Geometrizing the Fukaya Category via Lagrangian Cobordism and Immersions

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(Oberwolfach RIP activity report)

During our stay at MFO (April 30 - May 13, 2017) we focused on a particular aspect of this program. More precisely, we considered certain metric invariants associated to Lagrangian cobordism to deduce properties of symplectic invariants associated to Lagrangian submanifolds. The work describes here is, in part, also in collaboration with Egor Shelukhin (IAS and Université de Montréal). Our stay at Oberwolfach was very productive and we would like to use this opportunity to thank the Institute and its staff for providing us with exceptional working conditions and for their hospitality.

Here is the context and the specific problem we have considered. Let M be a symplectic manifold and $L_0 \subset M$ a Lagrangian submanifold. The space $\mathrm{Lag}(L_0)$ of Lagrangian submanifolds $L \subset M$ that are Hamiltonian isotopic to L_0 carries a natural metric induced by the Hofer metric on the group of Hamiltonian diffeomorphisms of M. There is yet another interesting metric on $\mathrm{Lag}(L_0)$ which bounds the Hofer-like metric from below and is called the spectral metric. This metric is defined via spectral invariants in Lagrangian Floer theory (the earliest version of this metric was introduced by Viterbo). The comparison between the spectral metric and the Hofer-like metric is a subject of ongoing research.

Our research at MFO was focused on studying the spectral-metric diameter of $\mathrm{Lag}(N)$ in the case when $M = U^*(N)$ is the *unit cotangent bundle* of a closed manifold N. A conjecture due to Viterbo asserts that this diameter is finite. This stands in contrast to the case of the Hofer-like metric for which the diameter is expected to be infinite (the latter has been established for several case of manifolds N). Below are more details on our approach to this problem.

There is a notion of spectral invariant associated to a Floer homology class α denoted by $\sigma(\alpha)$. This notion, inspired by classical min-max methods in non-linear analysis, can be easily understood by recalling that Floer complexes [Flo] are endowed with a filtration given by the action functional. The real number $\sigma(\alpha)$ is the lowest filtration level that contains a cycle representing the class α. Assuming, that L and L' are exact, Hamiltonian isotopic Lagrangians the Floer homology $HF(L, L')$ is isomorphic to the singular homology $H(L; \mathbb{Z})$. Therefore, one can consider the difference $\sigma([L]) - \sigma([pt])$ where $[L], [pt] \in HF(L, L')$ represent, respectively, the fundamental class and the class of the point. It is well-known that this difference provides a metric on the space $\mathrm{Lag}(L_0)$. This metric is called the spectral metric.

Fix a manifold N and a choice of a unit cotangent bundle $U^*(N) \subset T^*(N)$. In what follows we concentrate on the case $M = U^*(N)$ and spectral-metric diameter of $\mathrm{Lag}(N)$ - the space of Lagrangians in $U^*(N)$ that are Hamiltonian isotopic to the zero section N.

In work of Fukaya-Seidel-Smith [FSS] it is shown that exact Lagrangians inside $U^*(N)$ admit decompositions in a derived Fukaya category that is associated to a certain Lefschetz fibration whose total space extends $U^*(N)$. The decomposition in question involves a class of exact triangles - discovered earlier by Seidel - that are associated to a geometric operation called Dehn twist. In work of Biran-Cornea [BC2, BC3] it has been shown that Lagrangian cobordism provides another geometric source of exact triangles in the derived Fukaya category. In subsequent work of the same authors [BC1], as well as in independent work of Mak-Wu [MW], it is shown that the two types of exact triangles fit together. In particular, a number of cobordism type techniques can be exploited as in [BC1] to analyze the properties of the exact triangles associated to Dehn twists. A simple quantity associated to a Lagrangian cobordism is its shadow as introduced in [CS]. This notion is easily seen to be related to invariants of spectral type. One way to see this is that by infimizing the shadow of simple cobordisms with fixed ends one gets a metric that generalizes the Hofer metric. In turn, the Hofer metric admits as lower bound the spectral metric.

All these different ingredients were tied together in the following way during our stay in Oberwolfach: we developed a filtered version of the results and constructions from [BC1] together with the construction of [FSS] as well as some ideas inspired from the properties of the shadow of cobordisms, to prove the following result:

Theorem 1. *The spectral-metric diameter is finite among all exact Lagrangians $L \subset U^*(N)$ that have the additional property that there is some $x \in N$ (depending on L) so that L intersects the fibre $U^*_x(N)$ transversely in a single point.*

This is in fact a consequence of a stronger result we have proved, which appeals to the notion of boundary depth introduced by Usher [Ush]. This result bounds from above the spectral-distance between any exact Lagrangian L in $U^*(N)$ and the zero section N by an expression of the form $C_1\beta(CF(L, U^*(N))) + C_2$ where $C_i \in \mathbb{R}$ are positive constants depending only on $U^*(N)$ and $\beta(-)$ is the boundary depth of the respective chain complex. Along the way, our understanding of the interplay between filtered homological algebra and the triangulated structure of the derived Fukaya category improved substantially and this deeper understanding promises to have interesting further applications.

References

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