,$$

where $A_0 := 2\log((3-\sqrt{2})/2)+1 \approx 0.5358665582\ldots$.

4. Proof of Theorem 1 for $S = \mathbb{U}$

In this proof we will assume only that (1.2) holds for all x in the interval $X \le x \le X^{\sqrt{e}}$.

We may assume that $I(X)<1$ else by applying Lemmas 3 and 4 (as before) we get that $(1/\log X)|\sum_{n\le X} f(n)/n| \le (2-2/\sqrt{e})+o(1)$. We may also assume that $I(\sqrt{X}) \le \frac{2}{35}$ else, by Lemma 4, $\frac{1}{\log X}|\sum_{n\le X} f(n)/n| \le 1-I(\sqrt{X})/2+o(1) \le \frac{34}{35}+o(1)$.

Define

$$A=\sum_{p\le X}\frac{1-\text{Re } f(p)}{p} \qquad \text{and} \qquad B=\sum_{p\le X}\frac{1-\text{Re } f(p)}{p}\frac{\log(X/p)}{\log X}.$$

The second bound in Lemma 4 gives that

$$\frac{1}{\log X}\Big|\sum_{n\le X}\frac{f(n)}{n}\Big| \le 1-\frac{1-I(\sqrt{X})}{\log X}\int_1^X I(t)\frac{dt}{t}+o(1) \le 1-\frac{33}{35\log X}\int_1^X I(t)dt+o(1)$$

$$= 1-\frac{33}{35}\sum_{p\le X}\frac{1-g(p)}{p}\frac{\log(X/p)}{\log X}+o(1) \le 1-\frac{33B}{70}+o(1), \tag{4.1}$$

where the final inequality holds since $1-g(p)=2-|1+f(p)| \ge (1-\text{Re } f(p))/2$.

Now for $t \le X$ we have

$$\sum_{p\le t}\frac{1-\text{Re } f(p)}{p} = A-\sum_{t\le p\le X}\frac{1-\text{Re } f(p)}{p} \ge A-2\log\left(\frac{\log X}{\log t}\right)+o(1),$$

and also note that $\sum_{p\le t}\frac{1-\text{Re } f(p)}{p} \ge 0$. Hence, using (2.2), we deduce that

$$B=\frac{1}{\log X}\int_1^X \sum_{p\le t}\frac{1-\text{Re } f(p)}{p}\frac{dt}{t} \ge \frac{1}{\log X}\int_{X^{e^{-A/2}}}^X \left(A-2\log\left(\frac{\log X}{\log t}\right)+o(1)\right)\frac{dt}{t}$$
$$= A-2+2e^{-A/2}+o(1) \ge A_0-2+2e^{-A_0/2}+o(1),$$

for all $A \ge A_0 := 2\log(2(\sqrt{e}-1)) = 0.5207901030\ldots$ (since $A-2+2e^{-A/2}$ increases for all $A\ge 0$).

Put $h(n)=\sum_{ab=n} f(a)\overline{f(b)}$, so that $h(n) \ge d(n)(1-\sum_{p^k|n}(1-\text{Re} f(p)))$. Therefore

$$\sum_{n\le Y} h(n) \ge Y\log Y - 2Y\sum_{p\le Y}\frac{1-\text{Re } f(p)}{p}\log\frac{Y}{p}+o(Y\log Y).$$

Combining this with Lemma 5 we deduce that for $X \le Y \le X^{\sqrt{e}}$

$$\sum_{p\le Y} \frac{1-\text{Re } f(p)}{p}\log\frac{Y}{p} \ge \log\frac{Y}{X} + o(\log X).$$

Taking $Y = X^{1+\alpha}$ with $0 \le \alpha \le \sqrt{e}$, and using that $1 - \text{Re } f(p) \le 2$, we deduce that

$$\begin{aligned}\alpha + o(1) &\le \alpha A + B + 2\sum_{X\le p\le X^{1+\alpha}} \frac{\log(X^{1+\alpha}/p)}{p\log X}\\ &= \alpha A + B + 2(1+\alpha)\log(1+\alpha) - 2\alpha + o(1).\end{aligned}$$

If $A \le 1$ then taking $\alpha = e^{(1-A)/2} - 1$ we deduce from the above that $B \ge 2e^{(1-A)/2} + A - 3 + o(1) \ge 2e^{(1-A_0)/2} + A_0 - 3$, for $0 \le A \le A_0$ (since the function here is decreasing for $0 \le A \le 1$).

Either way we deduce that $B \ge 0.062284\ldots$ so that $1 - 33B/70 \le \frac{34}{35}$ which, by (4.1), proves the desired estimate for $\gamma(\mathbb{U})$.

Remark. The constant $34/35 = .9714285714\ldots$ may be replaced by $.9706838406\ldots$ in the above proof.

5. Upper bounds on average: Proof of Theorem 3

Let $k \ge 2$ be an integer, and suppose that $f \in \mathcal{F}(S_k)$ satisfies (1.3) for $X \le x \le X^{\varphi(k)+1}$. As in section 2 we can assume that $f(p) = 1$ for all $p \le y$, without loss of generality. Let $g(n)$ be the multiplicative function defined by

$$g(n) = \sum_{\prod_{(j,k)=1} a_j = n} \prod_{\substack{j=1\\(j,k)=1}}^{k} f(a_j)^j,$$

and $h(n) = \sum_{d|n} g(d)$. We consider

$$\sum_{n\le X^{\varphi(k)+1}} h(n) = \sum_{a_0\prod_{(j,k)=1} a_j \le X^{\varphi(k)+1}} \prod_{(j,k)=1} f(a_j)^j,$$

and distinguish two types of terms: when all the a_j with $(j,k) = 1$ are below X, and when one of them exceeds X. The first type contribute

$$\begin{aligned}&\sum_{\substack{a_j\le X\\(j,k)=1}} \prod_{(j,k)=1} f(a_j)^j \Big(X^{\varphi(k)+1} \prod_{(j,k)=1} \frac{1}{a_j} + O(1)\Big)\\ =&X^{\varphi(k)+1} \prod_{(j,k)=1} \Big(\sum_{n\le X} \frac{f(n)^j}{n}\Big) + O(X^{\varphi(k)})\end{aligned}$$

We next consider the contribution of the second type of terms. Suppose for example that $a_1 > X$ is the largest of the a_j's with $(j,k)=1$. For fixed a_0, a_j ($j \ge 2$ with $(j,k)=1$), we get, from our assumption on the range in which (1.3) holds, that the sum over a_1 is

$$o\bigg(\frac{X^{\varphi(k)+1}}{a_0}\prod_{\substack{j\ge 2\\ (j,k)=1}}\frac{1}{a_j}\bigg).$$

Writing $n = a_0\prod_{j\ge 2,(j,k)=1} a_j$ we see that the contribution of these terms is

$$\begin{aligned} o\bigg(X^{\varphi(k)+1}\sum_{n\le X^{\varphi(k)}}\frac{d_{\varphi(k)}(n)}{n}\bigg) &= o\left(X^{\varphi(k)+1}e^{\varphi(k)}\sum_{n=1}^{\infty}\frac{d_{\varphi(k)}(n)}{n^{1+1/\log X}}\right)\\ &= o\left(X^{\varphi(k)+1}e^{\varphi(k)}\zeta\left(1+\frac{1}{\log X}\right)^{\varphi(k)}\right)\\ &= o(X^{\varphi(k)+1}(\log X)^{\varphi(k)}). \end{aligned} \tag{5.1}$$

The same argument applies when any other a_j is the largest. Thus we conclude that

$$\frac{1}{X^{\varphi(k)+1}}\sum_{n\le X^{\varphi(k)+1}} h(n) = \prod_{(j,k)=1}\sum_{n\le X}\frac{f(n)^j}{n} + o\left((\log X)^{\varphi(k)}\right). \tag{5.2}$$

Since $h(n) = \sum_{d|n} g(d)$ and $|g(n)| \le d_{\varphi(k)}(n)$ we have

$$\frac{1}{X^{\varphi(k)+1}}\sum_{n\le X^{\varphi(k)+1}} h(n) = \sum_{d\le X^{\varphi(k)+1}}\frac{g(d)}{d} + O\bigg(\frac{1}{X^{\varphi(k)+1}}\sum_{n\le X^{\varphi(k)+1}} d_{\varphi(k)}(n)\bigg).$$

Writing $d_{\varphi(k)}(n) = \sum_{a|n} d_{\varphi(k)-1}(a)$, and arguing as in (5.1), we see that

$$\frac{1}{X^{\varphi(k)+1}}\sum_{n\le X^{\varphi(k)+1}} d_{\varphi(k)}(n) \le \sum_{a\le X^{\varphi(k)+1}}\frac{d_{\varphi(k)-1}(a)}{a} \ll (\log X)^{\varphi(k)-1}.$$

These observations and (5.2) give that

$$\begin{aligned} \prod_{(j,k)=1}\bigg|\sum_{n\le X}\frac{f(n)^j}{n}\bigg| &= \bigg|\sum_{d\le X^{\varphi(k)+1}}\frac{g(d)}{d}\bigg| + o\left((\log X)^{\varphi(k)}\right)\\ &\le \sum_{d\le X^{\varphi(k)+1}}\frac{|g(d)|}{d} + o\left((\log X)^{\varphi(k)}\right). \end{aligned}$$

Write $\delta = c/\log X$ for some positive constant $c > 0$ to be fixed later. Then

$$\begin{aligned} \sum_{d\le X^{\varphi(k)+1}}\frac{|g(d)|}{d} &\le e^{c(\varphi(k)+1)}\sum_{d=1}^{\infty}\frac{|g(d)|}{d^{1+\delta}}\\ &\ll e^{c\varphi(k)}\zeta(1+\delta)^{\varphi(k)}\prod_p\left\{1+\frac{|g(p)|}{p^{1+\delta}}+\frac{|g(p^2)|}{(p^2)^{1+\delta}}+\dots\right\}\left(1-\frac{1}{p^{1+\delta}}\right)^{\varphi(k)}\\ &\ll \left\{\frac{e^c}{c}\log X\exp\left(-\sum_p\frac{1-|g(p)|/\varphi(k)}{p^{1+\delta}}\right)\right\}^{\varphi(k)}. \end{aligned}$$

To justify this last step note that the pth term in the Euler product is 1 when $f(p) = 1$, which happens for all primes $p \le y$, and so the error term for the whole Euler product, in the transition from the penultimate bound to the last one, is $\prod_{p>y} \exp(O(\phi(k)^2/p^2)) = 1 + o(1)$. We deduce that

$$\left(\prod_{(j,k)=1} \frac{1}{\log X} \left| \sum_{n \le X} \frac{f(n)^j}{n} \right| \right)^{\frac{1}{\varphi(k)}} \le \frac{e^c}{c} \exp\left(- \sum_p \frac{1 - |g(p)|/\varphi(k)}{p^{1+\delta}} \right) + o_k(1). \tag{5.3}$$

For each prime $p \nmid q$ let l_p be such that $f(p)$ is a primitive l_p-th root of unity. Note that l_p is a divisor of k, and that as j varies over all reduced residue classes $\pmod k$, $f(p)^j$ runs over all primitive l_p-th roots of unity $\varphi(k)/\varphi(l_p)$ times. Thus

$$g(p) = \sum_{(j,k)=1} f(p)^j = (\varphi(k)/\varphi(l_p)) \sum_{\substack{a \pmod{l_p} \\ (a,l_p)=1}} e(a/l_p) = \mu(l_p)\varphi(k)/\varphi(l_p).$$

Define $l_p = k$ if $p|q$, and note that here $g(p) = 0$. From these remarks and (5.3) it follows that

$$\left(\prod_{(j,k)=1} \frac{1}{\log X} \left| \sum_{n \le X} \frac{f(n)^j}{n} \right| \right)^{\frac{1}{\varphi(k)}} \le \frac{e^c}{c} \exp\left(- \sum_{p \le X} \frac{1 - 1/\varphi(l_p)}{p^{1+\delta}} \right) + o_k(1). \tag{5.4}$$

To estimate the right hand side of (5.4) we employ the following result of Hildebrand [6] together with an idea of Vinogradov as exploited by Norton [7].

Lemma 6. *Fix $\theta > 0$. We have*

$$\lim_{x \to \infty} \inf \frac{1}{x} \sum_{\substack{n \le x \\ (n,P)=1}} 1 = \rho(e^\theta),$$

where inf *is taken over all subsets P of the primes up to x, such that $\sum_{p \in P} 1/p = \theta + o(1)$. Here $\rho(u)$ is the Dickman-de Bruijn function, defined by $\rho(u) = 1$ for $0 \le u \le 1$, and $u\rho'(u) = -\rho(u-1)$ for all $u \ge 1$. The lower bound is attained when P is the set of primes in $[x^{e^{-\theta}}, x]$.*

Let l be a divisor of k with $l < k$, and let P_l denote the product of those primes p below X for which $l_p \nmid l$. Observe that if $p \nmid P_l$ then $f^l(p) = 1$. Note that

$$\sum_{\substack{n \le X \\ f^l(n)=1}} 1 = \sum_{n \le X} \frac{l}{k} \sum_{v=1}^{k/l} f^{lv}(n) = \left(\frac{l}{k} + o(1) \right) X,$$

using (1.4) when $1 \le v < k/l$. On the other hand

$$\sum_{\substack{n\le X\\ f^l(n)=1}} 1 \ge \sum_{\substack{n\le X\\ (n,P_l)=1}} 1 \ge \rho\Big(\exp\Big(\sum_{p|P_l}\frac{1}{p}\Big)\Big)X + o(X)$$

by Lemma 6, so that

$$\exp\Big(\sum_{\substack{p\le X\\ l_p\nmid l}}\frac{1}{p}\Big) \ge \rho^{-1}(l/k) := \theta^{-1} \sim \frac{\log k}{\log\log k} \quad \text{if } l = k^{o(1)}, \tag{5.5}$$

since $\rho(u) = e^{-u\log u(1+o(1))}$. Here given $x \in [0,1)$, $\rho^{-1}(x)$ denotes the unique u with $\rho(u) = x$.

If $k = \prod p^{\alpha_p}$ then take $l = \prod p^{\beta_p}$ where $\beta_p = [\min\{\alpha_p, \log\log k/2\log p\}]$. Note that $p^{\beta_p} \le \sqrt{\log k}$ so that $l \le \prod_{p\le\sqrt{\log k}}\sqrt{\log k} \le \exp((\log k)^{\frac{2}{3}})$. Further if $l_p \nmid l$ then l_p is divisible by a prime power larger than $\sqrt{\log k}$ (as l_p divides k), and so $\phi(l_p) \ge \sqrt{\log k}/2$. From these remarks and (5.5) we see that

$$\begin{aligned}\exp\Big(-\sum_{p\le X}\frac{1-1/\varphi(l_p)}{p^{1+\delta}}\Big) &\le \exp\Big(-(1+O((\log k)^{-\frac{1}{2}}))\sum_{\substack{p\le X\\ l_p\nmid l}}\frac{1}{p^{1+\delta}}\Big)\\ &\le \{1+o_k(1)\}\exp\Big(-\sum_{X^\theta<p\le X}\frac{1}{p^{1+\delta}}\Big)\\ &\le \{1+o_k(1)\}\frac{\log\log k}{\log k}\exp\Big(\int_0^1\Big(\frac{1-e^{-ct}}{t}\Big)dt\Big).\end{aligned}$$

Combining this with (5.4) gives

$$\Big(\prod_{(j,k)=1}\frac{1}{\log X}\Big|\sum_{n\le X}\frac{f(n)^j}{n}\Big|\Big)^{\frac{1}{\varphi(k)}} \le \{1+o_k(1)\}\frac{\log\log k}{\log k}\exp\Big(c-\log c+\int_0^1\Big(\frac{1-e^{-ct}}{t}\Big)dt\Big).$$

The left side is minimized at $c = 0.5671432904\ldots$ (that is, where $e^{-c} = c$), giving an upper bound $< 2.8661e^\gamma\log\log k/\log k < (43/15)e^\gamma\log\log k/\log k$, proving Theorem 3.

6. Integral equations: Proofs of Theorems 2a and 2b

Proposition 1 of [3] shows how problems concerning the distribution of multiplicative functions (of absolute values ≤ 1), and problems concerning certain integral equations are essentially equivalent. In particular, our questions on $\gamma(S)$ and γ_k may be reformulated in terms of integral equations. Regarding the proofs of Theorems 1 and 3, this confers only a marginal advantage and so we did not pursue this approach in those contexts. However the integral equations approach considerably simplifies the treatment of the lower bounds for $\gamma(S_k)$ and γ_k claimed in Theorems 2a and 2b. We begin by recapitulating the relevant material from [3].

For a given closed, subset S of the unit disc, let $K(S)$ denote the class of measurable functions $\chi : [0,\infty) \to S^*$ (the convex hull of S) with $\chi(t) = 1$ for $0 \le t \le 1$. There is a unique (continuous) $\sigma : [0,\infty) \to \mathbb{U}$ satisfying

$$u\sigma(u) = \int_0^u \sigma(u-t)\chi(t)dt \quad \text{for} \quad u > 1, \tag{6.1}$$
$$\text{with the initial condition} \quad \sigma(u) = 1 \quad \text{for} \quad 0 \le u \le 1.$$

Define $\Lambda(S)$ to be the set of such values $\sigma(u)$.

Proposition 1. *Let f be a multiplicative function with $|f(n)| \le 1$ for all n, and $f(n) = 1$ for $n \le y$. Let $\vartheta(x) = \sum_{p\le x} \log p$ and define*

$$\chi(u) = \chi_f(u) = \frac{1}{\vartheta(y^u)} \sum_{p \le y^u} f(p) \log p.$$

Then $\chi(t)$ is a measurable function taking values in the unit disc with $\chi(t) = 1$ for $t \le 1$, and $\sigma(u)$, the corresponding unique solution to (6.1), satisfies

$$\frac{1}{y^u} \sum_{n \le y^u} f(n) = \sigma(u) + O\left(\frac{u}{\log y}\right).$$

The converse to Proposition 1 is also true.

Proposition 1 (Converse). *Let $S \subset \mathbb{U}$ and $\chi \in K(S)$ be given. Given $\epsilon > 0$ and $u \ge 1$ there exist arbitrarily large y and $f \in \mathcal{F}(S)$ with $f(n) = 1$ for $n \le y$ and*

$$\left| \chi(t) - \frac{1}{\vartheta(y^t)} \sum_{p \le y^t} f(p) \log p \right| \le \epsilon \quad \textit{for almost all } 0 \le t \le u.$$

If $\sigma(u)$ is the solution to (6.1) for this χ then

$$\sigma(t) = \frac{1}{y^t} \sum_{n \le y^t} f(n) + O(u^\epsilon - 1) + O\left(\frac{u}{\log y}\right) \quad \textit{for all } t \le u.$$

Theorems 2a and 2b will follow from the following result on integral equations.

Proposition 2. *For $0 \le \delta \le 1$ define $\chi_\delta(t) = 1$ for $0 \le t \le 1$, and $\chi_\delta(t) = -\delta$ for $t \ge 1$. Let σ_δ denote the corresponding solution in (6.1). We have $\sigma_\delta(u) = 1 - (1+\delta)\log u$ for $1 \le u \le 2$.*

(i) For $0 < \delta \le 1$ there exists a positive real root of $\sigma_\delta(u) = 0$. If U_δ is the smallest such root then U_δ is a decreasing function of δ with $U_\delta = e^{1/(1+\delta)}$ when $1/\log 2 - 1 \le \delta \le 1$, and $U_\delta \sim \log(1/e^\gamma \delta)/\log\log(1/e^\gamma \delta)$ as $\delta \to 0$.

(ii) There exists χ *with* $\chi(t) = -\delta$ *for* $1 \le t \le U_\delta$ *and* $\chi(t) \in [-\delta, 1]$ *for* $t > U_\delta$*, such that* $\sigma(u) = \sigma_\delta(u)$ *for* $0 \le u \le U_\delta$*, and* $\sigma(u) = 0$ *when* $u \ge U_\delta$*. The function* $\chi(t)$ *is continuous for all* $t > U_\delta$*. As* $\delta \to 0^+$ *we have that*

$$I_\delta := \frac{1}{U_\delta}\int_0^{U_\delta} \sigma(t)dt = \frac{e^\gamma}{U_\delta} + O(\delta).$$

Assuming Proposition 2 for the moment, we complete the proofs of Theorems 2a, b.

Proof of Theorem 2a. Taking $\delta = 1$ in Proposition 2 we know that there is a $\chi \in K(S_2)$ such that $\sigma(u) = 1$ for $u \le 1$, $\sigma(u) = 1 - 2\log u$ for $1 \le u \le \sqrt{e}$ and $\sigma(u) = 0$ for $u \ge \sqrt{e}$. By Proposition 1 (Converse) we can find $f \in \mathcal{F}(S_2)$ such that for $0 \le t \le A\sqrt{e}$,

$$\frac{1}{y^t}\sum_{n \le y^t} f(n) = \sigma(t) + O(\epsilon A + A/\log y).$$

Thus with $X = y^{\sqrt{e}}$ we find that f satisfies (1.2) in the range $X \le x \le X^A$, and further

$$\frac{1}{\log X}\sum_{n\le X}\frac{f(n)}{n} = \frac{1}{\sqrt{e}}\int_0^{\sqrt{e}}\frac{1}{y^t}\sum_{n\le y^t} f(n)dt + o(1) = 2 - \frac{2}{\sqrt{e}} + O(\epsilon A + A/\log y) + o(1).$$

From this it follows that $\gamma(S_2; A) \ge 2 - 2/\sqrt{e}$, and since A is arbitrary the same holds for $\gamma(S_2)$.

If $k \ge 3$ is odd then we apply Proposition 2 with $\delta = \delta_k (\ge 1/2)$. We thus find $\chi \in K([-\delta_k, 1]) \subset K(S_k)$ such that $\sigma(u) = 1 - (1+\delta_k)\log u$ for $1 \le u \le U_\delta = e^{1/(1+\delta_k)}$, and $\sigma(u) = 0$ for $u \ge U_\delta$. We now argue exactly as above, constructing f via Proposition 1 (converse), and noting that $(1/U_\delta)\int_0^{U_\delta}\sigma(t)dt = (1+\delta)(1 - e^{-1/(1+\delta)})$. This proves Theorem 2a.

Proof of Theorem 2b. Here we apply Proposition 2 with $\delta = 1/(k-1)$. So there is a $\chi \in K([-1/(k-1), 1])$ such that $\sigma(u) = 0$ for $u \ge U_\delta$. Write $\chi(t) = 1 - k\alpha(t)$ so that $0 \le \alpha(t) \le 1/(k-1)$ for all t; and note that $\alpha(t)$ is piecewise linear for $t \le U_\delta$ and continuous for $t > U_\delta$. For fixed $A \ge 1$, $\epsilon > 0$ and large y we may easily partition the set of primes below y^{AU_δ} as $\mathcal{P}_0 \cup \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_{k-1}$ such that for $0 \le t \le AU_\delta$ we have

$$\frac{1}{y^t}\sum_{\substack{p\le y^t\\ p\in\mathcal{P}_0}} \log p = 1 - (k-1)\alpha(t) + O(\epsilon),$$

and for $j = 1, 2, \ldots, k-1$ we have

$$\frac{1}{y^t}\sum_{\substack{p\le y^t\\ p\in\mathcal{P}_j}} \log p = \alpha(t) + O(\epsilon).$$

We now choose $f \in \mathcal{F}(S_k)$ by setting $f(p) = e(\ell/k)$ if $p \in \mathcal{P}_\ell$ for $\ell = 0, \ldots, k$. For such f we get that for $j = 1, \ldots, k-1$, and $t \le AU_\delta$,

$$\frac{1}{y^t} \sum_{p \le y^t} f(p)^j \log p = 1 - k\alpha(t) + O(k\epsilon) = \chi(t) + O(k\epsilon),$$

so that by Proposition 1 (Converse) we may conclude that

$$\frac{1}{y^t} \sum_{n \le y^t} f(n)^j = \sigma(t) + O(AkU_\delta \epsilon + AU_\delta/\log y).$$

Thus with $X = y^{U_\delta}$ we find that (1.3) holds in the range $X \le x \le X^A$, and further that

$$\frac{1}{\log X} \sum_{n \le X} \frac{f(n)^j}{n} = \frac{1}{U_\delta} \int_0^{U_\delta} \sigma(t) dt + o(1) = I_\delta + o(1).$$

This gives $\gamma_k(A) \ge I_\delta$ and, since A is arbitrary, Theorem 2b follows.

To prove Proposition 2 we require Lemma 3.4 from [3] which we quote below.

Lemma 7. *Let χ and $\hat{\chi}$ be two elements of $K(\mathbb{U})$, and let σ and $\hat{\sigma}$ be the corresponding solutions to* (6.1). *Then $\sigma(u)$ equals*

$$\hat{\sigma}(u) + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \int_{\substack{t_1, \ldots, t_j \ge 1 \\ t_1 + \ldots + t_j \le u}} \frac{\hat{\chi}(t_1) - \chi(t_1)}{t_1} \cdots \frac{\hat{\chi}(t_j) - \chi(t_j)}{t_j} \hat{\sigma}(u - t_1 - \ldots - t_j) dt_1 \ldots dt_j.$$

Proof of Proposition 2. First we apply Lemma 7 taking $\hat{\chi}(t) = 1$ for $t \le 1$ and $\hat{\chi}(t) = 0$ for $t > 1$ (so that $\hat{\sigma}(u) = \rho(u)$), and $\chi(t) = 1$ for all $t \ge 0$ (so that $\sigma(u) = 1$). This gives that

$$1 = \rho(u) + \sum_{j=1}^{\infty} \frac{1}{j!} \int_{\substack{t_1, \ldots, t_j \ge 1 \\ t+1 + \ldots + t_j \le u}} \rho(u - t_1 - \ldots - t_j) \frac{dt_1 dt_2 \ldots dt_j}{t_1 t_2 \ldots t_j},$$

and thus

$$0 \le \sum_{j=2}^{\infty} \frac{1}{j!} \int_{\substack{t_1, \ldots, t_j \ge 1 \\ t+1 + \ldots + t_j \le u}} \rho(u - t_1 - \ldots - t_j) \frac{dt_1 dt_2 \ldots dt_j}{t_1 t_2 \ldots t_j} \le 1. \tag{6.2}$$

Next we apply Lemma 7 taking $\hat{\chi}$ as above, and $\chi(t) = \chi_\delta(t)$. Hence

$$\sigma_\delta(u) = \rho(u) + \sum_{j=1}^{\infty} \frac{(-\delta)^j}{j!} \int_{\substack{t_1, \ldots, t_j \ge 1 \\ t_1 + \ldots + t_j \le u}} \rho(u - t_1 - \ldots - t_j) \frac{dt_1 dt_2 \ldots dt_j}{t_1 t_2 \ldots t_j}, \tag{6.3}$$

and in view of (6.2) we may conclude that

$$\left| \sigma_\delta(u) - \left(\rho(u) - \delta \int_1^u \rho(u-t) \frac{dt}{t} \right) \right| \le \delta^2. \tag{6.4}$$

If δ is sufficiently small then the asymptotics $\rho(u) = e^{-u\log u(1+o(1))}$, and $\int_1^u \rho(u-t)dt/t \sim (1/u)\int_0^\infty \rho(t)dt \sim e^\gamma/u$ enable us to deduce that $U_\delta \sim \log(1/e^\gamma\delta)/\log\log(1/e^\gamma\delta)$. Thus for sufficiently small δ we have established the existence of a positive real root of $\sigma_\delta(u) = 0$, and the asymptotic for U_δ claimed in part (i).

Suppose as above that δ is sufficiently small so that U_δ exists, and let $1 \geq \eta \geq \delta > 0$. Take $\chi = \chi_\delta$ and $\hat{\chi} = \chi_\eta$ in Lemma 7. Evaluating at $u = U_\delta$ we conclude that

$$0 = \sigma_\delta(U_\delta) = \sigma_\eta(U_\delta) + \sum_{j=1}^\infty \frac{(\eta-\delta)^j}{j!} \int_{\substack{t_1,\dots,t_j\geq 1\\ t_1+\dots+t_j\leq u}} \sigma_\eta(u-t_1-\dots-t_j)\frac{dt_1dt_2\dots dt_j}{t_1t_2\dots t_j}.$$

It follows at once that $\sigma_\eta(u)$ must change sign (and hence have a zero) in $(0, U_\delta]$. Thus U_η exists for all $\eta \in (0,1]$ and is a decreasing function of η.

Lastly note that for $1 \leq u \leq 2$ we have $\sigma_\delta(u) = 1-(1+\delta)\log u$, from which it follows that in the range $(0.4426\ldots=)1/\log 2 - 1 \leq \delta \leq 1$ we have $U_\delta = e^{1/(1+\delta)}$. This completes the proof of part (i).

We now turn to the proof of part (ii). First observe that $\sigma_\delta(u)$ satisfies the differential-difference equation $u\sigma_\delta'(u) = -(1+\delta)\sigma_\delta(u-1)$, for $u \geq 1$. Since $\sigma_\delta(u)$ is positive for $0 \leq u < U_\delta$ we conclude that $\sigma_\delta'(u) < 0$ for $1 \leq u < U_\delta + 1$. Further observe that

$$1 = \sigma_\delta(1) - \sigma_\delta(U_\delta) = \int_1^{U_\delta} (-\sigma_\delta'(t))dt. \tag{6.5}$$

We now take $\chi(t) = 1$ for $t \leq 1$, $\chi(t) = -\delta$ for $1 \leq t \leq U_\delta$, and for $t > U_\delta$ we define

$$\chi(t) = \int_1^{U_\delta} (-\sigma_\delta'(v))\chi(t-v)dv.$$

From this definition it is clear that χ is continuous for $t > U_\delta$. We shall first show that $\chi(t) \in [-\delta, 1]$ for all t. Plainly this holds for all $t \leq U_\delta$. From our definition of χ, the positivity of $-\sigma_\delta'(t)$ in $(1, U_\delta)$, and (6.5) we immediately glean that

$$\chi(t) \leq \max_{z\in[t-1,t-U_\delta]} \chi(z), \qquad \text{and} \qquad \chi(t) \geq \min_{z\in[t-1,t-U_\delta]} \chi(z).$$

Inductively it follows that $\chi(t) \in [-\delta, 1]$ for all t as desired.

Next we demonstrate that if $\sigma(u)$ denotes the solution to (6.1) for χ constructed above, then $\sigma(u) = 0$ for $u \geq U_\delta$. Note that for $u \leq U_\delta$ we have $\sigma(u) = \sigma_\delta(u)$, and so in particular $\sigma(U_\delta) = \sigma_\delta(U_\delta) = 0$. Now, since $\sigma'(t) = 0$ for $0 < t < 1$, we have for $u \geq U_\delta$,

$$\frac{d}{du}\int_{u-U_\delta}^u \sigma(u-t)\chi(t)dt = \int_{u-U_\delta}^u \sigma'(u-t)\chi(t)dt + \chi(u)\sigma(0) - \sigma(U_\delta)\chi(u-U_\delta) = 0$$

by definition of $\chi(u)$. Therefore $\int_{u-U_\delta}^u \sigma(u-t)\chi(t)dt$ is a constant for $u \geq U_\delta$, and at $u = U_\delta$ equals, by definition, $U_\delta\sigma(U_\delta) = 0$. Hence for $u \geq U_\delta$,

$$u\sigma(u) = \int_0^u \sigma(u-t)\chi(t)dt = \int_0^{u-U_\delta} \sigma(u-t)\chi(t)dt = \int_{U_\delta}^u \sigma(v)\chi(u-v)dv.$$

We claim that this gives $\sigma(u) = 0$ for all $u \geq U_\delta$. If not, select $u > U_\delta$ such that $|\sigma(u)| > 0$ and such that $|\sigma(u)| \geq |\sigma(v)|$ for all $v \in [U_\delta, u]$; then

$$u|\sigma(u)| \leq \int_{U_\delta}^{u} |\sigma(v)\chi(u-v)|dv \leq |\sigma(u)| \int_{U_\delta}^{u} dv = (u - U_\delta)|\sigma(u)|,$$

giving a contradiction.

We have thus constructed χ and σ as desired. For $u \leq U_\delta$ we have $\sigma(u) = \sigma_\delta(u) = \rho(u) + O(\delta/u + \delta^2)$ by (6.4), and so

$$\frac{1}{U_\delta} \int_0^{U_\delta} \sigma(t)dt = \frac{1}{U_\delta} \int_0^{U_\delta} \Big(\rho(t) + O\Big(\frac{\delta}{(t+1)} + \delta^2\Big)\Big)dt = \frac{e^\gamma}{U_\delta} + O(\rho(U_\delta) + \delta),$$

which completes the proof of Proposition 2.

We conclude this section with some numerical data pertaining to Proposition 2. We noted earlier that in the range $0.44269\ldots = 1/\log 2 - 1 \leq \delta \leq 1$ we have $U_\delta = e^{1/(1+\delta)}$ lying in the range $[1, 2]$.When $2 \leq u \leq 3$ we have

$$\sigma_\delta(u) = 1 - (1+\delta)\log u + \frac{(1+\delta)^2}{2} \int_1^{u-1} \frac{\log(u-t)}{t} dt.$$

From this we find that $U_{.061129446\ldots} = 3$, and therefore $2 \leq U_\delta \leq 3$ for $.061129446\ldots \leq \delta \leq .442695041\ldots$. Using Maple VI we computed, for each $u = 2, 2.1, \ldots, 2.9, 3$, the value of δ for which $U_\delta = u$, and then the value of $I_\delta := (1/U_\delta) \int_0^{U_\delta} \sigma_\delta(t)dt$:

$u = U_\delta$	δ	I_δ
$e^{1/2}$	1	.786938680
1.7	.884558536	.775994691
1.8	.701297528	.756132235
1.9	.557986983	.737993834
2.0	.442695041	.721347520
2.1	.353609191	.704809423
2.2	.286811221	.687757393
2.3	.234862762	.670734398
2.4	.193426306	.653994521
2.5	.159779207	.637653381
2.6	.132117433	.621755226
2.7	.109195664	.606305666
2.8	.090126952	.591288678
2.9	.074264622	.576675773
3.0	.061129446	.562431034

Using Maple, for $3 \leq k \leq 17$, we give the lower bounds on γ_k (and $\gamma(S_k)$) that arise from the above proof.

k	δ	U_δ	$\gamma_k \geq I_\delta =$	$\gamma(S_k) \geq$
4	.3333333333	2.127612763	.7002748427	.
5	.2500000000	2.268355860	.6773393732	.7682091384
6	.2000000000	2.382637377	.6601481027	.
7	.1666666667	2.477839089	.6471915206	.7776179102
8	.1428571429	2.558879516	.6372773420	.
9	.1250000000	2.629113171	.6295761905	.7813572891
10	.1111111111	2.690898725	.6235174605	.
11	.1000000000	2.745943649	.6187030892	.7832215162
12	.09090909091	2.795516633	.6148498476	.
13	.08333333333	2.840582242	.6117521137	.7842851149
14	.07692307692	2.881888814	.6092577703	.
15	.07142857143	2.920027494	.6072523556	.7849492382
16	.06666666667	2.955472829	.6056484342	.
17	.06250000000	2.988611474	.6043783304	.7853917172

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