CORRECTION TO "ZAPHOD BEEBLEBROX'S BRAIN AND THE FIFTY-NINTH ROW OF PASCAL'S TRIANGLE"

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In my paper [1], we studied Pascal's Triangle modulo 2, 4, 8 and 16; and, in particular, its self-similar structure. It is well-known that the number of entries $\equiv 1 \mod 2$ in the nth row of Pascal's Triangle is $2^{\#_2(n)}$, where $\#_2(n)$ is the number of '1's in the binary expansion of n. The proof, developed in our article, observed that if T_k denotes the top 2^k rows of Pascal's Triangle $\pmod 2$, then

and proceeded by induction.

We discovered by experiment that a similar rule works with Pascal's Triangle $\pmod 4$: if there *are not* two consecutive '1's in the binary expansion of n then there are $2^{\#_2(n)}$ entries $\equiv 1 \pmod 4$, and no entries $\equiv -1 \pmod 4$, in the nth row of Pascal's Triangle. On the other hand if there *are* two consecutive '1's in the binary expansion of n then there are $2^{\#_2(n)-1}$ entries $\equiv 1 \pmod 4$, and $2^{\#_2(n)-1}$ entries $\equiv -1 \pmod 4$, in the nth row of Pascal's Triangle. We proved this by noting that if

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where V_k^T is the transpose of V_k, and again proceeding by induction.

The main point of [1] was that there is a version of this "self-similarity" modulo 4 (with an easily understood change of states) modulo any prime power. This also allowed us to prove that the number of entries $\equiv a \pmod{2^b}$ with a odd, $b \leq 2$ in any row of Pascal's Triangle, is either 0 or a power of 2. This is also true for $b = 3$, though the proof that we gave was faulty. Surprisingly though, $b = 4$ is an exceptional case, since exactly six entries in Row 59 of Pascal's Triangle are $\equiv 1 \pmod{16}$.

In [1], we proved that if D_k denotes the top 2^k rows of Pascal's Triangle $\pmod 8$, with

We then attempted (see (3)) to write down all the cases for how the binary expansion of n determines the number of entries in row n that are $\equiv a \pmod 8$, for $a = 1, 3, 5$ and 7; and proceeded by induction on the binary digits of n, based on the evolution of D_k into E_k. Although this method is certainly valid, Fred Howard and Ken Davis, and Jim Huard, Blair Spearman and Ken Williams all observed that we failed to enumerate the cases correctly (thanks to all of them for such a careful reading of my paper). Here is a correct enumeration of those cases, which will follow by induction.

Let $(n)_2$ denote the binary expansion of n. If $(n)_2$ begins with "11", then we may need to "cut" the row up into four quadrants; we can discuss the first two quadrants only because of the horizontal symmetry of Pascal's Triangle. All of the statements preceding Figure 12 in [1] remain unchanged:

- If $(n)_2$ contains no 11 and no 101 then all odd entries are $\equiv 1 \pmod 8$
- If $(n)_2$ contains no 11 but does contain a 101 then there are an equal number of entries $\equiv 1 \pmod 8$ and $\equiv 5 \pmod 8$ and no other odd entries.

- If $(n)_2$ contains a 1111, or it contains both a 11 and a 101, then there are an equal number of entries $\equiv 1, 3, 5$ and $7 \pmod 8$, and similarly in each quadrant (when relevant).

If n does not belong to any of these cases then, in binary, it has the form

$$(n)_2 = \ 0\underbrace{11..1}_{t_1 1\text{'s}}\underbrace{00..0}_{u_1 0\text{'s}}\underbrace{1\dots1}_{t_2 1\text{'s}} \ \dots \ \underbrace{00..0}_{u_{m-1} 0\text{'s}}\ \underbrace{1\dots1}_{t_m 1\text{'s}}\underbrace{0\dots0}_{u_m 0\text{'s}}.$$

Here $1 \le t_j \le 3$ for each j, and $u_i \ge 2$ for $1 \le i \le m-1$. It is at this point that we part company from [1], where we failed to distinguish certain cases that do arise. Note that we have started $(n)_2$ with a single '0', followed by the usual '1'. For example, $(825)_2 = 01100111001$.

Henceforth, we may assume that $(n)_2$ contains no 1111 nor a 101, but does contain a 11.

- If $(n)_2$ contains no 111, and n is even or $n \equiv 1 \pmod 4$, then there are an equal number of entries $\equiv 1 \pmod 8$ and $\equiv 7 \pmod 8$ and no other odd entries.
- If $(n)_2$ contains no 111 nor a 0110, and $n \equiv 3 \pmod 8$, then there are an equal number of entries $\equiv 1 \pmod 8$ and $\equiv 3 \pmod 8$ and no other odd entries.
- If $(n)_2$ contains a 111 and no 0110, and $n \not\equiv 7 \pmod 8$, then there are an equal number of entries $\equiv 1 \pmod 8$ and $\equiv 3 \pmod 8$ and no other odd entries.
- Otherwise there are an equal number of entries $\equiv 1, 3, 5$ and $7 \pmod 8$.

It is possible to explain this induction proof succinctly: Let $S_n \subseteq \{1, 3, 5, 7\}$ be the set of residue classes $a \pmod 8$ such that there exists an integer j for which $\binom{n}{j} \equiv a \pmod 8$. First note that $S_{2^j} = \{1\}$ for all $j \ge 1, S_3 = \{1, 3\}$, and $S_{2^j-1} = \{1, 3, 5, 7\}$ for $j \geqq 3$. By studying the evolution of E_k from D_k we see that

whenever $(n)_2$ contains a 101 then $5S_n = S_n$,
whenever $(n)_2$ contains a 0110 then $7S_n = S_n$,
whenever $(n)_2$ contains a 01110 then $3S_n = S_n$ and
whenever $(n)_2$ contains a 1111 then $S_n = \{1, 3, 5, 7\}$.

These observations account for the composition of S_n. If $a, b \in S_n$, then the number of entries in the nth row that are $\equiv a \pmod 8$ is equal to the number of entries that are $\equiv b \pmod 8$.

Combining these observations by studying the binary expansion of n finishes the proof.

References

A. Granville, Zaphod Beeblebrox's brain and the fifty-ninth row of Pascal's Triangle, this Monthly 99 (1992), 318-331.

J.G. Huard, B.K. Spearman, and K.S. Williams, Pascal's triangle (mod 8), *European J. Combinatorics* (to appear).