ON A CLASS OF DETERMINANTS

Andrew Granville
Queen's University, Kingston, Ontario, Canada, K7L 3N6
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Recently,* D. H. Lehmer posed the following problem:

> If c_n is the coefficient of x^n in $(1 + x + x^2)^n$, then show that 2^n is the determinant of the matrix
>
> $$M_n = \begin{bmatrix} c_0 c_1 & \cdots & c_n \\ c_1 c_2 & \cdots & c_{n+1} \\ \vdots & & \\ c_n & \cdots & c_{2n} \end{bmatrix}.$$

He noted that the generating function for the c_n's is

$$(1 - 2x - 3x^2)^{-1/2} = 1 + x + 3x^2 + 7x^3 + 19x^4 + \cdots .$$

One might equally ask about the value of the same determinant where the c_n's are the coefficients of x^n in $(a + bx + cx^2)^n$ [note that these c_n's have generating function $(1 - 2bx + dx^2)^{-1/2}$, where $d = b^2 - 4ac$]; or perhaps where the c_n's are the coefficients of x^{n+r} in $(a + bx + cx^2)^n$ for some fixed integer r.

As an example, consider the case where the c_n's are the coefficients of x^{n+r} in $(1 + 2x + x^2)^n = (1 + x)^{2n}$, that is,

$$c_n = \begin{bmatrix} 2n \\ n + r \end{bmatrix}.$$

There does not seem to be an immediate combinatorial argument for finding the determinant even in this case.

In this paper we will answer all of these questions in a very simple way, by easy manipulations of the defining polynomials of the c_n's. We make the following definitions:

Let S be the set of sequences of polynomials $F = [F_n(x)]_{n \geq 0}$ such that each $F_n(x)$ has degree less than or equal to $2n$, and such that $F_n(x)/x^n$ is symmetric (about x^0). [Clearly $F_n(x) = (1 + x + x^2)^n$ and $F_n(x) = (1 + x)^{2n}$ are examples of such sequences.] We define the "elementary sequence" of S to be

$$I = [I_n(x)]_{n \geq 0},$$

where $I_0(x) = 1$ and $I_n(x) = x^{2n} + 1$ for each $n \geq 1$.

Suppose $F, G \in S$ and r is a fixed integer. For each integer $n \geq 0$, let $A_n(F, G)$ be the $(n + 1)$ by $(n + 1)$ matrix with $(i, j)^{\text{th}}$ entry

$$F_i(x)/x^i \cdot G_j(x)/x^j \quad (\text{for } 0 \leq i,\ j \leq n).$$

For any matrix A with entries in $\mathbb{Z}[x]$, we define $c_r(A)$ to be the matrix formed from A by replacing each entry with the coefficient of x^r. We let $D_r(A)$ be the determinant of $c_r(A)$.

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Finally, we let $B_n(F)$ be the $(n+1)$ by $(n+1)$ matrix with $(i, j)^{\text{th}}$ entry $b_{i,j}$ $(0 \le i, j \le n)$, where

$$F_i(x)/x^i = b_{i,0} + \sum_{j=1}^{i} b_{i,j}(x^j + x^{-j}).$$

We will see that the value $D_r[A_n(F, G)]$ is easily computed in terms of the determinants of $B_n(F)$, $B_n(G)$, and $D_r[A_n(I, I)]$.

Lemma 1: Suppose that A, U, and V are $n \times n$ matrices, where A has entries from $\mathbb{C}[x]$ and U and V from $\mathbb{C}$. Then, for any integer r,

$$c_r(UAV) = Uc_r(A)V.$$

The proof of this lemma follows immediately from the observation that, if $a(x)$, $b(x) \in \mathbb{C}[x]$ and α, $\beta \in \mathbb{C}$, then α times the coefficient of x^r in $a(x)$ plus β times the coefficient of x^r in $b(x)$ equals the coefficient of x^r in $\alpha a(x) + \beta b(x)$.

We also make the following trivial observation

Lemma 2: If F, $G \in S$, then for any positive integer n,

$$A_n(F, G) = B_n(F)A_n(I, I)B_n(G)^{\mathrm{T}}.$$

Combining Lemmas 1 and 2, we observe

Corollary 1: If F, $G \in S$ and r is a given integer, then

$$D_r[A_n(F, G)] = D_r[A_n(I, I)] \cdot \mathrm{Det}[B_n(F)] \cdot \mathrm{Det}[B_n(G)].$$

Observing that, by definition, $B_n(F)$ is a lower triangular matrix with diagonal entries $F_m(0)$, $0 \le m \le n$, we have

Lemma 3: If $F \in S$, then $\mathrm{Det}[B_n(F)] = \prod_{m=0}^{n} F_m(0)$.

We now compute the values of $D_r[A_n(I, I)]$.

Lemma 4: For integers r and n with $n \ge 0$, we have

$$D_r[A_n(I, I)] = \begin{cases} 2^n & \text{if } r = 0 \\ (-1)^{[(n+1)/2]} & \text{if } r \ne 0 \text{ and } 2r \text{ divides } n+1 \text{ or } n+r, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: $c_r[A_n(I, I)]$ has $(i, j)^{\text{th}}$ entry equal to the coefficient of x^r in $(x^i + x^{-i})(x^j + x^{-j})$ for $i, j \ge 1$. Thus,

$$c_r[A_n(I, I)] = c_{-r}[A_n(I, I)],$$

so we will assume henceforth that $r \ge 0$. Now, if $r = 0$,

$$[c_0(A_n(I, I))]_{i,j} = \begin{cases} 1 & i = j = 0, \\ 2 & i = j > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and so it is clear that $D_0[A_n(I, I)] = 2^n$.

Let $X = c_r[A_n(I, I)]$ and $D_n = D_r[A_n(I, I)]$. For $r \ge 0$,

$$(X)_{i,j} = \begin{cases} 1 & i + j = r, \\ 1 & |i - j| = r, \\ 0 & \text{otherwise.} \end{cases}$$

We will prove the result for fixed r by induction on n.

Now if $0 \le n \le r - 1$, then all entries of the top row of X are zero, and so $D_n = 0$. If $n = r$, then X has ones on the reverse diagonal and zeros everywhere else, so that

$$D_n = (-1)^{[(n+1)/2]}$$

For $r + 1 \le n \le 2r - 2$, observe that the $r - 1^{st}$ and $r + 1^{st}$ rows of X are both $(0, 1, 0, \ldots, 0)$ so that $D_n = 0$.

Now let K_r be the $2r$ by $2r$ matrix with $r \times r$ block structure

$$\left[\begin{array}{c|c} O_r & I_r \\ \hline I_r & O_r \end{array}\right]$$

so that Det $K_r = (-1)^r$.

If $n = 2r - 1$, then the i^{th} row of x has all zero entries except for ones in columns $r - i$ and $r + i$ if $i \le r - 1$, and in column $i - r$ if $i \ge r$. We subtract row $r + i$ from row $r - i$ for $i = 1, 2, \ldots, r - 1$, which are all determinant-preserving operations and get the matrix K_r. Thus,

$$D_n = \text{Det } K_r = (-1)^{(n+1)/2}.$$

Now suppose $n \ge 2r$. If $i \ge n - r + 1$, then row i has just one nonzero entry (in column $j = i - r$) and so we can subtract this row from all other rows with entries in the $(i - r)^{th}$ column. (This is clearly a determinant-preserving operation.) We perform the same action for each column j, with $j \ge n - r + 1$ and we are left with the matrix

$$\left[\begin{array}{c|c} Y & 0 \\ \hline 0 & K_r \end{array}\right], \text{ where } Y = c_r[A_{n-2r}(I, I)].$$

Thus,

$$D_n = D_{n-2r} \ \text{Det } K_r = (-1)^{[(n-2r+1)/2]}(-1)^r = (-1)^{[(n+1)/2]}$$

by the induction hypothesis.

So by combining Corollary 1 with Lemmas 3 and 4, we may state the main

Theorem: If $F, G \in S$ and A is the $(n + 1)$ by $(n + 1)$ matrix whose $(i, j)^{th}$ entry is the coefficient of x^{i+j+r} in $F_i(x) \cdot G_j(x)$, then the determinant of A equals

$$\left[\prod_{n=0}^{n} F_m(0)G_m(0)\right]. \begin{cases} 2^n & \text{if } r = 0, \\ (-1)^{[(n+1)/2]} & \text{if } r \ne 0 \text{ and } 2 \text{ divides } n + 1 \text{ or } n + r, \\ 0 & \text{otherwise.} \end{cases}$$

Some consequences are

Corollary 2: The determinant of M_n with c_n equal to the coefficient of x^n in $(1 + x + x^2)^n$ is 2^n.

Proof: Take $F_m(x) = G_m(x) = (1 + x + x^2)^m$ in the Theorem.

Corollary 3: The determinant of M_n with $c_n = \begin{bmatrix} 2n \\ n + r \end{bmatrix}$ is:

$$\begin{cases} 2^n & \text{if } r = 0, \\ (-1)^{[(n+1)/2]} & \text{if } r \neq 0 \text{ and } 2r \text{ divides } n+1 \text{ or } n+r, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Take $F_m(x) = G_m(x) = (1 + x)^{2m}$ in the Theorem.

We make an interesting combinatorial observation in

Corollary 4: If c_n is the coefficient of x^n in $(1 + tx + x^2)^n$, then the value of the determinant of M_n is independent of t.

Proof: Take $F_m(x) = G_m(x) = (1 + tx + x^2)^m$ in the Theorem and observe that each $F_m(0)$ is independent of t.

Corollary 5: The determinant of M_n with c_n equal to the coefficient of x^{n+r} in $(a + bx + cx^2)^n$ (with $a, b, c \neq 0$) is:

$$(a^{n-r}c^{n+r})^{(n+1)/2} = \begin{cases} 2^n & \text{if } r = 0, \\ (-1)^{[(n+1)/2]} & \text{if } r \neq 0 \text{ and } 2^n \text{ divides } n+1 \text{ or } n+r, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Let $\theta = (ac)^{1/2}$, $x = \theta y/c$, so that c_n is the coefficient of

$$\frac{\theta^{n+r}y^{n+r}}{c^{n+r}}$$

in $a^n[1 + (b/\theta)y + y^2]^n$. Let d_n be the coefficient of y^{n+r} in $[1 + (b/\theta)y + y^2]^n$ so that $c_n = (a^{n-r}c^{n+r})^{1/2}d_n$. Then

$$\begin{bmatrix} c_0c_1 & \cdots & c_n \\ c_1c_2 & \cdots & c_{n+1} \\ \vdots & & \\ c_n & \cdots & c_{2n} \end{bmatrix} = (c/a)^{r/2} \begin{bmatrix} 1 & & & 0 \\ & \theta & & \\ & & \theta^2 & \\ 0 & & & \ddots \ \theta^n \end{bmatrix} \begin{bmatrix} d_0d_1 & \cdots & d_n \\ d_1d_2 & \cdots & d_{n+1} \\ \vdots & & \\ d_n & \cdots & d_{2n} \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & \theta & & \\ & & \theta^2 & \\ 0 & & & \ddots \ \theta^n \end{bmatrix},$$

and so the result follows immediately from Corollaries 3 and 4.

Corollary 6: The Legendre polynomials $[P_n(t)]_{n\geq 0}$ are defined by

$$(1 - 2tx + x^2)^{-1/2} = \sum_{n\geq 0} P_n(t)x^n.$$

By taking $c_n = P_n(t)$, the determinant of M_n is

$$2^n\left(\frac{t^2 - 1}{4}\right)^{\binom{n+1}{2}}.$$

Proof: Use Corollary 5 with $b = t$ and $b^2 - 4ac = 1$.

Clearly, this technique of computing this class of determinants may be generalized to a number of different questions. The real keys to the method are that $(1, x + x^{-1}, x^2 + x^{-2}, \ldots)$ form an additive basis for $\mathbb{Z}[x + x^{-1}]$ over $\mathbb{Z}$; and that the action of taking the coefficients of x^r of the entries of a matrix of polynomials, commutes with multiplication by matrices with entries in $\mathbb{C}$ (i.e., Lemma 1).

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