Wide and Narrow Lagrangian Submanifolds

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I. Background

A.Underlying theme

$$\text{Alg. topology} \longleftrightarrow \text{Comb. of Morse traj.}$$

$$\downarrow$$

"quantization"

$$\downarrow$$

Combinatorics of Morse trajectories broken by J-hol. curves

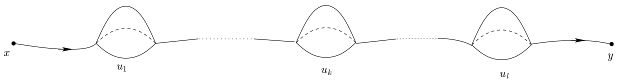


Figure: Pearls for instance.

Credit: Gromov, Floer, Witten, Donaldson, Fukaya, Oh and many others.

B. Current approaches. For $L \subset M$ general Lag. $\exists$ two algebraic machines, both based on counting elements in moduli spaces of objects modeled on trees such as:

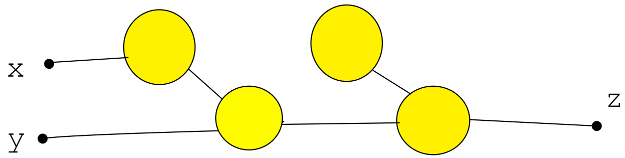


Figure: Circles represent J-disks; segments represent flow lines of a unique M. function f; the ends represent critical points of f.

Distinction due to the direction of the flow:

$\longrightarrow$	$\longleftarrow$
A_∞-structure	DGA-structure
Fukaya-Oh-Ohta-Ono, '00, '06,'08	C.- Lalonde, '04 - clusters (CDGA)

For even richer structures one uses more Morse functions along the edges.

Remarks:

- The two models are not precisely dual algebraically: hom$(-)$ misbehaves w.r. to ∞-sums ...very closely related though, in particular, $d^2 = 0$ in the DGA case corresponds to the relations among the m_k's in the A_∞ setting etc.
- In the *DGA*-model, homology is always defined, invariant w.r. to J; with "positive" coefficients, it never vanishes.
- The A_∞ model above is not the "standard" one. Rather it appears in *Canonical models of filtered A_∞-algebras and Morse complexes*, Fukaya - Oh - Ohta - Ono, arXiv:0812.1963. Cautionary note: the paper makes no mention whatsoever of clusters.
- Technical aspects (transversality !): considerable advances by Fukaya-Oh-Ohta-Ono, also remarkable work by Joyce as well as, in different (but likely adjustable) settings, by Hofer-Wisocki-Zehnder and Cieliebak-Mohnke. *Complete ???....not clear (to me at least).*

II. Monotone Lagrangians

A. Notation

- $L^n \subset (M^{2n}, \omega)$, μ=Maslov index,

$$N_L = \min\{\mu(\alpha) \ : \ \omega(\alpha) > 0\} \ .$$

- L is monotone if

$$\mu : \pi_2(M, L) \to \mathbb{Z}$$

  and

$$\omega : \pi_2(M, L) \to \mathbb{R}$$

  verify $\omega(-) = \rho\mu(-)$ for some $\rho > 0$ *and*

$$N_L \geq 2 \ .$$

- $\Lambda = \mathbf{k}[t, t^{-1}]$; $\Lambda^+ = \mathbf{k}[t]$, $|t| = -N_L$.

For a monotone Lagrangian, Floer hlgy. $HF(L) = HF(L, L; \mathbb{Z}/2)$ is well defined (Oh, '93).
All known monotone Lagr. are either:

- wide - in the sense that $HF(L) = H(L; \mathbb{Z}/2) \otimes \Lambda$ or
- narrow - in the sense that $HF(L) = 0$.

B. Some properties (Biran -C. '07-'08)

- Any two non-narrow L. in $\mathbb{C}P^n$ intersect (related results by Entov-Polterovich '07-'08, Tamarkin '08, Alston '08, earlier work by Albers).
- Normalize $\mathbb{C}P^n$ so that $w(\mathbb{C}P^n) = 1$.

$$L \subset \mathbb{C}P^n, \text{ wide} \Rightarrow w(\mathbb{C}P^n \backslash L) \leq \frac{n}{n+1}$$

(thus L is a barrier).

- $\exists$ many narrow tori in $\mathbb{C}P^n$, more with n (related to work of Chekanov-Schlenk '07).
- L monotone in $\mathbb{C}P^n$, then:

$$w(L) + w(\mathbb{C}P^n \backslash L) \leq 2 - \frac{1}{n+1}$$

(L is either a barrier or its width is $< w(\mathbb{C}P^n)$).

In short:

- Monotone L. form an interesting class of objects.
- Technically they are treatable in a reasonably direct and complete way - to be rapidly discussed next.
- They allow the discovery of phenomena which are interesting in themselves and are likely to admit generalizations to the general case. One such example - concerning enumerative invariants - will be discussed later in the talk.

C. A few technical points.

- $N_L \geq 2$ implies that in 0 and 1-dim. moduli spaces *side bubbling* is not possible. Thus, by Gromov compactification, the underlying type of trees does not change.

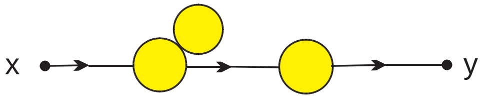


Figure: This type of bubbling is not possible.

- Important ingredient: structural results of Lazzarini ('00, 02, updated '09) (alternative results are due to Kwon-Oh '00): for instance, for generic J and $n \geq 3$ any J-hol disk is simple or multiply covered.
- As a consequence, the technical machinery necessary to work out this case is manageable - takes about 70 pages and has been described by Biran-C. ('07)).

D. Algebraic structures.

- Counting pearly trajectories leads to a chain complex

$$\mathcal{C}(L; f, J, g) = (\mathrm{Crit(f)} \otimes \Lambda, \mathrm{d})$$

whose homology $QH(L;\Lambda)$ is independent of J, the Morse function f and the R. metric on L g and $QH(L) \cong HF(L)$ (this complex was proposed by Oh, '94).

- $QH(L)$ comes with a product structure:

$$* : QH(L) \otimes QH(L) \to QH(L) \ .$$

Moreover, $QH(L)$ is an algebra over $QH(M)$ etc.

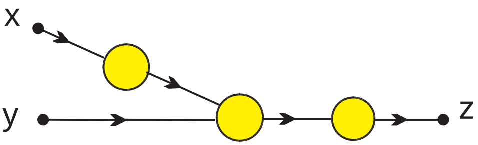


Figure: Configurations with two entries and one exit - as exemplified above - give the product (multiple functions are used here).

Additional useful notation:

$$\widetilde{\Lambda} = \mathbf{k}[\pi_2(M, L)] \ , \ \widetilde{\Lambda}^+ = \mathbf{k}[\pi_2(M, L)^+]$$

where $\pi_2(M, L)^+$ is the monoid generated by $\{\alpha \in \pi_2(M, L) : \omega(\alpha) > 0\}$; the grading is $|\alpha| = -\mu(\alpha)$,

Remarks.

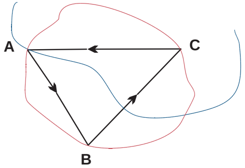
- All the structures above can be defined over the coefficient rings: $\widetilde{\Lambda}$ and $\widetilde{\Lambda}^+$ (at least when $\mathbf{k} = \mathbb{Z}/2$; if L has a fixed rel. spin structure, then one can use $\mathbf{k} = \mathbb{Z}$).
- The fact that one can work over $\widetilde{\Lambda}^+$ is crucial in applications as it allows for inductive arguments (note that this is *not* possible when working with Floer hlgy.).
- $QH(L; \widetilde{\Lambda}^+)$ is a very rich object, never vanishes and contains a lot of torsion.
- Any ring map $\widetilde{\Lambda}^+ \to \mathcal{R}$ can be used to change coefficients if convenient.

III. Enumerative invariants.

Wide Lagrangians are good test cases for enumerative invariants: $QH(-) \simeq H(-)$ as in the closed case; as GW invariants can be extracted from the quantum p. in the closed case, same can be expected here. *A bit naive !*

A.An example.

2-torus $\mathbb{T} \hookrightarrow M^4$,fixed spin str. , *wide*; J a.c str. (generic); $\Delta = ABC$ be a triangle on $\mathbb{T}$; $n_A = \#$ of J-disks of Maslov 2 going through A and crossing (transversely) the opposite edge; $n_\Delta = \#$ of J-disks of Maslov 4 going through A, B, C (in this order).



$$\Delta_L = n_A^2 + n_B^2 + n_C^2 - 2(n_A n_B + n_A n_C + n_B n_C) + 4n_\Delta$$

is invariant .... but n_A, n_B, n_C, n_Δ individually are not.

Remarks.

- In the monotone case, as there are no disks of Maslov class lower than N_L, the moduli space of disks of Maslov N_L produces, by evaluation, some obvious numerical invariants. For instance, if $N_L = 2$ the # of J-disks with $\mu = 2$ through some point $P \in L$ is independent of J and P (for generic J). The story is sharply different for higher Maslov numbers.
- Other variants of enumerative/GW invariants in the open case (different from what will be discussed further): Welschinger ('04), (relevant related work by Solomon ('07), Ceyhan ('07)), Joyce ('08), Fukaya ('09), Iacovino ('09) (much of this in the Calabi-Yau case)....

Next: *interpret Δ_L, why is it invariant ? why is it interesting ? are there other such polynomial expressions that are invariant ? NO !*

B. Varieties of representations.

Assumption: L relatively spin (with fixed structure); $\mathbf{k} = \mathbb{C}$. Any morphism $\rho : \pi_2(M, L) \to \mathbb{C}^*$ induces a ring morphism:

$$\rho' : \widetilde{\Lambda}^+ \to \mathbb{C}[t] \ , \alpha \to \rho(\alpha) t^{\mu(\alpha)/N_L} \ .$$

Let $QH^\rho(L)$ be the Q.H. obtained by changing coefficients via ρ'.

L is said *ρ-wide* if $QH^\rho(L) \cong H(L; \mathbb{C}) \otimes \mathbb{C}[t]$.

$$W(L) = \{\rho : \pi_2(M, L) \to \mathbb{C}^* \ : \ \rho \text{ group morphism}\}$$

$$W_2(L) = \{\rho \in W(L) \ : \ L \text{ is } \rho - \text{wide}\}$$

$$W_1(L) = \{\rho \in W_2(L) \ : \ \exists \hat{\rho}, \ \rho = \pi_2(M, L) \to \pi_1(L) \xrightarrow{\hat{\rho}} \mathbb{C}^*\}$$

Remarks

- $W_i(L)$ are (quasi)-algebraic varieties; their properties should reflect properties of L (reminiscent of techniques in 3-dim. topology initiated by Culler- Shalen '83).
- These varieties contain an interesting integral structure (given by the integral representations & because all defining equations have integral coefficients).
- Abelian case only here...

Let $\mathcal{O}_i = \mathcal{O}(W_i(L))$ be the ring of regular functions on $W_i(L)$ and

$$\psi_i(L) : \widetilde{\Lambda}^+ \to \mathcal{O}_i[t] \ , \alpha \to \tilde{\alpha}, \tilde{\alpha}(\rho) = \rho(\alpha) t^{\mu(\alpha)/N_L}.$$

Denote by $QH(L; \mathcal{O}_i[t])$ the resulting quantum homology.

Proposition

Any monotone Lagrangian L is $\psi_i(L)$-wide. In other words: there always is an isomorphism:

$$\eta : QH(L; \mathcal{O}_i[t]) \cong H(L; \mathbb{C}) \otimes \mathcal{O}_i[t] \ .$$

Remarks.

- The ring $\mathcal{O}_i[t]$ recovers much of the richness of $\widetilde{\Lambda}^+$ without the problems.
- With coefficients in $\mathcal{O}_i[t]$, all the further structures - quantum product, module structure over $QH(M)$ etc - are deformations (of formal deformation parameter t) of the corresponding singular structures.
- The identification η above is not canonical !

C. Deformation point of view.

We will drop the index i.

Fix a basis $\{a_r\} \in H_*(L;\mathbb{Z})$ and use

$$\eta : QH(L;\mathcal{O}[t]) \cong H(L;\mathbb{Z}) \otimes \mathcal{O}[t]$$

to write the quantum product (with coefficients in $\mathcal{O}[t]$) as:

$$a_r * a_s = \sum m_l^{r,s} a_l t^{\epsilon(r,s,l)}$$

with $\epsilon(r,s,l) = (n + |a_l| - |a_r| - |a_s|)/N_L$, $m_l^{r,s} \in \mathcal{O}[t]$.

The $m_l^{r,s}$ are formally the analogues of the (triple) GW invariants in the closed case - *but in this case they are not invariant* ! They depend on η and this depends on J and the rest of the data...

A natural question is: *do there exist some polynomial expressions in these coefficients that are invariant ?* We'll see:YES, but, essentially, just one !

Useful to rewrite the quantum product.

- clearly $H(L;\mathcal{O}(L)) = H(L;\mathbb{Z}) \otimes \mathcal{O}(L)$
- and $\mathcal{O}(L)[t] = \mathcal{O} \otimes \mathbb{C}[t]$

$x, y \in H(L;\mathbb{Z})$,

$$x \ *_{\eta} y = x \cdot y + \phi_1^{\eta}(x,y)t + \ldots + \phi_s^{\eta}(x,y)t^s + \cdots$$

Here $\phi_k^{\eta} : H(L;\mathcal{O}(L)) \otimes H(L;\mathcal{O}(L)) \to H(L;\mathcal{O}(L))$ are billinear maps (of the correct degree).
Thus our product becomes:

$$*_{\eta} : (H(L;\mathcal{O}(L)) \otimes \mathbb{C}[t]) \otimes (H(L;\mathcal{O}(L)) \otimes \mathbb{C}[t]) \to H(L;\mathcal{O}(L)) \otimes \mathbb{C}[t] \, .$$

This writing depends on $\eta = \eta_{J,f,g}$. A different choice of data J', f', g' together with homotopies comparing J, f, g to J', f', g' produces another expression of the q. product.
Any two such products are conjugated by an isomorphism

$$\psi : H(L;\mathcal{O}(L)) \otimes \mathbb{C}[t] \to H(L;\mathcal{O}(L)) \otimes \mathbb{C}[t]$$

with

$$\psi = id + \psi_1 t + \ldots \psi_k t^k + \ldots \ .$$

In short:

- each product $*_\eta$ is a deformation of the singular intersection product on $H(L; \mathcal{O}(L))$ (with formal parameter t).
- two such products are equivalent - again in the sense of classical deformation theory (see Gerstenhaber '64) - by an equivalence belonging to a a group $\mathrm{Iso}^G(L)$ formed by those deformation equivalences induced by changes in data $(J, f, g) \to (J', f', g')$.

Remark. The group $\mathrm{Iso}^G(L)$ is a subgroup, possibly proper, of the group $\mathrm{Iso}^A(L)$ of all *algebraic* equivalences (this only depends on $H(L; \mathcal{O}(L))$).

D. Hochschild co-homology and Quadratic Forms.

A associative graded, commutative algebra over a (non-graded) ring $\mathcal{R}$.

$$CH^{k,*}(A,A) = \hom((s^1A)^{\otimes_k}, A)$$

with $df(x_1,x_2,\dots x_{k+1}) = x_1 f(x_2,\dots x_{k+1}) + \dots \pm f(x_1 + \dots, x_i x_{i+1}, \dots x_{k+1}) + \dots \pm f(x_1,\dots,x_k)x_{k+1}$.
Hochschild cohomology of A: $HH(A,A) = H^*(CH(A,A))$.

A deformation $\widetilde{A}$ of A is an associative product structure on: $A \otimes \mathbf{k}[[t]]$ so that

$$x \odot y = xy + \phi_1(x,y)t + \dots \phi_s(x,y)t^s + \dots \ .$$

Our setting is graded: A is graded and we assume $A_* = 0$ for $* \notin \{0,\dots,n\}$, A_n is generated by the unit $1 \in A_n$ and t is of even degree $-N$ ($N \geq 2$).

Let $\mathcal{A}lg(A,N)$ = set of assoc. deformations of A of formal parameter of degree $-N$; $\mathrm{Iso}^A(A)$ = set of equivalences of such deformations.

By results of Gerstenhaber, the first $\phi_r \neq 0$, is a HH cycle. and its class $[\phi_k] \in HH^{2,kN-2}(A,A)$ "provides" (with some technical nuances) a map:

$$\Phi_A : \mathcal{A}lg(A,N)/\mathrm{Iso}^A(A) \to HH^2(A,A)\ ,\ \widetilde{A} \to [\phi_k]\ .$$

Let $Q^2(N,\mathcal{R})$ be the $\mathcal{R}$-valued quadratic forms defined on N.

Let $V \subset A_{n-\frac{kN}{2}}$ be an $\mathcal{R}$-submodule so that $\forall v \in V,\ v^2 = 0$. Set

$$\Psi_V : HH^{2,kN}(A,A) \to Q^2(V,\mathcal{R})\ ,\ \Psi_V(\phi)(x) = < 1^*, \phi(x,x) >$$

where $\phi : A \otimes A \to \mathcal{R}$ is of degree kN; $1^* \in A^*$ is the hom-dual of 1; $< -, - >$ is the evaluation.

Indeed, if $\phi' \sim \phi$, let $df = \phi - \phi'$ and $< 1^*, \phi(x,x) - \phi'(x,x) > = < 1^*, f(x) \cdot x - f(x \cdot x) + x \cdot f(x) > = 0.$

In short: There exists a "secondary squaring" map:

$$sq_V : \mathcal{A}lg(A, N)/\mathrm{Iso}^A(A) \to Q^2(V, \mathcal{R}) \ , \ sq_V = \Psi_V \circ \Phi_A \ .$$

Remarks.

- For instance, we may take $V = A_{n-s}$ whenever s is odd of the form $kN/2$ (of course, this requires that both $N/2$ and k be odd). We denote the resulting squaring operation by sq_s.
- The operation sq_1 is always defined (any deformation of formal deformation parameter of even degree $-N$ can be also seen as a deformation of formal def. parameter of degree -2).

E. Back to Lagrangians.

We apply the secondary squaring discussed before to the algebra $A = H(L; \mathcal{O}(L))$ and the deformation $\widetilde{A}^{\eta}(L)$ given by the q. p. on $QH(L; \mathcal{O}(L))$ together with the isomorphism η; $\mathcal{R} = \mathcal{O}(L)$; $N = N_L$.

- For a fixed monotone Lagrangian L and any $V \subset H_{n-\frac{kN_L}{2}}(L; \mathcal{O}(L))$ so that $\forall v \in V$, $v \cdot v = 0$, the form $sq_V(L) \in Q^2(V, \mathcal{O}(L))$, $sq_V(L) = sq_V(\widetilde{A}^{\eta}(L))$ is well defined and independent of η.
- For a fixed $N = 4r + 2$ and every monotone Lagrangian L with $N_L = N$ the quadratic forms $sq_s(L)$ (for s an odd multiple of $2r + 1$) verify the same properties.
- For every monotone Lagrangian the quadratic form $sq_1(L)$ is well-defined and invariant. This form only depends on the HHlgy class $\tilde{L} = \Phi(q.p.)$ associated to $QH(L; \mathcal{O}(L))$. It reduces to the form $x \to < [L]^*, x * x >$, $\forall x \in H_{n-1}(L)$.

It is not difficult to show that:

- In certain cases ($A = H(T^n; \mathbb{C})$ for instance) the map

$$sq_1 : \mathcal{A}lg(A, 2)/\mathrm{Iso}^A(A) \to HH^2(A, A) \to Q(A_{n-1}, \mathcal{R})$$

  is injective.
- There are tori (Clifford in $\mathbb{C}P^2$ for instance) so that $\mathrm{Iso}^A(L) = \mathrm{Iso}^G(L)$.

Denote by $\widetilde{\Delta}_L = \mathrm{discr}(sq_1(L)) \in \mathcal{O}$; $\Delta_L = \widetilde{\Delta}_L|_{tr}$ (if defined).

Corollary

Any enumerative invariant that:

- *is defined for all monotone Lagrangians which are wide with respect to the trivial representation;*
- *is polynomial in the coefficients of the quantum product (for the trivial representation);*
- *is integral;*

coincides with (a multiple of) Δ_L.

The reason for uniqueness is that such an invariant has to be read off the coefficients of $sq_1|_{tr}$. By work of D. Hilbert 1900, the only invariant polynomial in the coefficients of an integral quadratic form (up to multiples) is the discriminant.

With some work, $\Delta_L = \tilde{\Delta}_L|_{tr}$ is shown to coincide with the Δ_L associated to triangles...

<u>Remarks</u>

- For a fixed L the discriminant $\tilde{\Delta}_L \in \mathcal{O}(L)$ is an interesting object in itself as we shall see.
- The quadratic form $sq_1(L)|_{tr}$ also appears in a somewhat different context in work of Cho, '05 (who also did pioneering work in studying various toric fibers).

F.Free loop spaces.

Fix L monotone and J a.c.s. To simplify assume $N_L = 2$. Let $\mathcal{M}(2, J)$ be the moduli space of J-disks u with one marked point and $\mu(u) = 2$. Denote by ΛL the free loop space of L and denote by $C(L)$ the singular chains on L. There exists a representation:

$$\rho_J : \mathcal{M}(2, J) \to \Lambda L \quad \Rightarrow \quad \alpha_J \in H_n(\Lambda L; \mathbb{Z})$$

There exists a map constructed by Jones '87:

$$J : H_*(\Lambda L) \to HH^{\bullet, *-n}(C(L), C(L))$$

In favorable cases (for tori in particular) $HH^{s,*}(C(L), C(L)) \cong HH^{s,*}(H(L; \mathbb{Z}), H(L; \mathbb{Z}))$ and then for wide$|_{tr}$ tori

$$J(\alpha_J)|_{HH^{2,*}(H(L;\mathbb{Z}),H(L;\mathbb{Z}))} = \tilde{L}|_{tr}$$

Thus the loop representation suffices to determine $sq_1(L)|_{tr}$.

Remarks.

- There is related work of Fukaya '04.
- Jones' map can be enriched by the coefficients $\mathcal{O}_i(L)$ and the result extends to this situation as well as to $N_L > 2$.

G. Toric fibers.

We will assume that $L = T^n$ is a toric fiber, not necessarily monotone but in a Fano manifold.

No disks of Maslov class lower than 2 exist - by Cho - Oh '03 - so that the machinery described above can be applied. Thus $N_L = 2$; J is the standard a.c. structure (or close to it).

Fix a basis e_i of $\pi_1(L) \otimes \mathbb{C}$. The Landau-Ginzburg superpotential:

$$\xi_L(z_1, \ldots, z_n) = \sum_{\alpha} n(\alpha) z_1^{p_1} \ldots z_n^{p_n}$$

is defined by: $\alpha = \sum p_i e_i \in \pi_1(L) \otimes \mathbb{C}$ and $n(\alpha) = \#$ of J-disks of Maslov 2 in the class α.

Easy to see:

- $W_1(L) = \mathrm{Crit}(\xi_{\mathrm{L}})$
- $sq_1(L)$ is the quadratic form associated to the symmetric bilinear form $(z_1 \ldots z_n)^2 Hess(\xi_L)$;
  $\widetilde{\Delta}_L = (z_1 \ldots z_n)^2 det(Hess(\xi_L))$.

Due to Batyrev '93, Givental and Fukaya-Oh-Ohta-Ono '08 there is an isomorphism:

$$I : QH(M; \Lambda) \to Jac(\xi_L) \otimes \Lambda = \mathcal{O}_1(L) \otimes \Lambda$$

($Jac(-)$ is the Jacobian ring). Harder to see (and a number of technical points still left to check):

- $I(PD(e_Q)) = -\widetilde{\Delta}_L t^n$

where e_Q is the Euler quantum class of Abrams '97.

Remark. There is a closely related result announced by Fukaya '08 - it does not involve Δ_L but seems to be roughly equivalent to our statement. The proof appears to be somewhat similar.