Theorem 4 (Entov, Polterovich, Z.). *There is a constant $K > 0$ such that if $\{\rho_j\}_{j=1}^N$ is a partition of unity with* $\operatorname{supp}\rho_j$ *displaceable* $\forall j$, *then*

$$\max_{j,k} \|\{\rho_j, \rho_k\}\|_{C^0} \geq \frac{K}{N^3}\,.$$

For another application, consider the following model for simultaneous measurement of observables in classical mechanics. Let $F_1, F_2 \in C^\infty(M)$ be two observables to be measured. Couple the mechanical system (M, ω) with the measuring apparatus $(\mathbb{R}^4(p,q), dp \wedge dq)$ via the coupling Hamiltonian $H(x,p,q) = p_1F_1(x) + p_2F_2(x)$. It follows from Hamilton's equations with initial values $x(0) = y$, $p_i(0) = \varepsilon \geq 0$, $q_i(0) = 0$ that $p_i(t) = \varepsilon$ and $x(t) = g^{\varepsilon t}y$, where g^s is the Hamiltonian flow of the function $G = F_1 + F_2$. By definition, the output of a measurement which has duration $T > 0$ is the pair of functions

$$F_i'(y) := \frac{q_i(T)}{T}\,.$$

Note that if $\varepsilon = 0$ or if $\{F_1, F_2\} = 0$, we obtain $F_i' = F_i$, so the measurement is precise, the fact which also serves as a justification to use the presented model. In any case we can define the *error* of the measurement as the following quantity (which is independent of i):

$$\Delta(F_1, F_2, \varepsilon, T) := \|F_i - F_i'\|_{C^0}\,.$$

Using the above quasi-state ζ, we obtain the following lower bound for this error ($C \equiv C(M, \omega) > 0$ is a constant):

$$\Delta(F_1, F_2, \varepsilon, T) \geq \frac{1}{2}|\zeta(F_1 + F_2) - \zeta(F_1) - \zeta(F_2)| - \sqrt{\frac{C}{\varepsilon T}} \cdot \varphi(F_1, F_2)\,,$$

where φ is some nonnegative function independent of ε, T. Taking $T \to \infty$ we obtain

$$\Delta(F_1, F_2) := \liminf_{T\to\infty} \Delta(F_1, F_2, \varepsilon, T) \geq \frac{1}{2}|\zeta(F_1 + F_2) - \zeta(F_1) - \zeta(F_2)|\,;$$

here the left-hand side is independent of ε as long as $\varepsilon > 0$. $\Delta(F_1, F_2)$ can be interpreted as the error of a measurement performed on a fast-moving system. The inequality gives a (nontrivial) lower bound for this error if ζ is not additive on the pair F_1, F_2. Note that this also implies that F_1, F_2 do not commute.

Rigidity of Lagrangian submanifolds

Octav Cornea
(joint work with Paul Biran)

1. Setting

Given a symplectic manifold (M^{2n}, ω), a Lagrangian submanifold $L \subset (M, \omega)$ is an n-dimensional submanifold so that $\omega|_M = 0$. All Lagrangians discussed here are assumed closed. Such a Lagrangian is called monotone if the two morphisms $\omega : \pi_2(M, L) \to \mathbb{R}$ and $\mu : \pi_2(M, L) \to \mathbb{Z}$, the first given by integrating the symplectic form ω and the second by the Maslov class, are proportional with a positive constant of proportionality. All Lagrangians below are supposed to be monotone and we also assume that the minimal Maslov number

$$N_L = \min\{\mu(x) \mid x \in \pi_2(M, L),\ \omega(x) > 0\}$$

verifies $N_L \geq 2$. There are many examples of such Lagrangians: $\mathbb{R}P^n \subset \mathbb{C}P^n$, the Clifford torus $\mathbb{T}^n_{Cliff} \subset \mathbb{C}P^n$ and many others.

An important property of this class of Lagrangians is that Floer homology $HF(L, L)$ is defined (by early work of Oh [6]). It is easy to see that Floer homology is always "smaller" or equal than singular homology (for example, for the 0-section in a co-tangent bundle it equals singular homology) and, at the other end, $HF(L, L)$ can also vanish (for example if L can be disjoined from itself by a hamiltonian isotopy).

In case $HF(L, L)$ equals singular homology (tensored with an appropriate Novikov ring) L is called *wide* and if $HF(L, L)$ vanishes L is called narrow. All known monotone Lagrangians are either wide or narrow.

2. Prototype of Results.

In the work with Paul Biran [1, 2, 3] reviewed here we discovered some systematic forms of rigidity that apply to monotone Lagrangians.

While our results are considerably more general, in this report I will exemplify the types of results we get on some simple cases: monotone Lagrangians in $\mathbb{C}P^n$. First I recall the notion of *Gromov width*. If $U \subset M$ is open, the Gromov width of U is

$$w(U) = \sup_r\{\pi r^2 \mid \exists e : B^{2n}(r) \to M,\ \text{smpl. embedding}\}$$

(where $B^{2n}(r)$ is the ball of radius r endowed with the standard symplectic form) and for a Lagrangian submanifold $L \subset M$ we let the Gromov width of L be defined by:

$$w(L) = \sup_r\{\pi r^2 \mid \exists e : B^{2n}(2r) \to M,\ \text{sympl. embedding},\ e^{-1}(L) = \mathbb{R}^n \cap B^{2n}(r)\}.$$

Theorem 1 ([1, 2, 3]). *Assume that the Lagrangians below are all monotone with $N_L \geq 2$. We have:*

a. *Any two non narrow Lagrangians in $\mathbb{C}P^n$ intersect.*
b. *Any monotone Lagrangian in $\mathbb{C}P^n$ is either a barrier in the sense that $w(\mathbb{C}P^n \backslash L) < w(\mathbb{C}P^n)$ or is small in the sense that $w(L) < w(\mathbb{C}P^n)$.*
c. *For each n there exist monotone narrow Lagrangians in $\mathbb{C}P^n$.*
d. *Any monotone Lagrangian (in any ambient manifold) with singular cohomology generated (as algebra) by classes of degrees strictly less than N_L is either narrow or wide.*

3. Comments

3.1. **Remarks on the results.** Only a very short discussion of the results above is included here. See [1, 2, 3] for full bibliographic references as well as for the strongest forms of these statements.

a. Entov and Polterovich have first noticed that point a. of the Theorem follows from some spectral action estimates in our paper [1] together with some of the results in [5]. There is also a simple, direct proof for this result, again based on spectral invariant estimates, which we present in [2] as well as a more algebraically meaningful one which appears in [3] and which provides more information. A particular case of the statement a. is that there is no symplectomorphism of $\mathbb{C}P^n$ which disjoins the Clifford torus from $\mathbb{R}P^n$. This particular case has also been obtained by Tamarkin [8][1] by completely different methods.

b. Point b. can be strengthened in various ways. In particular, it is possible to show [2] that narrow Lagrangians in $\mathbb{C}P^n$ have width strictly smaller than that of $\mathbb{C}P^n$ and that wide Lagrangians verify the equation:

$$w(L) + 2w(\mathbb{C}P^n \backslash L) \leq 2w(\mathbb{C}P^n) \ .$$

c. It is remarkable that Lagrangians as at point c. were not known before. The examples we produce are constructed as lifts of Lagrangians in specific hypersurfaces to the associated normal circle bundle. The fact that hey are narrow follows as a consequence of a. and the fact that, by construction, these Lagrangians are disjoint from $\mathbb{R}P^n$.

d. The result at d. can also be strengthened to distinguish between the two cases - narrow or wide - as well as to include yet more general classes of Lagrangian submanifolds.

3.2. **General method of proof.** All our results are based on exploiting the qunatum structures of monotone Lagrangians. To summarize the ideas behind the definition of these structures recall first that all the results in classical algebraic topology can be recovered by Morse theory - more precisely by counting various configurations made up from negative gradient flow lines of one or more Morse-Smale gradient flows (of course, when restrictig to underlying spaces which are

[1]Coincidentally, the preprint of Tamarkin appeared on *Arxiv* precisely the day when I presented this work at Oberwolfach.

manifolds). To construct quantum structures we consider similar configurations but allow for a certain number of points along these flow lines to be replaced by J-holomorphic spheres, or in the Lagrangian case, disks. A classical such example is the quantum product of symplectic manifolds. Another example which is very relevant here is the *pearl* complex and the associated structures. This construction - only valid in the case of monotone Lagrangians - was initially proposed by Oh [7] following an idea of Fukaya and is a particular case of the more recent cluster complex of Cornea-Lalonde [4] which applies to general Lagrangians. There is a considerable amount of work necessary to establish the properties of the pearl complex as well as those of the resulting homology, $QH(L)$, which we call the *quantum homology* of L and the various technical points are treated in [1] (for a more compact presentation see also [2]). The key bridge between the properties of the ambient manifold and those of the Lagrangian is provided by the fact that $QH(L)$ has the structure of an augmented algebra over the quantum homology of the ambient manifold and, with adequate coefficients, is endowed with duality.

These quantum structures have an important property inherited from the fact that they are defined by making use of J-holomorphic objects: they are *positive* in the sense that all algebraic objects admit a filtration so that all the morphisms (or operations) of geometric origin can be written as a sum between a classical object which preserves filtration and a quantum contribution which strictly increases the filtration degree. This allows for a variety of sometimes delicate algebraic arguments which lead to results like those in Theorem 1.

It is important to note that with appropriate coefficients, $QH(L)$ is isomorphic to the Floer homology $HF(L, L)$ and many of the additional algebraic structures also have natural correspondents in Floer theory. However, the positivity mentioned above is lost by this isomorphism and so Floer homology as such is not the appropriate tool to directly approach the proofs of the results in Theorem 1.

References

[1] P. Biran and O. Cornea, *Quantum structures for Lagrangian submanifolds.* Preprint (2007). Can be found at `http://arxiv.org/pdf/0708.4221`.

[2] P. Biran and O. Cornea, *Rigidity and uniruling for Lagrangian submanifolds.* Preprint (2008). Can be found at `http://arxiv.org/pdf/0808.2440`.

[3] P. Biran and O. Cornea, *Lagrangian Quantum Homology.* Preprint (2008) Can be found at `http://arxiv.org/pdf/0808.3989`.

[4] O. Cornea and F. Lalonde, *Cluster homology: an overview of the construction and results*, Electron. Res. Announc. Amer. Math. Soc., **12** (2006), 1–12.

[5] M. Entov and L. Polterovich, *Rigid subsets of symplectic manifolds.* Preprint (2007), can be found at `http://arxiv.org/pdf/0704.0105`.

[6] Y.-G. Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I.*, Comm. Pure Appl. Math., **46** (1993), 949–993.

[7] Y.-G. Oh, *Relative floer and quantum cohomology and the symplectic topology of lagrangian submanifolds.* In C. B. Thomas, editor, *Contact and symplectic geometry*, volume 8 of *Publications of the Newton Institute*, pages 201–267. Cambridge Univ. Press, Cambridge, 1996.

[8] D. Tamarkin, *Microlocal condition for non-dsplaceability*, Preprint, Can be found at `hhtp://arxiv.org/pdf/0809.1584`.