Teoria dei numeri. — *Solution to a problem of Bombieri.* Nota(*) di ANDREW GRANVILLE, presentata dal Socio E. BOMBIERI.

— We solve a problem of Bombieri, stated in connection with the "prime number theorem" for function fields.

Distribution of Primes; Elementary Proofs; Prime Number Theorem.

— *Soluzione della probleme di Bombieri.*

In [1], Bombieri states that if $a_1, a_2, \ldots$ is a sequence of non–negative real numbers satisfying the Selberg–type formula

$$ma_m + \sum_{i=1}^{m-1} a_i a_{m-i} = 2m + O(1) \tag{1}$$

for each $m \geq 1$ then $a_m = 1 + o(1)$; however there is an error in the proof as may be seen by the counterexample $a_m = 1 - (-1)^m$. In [2], Bombieri shows that his original result may be recovered by also having the analogous formula to (1) for the sequence $a_2, a_4, \ldots$; and, in [4], Zhang improves the error term in this result to $a_m = 1 + O(1/m)$.

Herein we return to the original question and solve (slightly more than) a problem stated by Bombieri in [2]:

Theorem 1. *If $a_1, a_2, \ldots$ is a sequence of non–negative real numbers satisfying*

$$ma_m + \sum_{i=1}^{m-1} a_i a_{m-i} = 2m + o(m) \tag{2}$$

for each $m \geq 1$ then either (i) $a_m = 1 + o(1)$; or (ii) $a_m = 1 - (-1)^m + o(1)$.

In [3] (Theorem 2'), Erdös showed that for any sequence of non–negative real numbers satisfying (2) we have

$$\sum_{i=1}^{m} a_i = m + o(m). \tag{3}$$

We note also that as each $a_i \geq 0$, thus $ma_m \leq 2m + o(m)$ by (2), and so

$$0 \leq a_m \leq 2 + o(1). \tag{4}$$

Therefore, by taking $b_j = 1 - a_j$ for each j, we see that Theorem 1 follows immediately from

Theorem 2. *If $b_1, b_2, \ldots$ is a sequence of real numbers satisfying*

$$|b_m| \leq 1 + o(1) \tag{5}$$

and

$$mb_m = \sum_{i=1}^{m-1} b_i b_{m-i} + o(m) \tag{6}$$

for each $m \geq 1$ then one of the following cases holds:
(i) $b_m = o(1)$; (ii) $b_m = (-1)^m + o(1)$; (iii) $b_m = 1 + o(1)$.

Proof: We start by showing that either (i) holds or

$$B := \limsup_{m\to\infty} |b_m| = 1. \tag{7}$$

First note that $B \geq 0$ by definition, and $B \leq 1$ by (5). Now $m|b_m| \leq mB^2 + o(m)$ by (6) and so, choosing m with $b_m = B + o(1)$, we have $B \leq B^2$. Therefore either $B = 0$ (in which case (i) holds), or $B \geq 1$, so that $B = 1$.

Next we show that if (7) holds then

$$\max_{2m\leq n\leq 3m} |b_n| = 1 + o(1) \tag{8}$$

as $m \to \infty$. Suppose that (8) is false so that there exists $\delta > 0$ such that, for certain arbitrarily large m, we have $|b_n| < 1 - 10\delta$ for all n in the range $2m \leq n \leq 3m$. From this we can deduce (by induction on n) that $|b_n| < 1 - \delta$ for all $n > 3m$, if m is sufficiently large, which contradicts (7). The induction proceeds in a straightforward way, by using (6) in the form

$$n|b_n| \leq \sum_{i=1}^{n-1} |b_i||b_{n-i}| + o(n),$$

together with the bounds

$$|b_j| < \begin{cases} O(1) & \text{for } j \leq j_0; \\ 1 + \delta/10 & \text{for } j_0 < j < 2m; \\ 1 - 10\delta & \text{for } 2m \leq j \leq 3m; \\ 1 - \delta & \text{for } 3m < j \leq n - 1, \end{cases}$$

where j_0 is chosen so that $|b_j| < 1 + \delta/10$ for all $j \geq j_0$ (which is possible, by (5)).

We now prove two lemmas.

Lemma 1. *If* $|b_m| = 1 + o(1)$ *then* $|b_i| = 1 + o(1)$ *and* $b_i b_{m-i} = b_m + o(1)$ *for all but* $o(m)$ *values of* $i \le m$.

Proof: Let $c_i = b_i b_{m-i}/b_m$ so that $|c_i| \le 1 + o(1)$ by (5) (if $m - i \to \infty$) and $\sum_{i=1}^m c_i = m + o(m)$ by (6). Therefore $c_i = 1 + o(1)$ for all but $o(m)$ values of $i \le m$, which is equivalent to the second assertion of the lemma. Moreover $c_i = 1 + o(1)$ implies that $|b_i||b_{m-i}| = 1 + o(1)$ for such i, whereas both $|b_i|$ and $|b_{m-i}|$ are $\le 1 + o(1)$ by (5). Thus both $|b_i|$ and $|b_{m-i}|$ equal $1 + o(1)$.

Lemma 2. *Fix* $\varepsilon > 0$. *For any sufficiently large* m *and for any integers* k *and* n *in the ranges* $1 \le k \le \varepsilon m$, $m \le n \le 2m$, *for which* $|b_n|$, $|b_{n+k}| = 1 + o(1)$, *we have the estimate*

$$b_{m+k} = b_m b_{n+k}/b_n \; + \; O(\varepsilon),$$

where the constant implied by "O" is absolute.

Proof: Let $\sigma = 1$ if b_n and b_{n+k} have the same sign, and let $\sigma = -1$ otherwise. Now, as $|b_n|$ and $|b_{n+k}|$ both equal $1 + o(1)$, we see that $|b_i| = 1 + o(1)$, $b_i b_{n-i} = b_n + o(1)$ and $b_i b_{n+k-i} = b_{n+k} + o(1)$ for all but $o(m)$ values of $i \le n$, by Lemma 1. Therefore, by taking $j = n - i$, we see that $b_j = \sigma b_{j+k} + o(1)$ for all but $o(m)$ values of $j \le m$. Substituting this into (6) gives

$$mb_m - \sigma \left((m+k)b_{m+k} - \sum_{i=1}^{k} \; b_i b_{m+k-i} \right) \; = \; \sum_{j=1}^{m-1} (b_j - \sigma b_{j+k}) b_{m-j} + o(m) \; = \; o(m),$$

and the result follows from (5) as

$$k|b_{m+k}| + \sum_{i=1}^{k} \; |b_i b_{m+k-i}| \; = \; O(k).$$

Completion of the proof of Theorem 2: Fix $\varepsilon > 0$ and suppose that m is sufficiently large. By (8) there exists n in the range $2m \le n \le 3m$ with $|b_n| = 1 + o(1)$, and so $|b_i| = 1 + o(1)$ for all but $o(m)$ values of $i \le 2m$, by Lemma 1. Therefore there exists an integer k in the range $1 \le k \le \varepsilon m$ such that both $|b_{m+k}|$ and $|b_{m+2k}| = 1 + o(1)$. Taking $n = m + k$ in Lemma 2, we see that $b_m = b_{m+2k} + O(\varepsilon)$; and letting $\varepsilon \to 0$, we then get

$$|b_m| \; = \; 1 + o(1) \tag{9}$$

as $m \to \infty$.

Now take $k = 1$ and $n = m + 1$ in Lemma 2. By (9) this implies that $b_m b_{m+2} = b_{m+1}^2 + o(1) = 1 + o(1)$, so that b_m and b_{m+2} have the same sign if m is sufficiently large. Therefore there exist constants ν and η, equal to -1 or 1, such that

$$b_{2m} = \nu + o(1); \qquad \text{and} \;\; b_{2m+1} = \eta + o(1).$$

Substituting this into (6) for even m we obtain $\nu = (\nu^2 + \eta^2)/2 = 1$; therefore we get (ii) if $\eta = -1$, and (iii) if $\eta = 1$.

[1] E. Bombieri, *Sull'analogo della formula di Selberg nei corpi di funzioni*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **35** (1963), 252–257.

[2] E. Bombieri, *Correction to my paper "Sull'analogo della formula di Selberg nei corpi di funzioni"*, (preprint).

[3] P.Erdös, *On a tauberian theorem connected with the new proof of the prime number theorem*, Jour. Ind. Math. Soc., **13** (1949), 133–147.

[4] W.-B. Zhang, *Elementary proofs of the abstract prime number theorem for algebraic function fields*, (preprint).

School of Mathematics – Institute for Advanced Study,
Princeton, NJ 08540, USA.