Lagrangian submanifolds: their fundamental group and Lagrangian cobordism

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(joint work with Paul Biran)

1. setting

Given a symplectic manifold (M^{2n}, ω), a Lagrangian submanifold $L \subset (M, \omega)$ is an n-dimensional submanifold so that $\omega|_M = 0$. All Lagrangians discussed here are assumed closed. Such a Lagrangian is called monotone if the two morphisms $\omega : \pi_2(M, L) \to \mathbb{R}$ and $\mu : \pi_2(M, L) \to \mathbb{Z}$, the first given by integrating the symplectic form ω and the second by the Maslov class, are proportional with a positive constant of proportionality. Whenever refering to a monotone Lagrangian we implicitly also assume that the minimal Maslov number

$$N_L = \min\{\mu(x) \mid x \in \pi_2(M, L),\ \omega(x) > 0\}$$

verifies $N_L \geq 2$. There are many examples of such Lagrangians: $\mathbb{R}P^n \subset \mathbb{C}P^n$, the Clifford torus $\mathbb{T}^n_{Cliff} \subset \mathbb{C}P^n$ and many others.

An important property of this class of Lagrangians is that Floer homology $HF(L, L)$ is defined (by early work of Oh [7]). More recently, various quantum structures of monotone Lagrangians have been discussed in [3] and [4].

2. Lagrangian cobordism

Assume that $L_i \subset (M, \omega)$, $1 \leq i \leq k$ are closed connected Lagrangian submanifolds and consider also a second such set of Lagrangian submanifolds, $L'_j \subset (M, \omega)$, $1 \leq j \leq h$.

Definition. We say that $(L_i)_{1 \leq i \leq k}$ is cobordant to $(L'_j)_{1 \leq j \leq h}$ if there exists a smooth, connected, cobordism $(V; \coprod_i L_i, \coprod_j L'_j)$ and a Lagrangian embedding

$$V \hookrightarrow (M \times T^*[0,1], \omega \oplus \omega_0)$$

so that

$$V|_{M \times [0,+\epsilon)} = \coprod_i L_i \times [0, \epsilon) \times \{i\}\ ,\ V|_{M \times (1-\epsilon,1]} = \coprod_j L'_j \times (1-\epsilon, 1] \times \{j\}$$

where $T^*[0,1] = [0,1] \times \mathbb{R}$ and $\epsilon > 0$ is very small.

It is useful to imagine a cobordism V as extended - trivially - to $T^*\mathbb{R} \supset T^*[0,1]$ and viewed as a non-compact manifold with k-cyclindrical ends to the left and h-cyclindrical ends to the right. A Lagrangian cobordism with $h = k = 1$ will be called an *elementary cobordism.*

Lagrangian cobordisms have been introduced, in a slightly different setting, by Arnold [1]. They have been studied by Audin [2], Eliashaberg [6] as well as by Chekanov [5]. The results obtained on this topic have been somewhat contrasting. On one side, the results of Eliashberg together with the calculations of Audin and combined with the Lagrangian surgery technique (see for instance Polterovich

[8]) show that general Lagrangian cobordism is very flexible. The argument for this is roughly as follows: as shown by Eliashberg, if one considers the notion of immersed Lagrangian cobordism that corresponds to requiring V above to be only an immersion and not an embedding, then, by an application of the Gromov h-principle, classifying Lagrangians up to immersed cobordism is a purely algebraic topology question and is computable by classical homotopy theoretical techniques. At the same time, by surgery, any immersed cobordism between two embedded Lagrangians can be transformed in an embedded cobordism. On the other hand, using J-holomorphic techniques Chekanov's result shows a certain form of rigidity for monotone cobordisms. By definition, a cobordism V as above is monotone if V itself is a monotone Lagrangian. Chekanov's argument essentially shows that the number (mod 2) of J-holormophic disks passing through a point on any of the manifolds V_i or V'_j is the same, independently of i, j, J and the point in question, whenever $(V_i)_{1\leq i\leq k}$ is cobordant to $(V_j)_{1\leq j\leq h}$.

3. Results on cobordism

We list here a number of results of increasing degree of generality - we caution the reader that, at this time, this is still work in in progress.

Theorem 1. *Any monotone elementary Lagrangian cobordism $(V;L,L')$ is a quantum h-cobordism in the sense that for an appropriately defined relative quantum homology we have $QH(V,L)\cong QH(V,L')=0$. In particular, there exists a ring isomorphism (depending on V):*

$$QH(L)\cong QH(L')\ .$$

In this result the coefficients are in $\mathbb{Z}_2$ in general and in $\mathbb{Z}$ if the cobordism V is oriented and carries a spin structure. The same convention is implicitly understood for the statements below.

It is useful to note at this time that there are examples of Lagrangian cobordisms as above that are not topological h-cobordisms (in the sense that singular homology $H(V,L)\neq 0$).

We also recall here that the quantum homology of a Lagrangian is the homology of the "pearl" chain complex $\mathcal{C}(L;f,\langle,\rangle,J)$ where $f:L\to\mathbb{R}$ is a Morse function on L, $\langle -,-\rangle$ is a generic metric on L and J is a generic almost complex structure on M. Its generators are the critical points of f and the differential counts configurations that combine Morse trajectories with J-holomorphic disks [3], [4].

Theorem 2. *Let $(V;(L_i)_{1\leq i\leq k},L')$ be a monotone Lagrangian cobordism. Then for generic J and any Morse functions $f_i:L_i\to\mathbb{R}$ there are chain maps $\phi_i:\mathcal{C}(L_i;f_i,J)\to C_{i-1}$ where the chain complex C_{i-1} is the cone of the chain map ϕ_{i-1} for $i\geq 3$ and $C_1=\mathcal{C}(L_1;f_1,J)$. Moreover, the chain complex $\mathcal{C}(L';f,\langle,\rangle,J)$ is chain homotopy equivalent C_k.*

In other words, the chain homotopy type of L' can be recovered from that of the L_i's by an iterated cone-construction. A different and somewhat richer way to formulate this result is to say that the existence of the cobordism between L'

and the family $(L_i)_{1\leq i\leq k}$ translates - inside the adequate Fukaya derived category - into the fact that the class of L' belongs to the subcategory spanned by the L_i's, $[L'] \in [L_1, \ldots, L_k]$ (the meaning of the Fukaya category used here appears in Seidel [9]).

There are two wide-reaching extensions of this result that are worth mentioning here. First, one can extend the theory to cobordisms in the total space of a Lefschetz fibration with basis $\mathbb{C}$ and with finitely many singular fibres. In this case the decomposition in Theorem 2 has to take into account also the vanishing cycles. In other words, in the Fukaya derived category language we have $[L'] = [L_1, \ldots, L_k, S_1, \ldots, S_r]$ where S_j's are the vanishing cycles of the fibration. Secondly, it is expected that most of the results here remain valid outside of the monotone category by using the appropriate algebraic formalism.

4. Application to the study of Fundamental groups of Lagrangians.

Not much is known in a systematic way concerning the following natural problem: *given a symplectic manifold* (M, ω) *what can be said about the class of groups* G *so that there exists a Lgrangian submanifold* $L \subset (M, \omega)$ *with* $\pi_1(L) = G$.

One way to approach this question is to analyze how the fundamental group of Lagrangians changes along Lagrangian cobordism. A first rigidity result in this direction is available.

Theorem 3. *Assume* $(V; L, L')$ *is a Lagrangian cobordism with* L *and* L' *connected. If* $QH(L) \neq 0$*, then the maps:* $i : H_1(L; \mathbb{Z}_2) \to H_1(V; \mathbb{Z}_2)$ *and* $i' : H_1(L'; \mathbb{Z}_2) \to H_1(V; \mathbb{Z}_2)$ *have the same image.*

When $\dim(L) = 2$ it is possible to say more: both i and i' are isomorphisms.

References

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