NESTED STEINER n-CYCLE SYSTEMS AND PERPENDICULAR ARRAYS

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Abstract. We prove that for any odd positive integer $n > 1$ and for any sufficiently large integer $v > v_0(n)$, there exists a Nested Steiner n-Cycle System of order v if and only if $v \equiv 1 \pmod{2n}$. This gives rise to many new classes of perpendicular arrays.

1. Introduction.

In this paper, we are interested in a certain generalization of a Nested Steiner Triple System. A *Steiner Triple System*, STS(v), is a partition of the edge set of K_v into triangles (3-cycles); and is said to be *nested* if one can add a point to each triangle, obtaining a partition of the edges of $2K_v$ into K_4s. An *n-Cycle System of order* v, CS(v, n), is a partition of the edge set of K_v into n-cycles; and is said to be *nested* if one can add a point to each n-cycle in the system, obtaining a partition of the edges of $2K_v$ into '*wheels with n spokes*'(the original cycle being the rim and the added vertex, the *hub*).

These designs have been investigated by Lindner, Rodger and Stinson [3] and Stinson [5], [7]; and have been shown to exist in almost every case in which the necessary condition $v \equiv 1 \pmod{2n}$ holds.

A *Steiner n-Cycle System of order* v, SCS(v, n), is a CS(v, n) with the additional property that for each k with $1 \leq k < n/2$, any given pair of points is at distance k from one another in exactly one of the cycles: In other words, if $\{C_1, C_2, \ldots, C_m\}$ are the cycles of the CS(v, n) and $C_j^{(k)}$ is the graph defined by the vertices of C_j with edges between vertices that are at distance k in C_j, then the edges of $C_1^{(k)}, C_2^{(k)}, \ldots, C_m^{(k)}$ form a partition of the edge set of K_v for each k, $1 \leq k < n/2$ (in fact, a CS(v, r) where $r = n/gcd(n, k)$). For example, a SCS(v, 3) is just a STS(v), and a SCS(v, 4) is a CS(v, 4). A Steiner 5-cycle system is called a *Steiner Pentagon System* and is known to exist if and only if $v \equiv 1$ or $5 \pmod{10}$ and $v \neq 15$ (see

[2]). General Steiner n-cycle systems do appear in the literature as they are equivalent to cyclic perpendicular arrays: A *perpendicular array*, PA(v,n), is a $\binom{v}{2} \times n$ array, each cell containing an integer from the set $\{1,2,\ldots,v\}$, such that any given pair of columns contain all $\binom{v}{2}$ unordered pairs from the set $\{1,2,\ldots,v\}$. A *cyclic perpendicular array*, CPA(v,n), is a PA(v,n) with the extra property that $x_2,x_3,\ldots,x_n,x_1$ is a row of the array whenever $x_1,x_2,\ldots,x_n$ is. Thus, a CPA(v,n) has $\frac{1}{n}\binom{v}{2}$ *generator rows*, the entire array being formed by cyclically shifting each generator row n times.

Lemma 1.1. *For any odd integer $n > 1$, there exists a SCS(v,n) if and only if there exists a CPA(v,n).*

Proof: The $\frac{1}{n}\binom{v}{2}$ cycles of an SCS(v,n) can be viewed precisely as the $\frac{1}{n}\binom{v}{2}$ generator rows of a CPA(v,n); and vice-versa. ∎

Cyclic perpendicular arrays have what Stinson refers to as the *pair-column balanced* property, that is, among all the rows in the array containing a given pair x and y, each of x and y occurs $(n-1)/2$ times in each column. This is important in constructing certain optimal private-key cryptosystems (for a full discussion of the relationship between perpendicular arrays and theoretically secure codes, see Stinson [6]).

A SCS(v,n) is *nested* if we nest the underlying CS(v,n). Similarly, a CPA(v,n) is *nested* if we can adjoin a column to the array and so produce a PA$(v,n+1)$ with the property that $x_2,x_3,\ldots,x_n,x_1,y$ is a row of the array whenever $x_1,x_2,\ldots,x_n,y$ is (the resulting array is called *1-rotational*). We have the following analogue of Lemma 1.1.

Lemma 1.2. *For any odd integer $n > 1$, there exists a nested SCS(v,n) if and only if there exists a nested CPA(v,n).*

Example:

	1,2,4,0	
	2,4,1,0	
	4,1,2,0	5,6,1,4
1,2,4; 0	2,3,5,1	6,1,5,4
2,3,5; 1	3,5,2,1	1,5,6,4
3,4,6; 2	5,2,3,1	6,0,2,5
4,5,0; 3	3,4,6,2	0,2,6,5
5,6,1; 4	4,6,3,2	2,6,0,5
6,0,2; 5	6,3,4,2	0,1,3,6
0,1,3; 6	4,5,0,3	1,3,0,6
	5,0,4,3	3,0,1,6
	0,4,5,3	
A nested SCS(7, 3) (i.e., a nested STS(7))	Corresponding nested CPA(7, 3) (i.e., a 1-rotational PA(7, 4))	

In [7] it was shown that there exists a nested SCS$(v, 3)$ if and only if $v \equiv 1 \pmod 6$; and recently Stinson [5] has constructed SCS$(v, 4)$ for all $v \equiv 1 \pmod 8$ except $v = 57, 65, 97, 113, 185, 265$.

In this paper, we will construct a nested SCS(v, n) whenever n is an odd integer and v is a prime power congruent to 1 (mod $2n$). Since, for each n, the set $\{v$: there exists a nested SCS$(v, n)\}$ is PBD-closed, this will enable us to apply Wilson's theorem to obtain asymptotic results on the existence of these designs.

2. Direct constructions for nested SCS(v, n)s.

Theorem 2.1. *For any odd integer $n > 1$ and prime power v with $v \equiv 1$ (mod $2n$), there exists a nested SCS(v, n).*

Proof: Let g be a primitive element in the field F with v elements and let $t = g^{2m}$ where $m = (v-1)/2n$. Label the vertices of K_v with the elements of F. For each $a \in F$ and integer i, $0 \leq i \leq m-1$, let $C_{a,i}$ be the n-cycle with vertices $a + t^j g^i$, $0 \leq j \leq n-1$, where $a + t^j g^i$ is adjacent to $a + t^{j-1} g^i$ and $a + t^{j+1} g^i$; and let $B_{a,i}$ be the *star* in which vertex a is adjacent to the vertices of $C_{a,i}$.

We observe that if $d \neq 0$ and x and y are any two vertices of F then exactly one of $(x-y)/d$ and $(y-x)/d$ may be written in the form $t^j g^i$ where $0 \leq i \leq m-1$ (as $-1 = g^{nm} = t^{(n-1)/2} g^m$).

Fix d, and for any two vertices x_1 and x_2 let y, z be the permutation of x_1 and x_2 such that $y - z$ may be written in the form $dt^j g^i$ where $0 \leq i \leq m-1$.

For $d = 1$ we have $y = z + t^j g^i$ so that the edge (y, z) exists in $B_{z,i}$.

For each k, $1 \leq k < n/2$, let $C_{a,i}^{(k)}$ be defined from $C_{a,i}$ by joining the vertices at distance k, and let $d = t^k - 1$. Then $y = z + (t^k - 1)t^j g^i$, and if $a = z - t^j g^i$ then $y = a + t^{j+k} g^i$ and so the edge (y, z) exists in $C_{a,i}^{(k)}$.

Thus, for any pair of distinct vertices x_1, x_2 in K_v the edge (x_1, x_2) appears in each of the sets of graphs $\{B_{a,i}: a \in F, 0 \leq i \leq m-1\}$ and $\{C_{a,i}^{(k)}: a \in F, 0 \leq i \leq m-1\}$ for each k, $1 \leq k < n/2$. But each of these sets of graphs contain exactly $\binom{v}{2}$ edges, and so it is clear that no edge is counted twice and, therefore, they each partition the edge set of K_v. ∎

Remark 1: The nested SCS(v, n) constructed in the above theorem has the additive group of F as a point-transitive group of automorphisms.

Remark 2: We may replace the set $\{1, g, g^2, \ldots, g^{m-1}\}$ in the construction of the $C_{a,i}$s by any set of representatives of the cosets of the subgroup $\langle -t \rangle$ in F^* to get another, often non-isomorphic, construction.

Examples:

A nested SCS(11,5): 1, 4, 5, 9, 3; 0 (mod 11)

A nested SCS(31,5):
1, 2, 4, 8, 16; 0
3, 6, 12, 24, 17; 0 (mod 31)
5, 10, 20, 9, 18; 0

3. Asymptotic existence of nested SCS(v, n)s.

Lemma 3.1. *If there exists a nested CS(v, n) then $v \equiv 1$ (mod $2n$).*

Proof: As the cycles of a CS(v, n) form a decomposition of th edges of K_v, so every vertex appears in these cycles equally often; and so, as the edge set of the *wheels* forms a decomposition of $2K_v$, thus, every vertex appears as the *hub* of the wheel equally often, say t times. Therefore, vt = the number of wheels $= (1/n)\binom{v}{2}$ so that $t = (v-1)/2n$ and we see that $v \equiv 1 \pmod{2n}$. (This Lemma was stated, without proof, in [3]). ∎

We have already shown, in Section 2, that this condition is sufficient whenever v is a prime power. More examples of these designs can be obtained by applying MacNeish's Theorem [4]:

Theorem 3.2. *For any odd integer $n > 1$, and positive integer v, a product of prime powers, which are each congruent to 1 (mod $2n$), there exists a nested SCS(v, n).*

Proof: Let $v = q_1 q_2 \dots q_r$ be the prime power decomposition of v where $q_1 > q_2 > \dots > q_r$. By MacNeish's Theorem there is a transversal design with q_i groups of size $q_1 q_2 \dots q_{i-1}$ for each i, $2 \le i \le r$.

In this way we can construct a pairwise balanced design on v points with block sizes $q_1, q_2, \dots, q_r$. Constructing a nested SCS on each block yields a nested SCS(v, n), as desired. ∎

In the remainder of this section we will show that the necessary condition of Lemma 3.1 is sufficient, provided that v is large enough compared to n. We do this by applying Wilson's Theorem (see [1]):

Theorem 3.3. *[Wilson] Let K be any set of integers, and define $\alpha(K) = gcd\{k-1 : k \in K\}$ and $\beta(K) = gcd\{k(k-1) : k \in K\}$. There is an integer c_K such that if $v \ge c_K$, $v - 1 \equiv 0$ (mod $\alpha(K)$) and $v(v-1) \equiv 0$ (mod $\beta(K)$), then there exists a pairwise balanced design on v points having block sizes from the set K.*

Lemma 3.4. *Given any positive even integer m there exist primes p and q for which $p \equiv q \equiv 1$ (mod m), and $gcd\{p(p-1), q(q-1)\} = m$.*

Proof: By using Dirichlet's Theorem on the existence of primes in arithmetic progressions choose p to be any prime with $p \equiv m+1$ (mod m^2). Observe that $(p-1)/m \equiv p \equiv 1$ (mod m) so that $gcd\{p(p-1)/m, m\} = 1$; therefore, by the Chinese Remainder Theorem, we may select an integer r with $r \equiv 1$ (mod m) and $r \equiv -1$ (mod $p(p-1)/m$). By again applying Dirichlet's Theorem we choose q to be any prime satisfying $q \equiv r$ (mod $p(p-1)$) so that $q \equiv 1$ (mod m). It remains to be shown that $gcd\{p(p-1), q(q-1)\} = m$.

Now $q \equiv r \equiv -1$ (mod $p(p-1)/m$) so that $q(q-1) \equiv 2$ (mod $p(p-1)/m$). But $p \equiv (p-1)/m \equiv 1$ (mod m) so that $p(p-1)/m$ is odd and, therefore, $gcd\{q(q-1), p(p-1)/m\} = 1$. Recalling that $q \equiv 1$ mod m, we have $gcd(p(p-1), q(q-1)) = m$ as required. ∎

We can now prove

Theorem 3.5. *For any odd positive integer $n > 1$ there exists an integer c_n such that if $v \geq c_n$ then there exists a nested $SCS(v, n)$ if and only if $v \equiv 1$ (mod $2n$).*

Proof: From Lemma 3.4 we can chose primes p and q such that $p \equiv q \equiv 1$ (mod $2n$) and $gcd\{p(p-1), q(q-1)\} = 2n$. Applying Wilson's Theorem (3.3) with $K = \{p, q\}$, (so that $\alpha(K) = \beta(K) = 2n$), there exists an integer c_n such that whenever $v \geq c_n$ and $v \equiv 1$ (mod $2n$) then there is a pairwise balanced design on v points with block sizes p and q. Since $p \equiv q \equiv 1$ (mod $2n$) we can construct a nested SCS on each block (Theorem 2.1), to obtain a nested SCS(v, n) as desired. ∎

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