A note on sums of primes †

by

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Abstract: Under the assumption of the prime k–tuplets conjecture we show that it is possible to construct an infinite sequence of integers, such that the average of any two is prime.

Recently Pomerance, Sárközy and Stewart [2] constructed sets of integers A and B for which every element of $A+B$ is prime; and a set of odd integers A for which $\frac{1}{2}\ (a+a')$ is prime for any $a \neq a'$ in A. These sets were chosen from $\{1,2,\ldots,N\}$ so as to make them as large as possible. A natural question that arises is whether we can construct *infinite* sets with these properties, and we do this here under the assumption of Hardy and Littlewood's

Prime k–tuplets conjecture. *Suppose that $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$ are integers such that each $(a_j, b_j) = 1$ and, for each prime $p \leq k$, there exists an integer x for which none of $a_1x+b_1, \ldots, a_kx+b_k$ are divisible by p. Then there are arbitrarily large integers x for which each of $a_1x+b_1, \ldots, a_kx+b_k$ is prime.*

Actually we will prove a considerable (though technical) generalization of the above questions:

Theorem. *Let $c_1, c_2, \ldots, c_N$ be given positive integers and suppose that the prime k–tuplets conjecture is true. We can construct infinite sets $A_1, A_2, \ldots, A_N$ of distinct odd prime numbers such that every element of the set $\frac{1}{d}\ \{c_1A_1 + \ldots + c_NA_N\}$ is prime, where $g = gcd(c_1, c_2, \ldots, c_N)$ and $d = gcd(2g, c_1+c_2+\ldots+c_N)$.*

Remark: The set $\frac{1}{d}\ \{c_1A_1+\ldots+c_NA_N\}$ is defined to be the set whose elements are the sum of any c_1 elements of A_1, any c_2 elements of $A_2, \ldots$, and any c_N elements of A_N. Note that the element $c_1a_1+\ldots+c_Na_N$ of $c_1A_1+\ldots+c_NA_N$ must be divisible by d.

By taking $N=2, c_1=1, c_2=2, A=A_1$ and $B=\{2a : a \in A_2\}$ in the Theorem above, we have constructed infinite sets of integers A and B for which every element of

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$A+B$ is prime. By taking $N=1$ and $c_1=2$ we have constructed an infinite set of integers A for which $\frac{1}{2}\,(a+a')$ is prime for any $a,\ a' \in A$.

Before the Theorem we prove

Lemma. *For any given $B>0$ we can find, under the same hypothesis as in the Theorem, distinct primes $a_1,\ldots,a_N$, each greater than B, such that $\frac{1}{d}\,(c_1a_1+\ldots+c_Na_N)$ is prime.*

Proof: Without loss of generality we may assume that $g=1$. Let D be the product of the odd primes dividing $c_1c_2\ldots c_N$. We shall choose integers $r_1,\ldots,r_N$ which satisfy the following congruences:

$r_i \equiv 1 \pmod{p}$ for each prime $p=q$ dividing D, and also for $p=4$ and $q=2$, for each i,

unless

p divides $c_1+c_2+\ldots+c_N$ and i is the smallest index for which q does not divide c_i, in which case we take $r_i \equiv -1 \pmod{p}$.

Such integers exist because of the Chinese Remainder Theorem.

We now choose $a_2,\ldots,a_N$ to be distinct primes, greater than B, with each $a_i \equiv r_i \pmod{4D}$, which is certainly possible by Dirichlet's Theorem for primes in arithmetic progressions. We are thus left with having to find an arbitrarily large integer x such that both $a_1=4Dx+r_1$ and $2Dc_0x+e$ are prime, where $c_0=2c_1/d$ and $e=(c_1r_1+c_2a_2+\ldots+c_Na_N)/d$. However, by the choice of the r_i's, we know that both r_1 and e are coprime with $2Dc_0$ and so, by the prime k–tuplets conjecture, arbitrarily large such x exist.

Proof of the Theorem: By construction. The first elements of each set are given by taking $B=e^{c_1+c_2+\ldots+c_N}$ in the Lemma. We then continue to construct the sets by adding one new prime to each set in turn, starting $A_1,A_2,\ldots,A_N,A_1,\ldots$ etc.

We now show how to select a suitable prime p to add to the set F_j, once we have already constructed the subsets $F_1,F_2,\ldots F_N$ of $A_1,A_2,\ldots,A_N$ in this way:

Suppose that q was the last prime added to F_j and let m be the product of the primes $< q/2$. We shall be choosing p of the form $p=q+mx$ for some sufficiently large choice of x, so that p is larger than any prime previously chosen and also larger than $3c_j\ \prod_{i=1}^{n}\,(a_i+1)^{c_i}$, where a_i is the cardinality of F_i.

Let $G_j = F_j \cup \{p\}$ and $G_i = F_i$ otherwise. An element of $\frac{1}{d}\ \{c_1 G_1 + \ldots + c_N G_N\}$ that contains p in its sum can be seen to be equal to an element of $\frac{1}{d}\ \{c_1 F_1 + \ldots + c_N F_N\}$ that contains q in its sum plus some integer multiple of mx/d: Therefore it can be written in the form

$$(*) \qquad r \ + \ tmx/d$$

for some t in the range $1 \leq t \leq c_j$, where r is a prime, with $r > q/2$. By definition $(r, tm/d) = 1$, and so each such integer (*) is free of prime factors less than $q/2$.

There are $k-1 \ \leq c_j \ \prod_{i=1}^{n} \ a_i^{c_i}$ $(< q/3)$ such elements (*), and none of them is divisible by any prime $\leq k$ when we take $x = 0$. Therefore, by the prime k–tuplets conjecture, there are infinitely many integers x such that $q+mx$ and all of the integers (*) are prime. Hence, by choosing a sufficiently large such integer x, we can ensure that p is a suitable element to be added to F_j.

References

[1] G.H. Hardy and J.E. Littlewood, *Some problems of Partitio Numerorum III: On the expression of a number as a sum of primes*, Acta Math. **44** (1922), 1–70.

[2] C. Pomerance, A. Sárközy and C.L. Stewart, *On divisors of sums of integers III*, Pacific J. Math. **133** (1988), 363–379.

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