CLUSTER HOMOLOGY

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Introduction.

We associate, to a Langrangian submanifold L of a symplectic manifold, a new homology called the *cluster homology*, which provides an invariant of the Hamiltonian isotopy class of L. This leads to various applications concerning analytical, topological, and dynamical properties of Lagrangian submanifolds, and to a new universal Floer homology, defined without obstruction, for pairs of Lagrangian submanifolds.

Throughout this paper, we shall assume that all Lagrangian submanifolds are connected, orientable, and relatively spin (recall that a Lagrangian submanifold $L \subset (M, \omega)$ is relatively spin if the second Steifel-Withney class of L admits an extension to $H^2(M; \mathbb{Z}_2)$; a set of Lagrangian submanifolds is relatively spin if their

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second Stiefel-Whitney classes admit a common extension to $H^2(M;\mathbb{Z}/2)$). Actually, the notation L for a Lagrangian submanifold will always implicitely contain the information of a choice of an orientation and of a relatively spin structure - defined as in [10] (and the same applies for a set of such submanifolds). The ambient symplectic manifold (M^{2n},ω) is supposed to be compact or, if not, it should be either convex at infinity or geometrically bounded in the sense of [3], so that no sequence of Riemann surfaces with boundary lying on a set of compact Lagrangian submanifolds $L_1,\ldots,L_\ell\subset M$ can escape to infinity. In fact, the Lagrangian submanifolds need not all be compact, as long as the above control on sequences of Riemann surfaces is ensured.

Our construction starts with the remark that the usual coefficient rings used to define Floer type complexes are not rich enough to properly encode the bubbling of disks. Thus, for a relatively spin Lagrangian submanifold L in an a symplectic manifold (M,ω), we first introduce a rational differential graded algebra, *the cluster algebra* of L, which manages algebraically the bubbling of pseudoholomorphic disks with boundary on L. For two relatively spin Lagrangian submanifolds L, L', we essentially use the tensor product of the cluster algebras associated to L and to L' as a new coefficient ring of a Floer type homology which we call the *fine Floer* homology of the pair (L,L').

The cluster algebra of L, that we shall denote by $\mathcal{C}\ell_*(L,J,f)$, depends on auxiliary data; these data include, in particular, the choice of an almost complex structure J compatible with the symplectic form ω, and the choice of a Morse-Smale pair (f,g) consisting of a Morse function $f:L\to\mathbb{R}$ and of a Riemannian metric g on L. As an algebra over an appropriate (rational) Novikov ring, it is graded commutative and is freely generated, as a symmetric algebra, by the critical points of the Morse function f. Its homology $\mathcal{C}\ell H_*(L)$, called *the cluster homology*, is of interest in itself: we show that it is independent of J and (f,g) and is invariant with respect to symplectic diffeomorphisms. In particular, $\mathcal{C}\ell H_*(L)$ is an invariant of the Hamiltonian isotopy class of the Lagrangian embedding $L\subset M$.

The key idea in the definition of the cluster algebra has its origin in the observation that the main difficulty in modelling algebraically the bubbling of disks is that this is a non-localized co-dimension one phenomenon. In other words, the bubbling of disks produces boundary components - which, from our point of view, is not problematic - but in contrast with the usual breaking of Morse flowlines, the points where the bubbling appears are arbitrary. The solution that we propose to this problem is, in essence, to enlarge the moduli space of pseudo-holomorphic disks by allowing configurations formed of disks connected by negative gradient flow lines of an a priori fixed Morse function on our Lagrangian submanifold. These new moduli spaces, called *clustered moduli spaces*, are large enough to transform the bubbling of disks into an internal, co-dimension one phenomenon. The compactifications of these spaces certainly have boundaries but they consist only of pairs of configurations joined at some critical point of the fixed Morse function.

In short, in this way, the bubbling of disks is absorbed away from the boundary components of our big moduli spaces – the boundary components of the latter are now only associated to the ordinary breaking mechanism of flow lines of the fixed Morse function at some critical point. Since these points are known and are actually the algebra generators of our cluster complex, this enables us to define a differential d satisfying $d^2 = 0$ as well as the Leibniz rule. In § 2.1, we will describe in detail these moduli spaces. They are made of smaller "pieces" glued together along various codimension 0 strata of the boundaries, which motivates the name "cluster" .

Once the cluster algebra introduced and its main properties identified, it is not difficult to imagine how to define a universal Floer theory, the *fine Floer homology*, asssigned to the pair formed by two relatively spin Lagrangian submanifolds $L, L' \subset (M, \omega)$ in general position: it is the homology associated to a chain complex freely generated by (possibly, some of) the intersection points of L and L' over a ring roughly (but not exactly) of the form $\mathcal{C}\ell_*(L, f, J) \otimes \mathcal{C}\ell_*(L', f', J)$. As we shall see, this idea may be successfully pursued and the putative differential of the fine Floer chain complex has indeed a vanishing square. This vanishing, however, is due to a cancellation resulting from of subtle phenomenon that has appeared before just as a nuisance in standard Floer theory. In its simplest form, this phenomenon is seen in a one dimensional space of pseudoholomorphic strips resting on L and on L' and joining an intersection point $x \in L \cap L'$ to itself. It consists in the fact that such a moduli space might admit as boundary point a pseudoholomorphic disk with boundary on L and passing through x.

When L' is Hamiltonian isotopic to L, it is possible to identify the cluster algebras of L and L' and to use the resulting symmetry to simplify the construction. We call *symmetric fine Floer homology* the resulting homology which, in general, is different from the fine Floer homology. We establish a natural algebraic relation between the cluster and the symmetric fine Floer homologies, called the *strip-string symmetrization*. The definition of the fine Floer homology, symmetric or not, makes it obvious that it vanishes for a disjoint pair L, L'. Using this fact, we prove that, under additional purely algebraic topological restrictions, a relatively spin, orientable Lagrangian submanifold that can be displaced from itself by a Hamiltonian isotopy is uniruled. This means that through each point p of such a Lagrangian submanifold $L \subset (M, \omega)$, and for any ω-compatible almost complex structure J, there is a J-holomorphic disk whose boundary on L passes through p. In fact, there is also a natural action associated to the fine Floer complex and using it we show that the disks detected above are of symplectic area at most equal to the disjunction energy of L. An interesting geometric consequence of this fact, along ideas from Barraud-Cornea [4], is that the displacement energy of such a Lagrangian submanifold L admits as lower bound the square of the real Gromov radius of L (see §4.3) multiplied by $\pi/2$. We will also discuss a different application to the detection of periodic orbits of Hamiltonian flows. This is based on the construction of a chain morphism relating the cluster homologies of two Lagrangian submanifolds L and L' by making use of moduli spaces which integrate the clusters on L and L' with J-holomorphic cylinders with their ends on the two Lagrangian submanifolds, thus pursuing the work in Gatien-Lalonde [12].

A theory such as this one consists of a certain number of - by now familiar - levels:

a. a geometric phenomenon is modeled by introducing certain moduli spaces;
b. compactness properties of these moduli spaces are proved;

c. regularity properties of the moduli spaces are established - in general by proving certain transversality results;
d. the structure of the moduli spaces is used to define (hopefully efficient) algebraic invariants.

To make the paper as readable and, at the same time, as complete as possible, we will present here reasonably complete proofs covering the points a, b and d as well as those of a number of applications, but we will postpone the analytical theorems concerning the point c to a forthcoming article based on Hofer-Wisocki-Zehnder's new scale-polyfold Fredholm theory. Again, to improve readability, in the first section we outline many of our constructions and we also prove a number of results which follow rapidly from the general properties of the objects introduced. We come back to the technical details in later sections.

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1. Overview of the main constructions and results.

1.1. **Definition and properties of the cluster complex.** The cluster complex is associated to a triple formed by (1) a Lagrangian embedding $L^n \hookrightarrow (M,\omega)$, equipped with a choice of an orientation and of a relatively spin structure (which, we recall, is always assumed in the notation L), (2) a generic almost complex structure J on M compatible wth the symplectic form ω, and (3) a pair (f,g) with $f : L \to \mathbb{R}$ a Morse function and g a Riemannian metric on L so that (f,g) is Morse-Smale. The two conditions implicit in the notation L (i.e the orientation and the relative spin structure) are needed to orient the clustered moduli spaces - to be described below - in a coherent way (see § 2.1.5 for the discussion of the orientations).

This complex is denoted by $\mathcal{C}\ell_*(L,J;(f,g))$ and, if $\mathrm{Crit}(f)$ is the set of critical points of f, we set:

$$\mathcal{C}\ell_*(L,J;(f,g)) = (S\mathbb{Q} < s^{-1}\ \mathrm{Crit}(f) > \otimes \Lambda)^\wedge$$

where $s^{-1}\ \mathrm{Crit}(f)$ indicates that the natural index grading of $\mathrm{Crit}(f)$ is decreased by one unit, SV is the free, graded commutative algebra over the graded vector space V (as usual, the sign commutativity rule is $ab = (-1)^{|a||b|}ba$ for any two elements $a, b \in V$), $^\wedge$ indicates a certain completion described below and Λ is an appropriate group ring defined as follows: on $\pi_2(M,L)$, consider the equivalence relation $\lambda \sim \tau$ iff $\omega(\lambda) = \omega(\tau)$ and $\mu(\lambda) = \mu(\tau)$, where ω and τ are the area and Maslov classes respectively. The ring Λ is the rational group ring of $\pi_2(M,L)/\sim$. We write the elements of Λ under the form of finite sums $\sum_i c_i e^{\lambda_i}$, $c_i \in \mathbb{Q}$. Clearly, given the equivalence relation on $\pi_2(M,L)$, the variable e^λ could as well be replaced by $t_1^\mu t_2^a$ where μ, a are the Maslov and area numbers in the ranges of

$$\mu : \pi_2(M,L) \to \mathbb{Z} \qquad\qquad \omega : \pi_2(M,L) \to \mathbb{R}.$$

The grading in Λ is given by $|e^\lambda| = -\mu(\lambda)$ for $\lambda \in \pi_2(M,L)/\sim$. With this convention, the grading of the cluster complex is given by the usual tensor product formula. Thus for $x_i \in \text{Crit}(f)$, we have

$$|x_i| = \text{ind}_f(x_i) - 1 \ , \ |x_1 \dots x_k e^{\lambda}| = \sum_{i=1}^{k} |x_i| - \mu(\lambda) \ .$$

Finally, the completion $\wedge$ is obtained by first completing $S\mathbb{Q} < s^{-1}\text{Crit}(f) >$ in the sense of formal series thus getting a new algebra S'. We then consider $S' \otimes \Lambda$ and complete it in the usual Novikov sense: any sum $\sum a_\lambda e^\lambda$ with $a_\lambda \in S'$ has the property that, for any $k \in \mathbb{R}$, there are only finitely many $a_\lambda \neq 0$ when $\omega(\lambda) \leq k$.

In particular, notice that an element $m \in \mathcal{C}\ell\,(L, J; (f, g))$ can be written as a possibly infinite sum:

$$m = m_0 + m_1 e^{\lambda_1} + \ldots + m_i e^{\lambda_i} + \ldots$$

where m_i are monomials in the elements of $\text{Crit}(f)$ but if this sum is infinite, then any infinite subsequence with $\omega(\lambda_i)$ bounded above, must have its corresponding word length sequence converging to infinity (in the sense that any of its subsequences is unbounded). Conversely, any formal sum verifying this condition and belongs to the cluster complex if it also verifies the Novikov type condition concerning the lower bound for the area filtration.

In § 2.1 we will introduce in detail the moduli spaces on which is based the definition of the cluster differential. We now summarize the construction.

The generic data $J, (f, g)$ on L being given, fix an order on the critical points of f. Choose any integer $k \geq 0$, any $x \in \text{Crit}(f)$, and any sequence of critical points $x_1, \ldots, x_k$, and $\lambda \in \pi_2(M, L)/\sim$, with the sole constraint that the zero class $\lambda = 0$ is allowed only when k equals 1. Consider then the space $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ consisting of all configurations of the following kind. Let $\mathcal{T}$ be a finite connected, planar tree with oriented edges such that one and only one vertex has no ingoing edge (it is called the *root*) and all other vertices have exactly one ingoing edge. All vertices can have any number ≥ 0 of outgoing ones. For each edge $\alpha\beta$ (the writing means that the edge joins the vertex α to the vertex β in this order), let $t_{\alpha\beta}$ be a real number in $[0, \infty)$ and $z_{\alpha\beta}, z_{\beta\alpha}$ be two points on the boundary of the unit disk D in the $\mathbb{C}$-plane. Consider now the abstract topological space $\mathcal{T}'$ obtained by replacing each vertex γ by a copy D_γ of the unit disk D, view the point $z_{\alpha\beta}$ as a point of ∂D_α and likewise $z_{\beta\alpha}$ as a point of ∂D_β, and replace each edge $\alpha\beta$ by an edge joining the point $z_{\alpha\beta} \in \partial D_\alpha$ to the point $z_{\beta\alpha} \in \partial D_\beta$. The points $z_{\alpha\beta}, z_{\beta\alpha}$ are called *incidence points.*

Let $z, z_1, \ldots, z_k$ be $k+1$ additional *marked points* lying on the boundaries of the disks D_γ's so that z belongs to the boundary of the root disk. The special points on each disk (formed by the marked points and the incidence points) are all distinct. The ordering of all the special points on a single disk is the natural order coming from the orientation of the disk and the planarity of the tree. For the points z, z_i the order in which they appear on the respective disks is part of the structure. The order of the points $z_1, z_2, \ldots z_k$ reflects the natural order (from left to right when the root is up) induced by the planarity of the tree together with the orientation order around each disk.

Now consider the pair $(\mathbf{u}, \mathbf{z})$ where $\mathbf{z}$ is the sequence $z, z_1, \ldots, z_k \in \mathcal{T}'$ and $u : \mathcal{T}' \to M$ is a map which satisfies the following properties:

1. for each γ, $u_\gamma := u|_{D_\gamma}$ is a J-holomorphic map from $(D_\gamma, \partial D_\gamma)$ to (M, L) whose class is denoted by $\lambda_\gamma \in \pi_2(M, L)$;
2. the restriction of u to each edge $\alpha\beta$ is an integral curve of the negative gradient vector field of (f, g) when the edge is parametrized by the interval $[0, t_{\alpha\beta}]$;
3. the sum of the classes λ_γ over all vertices is equal to λ;
4. the point $u(z)$ belongs to the unstable manifold of the critical point x while each point $u(z_i), 1 \leq i \leq k$, belongs to the stable manifold of x_i (i.e one may view this as an extension of the tree by adding one semi-infinite outgoing flowline from x to the root disc at z, and k semi-infinite outgoing flowlines from the z_i's to the x_i's).

There are other more technical constraints related to (i) stability (ghost disks), (ii) codimension ≥ 2 phenomena - which we have neglected in the description above, and (iii) virtual perturbations with coherent orientations, see § 2.1 for more detail and § 2.1.5 where the orientations of these moduli spaces are described. It is worth noticing here that there are two transversality issues in this theory: the first one is the usual lack of transversality due to multiple coverings of disks; the second one arises when two special points meet. In this second case, the problem is that the two cluster constraints become identified at that meeting point, so this has the undesirable effect of increasing the dimension of the moduli space at that point. In both cases, the problem is solved by the Deligne-Mumford setting at the source, which produces ghost disks whenever needed (see § 2 for details). Note that, in contrast with the absolute case, the second transversality problem arises in generic one-dimensional families, hence can never be neglected in our theory.

The space $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ is then the topological space made of all pairs $(\mathbf{u}, \mathbf{z})$ quotiented out by the natural automorphisms. These moduli spaces are our basic building blocks. They admit natural compactifications $\bar{\mathcal{M}}^x_{x_1,\ldots,x_k}(\lambda)$. Their "boundary" points correspond either to a "break" of a flow line or to a crossing of two marked points (the bubbling of a disk being an internal phenomenon). A point of the second type is also a boundary point for some moduli space $\bar{\mathcal{M}}^x_{x_{\sigma(1)},\ldots,x_{\sigma(k)}}(\lambda)$ for some $\sigma \in \Sigma_k$. To write the resulting boundary formulas in a shorter more convenient way it is useful to form larger moduli spaces as follows (this is also useful when considering perturbations).

We fix a strict linear order on the set of critical points of f and for a partially ordered set of critical points S (in which repetitions are allowed) and which respects the fixed order, $S = (x_1, x_2, \ldots x_k)$, we let

$$\mathcal{M}^x_S(\lambda) = \mathcal{M}^x_{(x_1,\ldots,x_k)}(\lambda) = \bigcup_{\{x_{i_1},\ldots,x_{i_k}\}=\{x_1,\ldots,x_k\}} \mathcal{M}^x_{x_{i_1},\ldots,x_{i_k}}(\lambda) \ . \tag{1}$$

This union is not disjoint in general and admits a natural topology. If orientations are taken into account, signs will naturally occur here.

If S', S'' are two such ordered subsets, we will denote by $< S' \cup S'' >$ the ordered subset made of the elements in $S' \cup S''$. Letting S be the ordered set $(x_1, \ldots, x_k)$, the points in the top dimensional strata of $\bar{\mathcal{M}}^x_S(\lambda) \backslash \mathcal{M}^x_S(\lambda)$ can be identified with pairs belonging to some $\bar{\mathcal{M}}^x_{x_1,\ldots x_{i-1},y,x_{i+1},\ldots,x_t}(\lambda') \times \bar{\mathcal{M}}^y_{x'_1,\ldots x'_l}(\lambda'')$ where $\lambda' + \lambda'' = \lambda$, $y \in \mathrm{Crit}(f)$, and $< \{x_1, \ldots, x_t\} \cup \{x'_1, \ldots x'_l\} >= S$. The corresponding moduli spaces obtained by using the perturbed equation $\bar{\partial}_J u = \nu(u)$ are denoted by ${}^{\nu}\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$. If S does not contain a repetition of a critical point of odd degree it is possible to find perturbations so that the moduli spaces ${}^{\nu}\bar{\mathcal{M}}^{x}_{S}(\lambda)$ carry the structure of oriented singular manifolds with boundary, admitting a fundamental class over the rationals (with boundary) – see Definition 1 of § 2.1 for the definition of the orientations of these moduli spaces. The dimension of ${}^{\nu}\mathcal{M}^{x}_{x_1,\ldots,x_k}(\lambda)$ equals

$$|x| - \sum_{i=1}^{k} |x_i| + \mu(\lambda) - 1. \tag{2}$$

Moreover, for those moduli spaces of dimension 1, we have:

$$\begin{aligned}\partial({}^{\nu}\bar{\mathcal{M}}^{x}_{S}(\lambda)) &= \\ &= \bigcup_{S=<S'\cup S''>,y,\lambda'+\lambda''=\lambda} ({}^{\nu}\bar{\mathcal{M}}^{x}_{<S',y>}(\lambda')) \times ({}^{\nu}\bar{\mathcal{M}}^{y}_{S''}(\lambda'')).\end{aligned} \tag{3}$$

Here the summation is taken over all $y \in \mathrm{Crit}(f)$, all partitions of S into two subsets S', S'', all the splittings of λ as the sum of two classes λ', λ'' and all the ways y can be inserted among the elements of S' so as to produce $< S', y >$ (this last point is only significant when y is already present in S'; for example, if y appears precisely once in S' on position i, there are two ways to add y to S' so as to get $< S', y >$: it can be inserted on position i or on position $i+1$). When S does not contain a repetition of an element $x \in \mathrm{Crit}(f)$ with $|x|$ odd, the terms in the right hand side also admit a natural orientation and this equation holds with certain signs. The orientation on the left hand side is of course the one induced on the boundary.

Of course, by Gromov's compactification theorem, only a finite number of the moduli spaces appearing on the right hand side of the last equation are non-empty.

Given the moduli spaces described above, we define the cluster differential

$$d^{\nu} : (S\mathbb{Q} < s^{-1}\ \mathrm{Crit}(f) > \otimes\ \Lambda)^{\wedge}_{*} \to ((S\mathbb{Q} < s^{-1}\ \mathrm{Crit}(f) >) \otimes\ \Lambda)^{\wedge}_{*-1}$$

as the unique commutative, graded differential algebra extension of:

$$d^{\nu} x = \sum_{\lambda, k\geq 0, x_1,\ldots,x_k} a^{x}_{x_1,\ldots,x_k}(\lambda) x_1 \ldots x_k e^{\lambda} \tag{4}$$

where $x, x_1, \ldots, x_k \in \mathrm{Crit}(f)$ is an arbitrary sequence of points in $\mathrm{Crit}(f)$, $|x| - \sum_i |x_i| + \mu(\lambda) = 1$ and

$$a^{x}_{x_1,\ldots,x_k}(\lambda) = \#({}^{\nu}\mathcal{M}^{x}_{x_1,\ldots,x_k}(\lambda)) \in \mathbb{Q}.$$

In this definition we count the elements in ${}^{\nu}\mathcal{M}^{x}_{x_1,\ldots,x_k}(\lambda)$ with signs.

Note that, therefore, with our conventions, we have:

$$d^{\nu}(xy) = (d^{\nu}x)\cdot y + (-1)^{|x|} x \cdot (d^{\nu} y)\ .$$

Theorem 1. *The map d^{ν} satisfies $(d^{\nu})^2 = 0$.*

This is a consequence of formula (3) above together with the orientation conventions described in §2.1.5. The verification is left to the reader. We denote by $\mathcal{C}\ell\, H_*(L, (f, g), J, \nu)$ the homology of this complex, and call it the *cluster homology of L*. We prove in § 2.2 the invariance of this homology with respect to changes in the choices of the auxiliary data: $(f, g), J, \nu$. Notice that this implies that this homology is also invariant under symplectic diffeomorphism in the sense that if $\phi : M \to M$ is a symplectic diffeomorphism, then $\mathcal{C}\ell\, H_*(L) \simeq \mathcal{C}\ell\, H_*(\phi(L))$. However, it does depend in general on the choice of the orientation of L and of its relative spin structure.

1.1.1. *First remarks and an example.* We start with a simple observation which will play an important role later in this article.

Remark 1.1. A critical point y of index 0 can *never* appear as end in a 0-dimensional, non empty, moduli space of type

$$^{\nu}\mathcal{M}^{x}_{\dots,y,\dots}(\lambda)$$

(except for usual Morse flow lines). Indeed, consider the obvious forgetful map

$$^{\nu}\mathcal{M}^{x}_{x_1,\dots,y\dots x_k}(\lambda) \to^{\nu} \mathcal{M}^{x}_{x_1,\dots,x_k}(\lambda) \tag{5}$$

and notice that, due to (2), the domain of this map is of dimension greater by one unit than its image, a contradiction since this would imply that the right hand side is of negative dimension. Similarly, a critical point y of index $n = \dim(L)$ can not appear as the root in a 0-dimensional moduli space of the form $\mathcal{M}^{y}_{\dots}(\lambda)$ (except, again, for usual Morse flow lines). Of course, another, more intuitive way of seeing this is to realize that if there is a negative flowline from a non-constant loop in L to a local minimum, there must be a one-parameter family of these, and similarly for the relative maxima.

The next point is quite significant from an algebraic point of view.

Remark 1.2. The completion $\wedge$ is necessary, in particular because if $x_0 \in \mathrm{Crit}(f)$ verifies $|x_0| = 0$ and if $\dim(^{\nu}\mathcal{M}^{x}_{x_0,x_1,\dots,x_k}(\lambda)) = 0$, then we also have

$$\dim(^{\nu}\mathcal{M}^{x}_{x_0,\dots,x_0,x_1,\dots,x_k}(\lambda)) = 0$$

and so in our definition of d we need to allow formal powers in the critical points of f which are of index 1. A critical point of index at least 2 cannot appear infinitely many times as endpoint of a non-vanishing 0-dimensional moduli space because of formula (2). However, for algebraic reasons, it is preferable to complete using the word length filtration which counts all generators and not only those generators of degree 1.

An element in the cluster complex, $\tau \in \mathcal{C}\ell\,(L, J; (f, g))$, is written, in general, as a sum $\tau = (\sum_{\lambda} a(\lambda)e^{\lambda}) + m$ where $a(\lambda) \in \mathbb{Q}$ and m is a sum of words (in the letters consisting of the critical points of f) of length at least one. We call each of the terms $a(\lambda)e^{\lambda}$ with $a(\lambda) \neq 0$ *a free term* of τ and if there is a critical point x of f whose differential contains at least one free term, we say that the *complex has free terms.* A particular case will be significant: if the Morse index of x in the above definition is larger or equal to 1, we will say that *the cluster complex has high free terms.* Notice that, due to the fact that $\mu(\lambda)$ is even, if a critical point x verifies $dx = a_0 e^{\lambda} + \dots$, $a_0 \neq 0$, then $\mathrm{ind}_f(x)$ is even. Moreover, in view of Remark 1.1, $\mathrm{ind}_f(x) \neq \dim(L)$. To summarize, the complex has high free terms if and only if there is a critical point of even Morse index different from 0 and n whose cluster differential has a free term.

Proposition 1.3. *If the cluster complex is acyclic (that is $\mathcal{C}\ell\,H_*(L) = 0$), then the cluster complex has free terms. If the cluster complex $\mathcal{C}\ell\,(L, J, f)$ has high free terms, then there are some J-disks of non-positive Maslov class (i.e with Maslov index ≤ 0).*

Proof. If $\mathcal{C}\ell\, H_*(L) = 0$, then 1 is a boundary. But this is possible only if the cluster differential of a generator $x \in \mathrm{Crit}(f)$ has a non-vanishing free term.

For the second part of the statement: if $dx = a_0 e^{\lambda} + \ldots$, then $\mu(\lambda) = 2 - \mathrm{ind}(x)$ and, as $\mu(\lambda)$ is even, the claim follows. □

Remark 1.4. It is useful to make explicit the following property of the cluster complex $\mathcal{C}\ell\,(L, J, (f, g))$ where f is a Morse function with a single local minimum m and a single local maximum M (and any number of critical points of intermediate indices). By the last proposition, if $\mathcal{C}\ell\, H_*(L) = 0$, then $dm \neq 0$ (and there is therefore a J-holomorphic disk passing through m) or the cluster complex has high free terms. The proposition above also shows that the existence or not of free terms in the cluster complex provides a fundamental dichotomy on the algebraic side of our theory.

Example 1.5. If S^1 is a circle in $\mathbb{C}$ we have

$$\mathcal{C}\ell\, H_*(S^1) = 0 \ .$$

Indeed, take on S^1 the perfect Morse function with one minimum m and one maximum M. There exists one pseudoholomorphic disk passing through m, of Maslov index 2, with class in Λ that we will denote by λ_0. For the maximum, we have $dM = 0$. The differential of the minimum can be seen to be given by $dm = (1 + M + M^2 + M^3 + ...)e^{\lambda_0}$. We now notice $d(1 - M)e^{-\lambda} = 1$ which implies the claim.

Example 1.6. In the absence of bubbling (for example if $\omega|_{\pi_2(M,L)} = 0$), then

$$\mathcal{C}\ell\, H_*(L, J; (f, g)) \simeq S(s^{-1}H_*(L; \mathbb{Q})) \otimes \Lambda)^\wedge \ .$$

This happens, of course, because in this case the only component of the cluster differential is provided by the usual Morse differential.

1.1.2. *Relations to other constructions.*

A. Formally, at the algebraic level, our construction has some similarities with the construction of contact homology for Legendrian submanifolds which has been developed by Eckholm, Etnyre and Sullivan [9]. It is an interesting open problem at this time to see whether this analogy is in fact the reflection of a deeper relationship.

B. There are evident relations between our construction and the A^∞ - machinery of Fukaya, Oh, Ohta and Ono [10], at least due to the fact that both theories model algebraically the bubbling of disks. However, the underlying moduli spaces are quite different in the two theories, which reflects the fact that the solution that we propose here to lift the obstructions to the $d^2 = 0$ equation relies on a different, geometric, idea. While, at some point, some direct relation between the two approaches might be discovered, they remain, for the moment, complementary.

C. Spheres and disks with "spikes" have been used in various ways by other authors. Most notable is the announcement made by Oh in [18]: see our § 2.3.2 below in which we relate the cluster homology and Oh's version of Floer homology as described in [18] when the minimal Maslov number is larger or equal to 2. Other instances are the work of Schwarz [23] as well as Piunikin-Salamon-Schwarz [21] and more recently Bourgeois [7].

1.2. **Fine Floer Homology.** The purpose of this paragraph is to introduce a variant of Floer homology called *fine Floer homology* - denoted $I\!\!FH_*(-)$ - which is associated to a pair of transversal, orientable Lagrangian submanifolds L_0, L_1 of a symplectic manifold (M, ω), under the sole assumption that they be relatively spin in M. Let us recall that this means by definition that there is a common extension to M of the second Stiefel-Whitney classes of L_0 and L_1. This homology is defined, roughly, using $\mathcal{C}\ell\,(L_0)$ and $\mathcal{C}\ell\,(L_1)$ as coefficients rings via a Novikov ring that embraces all three terms L_0, L_1 and $L_0 \cap L_1$. Note that we do not require L_0 to be homeomorphic to L_1 nor any condition on the minimal Maslov numbers – in this sense, this is a universal Floer theory. As we shall see, its definition is entirely natural, up to two delicate points: (1) one must be careful in the definition of the right Novikov ring, and, more importantly, (2) to verify $d^2 = 0$, we will need to consider moduli spaces $\mathcal{W}^a_{\ldots;b}(\lambda)$ of strips joining intersection points a, b to which are attached clusters lying on L_0 and on L_1. These moduli spaces have undesirable boundary components consisting of of J-holomorphic strips joinning $a \in L_0 \cap L_1$ to $b \in L_0 \cap L_1$ with one gradient flow line in, say, L_0 stemming from, say, the point a and leading in L_0 to some cluster configuration. In general, this type of configuration has real codimension 1 and cannot therefore be neglected. We will see below how one can cancel them to get $d^2 = 0$, even in the most general case when L_0 and L_1 are not even homeomorphic.

1.2.1. *Coefficient ring and moduli spaces.* To define the fine Floer complex

$$I\!\!FC(L_0, L_1, \eta; J, (f_0, g_0), (f_1, g_1))$$

we first recall that the choices of orientations and of a relative spin structure for L_0, L_1 have been made and are included in the notation L_0, L_1. Besides this, we need auxiliary data as follows. First, as before, we need an almost complex structure J, Morse-Smale pairs (f_i, g_i) on L_i and coherent choices of perturbations. We also assume the f_i in generic position with respect to the intersection points $L_0 \cap L_1$ in the sense that these intersection points are included in the unstable manifolds of critical points of index 0 of f_i. We denote by $\Gamma = \{\alpha : [0,1] \to M : \alpha(i) \in L_i, i = 0,1\}$ the space of continuous paths from L_0 to L_1. Here, η is an element in Γ - its choice means that we fix a basepoint for this space. We denote by Γ_η the connected component of Γ which contains η. We denote by $I(L_0, L_1)$ the intersection points between L_0 and L_1 and we let I_η be those intersection points which, viewed as constant paths, belong to Γ_η. The generators of the fine Floer complex will be precisely the elements of I_η. Up to a shift in degrees, the resulting fine Floer homology will only depend on L_0, L_1, the connected component of η and the choice of orientations and relative spin structures of L_0, L_1.

To continue the construction, note that there are two group morphisms

$$\omega : \pi_1\Gamma_\eta \to \mathbb{R}\ ,\ \mu : \pi_1\Gamma_\eta \to \mathbb{Z}$$

the first given by integration of ω and the second obtained as follows. Fix a path l_t of Lagrangian subspaces of $T_{\eta(t)}M$ so that $l_i = T_{\eta(i)}L_i$, $i = 0, 1$. An element $[u] \in \pi_1\Gamma_\eta$ is represented by a map $u : [0,1] \times [0,1] \to M$ so that $u(s, 0) \in L_0$, $u(s, 1) \in L_1$, $\forall s$ and $u(0, t) = u(1, t) = \eta(t)$, $\forall t$. Therefore, u^*TM is trivial and, after fixing one such trivialization, define $\mu(u)$ as the Maslov index of the loop obtained by the concatenation of the following paths of Lagrangian subspaces: $u(s,0)^* \, TL_0$, $u(1,t)^* \, l_t$, $u(1-s,1)^* \, TL_1$ and $u(0,1-t)^* \, l_{1-t}$. It is easy to see that $\mu(u)$ only depends on the homotopy class of u. Let K be the kernel of the product morphism $\Psi = \mu \times \omega$ and let $\bar{\Gamma}_\eta$ be the regular covering of Γ_η corresponding to K. Let Π be the image of Ψ and let $\bar{\Lambda}$ be the rational group ring of Π. For any point $\xi \in \bar{\Gamma}_\eta$ we may define $\mu(\xi) = \mu(\xi, \eta) = \mu(\gamma)$ where γ is a path in Γ_η which joins η to $\pi(\xi)$ in the class defined by ξ (see Robbin-Salamon [22]). This number may be a half-integer.

We now define

$$\mathcal{R} = (S\mathbb{Q} < s^{-1}\mathrm{Crit}(f_i) : \ i = 0, 1 > \otimes \bar{\Lambda})^{\wedge} \tag{6}$$

where the completion is as in the cluster case except that we take into consideration both critical points of f_0 and of f_1.

Notice that there are injective group morphisms $\phi_i : \pi_2(M, L_i)/\sim \longrightarrow \Pi$ which are obtained by first assuming that η joins the base points in L_0 and L_1 and then viewing a disk with boundary in, say, L_0 as a cylinder whose end on L_1 is constant. Therefore, if we denote by Λ_i the group ring of $\pi_2(M, L_i)/\sim$, we have injective ring morphisms $\phi_i : \Lambda_i \to \bar{\Lambda}$. Thus $\mathcal{R}$ is isomorphic to the obvious completion of

$$\mathcal{C}\ell\,(L_0, f_0, J) \otimes \mathcal{C}\ell\,(L_1, f_1, J) \otimes_{\Lambda_0 \otimes \Lambda_1} \bar{\Lambda} \ .$$

In other words, $\mathcal{R}$ is obtained by replacing $\Lambda_0 \otimes \Lambda_1$ in the tensor product $\mathcal{C}\ell\,(L_0, f_0, J) \otimes \mathcal{C}\ell\,(L_1, f_1, J)$ by $\bar{\Lambda}$ by means of the ring morphism $\phi_0 \otimes \phi_1$ (all tensor products here are in the category of graded, commutative algebras). This also implies that, by the usual Leibniz formula, the cluster differentials on $\mathcal{C}\ell\,(L_i, f_i, J)$ induce a graded, commutative algebra differential δ on $\mathcal{R}$. Clearly, the differential algebra $(\mathcal{R}, \delta)$ depends of all the choices made till now but to ease notation we will only indicate these choices when necessary.

1.2.2. *The fine Floer complex.* This is the free differential, graded module over $(\mathcal{R}, \delta)$ given by

$$I\!\!FC(L_0, L_1, \eta; J, (f_0, g_0), (f_1, g_1)) = (\mathcal{R} \otimes \mathbb{Q} < I_\eta >, d_F) \ .$$

The grading of the elements in I_η is obtained as follows: we consider lifts $\bar{a} \in \bar{\Gamma}_\eta$ of the points $a \in I_\eta \subset \Gamma_\eta$ (here, as usual, the last inclusion means that we view intersection points as constant paths) and we define $|a| = \mu(\bar{a})$. Clearly, this grading depends on the choices of the lifts.

Remark 1.7. It is useful to note that, although our definition of the class $\lambda \in \Pi$ depends on the non-canonical choice of the lifts $\bar{a}$, different choices produce isomorphic complexes. Of course, we could as well have defined the complex by taking as generators pairs (a, α) made of a point $a \in I_\eta$ and of a lift of the constant path a to an element α of the covering $\bar{\Gamma}_\eta$. We would have then obtained a complex on which the group ring $\bar{\Lambda}$ acts in a natural way.

We now describe the differential d_F. We order the critical points in $\mathrm{Crit}(f_i)$. The differential d_F verifies the Leibniz formula and for an element $a \in I_\eta$ it is of the form:

$$d_F a = \sum_{\lambda, b, k \geq 0, l \geq 0, (x_1, \ldots, x_k, y_1 \ldots, y_l)} w^a_{x_1, \ldots, x_k, y_1, \ldots, y_l; b}(\lambda)\, x_1 \ldots x_k y_1 \ldots y_l b e^{\lambda}$$

where the x_i's belong to $\mathrm{Crit}(f_0)$, the y_j's to $\mathrm{Crit}(f_1)$, $\lambda \in \Pi$, and finally $b \in I_\eta$.

The coefficients $w^a_{x_1,\ldots,x_k,y_1,\ldots,y_l;b}(\lambda) \in \mathbb{Q}$ count the number of elements in certain 0-dimensional moduli spaces $\mathcal{W}^a_{x_1,\ldots,x_k,y_1,\ldots,y_l;b}(\lambda)$ (again after perturbation). These moduli spaces are defined in a way similar to the $\mathcal{M}_{\ldots}(\lambda)$'s of §1.1. We will describe them formally in more detail later in this paper in §3.1. We now only indicate the main idea in their construction. The starting point consists again of trees as in §1.1 but the root vertex v_0 of the tree corresponds to the closed unit disk D_0 with *two special distinct marked points* z_0 and z_- on its boundary. The restriction $u_0 := u|_{D_0}$ maps continuously D_0 to M in such a way that it sends z_0 to a, z_- to b. Moreover, on the punctured disk $D_0 \backslash \{z_0, z_-\}$, u_0 verifies the equation of a pseudoholomorphic strip $u_0 : \mathbb{R} \times [0,1] \to M$ with $u(\mathbb{R}, i) \subset L_i$, after the reparametrization

$$\mathbb{R} \times [0,1] \to D_0 \backslash \{z_0, z_-\}.$$

Except for codimension two phenomena, each of the other vertices correspond to pseudoholomorphic disks whith boundaries on one of the L_i's. Moreover, the gradient flows appearing in the construction correspond to one of the two functions f_i. In short, the elements of these moduli spaces are cluster trees on L_0 and L_1 that originate, *at finite points* from a single strip. Note that there may be (finitely) many such clusters attached to the boundary of the strip (and we still require that the corresponding incidence points on the boundary of the source of the strip u_0, i.e the points where the gradient flowlines of f_0 or f_1 start, be all distinct). We call them *cluster-strips*. We also need to associate a class $\lambda \in \Pi$ to such an object. Notice that, such a clustered strip may be viewed, topologically, as a strip joining a to b. Thus it lifts to a path in $\bar{\Gamma}_\eta$ which starts at $\bar{a}$ and ends at some element of the form $\bar{b}e^\lambda$. Precisely this λ is the class associated to the clustered-strip. With this definition we also see that λ equals the sum of the classes of the disks included in the clustered strip (these classes are well-defined as we have seen before) added to the class - defined as before - of the root J-strip that joins a to b. The space $\mathcal{W}^a_{x_1,\ldots,x_k,y_1,\ldots,y_l;b}(\lambda)$ is by definition the set of all unparametrized cluster-strips from $\bar{a}$ to $\bar{b}$ labelled, in the above sense, by the class λ.

In our formulae, it is important to distinguish the case in which the root is a constant strip and we fix the following convention: we will exclude form the moduli spaces $\mathcal{W}$'s, that serve to define our differential, all configurations in which the root is the constant strip at some point $a \in I_\eta$ and for which there is either no cluster attached to the strip, or more than one. In other words, when the strip is constant, we keep only those configurations in which *exactly one* cluster tree is attached to the strip a. In particular this cluster tree is *included in just one of the L_i's.* We obviously have two types of such elements: the first, grouped in the moduli spaces $T^a_{x_1,\ldots,x_k}(\lambda)$, consisting of configurations of one cluster tree in L_0 originating at a, and a similar moduli space $R^a_{y_1,\ldots,y_l}(\lambda)$ corresponding to trees in L_1. Thus, with this convention, the union $T^a_{x_1,\ldots,x_k}(\lambda) \cup R^a_{y_1,\ldots,y_l}(\lambda)$ consists of all the elements inside some of the $\mathcal{W}^a_{\ldots;a}(\lambda)$ for which the root is the constant strip. Note finally that, since the source map for the strip u_0 is $D_0 - \{z_0, z_-\} \simeq \mathbb{R} \times [0,1]$, our definition implies that a cluster tree in L_0 or in L_1 always originate from a flowline at a *finite* point of the source (i.e distinct from z_0, z_-). But, of course, configurations made of a strip with one cluster anchored at infinity, i.e anchored at the endpoints a or b of the strip will appear in the compactifications of our moduli spaces $\mathcal{W}^a_{x_1,\ldots,x_k,y_1,\ldots,y_l;b}(\lambda)$.

Clearly, the moduli spaces $\mathcal{W}$ depend on all of our choices and if we need to indicate these choices explicitly we will write: $\mathcal{W}^a_{\ldots;b}((L_0,f_0),(L_1,f_1),J;\lambda)$.

For generic choices of J, (f_i,g_i) and after pertubation, the dimension of the moduli space $\mathcal{W}^a_{x_1,\ldots,x_k,y_1,\ldots,y_l;b}(\lambda)$ is:

$$|a|-|b|-\sum|x_i|-\sum|y_j|-1\ .$$

For one-dimensional moduli spaces, there is a formula analogue to (3). It will be stated and proved in §3.1. As a consequence, we have:

Corollary 1.8. *With the notation above we have:* $d_F^2=0$.

Verifying this formula is less immediate than for the cluster differential because, besides the usual breaking of clusters and of strips, there is, as we just mentionned above, a third potential way for boundary points to emerge: they correspond to some cluster tree attached to a strip at some moving point p which "slides" along the boundary of the strip to one of the ends of the strip.

There are two reasons that make these boundary components disappear, one is purely algebraic and the other one is analytic and consists in the fact that (as remarked by Oh [19]) the usual gluing argument applies (under generic conditions) to a J-disk passing (transversally) through a and to a itself viewed as a constant strip. To see how these phenomena are related, we outline below the argument proving $d_F^2=0$.

1.2.3. *Verification of $d_F^2=0$, remarks and an example.* Let a be a generator of the fine Floer complex, i.e $a\in I_\eta$. We have $d_Fa=(A_0-A_1)a+\sum w^a_b b$ here A_0 represents the elements in $T^a_{\ldots}$, A_1 the elements in $R^a_{\ldots}$ and the w^a_b represent the other moduli spaces which involve only cluster-strips with a non-constant root (note that, in these non-constant strips, b might be equal to a). We rewrite this decomposition as $d_Fa=s(a)a+\delta a$. Note that $|A_i|=-1$ and so $(s(a))^2=(A_0-A_1)^2=0$, which is due to the graded-commutative algebraic setting that we chose (there is no geometric reason that would cancel these terms). We now write

$$(d_F)^2a=d(s(a))a+(s(a))(\delta a)+\sum(dw^a_b)b+\sum w^a_b(s(b)b)+\sum w^a_b(\delta b)\ .$$

To show that this vanishes, we need to identify, for each such element in $d_F^2(a)$, a way to include it as the 0-boundary end of some one-parameter family $C_{t\in[0,1]}$ sitting inside the compactification of one of our $\mathcal{W}$'s where $C_{t\in(0,1)}$ lies in the interior $\mathcal{W}$ of $\bar{\mathcal{W}}$, and where the configuration C_1 also corresponds to an element in the above expression of $d_F^2(a)$. It is easy to see that all the terms in the last four factors appear as boundary points of 1-dimensional moduli spaces $\mathcal{W}^a_{\ldots}$ of cluster-strips with a non constant root: the terms in $\sum(dw^a_b)b$ correspond to broken flowlines inside the cluster trees; the terms in $\sum w^a_b(\delta b)$ correspond to strip breaking; those in $\sum w^a_b(s(b)b)$ appear as boundaries when the point where some cluster tree is attached to a non-constant strip slides to b; similarly, the terms $(s(a))(\delta a)$ appear as boundaries when the point where some cluster tree is attached to a non-constant strip slides to a.

The terms of the fifth type - corresponding to $d(s(a))a$ - appear as top dimensional components in the boundary of the compactification of the moduli space

$T^a_{\dots} \cup R^a_{\dots} \subset \mathcal{W}^a_{\dots;a}$. This creates an apparent problem because these spaces have an additional type of top dimensional boundary elements: these appear when the length of the flowline joining a to the first disk in the cluster tends to 0. Thus, this boundary component of $T^a_{\dots} \cup R^a_{\dots}$ consists of cluster trees attached to a so that the first disk in the cluster goes through a. However, in view of the gluing result mentioned above, these elements are in fact interior points in the larger moduli space $\bar{\mathcal{W}}^a_{\dots,a}$. It is clear that the gluing construction produces a cluster-strip in which the cluster is attached at a finite point of the source of the non-constant strip joining a to a, i.e *away from a* (hence this one-parameter family will degenerate to an element of $d^2_F(a)$, not to an element of $d^3_F(a)$).

It is also useful to note that any 0-dimensional configuration corresponding to an element in the above formula for $d^2_F(a)$ which is of a "sliding form" may be included in a unique one-parameter family: this is because the union of a strip u from a to b with a cluster $T(a, x_1, \dots, x_k, \lambda)$ attached at a in L_0 say, is of dimension 0 iff both u and $T(a, x_1, \dots, x_k, \lambda)$ are 0-dimensional; but, by transversality, this means that there is locally in L_0 a n-dimensional family of such $T(p, x_1, \dots, x_k, \lambda), p \in L_0$. This implies that the pairs $(u, T(p, x_1, \dots, x_k, \lambda)), p \in \partial_{L_0} u$, form a 1-dimensional space. Conversely, and for the same reason, any one-dimensional family of cluster-strips that degenerates to a pair $(u, T(a, x_1, \dots, x_k, \lambda))$ by sliding must be such that both the class that contains u and the one that contains $T(a, x_1, \dots, x_k, \lambda)$ are 0-dimensional.

To summarize, this proves that the five types of boundary components exhaust the boundary points of a 1-dimensional moduli space – this concludes the sketch of the proof. $\square$

We denote the resulting homology by $I\!\!FH(L_0, L_1; \eta)$. We will show in §3.2 that it does not depend on the choices made in its construction (except the choice of the orientations and relative spin structure of L_0 and L_1) and that it is invariant with respect to Hamiltonian diffeomorphisms in the usual sense: if $\phi : M \to M$ is a Hamiltonian diffeomorphism, then we have isomorphisms

$$I\!\!FH(L_0, L_1; \eta) \simeq I\!\!FH(\phi(L_0), L_1; \eta') \simeq I\!\!FH(L_0, \phi(L_1); \eta'')$$

where η' and η'' correspond to η via the Hamiltonian diffeomorphism. As a corollary, we obtain that if L_0 and L_1 can be disjoined by a Hamiltonian isotopy, then $I\!\!FH(L_0, L_1; \eta) = 0$ for all connected components η.

Remark 1.9. For any $\xi \in \bar{\Gamma}_\eta$ we may define its action

$$\mathcal{A}(x) = -\int (u)^*\omega$$

where $u : [0, 1] \to \Gamma_\eta$ verifies $u(0) = \eta$, $u(1) = \pi(\xi)$ and belongs to the homotopy class defined by ξ. As mentioned before, by using an appropriate reparametrization, each element of the moduli spaces $\mathcal{W}^a_{\dots;b}(\lambda)$ may be viewed as a strip joining $\bar{a}$ and $\bar{b}e^\lambda$ (recall that $\bar{a}, \bar{b} \in \bar{\Gamma}_\eta$ are the fixed lifts of the elements $a, b \in I_\eta$). Clearly, this implies that, if $v \in \mathcal{W}^a_{\dots;b} \neq \emptyset$, then

$$\mathcal{A}(\bar{a}) - \mathcal{A}(\bar{b}e^\lambda) = \int v^*\omega$$

and this integral is non-negative and, obviously, equal to the sum of the symplectic areas of the strip and the disks appearing in v. Thus, as in the usual Floer theory, the fine Floer complex admits an action filtration.

Example 1.10. Let $L_0 = S$ be a circle in $\mathbb{C}$ and $L_1 = \mathbb{R} \subset \mathbb{C}$ so that $L_0 \cap L_1 = a, b$. We intend to describe the fine Floer complex for L_0, L_1. For this we fix a Morse function on S with a single minimum, m, and a single maximum, M both different from a and b. We take on L_1 a Morse function with no critical point. The relevant Novikov ring is the group ring of an abelian group with one generator e^{λ} with $\mu(\lambda) = 2$ where λ is the homotopy class of the disk whose boundary is the circle. This Novikov ring coincides with that appearing in the cluster complex of S. In this cluster complex we recall that $dM = 0$ and $dm = (1 + M + M^2 + ...)e^{\lambda}$. There are two pseudoholomorphic strips in the picture: one u joining a to b and one v joining b to a, both are of Maslov class 1.

We assume that M belongs to the boundary of v (if this is not the case, the resulting complex will be different even if, of course, the resulting homology will be the same). Here is the differential d_F of the fine Floer complex. First, $d_F a = ma + b$ - the factor ma corresponds to the flow line along the circle joining the constant strip a to the minimum; the term b corresponds to the strip u. Second, $d_F b = mb + (dm)a$ - the first term, mb, is as above; the term $(dm)a = (1 + M + M^2 + ...)ae^{\lambda}$ appears because the strip v itself passes through M ($|M| = 0$ makes it possible that all the powers of M appear). It is easy to check that $(d_F)^2 = 0$ and that the resulting homology is trivial.

1.3. **Symmetrization.** The fine Floer homology suffers from an algebraic defect: each generator of $I\!\!FC(L_0, L_1, (f_0, g_0), (f_1, g_1))$, $x \in I_{\eta}$, has a differential of the form $dx = (m - m')x + \ldots$ where m and m' are, respectively, minima of f_0 and f_1. This makes sometimes the fine Floer complex difficult to use. Moreover, this property implies that the fine Floer homology is trivial in many situations. This happens for example, in the absence of all bubbling, if L_0 is Hamiltonian isotopic to L_1 and the Morse functions f_0 and f_1 are both prefect with a trivial Morse differential. In this case the fine Floer differential is of the form $dx = (m - m')x$ for each generator x and the complex is therefore acyclic. However, in many interesting cases there is a natural, more symmetric version of the fine Floer complex which is more efficient and easier to use. While various versions of this are possible we will mostly use here one version which appears when L_1 is the image of L_0 by a Hamiltonian diffeomorphism.

We begin with the remark that we could have defined the fine Floer complex using the following, more general, setting: consider a generic time-dependent Hamiltonian $H : M \times [0, 1] \to \mathbb{R}$, and denote by $\phi_{t\in[0,1]}$ its flow. Take a generic family of almost complex structures $J_{t\in[0,1]}$ compatible with ω.

In the construction of the fine Floer complex, take as generators, instead of the intersection set $I_{\eta}(L_0, L_1)$, the set $I_{\eta}(L_0, L_1, H)$ of trajectories of $\phi_{t\in[0,1]}$ starting on L_0 and ending on L_1 in the component class of some path η from L_0 to L_1 (when $L_0 = L_1$ and no such choice is indicated, we work implicitly with η equal to the constant path; clearly, we may assume, generically, for each of these trajectories γ its i-end, $\gamma(i)$, belongs to the unstable manifolds of the minimum of f_i, $i = 0, 1$). Replace the condition requiring that the root be a J-holomorphic strip in the construction of the fine Floer complex by demanding that the root be a (J, H)-

semi-cylinder joining two such trajectories γ and γ', i.e a map

$$u : \mathbb{R} \times [0,1] \to M$$

satisfying the equation

$$\partial_s u + J_t(u)\partial_t u + \nabla H(u,t) = 0 \tag{7}$$

and the boundary conditions:

1) $u(s,i) \in L_i$ for $i = 0,1$
2) $\lim_{s\to -\infty} u(s,t) = \gamma(t)$ and $\lim_{s\to\infty} u(s,t) = \gamma'(t)$.

Note that the appropriate gluing of disks and constant strips used in the proof of $d_F^2 = 0$ in §1.2.3 remains valid in this context by replacing the constant strips by constant semi-cylinders, i.e by maps $u : \mathbb{R} \times [0,1] \to M$ that are of the form $u(s,t) = \gamma(t)$ or $\gamma'(t)$. Besides these modifications, the construction of this complex is identical to the fine Floer complex described before: the only change is that the root does not anymore correspond to a pseudo-holomorphic strip but rather satisfies the non-homogenous equation (7).

The notation for the underlying moduli spaces will be

$$\mathcal{W}^a_{\ldots;b}((L_0, f_0), (L_1, f_1), J_{t\in[0,1]}, H_{t\in[0,1]}; \lambda)$$

or, if the context is clear, $\mathcal{W}^a_{\ldots;b}(\lambda)$.

We will call the resulting complex and homology the *general fine Floer complex (homology)*. Although we will not show it in this paper (because we will not need it), it is not difficult to see that the the proofs concerning $d_F^2 = 0$ and the invariance of the fine Floer homology under variations of the auxiliairy data, that we give in § 3.2, may be generalized to the general fine Floer homology (as long as the data remain generic). Now the point is that the general fine Floer complex admits two special cases (which, therefore, for generic data, produce isomorphic homologies):

1) the first one, as seen before, is when H is the zero Hamiltonian and this leads to the fine Floer complex.
2) the second one is when $L_0 = L_1$.

Let us discuss the second case and put $L = L_0 = L_1$. The generators $I_\eta(L_0, L_1, H)$ are now elements in the set $I_\eta(L, H)$ that consists of trajectories of H_t starting and ending on L, while the configuration space that defines the differential is made of $(J_{t\in[0,1]}, H_{t\in[0,1]})$-semi-cylinders with the two ends coinciding with trajectories γ, γ' and the two side boundaries lying on L. Without loss of transversality, we may choose the generic family $J_{t\in[0,1]}$ so that $J_0 = J_1$ (we will denote this almost complex structure by J).

We can now define a new complex, the *symmetric fine Floer complex*, by considering this setting in which, additionally, we choose the pair (f_0, g_0) equal to the pair (f_1, g_1) (we denote both pairs by (f, g)). Since we have a differential graded algebra multiplication map:

$$\mathcal{C}\ell\,(L, (f,g), J) \otimes \mathcal{C}\ell\,(L, (f,g), J) \to \mathcal{C}\ell\,(L, (f,g), J)$$

and because $J_0 = J_1 = J$, we may replace the ring $\mathcal{R} = (\mathcal{C}\ell\,(L_0, (f_0, g_0), J_0) \otimes \mathcal{C}\ell\,(L_1, (f_1, g_1), J_1) \otimes \bar{\Lambda})^\wedge$ that appears naturally in the definition of the fine Floer complex by the ring $\hat{\mathcal{R}} = (\mathcal{C}\ell\,(L, (f,g), J) \otimes_\Lambda \bar{\Lambda})^\wedge$. This leads to a fine Floer complex $(\hat{I\!F}C(L, H, J, (f,g)), d_{\hat{F}})$ and a type of fine Floer homology $\hat{I\!F}H(L)$ which we call *symmetric*. If no additional notation appears the path η used in this case is just the constant path. Regarding the algebraic problem mentioned at the beginning of this section, note that if we choose f will a single local minimum, then the minima of f_0 and f_1 are identified in $\hat{\mathcal{R}}$ so that the term $(m-m')x$ disappears from the differential $d_{\hat{F}}(x)$. It is this algebraic identification of the clusters on the L_0-side with the clusters on the L_1-side that makes the symmetric fine Floer homology different, in general, from the fine Floer homology.

This homology has the same type of invariance properties as the non-symmetric version and, moreover, it is independent on the choice of $(H, J, (f, g))$ as long as it remains generic.

In the remainder of the section we will show how to express the symmetric fine Floer homology $\hat{I\!F}H(L)$ just in terms of the cluster complex.

Remark 1.11. If $\mathcal{C}\ell\, H_*(L) = 0$, then $\hat{I\!F}H_*(L) = 0$. Indeed, the Proposition 1.3 shows that $\mathcal{C}\ell\, H_*(L) = 0$ means that 1 is a boundary in the cluster complex. As the fine Floer complex is a module over the cluster complex the claim follows. In other words, if the cluster differential has free terms, then the symmetric fine Floer homology is trivial. This is also true of the non-symmetric version of the theory in which it is enough to assume that just one $\mathcal{C}\ell\, H_*(L_i)$ is trivial to deduce that $I\!FH_*(L_0, L_1)$ vanishes.

1.3.1. *An algebraic preliminary.* Consider a commutative, differential graded algebra of the form $\mathcal{A} = (SV, d)$ with V a rational vector space.

Consider the SV-module $SV \otimes V$ and write its elements under the form

$$x_1 \dots x_k \otimes v = x_1 \dots x_k \bar{v}$$

where $v, x_i \in V$. Define the following linear map

$$\alpha : SV \to SV \otimes V$$

by letting $\alpha(v) = \bar{v}$, $\alpha(1) = 0$ where 1 is the unit in SV and extending this map by the formula

$$\alpha(ab) = a\alpha(b) + (-1)^{|a||b|} b\alpha(a) \ .$$

It is easy to see by induction on the length of words that this map is well defined and that the formula above is verified for any homogenous monomials a, b. Explicitly, we have

$$\alpha(x_1 \dots x_k) = \sum_i (-1)^{\sigma_i} x_1 \dots \hat{x}_i \dots x_k \bar{x}_i$$

where σ_i is the product of the degree of x_i with the sum of the degrees of the $x_j, j > i$. We now define a map d on $SV \otimes V$ as the unique (SV, d)-module extension of

$$d\bar{v} = \alpha(dv)$$

which verifies the standard graded Leibniz rule.

One easily checks (by induction on word length) that, for any monomial m, this map verifies $\alpha(dm) = d(\alpha(m))$. This implies that the map d so defined is a differential and that α is a chain map. Denote by

$$\widetilde{\mathcal{A}} = (SV \otimes V, d)$$

the $\mathcal{A}$-differential module constructed in this way.

1.3.2. *String-strip symmetrization.* We return to our geometric setting. The algebraic construction above appears in the next proposition.

Proposition 1.12. *The symmetric fine Floer homology verifies:*

$$s^{-1}\,\hat{I\!F}H_*(L) \simeq H_*(\widetilde{\mathcal{C}\ell}\,(L, J_0; (f,g)))\ .$$

See § 4.1 for the proof of this proposition. It is essentially based on the following idea: suppose that in the definition of the symmetric fine Floer complex, instead of the Hamiltonian $H_{t\in[0,1]}$ and the loop $J_{t\in[0,1]}$, one takes the pull back to the Weinstein neighbourhood of L of a generic Morse function h on L (extended conveniently to all of M), and a single generic J. Then the generators, as $\hat{\mathcal{R}}$-module, of this new symmetric fine Floer complex are the critical points of h. This leads to a complex $\hat{\mathcal{C}}(L,J,f,h,g)$, defined in detail in § 4.1 where we also show that its homology coincides with the symmetric fine Floer homology defined above. Finally, by letting h be equal to f in $\hat{\mathcal{C}}(L,J,f,h,g)$, one gets a complex that can be easily related, algebraically, to the cluster complex – this is what produces the isomorphism of the above proposition (it is therefore produced in two steps: the first one is a PSS type of comparison map that relates cluster-strips to string-strips, i.e replace the semi-cylinder by (J,h)-linear clusters; the second one is based on the fact that the identification of h with f leads to a symmetrization of the kind described in the above algebraic construction – this is why it is natural to refer to this isomorphism as the *string-strip symmetrization*).

An immediate corollary of this proposition gives the description of the symmetric Floer homology if no bubbling is present.

Corollary 1.13. *If* $\omega|_{\pi_2(M,L)} = 0$ *then*

$$\hat{I\!F}H_*(L) \simeq (S(s^{-1}H_*(L;\mathbb{Q}))\otimes\Lambda)^\wedge \otimes H_*(L)\ .$$

Clearly, if a Lagrangian is displaceable, then both the fine Floer homology and its symmetric version vanish.

1.4. **Applications.** We will describe three consequences of our machinery.

1.4.1. *The Gromov-Sikorav problem.* As a first consequence, we will study a plausible conjecture going back to Gromov's original paper [13] on pseudo-holomorphic curves, stated orally by Sikorav in the late eighties in the following way: given any compact Lagrangian submanifold of $\mathbb{C}^n$, there is a holomorphic disc passing through each point of L [1]

To state our first result in this direction, have in mind the notion of *free terms* and the statement of Proposition 1.3.

Corollary 1.14. *Let $L \subset M$ be a compact, orientable, relatively spin Lagrangian submanifold of any symplectic manifold M. Assume that $\hat{I\!F}H_*(L) = 0$ (for example if L is displaceable by Hamiltonian isotopy). Then, for any generic almost complex structure J compatible with the symplectic form, one of the following holds:*

[1]The word "conjecture" should not be understood in the sense that Sikorav claimed that this is true, but in the milder sense of asking the question and noting that it is not a consequence of Gromov's or Floer's theories. Obviously, what is at stake here, is the question of the degree of the evaluation maps on the boundaries of J-holomorphic disks.

i. *there are J-holomorphic disks with boundary on L passing through a dense subset of points of L.*
ii. *the cluster complex $\mathcal{C}\ell(L, J, f)$ has high free terms for some Morse function f with a single local minimum and a single local maximum (in particular, there are J-disks of non-positive Maslov index).*

Proof. Assume that we are not in the case ii, i.e assume that for any Morse function with a single local minimum and local maximum, the associated cluster complex does not have high free terms. Fix such a function f and denote its minimum by m. By Proposition 1.12, $s^{-1}\hat{I\!\!F}H_*(L)$ is isomorphic to $H_*(\widetilde{\mathcal{C}\ell}(L, J; (f, g)))$, which means that $H_*(\widetilde{\mathcal{C}\ell}(L, J; (f, g)))$ vanishes. Notice that if $d\bar{m} \neq 0$, then we also have $dm \neq 0$ in the cluster complex. Assume now that

$$\bar{m} \in \widetilde{\mathcal{C}\ell}(L, J; (f, g))$$

is a cycle. We want to show that this also leads to $dm \neq 0$. Using Remark 1.1, we see that $\bar{m}$ can be a boundary only if $\mathcal{C}\ell(-)$ has free terms: indeed, by this remark, the only possible primitive of $\bar{m}$ must have the form $\tau\bar{m}$ where τ is a primitive of the unit 1 in the cluster complex. This means that there is a free term in some dx for some $x \in \mathrm{Crit}(f)$. Once again by Remark 1.1, the index of this x cannot be n and it cannot be stricly between 0 and n by our assumption. Therefore, $x = m$ and $dm \neq 0$.

The fact that $dm \neq 0$ means that there exists a moduli space ${}^{\nu}\mathcal{M}^{m}_{x_1,\ldots,x_k}(\lambda)$ of dimension 0 and non empty. But for a cluster tree to originate at the minimum m, the root disk must go through m. As we may use a different function f to place m in any generic point in L, this implies the claim. □

The dichotomy in the statement of the previous corollary can be sometimes resolved by homological restrictions.

Corollary 1.15. *Suppose that L is orientable, relatively spin and that $H_{2k}(L; \mathbb{Q}) = 0$ for $2k \notin \{0, \dim(L)\}$. If $\hat{I\!\!F}H_*(L) = 0$, then L verifies i. of the Corollary 1.14 above.*

Proof. We postpone the full proof of this to §4.2. We only give here the argument in the case when the manifold L as above admits a perfect Morse function. Obviously, all the critical points of such a perfect Morse function f different from its minimum m and from its maximum are of odd index. It follows that the cluster complex $\mathcal{C}\ell(L, J, f)$ does not have any high free terms and, as in the corollary above, this means that $dm \neq 0$ in $\mathcal{C}\ell(L, J, f)$ and implies that there is a J-disk passing through m. By conjugating the given Morse function by a diffeomorphism we may construct prefect Morse functions with the minimum in any desired point of L which shows the claim in this case. □

Remark 1.16. The statement in Corollary 1.15 is most relevant in the case when the minimal Maslov index of J-disks is non-positive. If this minimal Maslov index is greater or equal than 2, we shall see that the Corollary is valid without the orientability, relative spin and homological assumptions by a simpler argument - see our Remark 2.9 c., as well as the end of §4.3. In the monotone case a different proof is due to Peter Albers [1].

Example 1.17. The homological restriction in the corollary above is serious but, still, there are many examples of such manifolds: $S^1 \times S^{n-1}$ and its connected sums with itself provides maybe the simplest examples.

In §4.3 we will improve these results in the displaceable case by also bounding from above the area of the disks detected in terms of the displacing energy. This upper bound and Gromov's compactness theorem then imply:

Corollary 1.18. *Suppose that the relatively spin, orientable Lagrangian submanifold L is displaceable by a Hamiltonian isotopy and let $E(L)$ be its Hofer displacement energy. Any ω-tame almost complex structure J has the property that one of the following is true:*

i. *for any point $x \in L$ there exists a J-pseudo-holomorphic disk of symplectic area at most $E(L)$ whose boundary rests on L and which passes through x.*
ii. *there exists a J-disk of Maslov index at most*
$$2 - \min\{2k \in \mathbb{N}\backslash\{0, \dim(L)\} : H_{2k}(L;\mathbb{Q}) \neq 0\}$$
and of symplectic area at most $E(L)$.

If the set $\{2k \in \mathbb{N}\backslash\{0, \dim(L)\} : H_{2k}(L;\mathbb{Q}) \neq 0\}$ is empty, the statement should be understood as saying that (i) automatically holds. In §4.3 we also deduce an interesting geometric consequence of this.

1.4.2. *Constraints on Maslov indices.* The cluster complex setting provides straighforward proofs of various constraints regarding Maslov indices of Lagrangian submanifolds. For instance, we give in § 4.4 a rapid proof of Fukaya's recent result in [11] that we may state in the following general form: if $S^1 \times S^{n-1}$ admits a Lagrangian embedding in a symplectic manifold in such a way that it is displaceable by a Hamiltonian isotopy, then $\{2, n\} \cap \mathrm{Im}(\mu) \neq \emptyset$ for n even and $\{2, 3-n\} \cap \mathrm{Im}(\mu) \neq \emptyset$ for n odd.

1.4.3. *Detection of periodic orbits.* In §4.5 we discuss another application. In the presence of an orientable relatively spin pair of Lagrangian submanifolds L, L', we show that, by replacing in the definition of the clustered moduli spaces anyone (one and only one) of the J-disks by a pseudo-holomorphic cylinder with one boundary on L and the other boundary on L', one can construct a chain map:
$$\mathrm{cyl} : \mathcal{C}\ell\, C(L) \to \mathcal{C}\ell\, C(L) \otimes \mathcal{C}\ell\, C(L') \otimes \Lambda_{\Phi_0}$$
where Λ_{Φ_0} is an appropriate Novikov ring.

This map induces a morphism in homology whose non-triviality is used to detect the existence of periodic orbits of Hamiltonian diffeomorphisms that *separate* L from L'. These results generalize considerably those in [12].

2. The cluster complex and its homology.

We now discuss in more detail the construction of the cluster complex - in particular that of the underlying moduli spaces. We then pursue this section with the proof of the invariance of the cluster homology and with some other of its properties.

2.1. **Clustered moduli spaces.** As we mentionned in § 1.1, the definition of the cluster differential

$$d : \mathcal{C}\ell\,(L, J; (f, g)) \to \mathcal{C}\ell\,(L, J; (f, g))$$

depends on certain moduli spaces which combine Morse theory with the main features of the standard construction of moduli spaces of stable maps. We gave an idea of what these spaces look like in § 1.1. We now describe them in detail.

We fix a compact oriented Lagrangian submanifold $L \subset M$ as well as an almost complex structure J compatible with ω and we also fix $f : L \to \mathbb{R}$ a Morse function and g a Riemannian metric on L so that the pair (f, g) be Morse-Smale. Finally, we fix as well a relative spin structure on L.

2.1.1. *Trees.* We start with the notion of $\mathbf{n}$-labeled, oriented, planar, metric trees where $\mathbf{n} = (n_1, n_2) \in \mathbb{N} \times \mathbb{N}$.

These are couples $(\mathcal{T}, \psi)$ where: $\mathcal{T} = (T, V_D, \phi_T, \mu_T)$ with T a (connected) planar tree with oriented edges and with exactly one edge entering each vertex except for one - the root of the tree $v_0 \in Vertex(T)$ - in which no edge enters; $V_D \subset Vertex(T)$ is a subset of the set of vertices of T which generates a connected subtree of T and $v_0 \in V_D$; $\phi_T : \{0, 1, \ldots, n_1\} \to V_D$, $\phi_T : \{n_1 + 1, \ldots, n_1 + n_2\} \to Vertex(T)\backslash V_D$ assigns markers to the vertices of T so that $0 \in \phi_T^{-1}(v_0)$; $\mu_T : Vertex(T) \to \Lambda$ assigns classes in $\pi_2(M, L)/\sim$ to each vertex of T; $\psi : Edges(T) \to [0, +\infty)$ assigns lengths to the edges of T so that if $e \in Edges(T)$ relates two vertices so that one of which (at least) does not belong to V_D, then $\psi_T(e) = 0$. Below we denote $V_S = Vertex(T)\backslash V_D$.

We denote the fact that α is related by an edge to β by $\alpha E \beta$. By a slight abuse of notation we shall denote the edge joining α to β also by $\alpha E \beta$. For further use, we let for each $\alpha \in Vertex(T)$

$$n_\alpha = \#(\phi_T^{-1}(\alpha)) + \#\{\ \beta \in Vertex(T) : \alpha E \beta \text{ or } \beta \mathrm{E} \alpha\ \}.$$

2.1.2. *Pseudo-holomorphic disks and flow lines.* For a Morse-Smale pair (f, g), we denote by $\gamma_t(-)$ the negative gradient flow induced by $-\nabla f$. We understand $\gamma_\infty(x)$ as $\lim_{t\to\infty} \gamma_t(x)$.

For our fixed Lagrangian submanifold L, almost complex structure J and Morse-Smale pair (f, g), we define the clustered moduli space modeled on the metric tree $(\mathcal{T}, \psi)$:

$$\mathcal{M}_{\mathcal{T},\psi}(L, J, f) = \{[(\mathbf{u}, \mathbf{z})] : (\mathbf{u}, \mathbf{z}) = (\{u_\alpha\}_{\alpha \in Vertex(T)}, (\{z_{\alpha\beta}\}_{\alpha E \beta \text{ or } \beta \mathrm{E}\alpha}, \{z_i\}_{1 \le i \le n_1 + n_2}))\}$$

where the pairs $(\mathbf{u}, \mathbf{z})$ are so that:

i. $u_\alpha : D^2 \to M$ is a J-holomorphic disk with boundary on L and with class $\mu_T(\alpha)$ when $\alpha \in V_D$.
ii. when $\alpha \in V_S$, $u_\alpha : S^2 \to M$ is a pseudoholomorphic sphere with class $\mu_T(\alpha) = 2c_1(\alpha)$.
iii. If $\alpha \in V_D$, then $z_{\alpha\beta}$ are points in D^2 with the property that if $\beta \in V_D$, $\alpha E \beta$, then $z_{\alpha\beta} \in S^1$, $z_{\beta\alpha} \in S^1$ and $\gamma_{\psi(\alpha E \beta)}(u_\alpha(z_{\alpha\beta})) = u_\beta(z_{\beta\alpha})$. If $\alpha \in V_S$, then $z_{\alpha\beta} \in S^2$ and if $\beta \in Vertex(T)$ so that $\alpha E \beta$ or $\beta E \alpha$, then $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$.

iv. We have $z_i \in S^1 = \partial D^2$ if $\phi_T(i) \in V_D$ and $z_i \in S^2$ if $\phi_T(i) \in V_S$. For $\alpha \in Vertex(T)$ fixed, the points $z_{\alpha\beta}$, $z_{\delta\alpha}$ and z_i, $i \in \phi_T^{-1}(\alpha)$ are pairwise distinct. The union of all these points is the set of *special points* on the disk (or sphere) u_α. Their number is equal to n_α. The order in which the incidence points $z_{\alpha\beta}$ appear on the disk $\alpha \in V_D$ respects the order of the edges induced by the planarity of the graph.

v. If u_α is the constant map, then $n_\alpha \geq 3$ and there exists a chain of edges of length 0 connecting α to a vertex β with u_β non-constant.

The points $z_{\alpha\beta}$ will be called the *incidence points* corresponding to the disk (or sphere) α and the points $z \in \phi_T^{-1}(\alpha)$ are the *marked points*. Notice that the planarity of the trees used in this construction induce a natural order on the set of marked points $\{z_i\}$ with $\phi_T(i) \in V_D$: if two such z_i, z_j are on the same disk they are naturally ordered. If not, we notice that there is a vertex $\alpha \in V_D$ and two incidence points $z_{\alpha\beta}$ and $z_{\alpha\beta'}$ from which starts respectively a chain of edges ending in z_i and, respectively, in z_j. The order between z_i and z_j is the same as the one between $z_{\alpha\beta}$ and $z_{\alpha\beta'}$. We shall call this order the *planar order*.

The obvious reparametrization group acts on these objects and $[(\mathbf{u}, \mathbf{z})]$ are the equivalence classes obtained by taking the quotient with respect to this action as well as with respect to the following additional type of identification: any two configurations that differ by a permutation of the special points corresponding to a vertex α with $\mu(u_\alpha) = 0$ are declared equivalent.

We call such an equivalence class a *cluster configuration* or, sometimes, a cluster element or cluster tree.

We now define other associated moduli spaces

$$\mathcal{M}_{\mathcal{T}}(L, J, f) = \bigcup_{\psi} \mathcal{M}_{\mathcal{T},\psi}(L, J, f)$$

and

$$\mathcal{M}(L, J, f; \lambda, \mathbf{n}) = \bigcup_{\mathcal{T} \in \mathcal{T}(\mathbf{n},\lambda)} \mathcal{M}_{\mathcal{T}}(L, J, f) \ .$$

where the set $\mathcal{T}(\mathbf{n}, \lambda)$ consists of all trees $\mathcal{T}$ so that:

$$\#(Dom(\phi_T)) = n_1 + n_2 \ , \quad \sum_{\alpha \in Vertex(T)} \mu_T(\alpha) = \lambda \ .$$

The moduli spaces $\mathcal{M}_{\mathcal{T};\psi}(L, J, f)$ are endowed with the Gromov topology and this induces topologies on the other moduli spaces introduced above (except for the minor modifications that we introduced, this is by now a standard construction; for an exposition see, for example, McDuff-Salamon [17]). Using this topology, notice also that the last union above is certainly not disjoint. Indeed, if the length of an edge joining the vertices a and b in a tree shrinks to zero, the corresponding cluster element is also obtained by bubbling from an element for which the underlying tree has a single vertex instead of a, b (and the edge aEb is absent). Another important remark is that, obviously, two special points might cross. When this happens a constant bubble appears in our moduli space. There are two sides for each crossing and thus these configurations appear in pairs which are different only by the order of some special points on a constant disk. In view of the equivalence relation used in constructing $\mathcal{M}(\ldots)$ these two points are also identified (and are interior points).

2.1.3. *Compactification.* We now discuss the compactification of $\mathcal{M}(L, J, f; \lambda, \mathbf{n})$, $\bar{\mathcal{M}}(L, J, f; \lambda, \mathbf{n})$. We claim that

$$\begin{aligned}&\bar{\mathcal{M}}(L, J, f; \lambda, \mathbf{n}) \backslash \mathcal{M}(L, J, f; \lambda, \mathbf{n}) = \\ &= \bigcup_{x \in \mathrm{Crit}(f)} \bar{\mathcal{M}}_x(L, J, f; \lambda', \mathbf{n}') \times \bar{\mathcal{M}}^x(L, J, f; \lambda'', \mathbf{n}'')\end{aligned} \tag{8}$$

where $\lambda' + \lambda'' = \lambda$, $\mathbf{n}' + \mathbf{n}'' = \mathbf{n}$ and the space $\mathcal{M}_x(L, J, f; \lambda', \mathbf{n}')$ is defined similarly to $\mathcal{M}(L, J, f; \lambda, \mathbf{n})$ except that we modify the definition of $\mathcal{M}_{T,\psi}(L, J, f)$ in the following way: there exists $\alpha \in V_D$ and one special point $z_x \in S^1$ so that $\lim_{t\to\infty} \gamma_t(u_\alpha(z_x)) = x$. The definition of $\mathcal{M}^x(L, J, f; \lambda'', \mathbf{n}'')$ is analogous with the only change that $\lim_{t\to-\infty} \gamma_t(u_{v_0}(z_0)) = x$ (recall that v_0 is the root of the tree).

This formula is immediate as the only possible reason for non-compactness of the space $\mathcal{M}(L, J, f; \lambda, \mathbf{n})$ is due to the fact that the length of some gradient flow line of f may tend towards ∞.

Remark 2.1. The role of the flow lines that appear in the definition of our moduli spaces becomes apparent from this product formula as it implies that the bubbling of disks is an *internal* "codimension one" phenomenon here. For example, a configuration of two disks joined in a point may occur as the limit of bubbling. But then it is also the limit of a configuration formed by two disks connected by a flow line whose length decreases to 0.

Some related moduli spaces play a key role in the construction of the differential of the cluster complex.

For $x, x_1, \ldots, x_k \in \mathrm{Crit}(f)$ we let $\mathcal{M}^x_{x_1,\ldots,x_k}(L, J, f; \lambda, \mathbf{n})$ be defined in the same way as above except that $\mathcal{M}_{\mathcal{T},\psi}(L, J, f)$ is replaced with $(\mathcal{M}^x_{x_1,\ldots,x_k})_{\mathcal{T},\psi}(L, J, f)$ where the definition of this last moduli space is the same as that of the moduli space $\mathcal{M}_{\mathcal{T},\psi}(L, J, f)$ with one additional condition:

vi. We have $\lim_{t\to-\infty} \gamma_t u_{v_0}(z_0) = x$ (we recall that v_0 is the root of the tree T and that $0 \in \phi_T^{-1}(v_0)$) and there is an ordered set of marked points $(z_1, \ldots, z_k)$, $\phi_T(i) \in V_D$, called *terminations* respecting the planar order and so that
$$\lim_{t\to+\infty} \gamma_t(u_{\phi_T(j)}(z_j)) = x_j \ .$$

This is clearly an extension of the moduli spaces $\mathcal{M}_x(L, J, f; \lambda, \mathbf{n})$, $\mathcal{M}^x(L, J, f; \lambda, \mathbf{n})$ that appeared above.

When $\mathcal{T}$ is the void graph we let $(\mathcal{M}^x_{x_1})_\emptyset(L, J, f)$ be the set of negative gradient flow lines joining the critical points x and x_1.

2.1.4. *Justification for equation (3).* As mentioned in §1.1, formula (3) together with the analysis of orientations imply the vanishing of the square of the cluster differential. In this subsection, we justify this formula and, in the next one, we discuss orientations.

First, we define unparametrized moduli spaces denoted by
$$\mathcal{M}^x_{x_1,\ldots,x_k}(L, J; f, \lambda)$$
(and similarly for the other moduli spaces involved - the difference in notation is that we do not indicate the marked points anymore). These are subspaces in

$\cup_{\mathbf{n}}\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda,\mathbf{n})$ defined by the additional condition that $\mathbf{n}=(n_1,n_2)$ verifies $n_1=k+1$, $n_2=0$.

We claim that the natural compactification $\bar{\mathcal{M}}^x_{x_1,\ldots,x_k}(\lambda)$ of the moduli spaces $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ verifies:

$$\begin{aligned}(9)\qquad &\bar{\mathcal{M}}^x_{x_1,\ldots,x_k}(\lambda)\backslash\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)\\ &=\textstyle\bigcup_{y\in\mathrm{Crit}(f),i,s,\alpha=\lambda'+\lambda''}\bar{\mathcal{M}}^x_{x_1\ldots,x_{i-1},y,x_{i+s},\ldots,x_k}(\lambda')\times\bar{\mathcal{M}}^y_{x_i,\ldots,x_{i+s-1}}(\lambda'')\cup\mathcal{B}\end{aligned}$$

where $\mathcal{B}$ consists of configurations corresponding to the crossing of two terminations (or of two incidence points linearly related to terminations).

To see that this formula is correct, first notice that the bubbling of a non-trivial disk in a family of elements in $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ leads to an element which is, in fact, an interior point in $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ because the planar order is respected by this type of bubbling and this simply corresponds to the appearance of a new edge in the tree. Moreover, use (8) and notice that if one flow line corresponding to an edge $\alpha E\beta$ in a tree associated to an element of $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ breaks, then the resulting element is represented by two planar trees T' and T'' so that T'' is the full subtree generated by β and T' is the tree obtained by eliminating from T all the vertices of T''. In this process, some incidence point of T becomes a termination of T'. Finally, we also need to discuss crossing in this case. There are two types of crossing to be discussed and both are seen in our moduli spaces as the bubbling of a trivial bubble. The first type corresponds to some incidence point $z_{\alpha\beta}$ which is not related linearly to a termination point and which crosses some special point. Clearly, a similar configuration appears as a limit from the other side of the crossing. Because all Maslov classes of J-disks are even here we shall see that the natural orientations at such a point match so that it is natural to consider this type of configuration as an interior point of $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$. The second type of crossing corresponds to two terminations that cross (or, similarly, to a crossing of two incidence points linearly related to terminations). These are precisely the points assembled in the set $\mathcal{B}$.

Each such configuration also appears a second time in the compactification of a space of type $\mathcal{M}^x_{x_{\sigma(1)},\ldots,x_{\sigma(k)}}(\lambda)$ for some permutation $\sigma\in\Sigma_k$ (this again happens because we identify any two configurations differing only by a permutation of the special points on a constant disk).

In view of this it is natural to enlarge our moduli spaces even more.

We shall assume from now on that *the critical points of f are (strictly) ordered.*

Let $S=(x_1,\ldots,x_k)$ be a set consisting of critical points of $\mathrm{Crit}(f)$ (repetitions being allowed) so that S is a partially ordered set whose elements respect the order fixed on $\mathrm{Crit}(f)$ as in § 1.1. We will assume that there are no repetitions of critical points of odd degree (obviously, algebraically such repetitions are irrelevant).

We now define:

$$(10)\qquad \mathcal{M}^x_S(\lambda)=\mathcal{M}^x_{(x_1,\ldots,x_k)}(\lambda)=\bigcup_{\{x_{i_1},\ldots,x_{i_k}\}=\{x_1,\ldots,x_k\}}\mathcal{M}^x_{x_{i_1},\ldots,x_{i_k}}(\lambda)\ .$$

When taking into account orientations each of the factors on the right appears with a certain sign $\epsilon(x_{i_1},\ldots x_{i_k})$ which equals the sign needed to rearrange the monomial $x_{i_1}\ldots x_{i_k}$ in the form $x_1,\ldots,x_k$ without permuting any two terms x_{i_j}, x_{i_l} which represent the same critical point (this last condition is needed to make sure the sign is well defined even if in the monomial appears a repetition of a critical point of odd degree).

This union is again topologized with the Gromov topology. As we shall see later there are natural orientations on each of the top dimensional moduli spaces appearing in the union above as long as there are no two points x_i, x_j which are equal and of odd degree and $\epsilon(\dots)$ are the signs necessary to orient coherently the union: the key issue is that inside such a space the natural orientations have to match at crossings and at a crossing of two special points related to the same critical point of odd degree these orientations do not match. Clearly, this issue does not apply to the moduil spaces which are of dimension 0 so that in that case the orientations are well defined.

The compactifications of these moduli spaces verify the following product formula:

$$\begin{aligned}(11)\qquad & \bar{\mathcal{M}}^x_S(\lambda)\backslash\mathcal{M}^x_S(\lambda) = \\ & \textstyle\bigcup_{S=<S'\cup S''>,y,\lambda'+\lambda''=\lambda} \bar{\mathcal{M}}^x_{<S',y>}(\lambda')\times\bar{\mathcal{M}}^y_{S''}(\lambda'')\end{aligned}$$

This is interpreted again, as in (3), in the sense that the sum is also taken over all ways to add y to S' so as to obtain $< S', y >$.

Our moduli spaces may be viewed as intersections of moduli spaces of disks and appropriate moduli spaces of flow lines by means of evaluation maps at the special points. Under ideal transversality conditions, one expects that these moduli spaces have the structure of singular manifolds with boundary and that their dimension is given by formula (2). Here a singular closed manifold of dimension n is a stratified space A with two strata $A = A_1 \cup A_2$ where A_1 is an open, dense manifold of dimension n while A_2 is a closed stratified space of dimension at most $n-2$. The stratification condition means that there is a neighbourhood of A_2 in A which admits A_2 as a deformation retract. This implies, in particular, that A admits a fundamental class. Similarly, a singular manifold with boundary A is a stratified space with three strata, $A = A_1 \cup A_2 \cup A_3$, where A_1, A_2 are as above and A_3 is a closed singular manifold of dimension $n-1$ so that $\partial A\backslash A_2 = A_3$. We will denote $\partial A = A_3$.

It is well-known that the ideal transversality conditions mentioned above are not in fact satisfied even for generic choices of J (this is seen in our case in particular when two terminations cross - this situation also indicates the need for ghost bubbles in our model). However, following the methods recently developped by Hofer, Wysocki and Zehnder in [15], it is possible to construct a system of perturbations ν so that for generic J and (f,g), enough transversality is achieved to obtain, at least, a moduli space carrying a rational cycle. More precisely we have:

Basic analytic result. *For generic choices of (f,g) and J which is compatible with ω and for each $\lambda \in \pi_2(M,L)/\sim$, $k \geq 0$ and $x, x_1, \dots x_k \in \mathrm{Crit}(f)$ without repetitions of a critical point of odd degree, there is a system of perturbations:*

$$\nu = \nu(f, g, J, x, x_1, \dots, x_k, \lambda)$$

of the equation $\bar{\partial}_J = 0$ on each pseudoholomorphic disk of the tree $\mathcal{T}'$, such that each moduli space ${}^{\nu}\mathcal{M}^x_{x_1,\dots,x_k}(\lambda)$ of dimension 1, consisting by definition of the configurations satisfying the perturbed equations $\bar{\partial}_J(u) = \nu(u)$ admits a natural *compactification* ${}^{\nu}\bar{\mathcal{M}}^{x}_{x_1,\ldots,x_k}(\lambda)$ *that carries the structure of an oriented rational singular cycle with boundary of dimension*

$$|x| - \sum_{i=1}^{k} |x_i| + \mu(\lambda) - 1 \ .$$

A similar result is valid for the spaces $\mathcal{M}^{x}_{S}(\lambda)$. As mentioned in the introduction, the proof of this regularity property of the perturbed, clustered moduli spaces, is postponed to a subsequent paper. Variants of this result will be used below in the description of other moduli spaces similar to the clustered ones.

If $|x| - \sum |x_i| + \mu(\lambda) = 2$, then, as a consequence of (11) we deduce formula (3) up to the discussion on orientation which follows next.

2.1.5. *Orientation.* We will show in this section that, although the orientation is not the one that one would expect a priori – see Definition 1 – the fact that L is orientable and relatively spin implies that we may assign coherent orientations to our clustered moduli spaces. This will be done in three steps:

1) It follows from [10] that the moduli space of perturbed pseudo-holomorphic disks in a given class λ inherits a natural orientation associated to a choice of (i) an orientation of L, (ii) the choice of an extension $st \in H^2(M;\mathbb{Z})$ of the second Stiefel-Withney class of L, and (iii) the choice of a relative spin structure on (M, L) (which is made possible by the last two conditions on L).

2) This gives an orientation on the "root" cells of our moduli spaces

$${}^{\nu}\mathcal{M}(L, J, (f, g); \lambda, (n_1, n_2) = (0, 0))$$

without marked points, corresponding to the trees consisting of a single vertex; we then show how this can be used to orient, in a coherent way, all other cells of any codimension in ${}^{\nu}\mathcal{M}(L, J, (f, g); \lambda, (n_1, n_2) = (0, 0))$ (i.e correponding to any other tree).

3) Given coherent orientations on ${}^{\nu}\mathcal{M}(L, J, (f, g); \lambda, \mathbf{n} = (0, 0))$, we get orientations on ${}^{\nu}\mathcal{M}(L, J, (f, g); \lambda, \mathbf{n})$ for general $\mathbf{n}$, and finally on the moduli spaces ${}^{\nu}\mathcal{M}^{x}_{x_1,\ldots,x_k}(L, J, (f, g); \lambda)$ (see Definition 1)

We will then look at the compactifications of these moduli spaces and prove that, with these orientations, the formula $d^2 = 0$ for the cluster differential is valid.

For the convenience of the reader, we are going to briefly review the relevant arguments from [10] in (1) above. So let us assume that an orientation of L as well as an extension st of the second Stiefel-Withney class have been chosen, and let us define the orientation on the root cell of ${}^{\nu}\mathcal{M}(L, J, (f, g); \lambda, \mathbf{n} = (0, 0))$, which are just maps u satisfying the equation $\bar{\partial}_J = \nu$ for generic choices of J and ν. More precisely, denote by $\mathcal{B}$ the space of all $W^{1,p}$-maps $u : (D^2, \partial D^2) \to (M, L)$ with fixed integer $p > 2$ and by $\mathcal{E}$ the infinite dimensional vector space over $\mathcal{B}$ whose fiber F_u at u is the space of $(0, p)$-sections $\Gamma^{0,p}(\Lambda^{0,1} D^2 \otimes u^* TM)$. The operator $\bar{\partial}_J$ is a section of this bundle. For a small generic section ν of this bundle, our moduli space is the inverse image of ν by the $\bar{\partial}_J$-section. In order to linearize this operator at a solution u, one needs a Hermitian connection on this bundle. Indeed, if $\mathcal{H}$ is such a connection, viewed as a horizontal distribution, the linearisation is given by definition by:

$$D_u^{\mathcal{H}}(\bar{\partial}_J) = \Pi_{\mathcal{H}} \circ d_u(\bar{\partial}_J) : \Gamma^{1,p}(D^2, \partial D^2; u^*TM, u^*TL) \to \Gamma^{0,p}(\Lambda^{0,1}D^2 \otimes u^*TM)$$

where

$$d_u : T_u\mathcal{B} = \Gamma^{1,p}(D^2, \partial D^2; u^*TM, u^*TL) \to T_{\bar{\partial}_J(u)}\mathcal{E}$$

is the ordinary differential and $\Pi_{\mathcal{H}}$ is the linear projection, induced by the connection, on the tangent space to the fiber at $\bar{\partial}_J(u)$, identified with the fiber itself. Such a connection is most naturally induced by a connection on the ambient manifold M. Let ∇ be the Levi-Civita connection associated to the metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ and define, as in [17], the following connection

$$\tilde{\nabla}_v X = \nabla_v X - \frac{1}{2} J(\nabla_v J)X.$$

The point is that, with respect to this new connection, J is parallel (but not necessarily g anymore). (Recall that if J is not integrable, the Levi-Civita connection need not preserve J). Then, as showed in [17], the explicit expression for $D_u^{\mathcal{H}}\bar{\partial}_J$ is:

$$D_u^{\mathcal{H}}\bar{\partial}_J(\xi) = (\tilde{\nabla}\xi)^{0,1} + \frac{1}{4}N_J(\xi, \partial_J(u))$$

where the first term is the complex-anti-linear part of $\tilde{\nabla}\xi$, equal by definition to

$$\frac{1}{2}(\tilde{\nabla}\xi + J(u) \circ (\tilde{\nabla}\xi) \circ j_{D^2}$$

(here j_{D^2} is the standard complex structure on D^2 equal to the product by $\sqrt{-1}$), and where in the second term, N is the integrability tensor of J: $N(X,Y) = [JX, JY][X,Y] - J[JX,Y] - J[X,JY]$. It is easy to check that the first term, of order 1, of $D_u^{\mathcal{H}}\bar{\partial}_J(\xi)$ is the $(j_{D^2}, J(u))$-complex linear part while the second term, of order 0, is the $(j_{D^2}, J(u))$-anti-complex linear part. By hypothesis, we may assume that the differential $D_u = D_u^{\mathcal{H}}\bar{\partial}_J - D_u^{\mathcal{H}}\nu$ is onto, where of course the new term $D_u^{\mathcal{H}}\nu$ is of order 0.

The argument for the point (1) above breaks in two parts: one first shows that the choice of a trivialization of the real oriented bundle $u^*(TL) \to \partial D^2$ naturally induces an orientation on the tangent space of ${}^{\nu}\mathcal{M}(\lambda) = {}^{\nu}\mathcal{M}(L, J, (f,g); \lambda, \mathbf{n} = (0,0))$ at u, i.e an orientation on the kernel of D_u, and one then shows that the choices made of an orientation and a relative spin structure on L enables one to define the trivializations coherently along any loop in the moduli space. Let us explain the first part. Pick a $SO(n)$-trivialisation

$$\Phi : \partial D^2 \times \mathbb{R}^n \to u^*(T_*L)|_{\partial D^2}$$

and consider the $U(n)$-trivialization

$$\Phi^{\mathbb{C}} : \partial D^2 \times \mathbb{C}^n \to u^*(T_*M)|_{\partial D^2}$$

that it induces by complexification (using $J(u)$). Let us denote by $E \to D^2$ the bundle obtained by attaching $u^*(TM) \to D^2$ to the bundle $\partial D^2 \times \mathbb{C}^n$ via the map $\Phi^{\mathbb{C}}$. Because the above Hermitian connection is flat on the boundary, one may assume that both the bundle E and the connection have been trivialized on a closed collar neighbourhoud N of the boundary of the disk whose inner circle will be denoted by C. By shrinking C to a point, and using standard gluing techniques (see Appendix A in [17]), one gets an isomorphism

$$\ker D_u \simeq \ker(Hol_{\nu,J}(D^2, \partial D^2 : \mathbb{C}^n, \mathbb{R}^n) \times Hol_{\nu,J}(S^2; E'_{tr}) \overrightarrow{\text{ev}}\, \mathbb{C}^n)$$

where the first term is the space of (ν, J)-holomorphic maps from $(D^2, \partial D^2)$ to $(\mathbb{C}^n, \mathbb{R}^n)$, the second term is the space of (ν, J)-holomorphic sections of the linearized operator in the absolute case (that is to say defined on a closed Riemann surface) of the resulting Hermitian fiber bundle E' in which the fiber at the point $p \in S^2$ is identified with $\mathbb{C}^n$ though $\Phi^{\mathbb{C}}$, and where the evaluation map sends (v, w) to $v(0) - (\Phi^{\mathbb{C}})^{-1}(w(p))$. But the first space consists only of constant maps, and can be identified with $\mathbb{R}^n$ while the second space carries a canonical orientation (see [17] for this). Therefore the right hand side is oriented, and the left hand side inherits this orientation.

Remark 2.2. For the convenience of the reader, let us mention that one can see this canonical orientation of the second space by considering the above equation $D_u = D_u^{\mathcal{H}} \bar{\partial}_J - D_u^{\mathcal{H}} \nu$ which applies as well to the case of the sphere. Since the order 1 term is $(j_{D^2}, J(u))$-complex linear, one may deform the operator, within the space of Fredholm operators, to the operator consisting of the first term only, which then carries a natural complex structure (being the kernel of a complex map) and therefore a natural orientation. But this orientation can then be carried back, along the one-parameter family of Fredholm operators, to an orientation of the original kernel. Note that along this generic deformation, one may assume that only real dimension one cokernels may appear; in any case, it is an easy exercise to check that the orientation on the determinant bundle ker $\otimes$coker can be carried over the family.

This shows how to orient the tangent space of the root cell of

$$^{\nu}\mathcal{M}(\lambda) =: \, ^{\nu}\mathcal{M}(L, J, (f, g); \lambda, \mathbf{n} = (0, 0))$$

at a given point u_0. To prove that this defines an orientation of the the root cell, we now come back to the argument in [10]. Starting from a given map u_0, we may transport the orientation at u_0 along any path to an orientation at some other map u_1 by noting that a trivialisation over ∂D^2 extends canonically to a trivialisation over $\partial D^2 \times [0, 1]$. To show that the orientation defined above at two different maps u and u' does not depend on the choice of the path, we must show that this orientation carries back to itself along any loop $u_{t \in [0,1]}$ based at some map $u = u_0 = u_1$. Assuming (i) and (ii) of the above point (1), one first triangulate L and extend this triangulation to M. Let V be an oriented real fiber bundle over the 3-skeleton $M^{(3)}$ of M such that $w_2(V) = st|_{M^{(3)}}$. By the choice of V, the bundle $TL \oplus V \to L^{(3)}$ has both of his first and second Stiefel-Withney classes equal to zero, and is therefore spin. We can thus make the choice of a spin structure on $TL \oplus V \to L^{(3)}$, i.e the choice of a trivialization of $TL \oplus V$ over $L^{(2)}$. If u_t is a loop based at u, consider the map

$$U : (D^2, \partial D^2) \times S^1 \to (M, L)$$

sending (x, t) to $u_t(x)$, which can be homotoped to $(M(3), L(2))$. The spin structure gives a trivialization of $TL \oplus V$ over $L(2)$, i.e a trivialization of $U^*(TL \oplus V) \to \partial D^2 \times S^1$. But $U^*(TL \oplus V) = U^*(TL) \oplus U^*(V)$ and the second term extends to $D^2 \times S^1 \simeq S^1$ and is therefore trivialized (because V is oriented). Thus, this gives a stable trivialization of $u_t^*(TL)$ that varies continuously over time. But over S^1, the $SO(n)$-trivializations coincide with the stable $SO(n)$-trivialisations. This completes the proof of 1) above.

(2) The orientation on the root cell of ${}^{\nu}\!\mathcal{M}(\lambda) = {}^{\nu}\!\mathcal{M}(L, J, (f, g); \lambda, \mathbf{n} = (0, 0))$ gives rise in the obvious way to an orientation on the same moduli space with m distinct marked points that we will denote ${}^{\nu}\!\mathcal{M}_k(\lambda)$. Given a partition of λ into a sum $\lambda_1 + \ldots + \lambda_m$, and a tree T with m vertices and incidence points $z_{\alpha\beta} \in \partial D^2$, we consider, for each $1 \le \alpha \le m$, the moduli space ${}^{\nu}\!\mathcal{M}_\alpha(\lambda_\alpha)$ equal to ${}^{\nu}\!\mathcal{M}_{m_\alpha}(\lambda_\alpha)$ where m_α is the valence of the vertex α and λ_α the class corresponding to the vertex α. Consider the evaluation map:

$$\Phi : (\Pi_\alpha \ {}^{\nu}\!\mathcal{M}_\alpha(\lambda_\alpha)) \times (\mathbb{R}^+ \times \ldots \times \mathbb{R}^+) \to L^2 \times \ldots \times L^2$$

where the number of copies of $\mathbb{R}^+ = [0, \infty)$, as well as the number of copies of L^2, is equal to the number of edges in T and where, for each edge $\alpha E \beta$ with $\alpha < \beta$, Φ maps

$$((u_\alpha, z_{\alpha\beta}), (u_\beta, z_{\beta\alpha}), t) \in {}^{\nu}\!\mathcal{M}_\alpha(\lambda_\alpha) \times {}^{\nu}\!\mathcal{M}_\beta(\lambda_\beta) \times \mathbb{R}^+$$

to $(\phi_t(u_\alpha(z_{\alpha\beta})), u_\beta(z_{\beta\alpha})) \in L^2$. Clearly, the parametrised moduli space

$${}^{\nu}\!\mathcal{M}_{T,\psi}(L, J, (f, g))$$

is equal to the inverse image by Φ of the products of diagonals $\Delta \times \ldots \Delta \subset L^2 \times \ldots \times L^2$. Since L is oriented, Δ is co-oriented in L^2 and this induces a co-orientation, and therefore an orientation, on ${}^{\nu}\!\mathcal{M}_{T,\psi}(L, J, (f, g))$. The order of the evaluation maps here is given by the planar order of the exiting edges.

We must now show that these orientations are compatible at the codimension 0 strata of the boundaries of the compactifications of ${}^{\nu}\!\mathcal{M}_{T,\psi}(L, J, (f, g))$. Generically, the first case to consider corresponds to the elementary surgery on a tree T in which an edge $\alpha E \beta$ is contracted to a point while the two vertices α, β are identified. Let's denote by T' the tree after this surgery. This corresponds to the gluing of two (ν, J)-holomorphic disks, while of course the reverse surgery corresponds to the bubbling-off of a given disc. Denote by ${}^{\nu}\!\mathcal{M}_{T,\psi,\alpha\beta}(L, J, (f, g))$ the codimension 1 stratum of ${}^{\nu}\!\mathcal{M}_{T,\psi}(L, J, (f, g))$ formed of configurations in which the time-flow $\psi(\alpha E \beta)$ is zero. Consider the corresponding attaching map

$${}^{\nu}\!\mathcal{M}_{T,\psi,\alpha\beta}(L, J, (f, g)) \to {}^{\nu}\!\bar{\mathcal{M}}_{T',\psi'}(L, J, (f, g))$$

given by gluing. Equipping the domain with the orientation induced as the boundary of ${}^{\nu}\!\mathcal{M}_{T,\psi}(L, J, (f, g))$ (which of course is simply the one given by the inverse image of the above map Φ with the modification that Φ now maps the factor $((u_\alpha, z_{\alpha\beta}), (u_\beta, z_{\beta\alpha}), t) \in {}^{\nu}\!\mathcal{M}_\alpha(\lambda_\alpha) \times {}^{\nu}\!\mathcal{M}_\beta(\lambda_\beta)$ to $(u_\alpha(z_{\alpha\beta}), u_\beta(z_{\beta\alpha})) \in L^2$), and equipping the codomain by the natural orientation, we must show that this attaching map preserves the orientations. It is evident that this reduces to prove the following statement.

Lemma 2.3. *Let $\mathcal{M}(\lambda_1, \lambda_2)$ denote the space of pairs $((u_1, z_1), (u_2, z_2))$ made of discs u_i with boundary on L satisfying the equation $\bar{\partial}_J u_i = \nu(u_i)$ in non-trivial classes λ_1, λ_2, and with marked points $z_i \in \partial D^2$ that satisfy $u_1(z_1) = u_2(z_2)$. Let $\mathcal{O}(\lambda_1, \lambda_2)$ be the orientation on $\mathcal{M}(\lambda_1, \lambda_2)$ defined above through Φ, and $\mathcal{O}(\lambda)$ be the orientation of ${}^{\nu}\!\mathcal{M}(\lambda)$, the space of (ν, J)-holorphic discs in class $\lambda = \lambda_1 + \lambda_2$. Then, near any such Fredholm regular pair, the gluing at $p = u_1(z_1) = u_2(z_2)$ maps the orientation $\mathcal{O}(\lambda_1, \lambda_2)$ to the boundary-orientation of $\mathcal{O}(\lambda)$.*

This lemma is, by now, a standard result in the theory of J–holomorphic disks with Lagrangian boundary conditions: see for instance [10] for a proof.

To obtain an orientation on the full moduli space ${}^{\nu}\mathcal{M}(L, J, (f, g); \lambda, \mathbf{n})$ (which is made, by definition, of the above cells ${}^{\nu}\mathcal{M}_{T,\psi}(L, J, (f, g))$ glued together) the next step is to check that, at crossing points, the orientations match. This is equivalent to showing that changing the order of the evaluation maps above preserves the orientation. In turn, this reduces to showing that for just a single permutation of two evaluation maps the orientation is not change. Explicitly, we consider a configuration made of a root disk in class λ_0 with two incidence points z_i and $i = 1, 2$ each leading, respectively, to a disk in class λ_i and we need to show that the orientation obtained by evaluating first on z_1 and then on z_2 is the same as that obtained by evaluating first on z_2 and then on z_1. We denote $\mathcal{M}_1(\lambda)$ the moduli space of disks in the class λ with one marked point and we notice that the first orientation is given by a map:

$$\psi : \mathcal{M}(\lambda_0) \times S^1 \times S^1 \times \mathcal{M}_1(\lambda_1) \times \mathcal{M}_1(\lambda_2) \times \mathbb{R} \times \mathbb{R} \to L \times L \times L \times L$$

given by

$$(u, z_1, z_2, u_1, u_2, t, \tau) \to ((\gamma_t u(z_1), \gamma_\tau u(z_2), ev(u_1), ev(u_2))$$

where $ev : \mathcal{M}_1(\lambda) \to L$ is the canonical evaluation. The moduli space we need to orient is obtained by the preimage via this map of the product of diagonals $\Delta_{13} \times \Delta_{24} : L \times L \to L \times L \times L \times L$. If we change the order of the evaluations we obtain a map ψ' which describes geometrically the same moduli space except that in the domain the second and third factors are permuted - this contributes to the change of sign by -1 - as are the fourth and fifth factors - with a contribution of $(n + \nu(\lambda_1) - 2)(n + \nu(\lambda_2 - 2) = n^2 \bmod 2$ and are also permuted the sixth and the seventh factors - with a contribution of -1. In the target, the first and second factors are permuted - another n^2 - as well as the third and the fourth contributing again by n^2. Moreover the two terms in the domain of $\Delta_{13} \times \Delta_{24}$ are also permuted for a final contribution of n^2. Thus, the orientations match.

Finally, it is evident that the boundary of ${}^{\nu}\mathcal{M}(L, J, (f, g); \lambda, \mathbf{n})$ is given by formula (8) understood with signs: indeed, the boundary only consists of breaking of gradient trajectories of the ordinary Morse function f and it is an elementary exercise to check that, near any breaking point, the orientation of the right hand side in formula (8) coincides with the boundary-orientation of the left hand side.

(3) We now discuss how the above orientations on the moduli spaces

$${}^{\nu}\mathcal{M}(L, J, (f, g); \lambda, \mathbf{n})$$

for general $\mathbf{n}$, induces orientations on the moduli spaces ${}^{\nu}\mathcal{M}^{x}_{x_1,\ldots,x_k}(L, J, (f, g); \lambda)$.

As before, we assume that L is oriented and we pick orientations of the stable manifolds so that the unstable and stable manifolds (in that order) of the same critical point have intersection number equal to 1 (with respect to the orientation of L).

Consider now the following commutative diagram:

$$\begin{array}{c}\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)\\ \downarrow \iota\\ \mathcal{M}_{k+1}(\lambda)\\ \downarrow \mathrm{ev}\\ L^{k+1}\\ \uparrow \iota_1\\ W^u_x \times W^s_{x_1}\times \ldots W^s_{x_k}\end{array}$$

where $\mathcal{M}_{k+1}(\lambda)$ denotes the moduli spaces of J-holomorphic discs with boundary on L in class λ with $k+1$ ordered marked points z_0 (the root), $z_1,\ldots,z_k$, where ι,ι_1 denote the various inclusions and ev the evaluation map. Up to perturbations of (f,g), the map ev is transverse to the product of the unstable manifold of x and the stable manifolds of the x_i's in the target space L^{k+1}. The order of the points $z_1,\ldots,z_k$ is fixed by the planar order.

Definition 1. *When n is odd, we give the moduli space $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ the orientation induced by* ev*; more precisely, we set the orientation of $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ so that*

$$\mathcal{O}(\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)) \;+\; \mathrm{coorientation}(\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)) \;=\; \mathcal{O}(\mathcal{M}_{k+1}(\lambda))$$

where the co-orientation is induced by the pull-back by ev *of the co-orientation of $W^u_x\times W^s_{x_1}\times\ldots W^s_{x_k}$ in L^{k+1}.*

When n is even we change this orientation by multiplying it by

$$(-1)^{\Pi_{1\leq i\leq k}(i+1)|x_i|}\;.$$

We now need to check that with these choices our orientations match at crossing points so that the moduli space defined in formula (10) is oriented. In short, denote by ${}^a\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ the moduli space obtained by taking the z_i, $i\geq 1$ in an order which is arbitrary (and so, possibly different from the planar one). It is easy to see that our verification reduces to checking that the following formula is verified (where the orientations are provided again by the definition above).

Lemma 2.4. $\mathcal{O}({}^a\mathcal{M}^x_{x_1,\ldots,x_{i+1},x_i,\ldots,x_k}(\lambda)) = (-1)^{|x_i||x_{i+1}|}\mathcal{O}({}^a\mathcal{M}^x_{x_1,\ldots,x_i,x_{i+1},\ldots,x_k}(\lambda))$.

Remark 2.5. This shows that the formula (10) is correct and in the sense that the resulting moduli space is oriented at least in the absence of a repetition of a critical point x_i which is of odd degree.

Proof. Let

$$W_1 = W^u_x\times W^s_{x_1}\times\ldots\times W^s_{x_i}\times W^s_{x_{i+1}}\times\ldots W^s_{x_k}\;,$$
$$W_2 = W^u_x\times W^s_{x_1}\times\ldots\times W^s_{x_{i+1}}\times W^s_{x_i}\times\ldots W^s_{x_k}$$

and consider the following commutative diagram:

$$\begin{array}{ccc}{}^a\mathcal{M}^x_{x_1,\ldots,x_i,x_{i+1},\ldots,x_k}(\lambda) & & {}^a\mathcal{M}^x_{x_1,\ldots,x_{i+1},x_i,\ldots,x_k}(\lambda)\\ \downarrow\iota & & \downarrow\iota\\ \mathcal{M}_{k+1}(\lambda) & \xrightarrow{\tau} & \mathcal{M}_{k+1}(\lambda)\\ \downarrow\mathrm{ev} & & \downarrow\mathrm{ev}\\ L^{k+1} & \xrightarrow{\bar\tau} & L^{k+1}\\ \uparrow\iota_1 & & \uparrow\iota_2\\ W_1 & & W_2\end{array}$$

Here ι,ι_1,ι_2 are the various inclusions, τ permutes z_i and z_{i+1} and $\bar\tau$ permutes the two corresponding factors in L^{k+1}. Endow the moduli space $\mathcal{M}^x_{x_1,\ldots,x_i,x_{i+1},\ldots,x_k}(\lambda)$

with the orientation induced by $\mathrm{ev}^{-1}(W_1)$ (ie so that the co-orientation of W_1 in L^{k+1} lifts to the co-orientation of ${}^a\mathcal{M}^x_{x_1,\ldots,x_i,x_{i+1},\ldots,x_k}(\lambda)$ in $\mathcal{M}_{k+1}(\lambda)$) and similarly endow ${}^a\mathcal{M}^x_{x_1,\ldots,x_{i+1},x_i,\ldots,x_k}(\lambda)$ with the orientation induced by $\mathrm{ev}^{-1}(W_2)$. Obviously, τ identifies ${}^a\mathcal{M}^x_{x_1,\ldots,x_i,x_{i+1},\ldots,x_k}(\lambda)$ with ${}^a\mathcal{M}^x_{x_1,\ldots,x_{i+1},x_i,\ldots,x_k}(\lambda)$. We claim that it preserves or reverses the orientation if $\dim L + \dim W^s_{x_i} \dim W^s_{x_{i+1}} + 1$ is even or odd respectively. Denote by $\mathcal{O}_1$ the co-orientation of ${}^a\mathcal{M}^x_{x_1,\ldots,x_i,x_{i+1},\ldots,x_k}(\lambda)$ in $\mathcal{M}^x_{k+1}$ and by $\mathcal{O}_2$ the co-orientation of ${}^a\mathcal{M}^x_{x_1,\ldots,x_{i+1},x_i,\ldots,x_k}(\lambda)$ in $\mathcal{M}^x_{k+1}$. To prove the claim, and since the map τ clearly reverses the orientation of $\mathcal{M}^x_{k+1}$ (because it permutes two successive S^1-factors), it suffices to show that the co-orientation of ${}^a\mathcal{M}^x_{x_1,\ldots,x_{i+i},x_i,\ldots,x_k}(\lambda)$ in $\mathcal{M}^x_{k+1}$, defined as the image $\tau(\mathcal{O}_1)$, agrees with $\mathcal{O}_2$ iff $\dim L + \dim W^s_{x_i}) \dim W^s_{x_{i+1}}$ is even. But, because the diagram is commutative, the co-orientation $\tau(\mathcal{O}_1)$ can as well be obtained throught the map $\bar{\tau}^{-1} \circ ev$. Thus $\tau(\mathcal{O}_1)$ agrees with $\mathcal{O}_2$ if and only if the image by $\bar{\tau}$ of the co-orientation of W_1 inside L^{k+1} agrees with the co-orientation of W_2 inside L^{k+1}. But this is the case exactly when the parity of $\dim L + \dim W^s_{x_i} \dim W^s_{x_{i+1}}$ is even since the first term takes into account the change in orientation of the total space L^{k+1} while the second takes into account the discrepancy between the orientation of the $\bar{\tau}$-image of W_1 and the orientation of W_2. Summarizing, τ identifies ${}^a\mathcal{M}^x_{x_1,\ldots,x_i,x_{i+1},\ldots,x_k}(\lambda)$ with ${}^a\mathcal{M}^x_{x_1,\ldots,x_{i+1},x_i,\ldots,x_k}(\lambda)$ and the change in orientation is given by the parity of

$$\begin{aligned}
& n + (n - \mathrm{ind}(x_i))(n - \mathrm{ind}(x_{i+1})) + 1 \\
= \quad & n + n^2 - n(\mathrm{ind}(x_i) + \mathrm{ind}(x_{i+1})) + \mathrm{ind}(x_i)\mathrm{ind}(x_{i+1}) + 1 \\
= \quad & n(\mathrm{ind}(x_i) + \mathrm{ind}(x_{i+1})) + \mathrm{ind}(x_i)\mathrm{ind}(x_{i+1}) + 1 \\
= \quad & n(|x_i| + |x_{i+1}|) + |x_i||x_{i+1}| + |x_i| + |x_{i+1}| \qquad .
\end{aligned}$$

When n is odd, this is equal to $|x_i||x_{i+1}|$ as claimed in the statement of this lemma, and when n is even, the formula of the lemma remains valid by our particular choice of orientations. □

With these conventions, we can now identify the relevant signs in formula (3). First, formula (8) holds without any additional signs. It then follows that in (3) the sign of the term $({}^\nu\bar{\mathcal{M}}^x_{<S',y>}(\lambda')) \times ({}^\nu\bar{\mathcal{M}}^y_{S''}(\lambda''))$ is: $\epsilon(\{y\}, <S', y>)\epsilon(S'', S)$. Here, $<S', y>$ is, as before, the ordered set obtained by the rearrangement of the elements of $S' \cup \{y\}$ by following the order in $\mathrm{Crit}(f)$ and $\epsilon(T', T)$ is the sign necessary to move the ordered subset $T' \subset T \subset \mathrm{Crit}(f)$ so that it precedes $T \backslash T'$ (in counting these signs we always use the convention $ab = (-1)^{|a||b|}ba$). Of course, if y appears also in S', then there will be as many terms in the union (3) as the number of repetitions of y in $<S', y>$ and the sign of each such term is the one needed to move y from the position it appears in to the first position.

Using these conventions, it is an easy exercise to verify $d^2 = 0$ where d is defined by formula (4). An important point in this verification is that, as mentioned before, in formula (3) we might have in some of the terms on the right hand side the same critical point of odd degree that appears twice. However, by the description of the union in (3) and due to the sign conventions described before, each such term will appear twice in the union and with opposite signs so that the only remaining terms are those relevant for d^2.

2.2. **Invariance of the cluster homology.** Recall that for our choices of almost complex structure J, Morse-Smale pair (f,g) and perturbations ν, the cluster homology of L is defined by:

$$\mathcal{C}\ell\, H_*(L,J,(f,g),\nu) = H_*((S\mathbb{Q} < s^{-1}\ \mathrm{Crit}(f) > \otimes\ \Lambda)^\wedge, d^\nu)\ .$$

Theorem 2. *This homology is independent up to isomorphism of these choices and thus we define the cluster homology of L by:*

$$\mathcal{C}\ell\, H_*(L) = \mathcal{C}\ell\, H_*(L,J,(f,g),\nu)$$

where $J,(f,g),\nu$ is any generic choice.

Proof. The proof is based on the usual argument in Floer theory in which two complexes associated to two choices of data are compared by relating the moduli spaces defining the differentials of the two complexes through cobordisms consisting of moduli spaces associated to "homotopies" between the two selections of data. For this, we need the following notions.

2.2.1. *Clustered moduli spaces for Morse cobordisms.* We shall use the notion of Morse cobordism between two Morse-Smale pairs (f,g) and (f',g'). This is a homotopy

$$F : L \times [0,1] \to \mathbb{R}$$

together with a metric G on $L \times [0,1]$ with the following properties:

- $(F,G)|_{L\times\{0\}} = (f,g)$, $(F,G)|_{L\times\{1\}} = (f'+k,g')$ for some constant k.
- $\frac{\partial F}{\partial t}(x,0) = 0 = \frac{\partial F}{\partial t}(x,1)$ for all $x \in L$, $\frac{\partial F}{\partial t}(x,t) > 0$ for all $x \in L$, $t \in (0,1)$.
- (F,G) is a Morse-Smale pair so that

$$\mathrm{Crit}_i(F) = \mathrm{Crit}_i(f) \times \{0\} \cup \mathrm{Crit}_{i-1}(f') \times \{1\}\ .$$

As described in Cornea-Ranicki [8], given that L is compact, Morse cobordisms exist and are easy to contruct between any two (f,g), (f',g').

Consider two sets of data, $J,(f,g)$ and $J',(f',g')$, so that are defined the moduli spaces

$$\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda) = \mathcal{M}^x_{x_1,\ldots,x_k}(L,J,(f,g);\lambda)\ ,\ x,x_1,\ldots,x_k \in \mathrm{Crit}(f)$$

and

$$(\mathcal{M}')^{x'}_{x'_1,\ldots,x'_{k'}}(\lambda) = \mathcal{M}^{x'}_{x'_1,\ldots,x'_{k'}}(L,J',(f',g');\lambda)\ ,\ x',x'_1,\ldots,x'_{k'} \in \mathrm{Crit}(f')\ .$$

We also fix a Morse cobordism (F,G) between (f,g) and (f',g') as well as a smooth one-parameter family of almost complex structures J_t with $J_0 = J$ and $J_1 = J'$. We denote by $\bar{\gamma} : (L\times[0,1])\times\mathbb{R} \to L\times[0,1]$ the negative gradient flow induced by (H,G).

Obviously, at the heart of the construction will be certain moduli spaces that we shall denote by $\mathcal{N}^x_{y_1,\ldots,y_k}(\lambda)$ (or by $\mathcal{N}^x_{y_1,\ldots,y_k}(\lambda,F)$ in case the Morse cobordism F needs to be explicitly mentioned) where $x \in \mathrm{Crit} f'$, $y_i \in \mathrm{Crit}(f)$.

The definition of $\mathcal{N}^x_{y_1,\ldots,y_k}(\lambda)$ is perfectly analogous to that of $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ from § 2.1.2 with the modification that the moduli spaces $\mathcal{M}_{\mathcal{T},\psi}(L,J,f)$ are replaced with moduli spaces $\mathcal{N}_{\mathcal{T},\psi}$ in whose definition conditions i. ii. and iii. from §2.1.2 are replaced by i'. ii'. iii'. below (of course, iv. and v. are still required):

i'. $u_\alpha : D^2 \to M \times \{d(\alpha)\} \subset M \times [0,1]$ is a $J_{d(\alpha)}$-holomorphic disk with boundary on $L \times \{(d\alpha)\}$ and with class $\mu(\alpha)$ when $\alpha \in V_D$.

ii'. when $\alpha \in V_S$, $u_\alpha : S^2 \to M \times \{d(\alpha)\}$ is a $J_{d(\alpha)}$-pseudo-holomorphic sphere with class $\mu(\alpha)$.

iii'. If $\alpha \in V_D$, then $z_{\alpha\beta}$ are points in D^2 with the property that if $\beta \in V_D$, $\alpha E\beta$, then $z_{\alpha\beta} \in S^1$, $z_{\beta\alpha} \in S^1$ and $\bar{\gamma}_{\psi(\alpha E\beta)} u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$. If $\alpha \in V_S$, then $z_{\alpha\beta} \in S^2$ and if $\beta \in Vertex(T)$ so that $\alpha E\beta$ or $\beta E\alpha$, then $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$.

We proceed again as in § 2.1 to construct $\mathcal{N}^x_{y_1,\ldots,y_k}(\lambda)$ out of $\mathcal{N}_{\mathcal{T},\psi}$ except that the analogue of condition vi. uses $\bar{\gamma}$ - the negative gradient flow of the Morse cobordism F - instead of the flow γ.

The expected dimension of these moduli spaces is:

$$\dim(\mathcal{N}^x_{y_1,\ldots,y_k}(\lambda)) = \mathrm{ind}_{f'}(x) - (\sum_i (\mathrm{ind}_f(y_i) - 1) + \mu(\lambda) - 1 \ .$$

As in the case of the moduli spaces $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$, there are perturbed moduli spaces ${}^\nu\mathcal{N}^x_{y_1,\ldots,y_k}(\lambda)$ so that at least when $\mathrm{ind}_{f'}(x) - (\sum_i(\mathrm{ind}_f(y_i)-1)+\mu(\lambda)-1 = 1$, ${}^\nu\mathcal{N}^x_{y_1,\ldots y_k}(\lambda)$ is a rational 1-dimensional cycle. Moreover these perturbations may be chosen in a way compatible with the perturbations associated to the data $(J,(f,g))$ and $(J',(f',g'))$. The resulting perturbed moduli space admits a compactification whose boundary is described by the formula below by using the same writing as in (3) (see also (11); to avoid carrying a notation that would be too heavy, we omit the perturbations from the writing of the moduli spaces):

$$\begin{aligned} \partial\bar{\mathcal{N}}^x_S(\lambda) = & \bigcup_{S'\cup S''=S,y,\lambda'+\lambda''=\lambda} \bar{\mathcal{N}}^x_{<S',y>}(\lambda') \times \bar{\mathcal{M}}^y_{S''}(\lambda'') \ \bigcup \\ & \bigcup_{\lambda'+\lambda''=\lambda,x_1,\ldots,x_s} [(\bar{\mathcal{M}}')^x_{x_1,\ldots x_s}(\lambda') \times_{\cup S_i=S,\sum \lambda_i=\lambda''} \ (\bar{\mathcal{N}}^{x_i}_{S_i}(\lambda_i))] \ . \end{aligned} \tag{12}$$

Here $y, y_i \in \mathrm{Crit}(f)$ and $x, x_i \in \mathrm{Crit}(f')$.

Using the moduli spaces above and, as before, for $x \in \mathrm{Crit}(f')$, $y_i \in \mathrm{Crit}(f)$ with $\mathrm{ind}_{f'}(x) - \sum_i(\mathrm{ind}_f(y_i) - 1) + \mu(\lambda) - 1 = 0$, we let

$$b^x_{y_1,\ldots,y_k}(\lambda) = \#({}^\nu\mathcal{N}^x_{y_1,\ldots,y_k}(\lambda))$$

where the elements of ${}^\nu\mathcal{N}^x_{y_1,\ldots,y_k}(\lambda)$ are counted with signs (the orientations are defined in a way similar to that used in §2.1.5).

We now define:

$$\phi^F : \mathcal{C}\ell\,(L, J'; (f', g')) \to \mathcal{C}\ell\,(L, J; (f, g))$$

by the formula

$$\phi^F(x) = \sum b^x_{y_1,\ldots,y_k}(\lambda) y_1 \ldots y_k e^\lambda \tag{13}$$

and from (12), we immediately obtain (again, up to a sign verification) that ϕ^F is a morphism of chain complexes.

Remark 2.6. In order to get the relation $d\phi^F = \phi^F d$ that makes ϕ^F into a complex morphism, it is required that the top dimensional stratum of the boundary of the moduli space $\bar{\mathcal{N}}^x_S(\lambda)$ be asymmetric: indeed, applying first d to some critical point x and then ϕ^F means that there might be many breaking points on the $M \times \{1\}$ side, whereas applying first ϕ^F to some critical point x and then d makes appear only one breaking point of the same moduli space $\bar{\mathcal{N}}^x_S(\lambda)$ (this asymmetry is of course due to the fact that there is only one root but many ends). The fact that, indeed, the top dimensional stratum of $\bar{\mathcal{N}}_S^x(\lambda)$ behaves in this way can be seen as follows. Notice first that one may regard an element of $\mathcal{N}^x_{y_1,\ldots,y_k}$ as a "tree" in $M\times[0,1]$ having disks (or spheres) instead of vertices and with its edges represented by flow lines of $\bar{\gamma}$. The origin of the tree is at y and it has terminal vertices in $x_1,\ldots x_k$. The boundary of the compactification of this moduli space consists of broken trees, the breaking points occuring in critical points of F. Therefore, these breaking points appear on $L\times\{1\}$ or on $L\times\{0\}$ and if a broken tree appears in the top dimensional stratum of the boundary of $\mathcal{N}^x_{y_1,\ldots,y_k}$, then it has breaking points only on one of these. The interesting point is that such a top dimensional stratum tree may be broken in *multiple* points on $L\times\{1\}$ (but brakes at only *one* critical point on $L\times\{0\}$). In essence, this happens because if a flow line inside the tree breaks on $L\times\{1\}$ then the disk (or sphere) at the origin of this flow line is contained in $M\times\{1\}$ and the only way to leave $M\times\{1\}$ via a flow line of $\bar{\gamma}$ is through a critical point of f'. A simple dimension computation shows that indeed this multiple breaking points phenomenon appears in the top dimensional stratum of the boundary.

To show that ϕ^F induces an isomorphism in homology, one could apply a variant of the standard approach in Morse-Floer theory which is to show that a cobordism of Morse cobordisms induces a chain homotopy. However, things are more delicate here because, while this approach provides the definition of the chain-homotopy on the generators of $\mathcal{C}\ell(-)$, it is not clear how to directly extend this chain homotopy to the words of length longer than one in the commutative, graded algebra $S\mathbb{Q}<Crit(f)>\otimes\Lambda$ (as an interesting side comment, this is easy to do for non-commutative free algebras [2]). In view of this, our proof of the existence of an isomorphism between $\mathcal{C}\ell\, H(L,J;(f,g))$ and $\mathcal{C}\ell\, H(L,J';(f',g'))$ will be based (besides the construction of ϕ^F) on the properties of a differential filtration which is of independent interest.

2.2.2. *The word-area filtration.* Denote by ϵ_D the infimum of $\int_{D^2}u^*\omega$ over the set of maps $u:(D^2,S^1)\to(M,L)$ which are J_t-holomorphic for some $t\in[0,1]$ and non-constant. This number is strictly positive and we use it to define a filtration of the cluster complex $\mathcal{C}\ell\,(L,J;(f,g))$:

$$F^k\mathcal{C}\ell\,(f)=\mathbb{Q}<\ x_1x_2\ldots x_me^{\lambda}\ |\ \ m\frac{\epsilon_D}{2}+\omega(\lambda)\geq k\frac{\epsilon_D}{2}\quad>\ .$$

Notice that the cluster differential preserves this filtration. Indeed, if a term in the differential of $x_1\ldots x_me^{\lambda}$ is of the form $y_1\ldots y_le^{\lambda'}$ and $l<m$, then $l=m-1$ and $e^{\lambda'}=e^{\lambda}e^{\lambda''}$ with λ'' represented by a sum of pseudoholomorphic disks. Therefore, $\omega(\lambda'')\geq\epsilon_D$ and our filtration is differential. This remark implies in fact a little more: if we set the *weight* of a monomial $x_1\ldots x_ke^{\lambda}$ to be

$$w(x_1\ldots x_ke^{\lambda})=k+2\frac{\omega(\lambda)}{\epsilon_D}$$

and if $m\in\mathcal{C}\ell\,(L,J;(f,g))$ is a monomial, then we may write $dm=d_0m+\sum_i m_i$ with d_0 the Morse differential and the m_i are monomials with $w(m_i)\geq w(m)+1$.

We now consider the spectral sequence $E^r(f)$ associated to the filtration $F^k\mathcal{C}\ell\,(f)$ which we shall call further the *word-area spectral sequence*. The remark above implies that the total vector space of the term $E^1(f)$ is isomorphic to $(S(s^{-1}H_*(L;\mathbb{Q}))\otimes$

$\Lambda)^\wedge$ because the 0-order differential in the spectral sequence coincides with the Morse differential.

We obviously have a similar filtration $F^k\mathcal{C}\ell\,(f')$ and an associated spectral sequence $E^r(f')$. It is easy to notice (by an argument similar to the one applied above to the differential) that the chain morphism ϕ^F preserves the word-area filtration and thus induces a morphism of spectral sequences $E^r(\phi^F)$. Moreover, $E^0(\phi^F)$ coincides with the morphism induced by the Morse comparison morphism associated to the Morse cobordism F and, as this morphism induces an isomorphism in homology, we deduce that $E^1(\phi^F)$ is an isomorphism. This means that $E^r(\phi^F)$ is an isomorphism for all $r \geq 1$.

The cluster differential of a term $x \in \mathrm{Crit}(f)$ may contain infinitely many terms (even if only finitely many terms of weight bounded by some k can appear). Consequently, knowing that $E^\infty(\phi^F)$ is an isomorphism does not imply directly that $H_*(\phi^F)$ is an isomorphism. We will now prove this last fact under the additional assumption that ϕ_0, the Morse-comparison map relating the Morse complexes of f' and of f and induced by F, is surjective.

We start by noting that if ϕ_0 is surjective, then ϕ^F is surjective too. Indeed, if we denote by $x_1, \ldots, x_n$ the critical points of f, we may write $x_i = \phi_0(z_i)$ with $z_i \in \mathbb{Q} < \mathrm{Crit}(f) >$. We also have $\phi_0(z_i) = \phi^F(z_i) - \sum_{\omega(\lambda)>0} m_\lambda e^\lambda$ where m_λ are monomials in the generators x_k's. In all the monomials m_λ for which $\omega(\lambda)$ is minimal, we replace the x_k's by the expression $\phi^F(z_k) - \sum m_\lambda e^\lambda$. This process replaces these monomials with monomials written in the $\phi^F(z_i)$'s summed with monomials (in the x_i's) which are multiplied with some $e^{\lambda'}$ with $\omega(\lambda') > \omega(\lambda)$. Recursively, this shows that x_i can be written (as formal series) only in terms of the $\phi^F(z_i)$'s and thus ϕ^F is surjective. This argument does also imply that the restriction of ϕ^F to $F^k\mathcal{C}\ell\,(f')$ is onto $F^k\mathcal{C}\ell\,(f)$.

Now let $K = \ker(\phi^F)$. We intend to show that K is acyclic. Let

$$A^n = \mathcal{C}\ell\,(f')/(F^n\mathcal{C}\ell\,(f')) \ \text{ and let } \ B^n = \mathcal{C}\ell\,(f)/(F^n\mathcal{C}\ell\,(f))\ .$$

Clearly, our filtrations induce filtrations on both A^n and B^n and the map ϕ^h induces chain morphisms $\phi^n : A^n \to B^n$ which respect the filtrations. As before, we see that at the level of the induced spectral sequences, $E^1(\phi^n)$ is an isomorphism. Thus $E^\infty(\phi^n)$ is an isomorphism. The differential of the algebra generators in A^n and B^n are *finite* sums of monomials and so we get that $H_*(\phi^n)$ is an isomorphism for all $n \geq 1$. Thus the kernel $K_n = Ker(\phi^n)$ is acyclic. This kernel fits into an exact sequence

$$0 \to K \cap F_n\mathcal{C}\ell\,(f') \longrightarrow K \xrightarrow{p_n} K_n \to 0$$

and the projections p_n commute with the obvious projections $K_{n+1} \xrightarrow{q_n} K_n$ which are thus surjective. We have

$$K = \lim_{\leftarrow} K_n\ .$$

Let $\sigma \in K$ so that $d\sigma = 0$. As K_n is acyclic, there are elements $\xi_n \in K_n$ so that $d\xi_n = p_n\sigma$, $\forall n \geq 1$. Therefore, $d(q_{n-1}\xi_n - \xi_{n-1}) = 0$. Using the acyclicity of K_{n-1} again there is $\tau \in K_{n-1}$ so that $d\tau = q_{n-1}\xi_n - \xi_{n-1}$. But q_{n-1} is surjective, so there is $\tau' \in K_n$ with $q_{n-1}\tau' = \tau$. Now define a preturbation of ξ_n by $\xi'_n = \xi_n - d\tau'$. Notice that $d\xi'_n = p_n\sigma$ and $q_{n-1}\xi'_n = \xi_{n-1}$. By applying this process recursively, we get a sequence of elements ξ'_n so that $q_{n-1}\xi'_n = \xi'_{n-1}$, $d\xi'_n = p_n\sigma$, $\forall n$. This implies that we may define $\xi = \lim_{\leftarrow} \xi'_n$ and that we have $d\xi = \sigma$, which shows that K is acyclic and that ϕ^F induces an isomorphism in homology.

To prove that $\mathcal{C}\ell\, H(L, J; (f, g))$ and $\mathcal{C}\ell\, H(L', J'; (f', g'))$ are isomorphic without any surjectivity condition for ϕ_0, it is sufficient to notice that for any two Morse-Smale pairs (f, g) and (f', g'), there is a chain of Morse cobordisms:

$$(f, g) \leftarrow \odot \rightarrow \odot \leftarrow \odot \ldots \odot \rightarrow (f', g')$$

so that each Morse cobordism in the chain induces a surjective chain map. This is quite easy to show either as a consequence of bifurcation analysis or by the rigidity results in [8].

This completes the proof of Theorem 2.

Remark 2.7. We expect the comparison isomorphism $H_*(\phi^F)$ to be canonical, as it is in Morse theory or in Floer theory. Unfortunately, the argument above does not provide this stronger result. However, it does imply that the morphisms induced by ϕ^F on the pages of order 1 and more of the area spectral sequence are canonical. Indeed, $E^1(\phi^F)$ is induced by the morphism in Morse homology which is canonical. So $E^1(\phi^F)$ does not depends of the choice of h. But this means that $E^r(\phi^F)$ does not depends of this choice for $r \geq 1$.

2.3. **Various special cases.** We start by recalling from Example 1.6 that in the absence of bubling we have:

$$\mathcal{C}\ell\, H_*(L, J; (f, g) \simeq S(s^{-1}H_*(L; \mathbb{Q})) \otimes \Lambda)^\wedge \ .$$

2.3.1. *Higher dimensional cluster moduli spaces.* It is an interesting question to see, in the spirit of [4], how to encode the information, in moduli spaces of clusters of any dimension (not only 0-dimensional), that has to do with the loop structure of the "boundary" of each cluster tree. By this, we mean that we could try to assign, to each cluster configuration in $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$, a loop in L that starts at x and goes round the tree (i.e round each J-holomorphic disk once, and twice (back and forth) on each gradient flowline) and comes back to x. Because special marked points may cross each other (not directly, but through ghost bubbles), this loop structure is not well-defined, but there is certainly a tree-structure. For now just notice that, in fact, these higher dimensional moduli spaces appear implicitly in the cluster differential because we may view $\mathcal{M}^x_{x_1,\ldots,x_k}$ as the intersection of $\mathcal{M}^x_{x_1,\ldots,x_{k-1}}$ with the unstable manifold of x_k and thus high dimensional moduli spaces become visible in the cluster differential once they are reduced by intersection to 0-dimensional spaces. This is, of course, similar to the process leading to Gromov-Witten invariants if the moduli spaces of disks are replaced by moduli spaces of pseudoholomorphic spheres.

2.3.2. *Relation to Floer homology.* In this subsection we assume that the minimal Maslov index of a pseudo-holomorphic disk with boundary on L is at least equal to 2. We shall denote this minimal Maslov index of some pseudo-holomorphic disk by μ_{min}. For $x, y \in \mathrm{Crit}(f)$ and $\lambda \in \Lambda$, let $\mathcal{F}^x_y(\lambda)$ be the moduli space constructed in the same way as the moduli spaces $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ except that all graphs $\mathcal{T}$ that are used in the construction are *linear* chains (they consist simply of chains of edges $xEy_1, y_1Ey_2, \ldots, y_sEy$). We shall call further these cluster configurations *linear cluster elements.* The key point is that, because $\mu_{min} \geq 2$, the compactification of

$\mathcal{F}^x_y(\lambda)$, $\bar{\mathcal{F}}^x_y(\lambda)$, has the property that (after appropriate perturbations) if $|x|-|y|+\mu(\lambda)-1=1$, then

$$\partial\bar{\mathcal{F}}^x_y(\lambda)=\sum_{z,\lambda'+\lambda''=\lambda}\bar{\mathcal{F}}^x_z(\lambda')\times\bar{\mathcal{F}}^z_y(\lambda'')\ . \tag{14}$$

This happens because bubbling in $\mathcal{F}^x_y(\lambda)$ can be transformed into an interior point just by using linear trees. Indeed, fix one disk D inside a linear chain κ of flow lines and disks joining x to y. As κ is a linear chain, there are precisely two double points on D. Assume now that, by bubbling off inside $\bar{\mathcal{F}}^x_y(\lambda)$, the disk D splits into two other disks D' or D''. In this case, for dimensional reasons, each of D' and D'' will carry one of these double points. More precisely, suppose that a linear tree from $x\in \mathrm{Crit}(f)$ to $y\in\mathrm{Crit}(f)$ is the starting point of a 1-dimensional cobordism of configuartions, and suppose that at some time $t_0>0$, it reaches a non-linear tree by bubbling off from some disk D_0 at a point which is disctinct from the two special points of D_0. The class D_0 would then split in $D'_0+D''_0$ where say D'_0 would still be part of a new linear tree κ' (but not D''_0). But this is impossible since the new linear tree κ' would then have dimension at least 2 less that the original tree κ (by our assumption on the minimal Maslov index). Since κ was assumed to be of dimension 0, this cannot happen generically in a one-parameter family.

This immediately implies (14).

Clearly, formula (14) may be used to define a chain complex

$$\mathcal{C}(L,J;(f,g))=(\mathbb{Q}<\mathrm{Crit}(f)>\otimes(\Lambda)^{\wedge},d)$$

with

$$d(x)=\sum_{y,\lambda}(\#\mathcal{F}^x_y)ye^{\lambda}$$

where $x\in\mathrm{Crit}(f)$, the sum is over all $y\in\mathrm{Crit}(f)$, $\lambda\in\Lambda$ so that $\dim(\mathcal{F}^x_y(\lambda))=0$ and the elements in $\mathcal{F}^x_y(\lambda)$ are counted with signs. The completion $^{\wedge}$ only applies now to the group ring Λ which reduces here to the usual Novikov ring.

Essentially, this same complex has appeared before in an annoucement by Oh (see the very end of the paper [18]) who has indicated that its homology should coincide with the standard Floer homology. Indeed, it is easy to show first that the homology of this complex is independent of $J,(f,g)$ (for example, by the "chain homotopy" method typical in Floer theory) and then it is not difficult to find a chain map defined on the Floer complex

$$\psi: FC(L,J,H)\to\mathcal{C}(L,J;(f,g))\ .$$

Here we use the Floer complex associated to the action functional

$$\mathcal{A}:\widetilde{\mathcal{P}_0(L,L)}\to\mathbb{R}$$

where $\mathcal{P}_0(L,L)$ is the space of paths

$$\gamma:[0,1]\to M,\ \gamma(0),\gamma(1)\in L$$

which represent the trivial element in $\pi_1(M,L)$,

$$\mathcal{A}(\gamma)=-\int u^*\omega+\int_0^1 H(\gamma(t)dt$$

and $\widetilde{\ }$ represents the universal cover. The Floer trajectories associated to this setting are semi-cylinders with their boundary on L which join orbits of the Hamiltonian flow which start from L and arrive in L in time 1. The comparison map

ψ may be defined by means of moduli spaces of objects that ressemble semi-disks connected by a flow line to a chain of flow lines and disks (this is the Piunikin, Salamon, Schwarz method [21]). The same argument with one additional parameter shows that any two comparison maps obtained in this way are chain homotopic. Comparison maps $\psi' : \mathcal{C}(L, J; (f, g)) \to FC(L, J, H)$ can also be constructed and they verify similar properties. It is then easy to see that this implies that ψ induces an isomorphism in homology.

Finally, we discuss the relation between the complex $\mathcal{C}(L, J; (f, g))$ and the cluster complex.

Proposition 2.8. *The natural projection*

$$l : S(s^{-1}\mathbb{Q} < Crit(f) >) \to s^{-1}\mathbb{Q} < Crit(f) > ,$$

which sends to 0 *all the words of length longer or equal than* 2 *as well as the unit, induces a morphism of chain complexes:*

$$l : \mathcal{C}\ell\,(L, J; (f, g)) \to s^{-1}\mathcal{C}(L, J; (f, g)) \ .$$

Proof. The only reason that could prevent the map l to be a morphism of chain complexes is that a priori there might exist some element in a 0-dimesional moduli space $\mathcal{M}^x_y(\lambda)$ whose corresponding tree $\mathcal{T}$ is not a linear tree but rather a tree with some branches. As this tree has one root and one single end, we may consider the linear subtree $\mathcal{T}'$ which connects the root to the end. Let λ' be the total Maslov class of the disks corresponding to the vertices which are not on $\mathcal{T}'$. Then the cluster configuration corresponding to $\mathcal{T}'$ provides an element of $\mathcal{M}^x_y(\lambda - \lambda')$. But the dimension of this last moduli space is $|x| - |y| + \mu(\lambda) - \mu(\lambda') - 1$. As $|x| - |y| + \mu(\lambda) - 1 = 0$ and $\mu(\lambda') \geq 2$, this is impossible (for a generic J). □

Remark 2.9. a. There is obviously an area filtration also on the complex $\mathcal{C}(L, J; (f, g))$. The spectral sequence associated to this filtration is essentially a variant of Oh's spectral sequence from [20]. It is useful to notice also that the map l defined above preserves this filtration and we have therefore an induced morphism of spectral sequences which compares our area spectral sequence to Oh's spectral sequence.

b. One may view the definition of cluster homology as the result of an attempt to define a quantum product for Lagrangian submanifolds to be defined on Floer homology. Without the condition $\mu_{min} \geq 2$, such a product cannot be defined. However, if this condition is satisfied, it is easy to define a bilinear map which descends to Floer homology and which is just the dual of the linear map

$$s\mathcal{C}(L, J; (f, g)) \to s\mathcal{C}(L, J; (f, g) \otimes s\mathcal{C}(L, J; (f, g))$$

given by the quadratic part of the cluster differential (for a generator $x \in \mathrm{Crit}(f)$, this is the part of dx which is written as a sum of words of length two). Indeed, the fact that the linear part of the cluster differential is itself a differential implies that this bilinear map is a chain map. Here s indicates raising the degrees of the generators in $\mathcal{C}$ by one unit. This bilinear map is associative but is not a product due to the lack of a unit (as Paul Biran pointed out to us). Indeed, our cluster construction does not "integrate" the usual cup-product (this is done on purpose).

c. In the case when $\mu_{min} \geq 2$, it is easy to use directly the complex $\mathcal{C}(L, f)$ described above to show that the conclusion (i) of Corollary 1.14 holds: the argument is exactly the one in Corollary 1.14. Moreover, in this case the result remains true without any relatively spin or orientability restrictions because it can be seen that the complex $\mathcal{C}(L,f)$ is well defined over $\mathbb{Z}/2$.

3. The fine Floer complex and its homology

3.1. **Moduli spaces for the fine Floer Homology.** The purpose of this section is to give a precise definition of the moduli spaces used in the definition of the differential of the fine Floer complex. They were very briefly described in § 1.2.2. As in that subsection, we assume that we have a transversal pair of orientable, relatively spin Lagrangian submanifolds L_0 and L_1, as well as an almost complex structure J, and Morse-Smale pairs (f_i, g_i) on L_i. We denote by γ^i the negative gradient flow of f_i. We also assume that $\mathrm{Crit}(f_j)\cap L_0\cap L_1=\emptyset$ for $j=0,1$ and that the points in $L_0\cap L_1$ belong to the unstable manifolds of the minima of f_j.

Th readers should have in mind the large Novikov ring $\bar{\Lambda}$ as well as the desired coefficient ring for the fine complex $\mathcal{R}$ from equation (6).

Recall also that the moduli spaces to be described are associated to $a,b\in I_\eta$, $\lambda\in\bar{\Lambda}$, $x_1,\ldots,x_k\in\mathrm{Crit}(f_0)$, $y_1,\ldots,y_l\in\mathrm{Crit}(f_1)$ and are denoted by $\mathcal{W}^a_{x_1,\ldots,x_k,y_1,\ldots,y_l;b}(\lambda)$. They are defined in a way similar to the moduli spaces $\mathcal{M}^a_{x_1,\ldots,x_i}(\lambda)$ from §2.1, but there are a number of modifications that we now describe.

As in §2.1, we start with trees and consider couples $(\mathbf{u},\mathbf{z})$ with the difference that, in this case, there are at least two marked points z_0, z_{-1} on the root of the tree (in other words $0,-1\in\phi_T^{-1}(v_0)$). Besides this, the properties (i.-vi.) become:

i". The map u_{v_0} verifies:

$$u_{v_0}:\mathbb{R}\times[0,1]\to M\ ,\ \frac{\partial u}{\partial s}+J\frac{\partial u}{\partial t}=0\ , \tag{15}$$

$\lim_{s\to-\infty}u(s,t)=a$, $\lim_{s\to+\infty}u(s,t)=b$; $u_{v_0}(\mathbb{R}\times\{i\})\subset L_i$. We view u_{v_0} as a continuous map defined on D^2 and which sends the marked point z_0 to a and sends the second marked point, z_{-1}, to b. The planar order on the root coincides with the order of the special points along the strip (descending from z_0 to z_{-1} along L_0 and going up along L_1).

ii". For $\alpha\neq v_0$, $\alpha\in V_D$, u_α are disks so that $u_\alpha(\partial D^2)\subset L_i$ (for some $i\in\{0,1\}$) and u_α is a J-pseudoholomorphic disk. In both cases, these disks are of class $\mu(\alpha)$.

iii". For $\alpha\in V_S$, u_α is a J-holomorphic sphere of class $\mu(\alpha)$.

iv". Properties iii., iv. from §2.1 are verified with the only modification that we require $\gamma^j_{\psi(\alpha E\beta)}(u_\alpha(z_{\alpha\beta}))=u_\beta(z_{\beta\alpha})$ in case $u_\alpha(z_{\alpha\beta})\in L_j$.

v". If u_α is constant, then $n_\alpha\geq 3$. Moreover, if $\alpha\neq v_0$, there is a chain of edges of zero length that connects α to a vertex β with u_β non-constant.

vi". Instead of vi, we have the following property: there exists an ordered subset $(z_1,\ldots,z_k,z'_1,\ldots,z'_l)$ of marked points (again called terminations) $i\in\phi_T^{-1}(V_D)$ which respect the planar order and so that $u_{\phi_T(i)}(z_i)\in L_0$, $u_{\phi_T(i')}\in L_1$, and

$$\lim_{t\to+\infty}\gamma^0_t(u_{\phi_T(i)}(z_i))=x_i$$

$$\lim_{t\to+\infty}\gamma^1_t(u_{\phi_T(i')}(z'_i))=y_i$$

An element $(\mathbf{u},\mathbf{z})$ as defined above may be viewed roughly as a pseudoholomorphic strip (of ends a and b) corresponding to the root of the tree v_0 together with a number of cluster (sub)trees attached at some incidence points $z_{v_0\eta_i}$ (and maybe some additional pseudoholomorphic spheres). Clearly, according to whether $u_{\eta_i}(S^1) \subset L_0$ or $u_{\eta_i}(S^1) \subset L_1$, the corresponding cluster subtree is included in L_0 or in L_1. As in the construction of the moduli spaces $\mathcal{M}$ in §2.1, the number of marked points for a moduli space $\mathcal{W}^a_{r_1,\dots,r_k;b}(\lambda)$ is the minimal possible and is equal to $k+2$ (because, in this case, the root carries two special marked points). This means, in particular, that a constant strip by itself does not belong to our moduli spaces.

Assume now that we fix an order on $\mathrm{Crit}(f_j)$ for $j \in 0, 1$. For an ordered subset $S = (x_1, \dots, x_k, y_1, \dots, y_s)$, $x_i \in \mathrm{Crit}(f_0)$, $y_i \in \mathrm{Crit}(f_1)$, we set

$$\mathcal{W}^a_{S,b}(\lambda) = \bigcup \mathcal{W}^a_{x_{i_1},\dots,x_{i_k},y_{i_1},\dots y_{i_s};b}(\lambda)$$

where, when taking into account orrientations, the sign in front of the terms on the right hand side is as in (10).

3.1.1. *Compactification.* The moduli spaces above admit compactifications

$$\bar{\mathcal{W}}^a_{S,b}(\lambda)$$

and after appropriate perturbations these are singular cycles (with boundary) of dimension

$$|b| - |a| - \sum_i |x_i| + \mu(\lambda) - 1$$

and, at least if they are of dimension 1, their boundaries satisfy a formula analogous to (3).

To write this formula, we recall that S denotes the ordered set $(x_1, \dots, x_k, y_1, \dots, y_s)$. We set $S_0 = (x_1, \dots, x_k)$, $S_1 = (y_1, \dots, y_s)$. We need also to consider the clustered moduli spaces $\mathcal{M}$ for each of $(L_i, (f_i, g_i), J)$. To simplify notation, is it useful to define $\mathcal{M}^r_R(\lambda)$ for any $r \in \mathrm{Crit}(f_i)$ and for R any ordered subset of $\mathrm{Crit}(f_0) \cup \mathrm{Crit}(f_1)$ as follows: if $R \not\subset \mathrm{Crit}(f_i)$, then $\mathcal{M}^r_R(\lambda) = \emptyset$ and, if $R \subset \mathrm{Crit}(f_i)$, then $\mathcal{M}^r_R(\lambda) = \mathcal{M}^r_R(L_i, (f_i, g_i), J; \lambda)$. With this notation and when $a \neq b$, it is easy to see that the formula is:

$$\begin{aligned}\partial \bar{\mathcal{W}}^a_{S,b}(\lambda) = &\textstyle\bigcup_{y,\lambda=\lambda'+\lambda'',S=<S'\cup S''>} \bar{\mathcal{W}}^a_{<S',y>,b}(\lambda') \times \bar{\mathcal{M}}^y_{S''}(\lambda'') \cup \\ &\textstyle\bigcup_{c,\lambda=\lambda'+\lambda'',S=<S'\cup S''>} \bar{\mathcal{W}}^a_{S',c}(\lambda') \times \bar{\mathcal{W}}^c_{S'',b}(\lambda'') \ .\end{aligned} \tag{16}$$

This formula is again interpreted as in (11).

Notice that, in a one-dimensional moduli space as above, one and only one cluster tree attached to a strip might slide to one of the ends of this strip thus creating a boundary point. This type of boundary point is taken into account above by the terms $\bar{\mathcal{W}}^a_{S',a}(\lambda') \times \bar{\mathcal{W}}^a_{S'',b}$ and $\bar{\mathcal{W}}^a_{S',b}(\lambda') \times \bar{\mathcal{W}}^b_{S'',b}$. Thus, to take into account these terms, the only moduli spaces that need to be considered are those which satisfy the auxiliary condition that:

vii". if u_{v_0} is constant then $n_{v_0} = 3$.

As we only need to deal with one parametric moduli spaces here, we will assume from now on that this condition is verified. This condition means that a constant strip can only contain the attaching point of a single cluster tree. Thus the moduli spaces in which u_{v_0} is the constant strip equal to $a \in L_0 \cap L_1$ are of two types: $T^a_{S_0}(\lambda) \subset \mathcal{W}^a_{S_0,a}(\lambda)$ with $S_0 \subset \mathrm{Crit}(f_0)$ and $R^a_{S_1} \subset \mathcal{W}^a_{S_1,a}(\lambda)$ where $S_1 \subset \mathrm{Crit}(f_1)$

(an element in these spaces corresponds to one cluster tree attached to a contained in L_0 in the case of $T^a_{\dots}$ and in L_1 in the case of $R^a_{\dots}$).

We now show that the boundary formula (16) remains true even in the case when $a = b$. Indeed, there are only two possibilities which need to be explicited in this case as they are different from the case of the clustered moduli spaces. The first one is the case when z_0 and z_{-1} become identified. This can be viewed as the bubbling of a disk with boundary on just one L_i, say L_0, which is attached to the constant strip a (of course some cluster trees might be attached to this disk). This means that this bubbling can only happen in a space $\mathcal{W}^a_{S_0,a}(\lambda)$ with $S_0 \subset \mathrm{Crit}(f_0)$. Moreover, this element is not a boundary in $\mathcal{W}^a_{S_0,b}(\lambda)$ because this configuration also appears in the compactification of $T^a_{S_0}(\lambda)$ (when the flowline joining a to the root of some cluster tree tends to 0). The second case already appears in the discussion above: inside $T^a_{S_0}(\lambda)$ (or similarly, inside $R^a_{S_1}(\lambda)$) the length of the flow line joining the root of some cluster tree to a tends to 0. Thus the previous discussion together with the gluing result described in §1.2.3 takes care of both of these cases and hence provides the formula.

As in §2.1.5, the moduli spaces constructed above also admit orientations. The only significant remark here is that for the clusters on L_1, we must take the inverse orientations compared of those described in §2.1.5. We take the orientation described there for the clusters on L_0. With these choices, as in §1.2.3, it is easy to check that $d_F^2 = 0$.

3.2. **Invariance of the fine Floer Homology.** The goal of this section is to prove:

Theorem 3. *The homology $I\!\!FH_*(L_0, L_1, \eta; J, (f_i, g_i))$ is independent up to isomorphism of J, (f_i, g_i) and if $\psi : M \to M$ is a Hamiltonian diffeomorphism, then $I\!\!FH_*(L_0, \psi(L_1), \eta')$ is isomorphic to $I\!\!FH_*(L_0, L_1, \eta)$ (here $\eta' = \psi(\eta)$).*

Proof. First note that, since loops of Hamiltonian diffeomorphisms act trivially on the homology of M (see for instance Lalonde, McDuff, Poletorvich [16]), two different Hamiltonian isotopies from the identity to ψ lead to the same class η' given by concatenation of η and the Hamiltonian isotopy applied to the endpoint of η. Thus the class η' in the statement of the Theorem is well-defined. So fix L_i, f_i, g_i as before and consider also any Hamiltonian isotopy ψ^t, $t \in [0, 1]$ from the identity to ψ. We denote $L'_1 = \psi^1(L_1), L'_0 = L_0$ and we also fix (f'_i, g'_i) Morse-Smale pairs on L'_i. We also need almost complex structures J and J'. We assume that we are in a generic case and our purpose is to construct a comparison morphism:

$$\Phi : I\!\!FC(L_0, L_1, \eta; J, (f_i, g_i)) \to I\!\!FC(L'_0, L'_1, \eta'; J', (f'_i, g'_i)) \ .$$

The construction of this morphism is quite similar to the construction of the comparison morphism ϕ^F in the case of the cluster complex (see §2.2.1). The second step will be to use such morphisms to show the desired isomorphism.

To construct Φ, we fix Morse cobordisms $(F_0 : L_0 \times [0, 1] \to \mathbb{R}, G_0)$ between (f_0, g_0) and (f'_0, g'_0) and $(F_1 : L_1 \times [0, 1] \to \mathbb{R}, G_1)$ between (f_1, g_1) and $(f'_1 \circ$

$\psi^1, (\psi^1)_*^{-1}(g_1'))$ (see 2.2). It is preferable here to assume that the Morse cobordisms are in fact defined on $L_i \times \mathbb{R}$ but this is easy to arrange. We also extend ψ to $M \times \mathbb{R}$ so that $\psi^s = id$ for $s \leq 0$ and $\psi^s = \phi^1$ for $s \geq 1$. We fix a time-dependent almost complex structure $\bar{J}$ which interpolates between J and J' again in infinite time. As before we denote the generators of the fine Floer complex associated to (L_0, L_1) by I_η and we denote by $I'_{\eta'}$ the similar generators of the complex associated to (L_0', L_1'). We also fix the notation $\mathcal{R}$ for the coefficient ring of $I\!FC(L_0, L_1)$ and $\mathcal{R}'$ for the coefficient ring of $I\!FC(L_0', L_1')$. The morphism Φ is a module morphism whose restriction to the coefficient ring $\mathcal{R}$ is the morphism $\Phi : \mathcal{R} \to \mathcal{R}'$ induced by $\phi^{F_0} \otimes \bar{\phi}^{F_1}$. Here ϕ^{F_0} is induced by F_0 and $\bar{J}$ and is defined as in (13). We need to be a bit more precise about the morphism $\bar{\phi}^{F_1}$. We consider the map

$$\bar{F}_1 = F_1 \circ ((\psi^t)^{-1}, t) : \bar{L}_1 = \{(\psi^t(L_1), t)\} \subset M \times \mathbb{R} \to \mathbb{R}$$

and define the comparison morphism $\bar{\phi}^{F_1}$ with respect to this Morse cobordism and the almost complex structure $\bar{J}$.

On the generators of the fine Floer complex, I_η, Φ has the form:

$$\Phi(a) = \sum v^a_{x_1,\ldots,x_k,y_1,\ldots,y_s;b}(\lambda) x_1 \ldots x_k y_1 \ldots y_s b e^\lambda$$

where $x_i \in \mathrm{Crit}(f_0')$, $y_j \in \mathrm{Crit}(f_1')$, $a \in I_\eta$, $b \in I'_{\eta'}$, $\lambda \in \bar{\Lambda}$.

The coefficients $v^a_{\ldots;b}(\lambda)$ are again obtained by counting the elements of some appropriate moduli spaces. For $r_i \in \mathrm{Crit}(f_0') \cup \mathrm{Crit}(f_1')$ and with the rest of the notation as before we denote these moduli spaces by:

$$\mathcal{V}^a_{r_1,\ldots,r_k;b}(\lambda) \ .$$

They are obtained by an obvious adaptation of the definition of the moduli spaces $\mathcal{N}$ (from §2.2) following the exact same procedure that gave the moduli spaces $\mathcal{W}$ (in §3.1) when starting with the moduli spaces $\mathcal{M}$: in short, the root disk is replaced by a pseudoholomorphic $\bar{J}$-strip u which verifies instead of (15) the moving boundary condition

$$u_{v_0}(\mathbb{R}, 0) \subset L_0 \ , \ u_{v_0}(s, 1) \in \psi^s(L_1)$$

and, in the analogue of property iv", (of §3.1) we use the gradient flow of F_0 on $L_0 \times \mathbb{R}$ and the gradient flow of F_1 on $\{(\psi^t(L_1), t)\} \subset M \times \mathbb{R}$. Verifying that Φ is a chain morphism is now a simple exercise by the same arguments as in §2.2.

We will now use this construction to show the existence of an isomorphism between the two fine Floer complexes.

First we assume that the Hamiltonian diffeomorphism ψ is constant equal to the identity. Therefore $I_\eta = I'_{\eta'}$. We introduce a word-area filtration as in §2.2.2 (with the obvious adaptation that the area of the strips involved is also taken into account). In the associated spectral sequence the term E^1 is an isomorphism. This happens because the only connecting orbits that are taken into account at the level E^0 are the Morse orbits and the constant strips joining a point in $I_\eta \times -\infty$ with the same point in $I_\eta \times +\infty$. Applying the same type of algebraic argument as in §2.2.2 we deduce in this case that $H_*(\Phi)$ is an isomorphism.

Now we fix F_0 and F_1 (of course, this means that also (f_i, g_i) as well as (f_i', g_i') are fixed) and we also pick $\bar{J}$ (which is in general time-dependent) so that $\psi^s_*(\bar{J}_{s,0}) = \bar{J}_{s,1}$. This means that $\bar{\phi}^{F_1} = \phi^{F_1}$. We also assume that L_i, L_i', $i \in \{0, 1\}$ are fixed.

Clearly, the resulting chain morphism Φ continues to depend on the Hamiltonian diffeomorphims ψ, on $\bar{J}$ and on various other data which we summarize by ν. We denote this chain morphism by $\Phi = \Phi(\psi, \bar{J}, \nu)$. We claim that any two such chain morphisms are chain homotopic. The key reason for this is that if Φ' is a second such morphism, then $\Phi|_{\mathcal{R}} = \Phi'|_{\mathcal{R}}$. To simplify notation, we denote $k = \Phi|_{\mathcal{R}}$. Now, as usual in Floer theory, we may repeat the construction of the moduli spaces $\mathcal{V}$ with one additional parameter: the construction is simply the analogue in our setting (with our big coefficient rings) of the usual proof that the comparison morphisms in Floer theory induce a canonical map in homology. This provides a linear map

$$\xi : \mathbb{Q}_* < I_\eta > \to (\mathcal{R}' \otimes \mathbb{Q} < I'_{\eta'} >)_{*+1}$$

so that if we now define $\xi : I\!\!FC(L_0, L_1) \to I\!\!FC_{*+1}(L'_0, L'_1)$ by the formula $\xi(r \otimes a) = k(r)\xi(a)$ this provides a chain homotopy between Φ and Φ'.

Finally, by using the inverse Hamiltonian diffeomorphism we obtain that the composition $\Phi(\psi^{-1}, \bar{J}', \nu') \circ \Phi(\psi, \bar{J}, \nu)$ is chain homotopic to a chain morphism of the form $\Phi(id, \bar{J}'', \nu'')$ but we have seen before that such a chain morphism induces an isomorphism in homology and this ends the proof.

□

We formulate the corresponding statement for the symmetric fine Floer homology as a Corollary as its proof follows exactly the same scheme.

Corollary 3.1. *The isomorphism type of the symmetric fine Floer homology*

$$(\hat{I\!\!F}H(L, H, J, (f, g)), d_{\hat{F}})$$

defined in §1.3 does not depend on the generic choice of $H, J, (f, g)$.

Remark 3.2. In the symmetric case, it is be useful to make explicit the following situation. Assume that we have two Hamiltonians $H, H' : M \times S^1 \to \mathbb{R}$. Fix the almost complex structure $\bar{J}$ which interpolates between J and J' (we might even take $J = J'$ and $\bar{J}_s = J$, $\forall s$) as well as the Morse-Smale pair (f, g). Generically, we have two symmetric fine Floer complexes:

$$\hat{I\!\!F}C(L, H, J, (f, g)) \ , \ \ \hat{I\!\!F}C(L, H', J', (f, g)) \ .$$

Clearly, the coefficient rings $\hat{\mathcal{R}}$, $\hat{\mathcal{R}}'$ are just the cluster algebras $\mathcal{C}\ell\,(L, J, (f, g))$, $\mathcal{C}\ell\,(L, J', (f, g))$. Fix also a trivial Morse cobordism between (f, g) and itself. As in §2.2 this data provides a chain morphism $\phi : \mathcal{C}\ell\,(L, J, (f, g)) \to \mathcal{C}\ell\,(L, J', (f, g))$ (it is easy to see that this is even an isomorphism). The argument in the proof above shows that, in conjunction with this fixed data, any two homotopies G, G' between H and H' provide a chain morphisms:

$$\Phi(G), \Phi(G') : \hat{I\!\!F}C(L, H, J, (f, g)) \to \hat{I\!\!F}C(L, H', J', (f, g))$$

which are chain homotopic and, thus, induce the same morphism at the level of the symmetric, fine Floer homology. Moreover, this morphism is an isomorphism.

4. Some applications and comments.

It is natural to expect that the symmetric fine Floer homology has a simple expression when the Hamiltonian H is a very small time-independent Morse function. We start in this section by discussing this point.

4.1. **The symmetric fine Floer complex and the cluster complex.** Assume that, as before, we have a fixed Morse-Smale pair on L (f,g) and an almost complex structure J so that the cluster complex $\mathcal{C}\ell\,(L,(f,g),J)$ is defined.

Let $h : L \to \mathbb{R}$ be another Morse function so that the pair (h,g) is Morse-Smale. By using h we may construct a complex $\hat{\mathcal{C}}(L,J,f,h,g)$ similar to the fine Floer complex except that the intersection points I_η are replaced by the critical points of h and the strips at i". in equation (15) are replaced by linear cluster elements corresponding to $L, J, (h,g)$, see § 2.3.2. While this construction is quite similar to many of our other constructions discussed before, one significant point is worth explicit notice: in the moduli spaces which provide the differential in this new complex the cluster elements of f can only be anchored at points on the disks appearing in the linear cluster element of h (and not on points appearing on the flow lines of h). Clearly, all bubbling is dealt with by using cluster elements of f. Given these remarks, the definition of the differential of this complex is transparent as well as the facts that its square vanishes and that its homology is independent of the choice of J, h, f, g.

Moreover, it is not difficult to define a comparison morphism relating the symmetric fine Floer complex to $\hat{\mathcal{C}}(L,J,f,h,g)$ as well as a morphism in the other direction. The construction of these morphisms is based on the PSS [21] idea which was already mentioned in §2.3.2. More precisely, to define these morphisms, we fix a bump function $\beta : \mathbb{R} \to [0,1]$ (as, for example, in Schwarz [23]) so that $\beta(s) = 0$ for $s \le 1/2$, $\beta(s) = 1$ for $s \ge 1$ and β is increasing. We consider moduli spaces which, up to codimension two elements, are made of objects consisting of: a chain of J-disks $u_1, \ldots, u_k$ so that a critical point x of h is connected by a negative gradient flow line of h to u_1 and each disk u_i is related to the disk u_{i+1} by a negative gradient flow line of h; a finite energy solution u of the equation

$$\partial_s u + \partial_t u + \beta(s)\nabla H = 0$$

where

$$u : (\Sigma, \partial\Sigma) \to (M, L)$$

is defined on $\Sigma = \mathbb{R} \times [0,1]$; we ask that a point on the disk u_k be related to $u(-\infty)$ by a negative gradient flow line of h and that $\lim_{x\to\infty} u = \gamma$ where γ is a trajectory of H; f-clusters possibly attached to the disks u_i as well as to u. Under the usual genericity conditions and by the same arguments used in describing the comparison morphisms for the fine Floer homology, it is easy to see that by counting the elements of such 0-dimensional moduli spaces, we get a morphism:

$$\hat{\mathcal{C}}(L,J,f,h,g) \to \hat{I\!\!F}C(L,H,J,(f,g)\ .$$

The morphism in the other direction is constructed in a similar way. Both morphisms are $\mathcal{C}\ell\,(L,(f,g),J)$-morphisms. By treating both of these maps as module morphisms over the ring $\hat{\mathcal{R}} = \mathcal{C}\ell\,(L,(f,g),J)$, it is not difficult to show, along the lines in (3.2) and by making use of the PSS method, that the composition

$$\hat{\mathcal{C}}(L,J,f,h,g) \to \hat{I\!\!F}C(L,H,J,(f,g) \to \hat{\mathcal{C}}(L,J,f,h,g)$$

is chain homotopic to the identity.

Thus we obtain:

Proposition 4.1. *With the notation above, we have:*

$$\hat{I\!\!F}H_*(L) \simeq H_*\hat{\mathcal{C}}(L,J,f,h,g)\ .$$

The construction of $\hat{\mathcal{C}}(L,J,f,h,g)$ is possible even if $h=f$. A simple exercise shows that, in this case, $\hat{\mathcal{C}}(L,J,f,f,g)$ coincides with $s\widetilde{\mathcal{C}\ell}(L,J;(f,g))$, which provides the proof of Proposition 1.12 in §1.3.2 (the suspension s appears here because, in $\mathcal{C}(L,J,f,h,g)$, the degrees of the generators represented by the critical points of h are the same as their Morse indices).

4.2. **The cluster complex in minimal form.** As we shall see further in this section, it is useful to reduce algebraically the cluster complex to a minimal form that we now describe.

Proposition 4.2. *Assume that L, (f,g), J are so that the cluster complex*

$$\mathcal{C}\ell(L,(f,g),J)$$

is defined. There exists a commutative differential graded algebra $\mathcal{C}\ell_{min}(L,(f,g),J)$ and a surjective differential algebra morphism

$$P:\mathcal{C}\ell(L,(f,g),J)\to\mathcal{C}\ell_{min}(L,(f,g),J)$$

so that, as an algebra, $\mathcal{C}\ell_{min}(L,(f,g),J)=(S(s^{-1}\ H_*(L;\mathbb{Q}))\otimes\ \Lambda)^\wedge$, the differential d_{min} of $\mathcal{C}\ell_{min}(L,(f,g),J)$ increases strictly the word-area filtration and P induces an isomorphism in homology.

Remark 4.3. If f happens to be a perfect Morse function and thus its Morse differential vanishes, then $\mathcal{C}\ell(L,(f,g),J)$ itself may be taken as $\mathcal{C}\ell_{min}(L,f,J)$.

Proof. Let d be the differential of the cluster complex $\mathcal{C}\ell(L,f,J)$. We decompose $d=d_0+d'$ where d_0 is induced by the Morse differential (in other words, this is the part that does not increase the word-area filtration). By a change of basis inside $\mathbb{Q}<\mathrm{Crit}(f)>$ we may assume that each generator x verifies either $d_0x=0$ or $d_0x=y$ with y another generator of the complex. We fix such a pair of generators $x, y=d_0x$ and consider the element $y'=dx$. We want to replace, in the cluster complex, the generator y by the element y'. For this we only need to show that y can be expressed in terms of the other generators of $\mathcal{C}\ell(L,f)$ together with y'. Write $y'=y+b$. Certainly, b is of higher word-area filtration than 0. We now express $y=y'-b$. For further use we put $u_0=y'$ and $-b=r_0$ so that $y=u_0+r_0$. The term b might still contain some y's and we write $b=a+y^ib'+c$ where a and b' do not contain y and c might contain y but is of higher word-area filtration than y^ib'. Replace y by $y'-b$ in the expression of b. This gives $y=y'-a+(y'-b)^ib'+c$. This means that we have now written y as a sum $y=u_1+r_1$ so that u_1 is only expressed in y' and in the generators of $\mathcal{C}\ell$ which are different from y and r_1 has higher word-area filtration than r_0. We apply iteratively this process and, with respect to our completion, this does express y as the limit of the u_i's.

Now let $I(x,y')$ be the differential ideal generated by x and $y'=dx$. We consider the projection

$$P_1:\mathcal{C}\ell(L,f,J)\to\mathcal{C}\ell(L,f,J)/I=\mathcal{C}\ell_1\ .$$

We notice that $\mathcal{C}\ell_1$ is a free commutative differential algebra generated by the same generators as $\mathcal{C}\ell(L,f,J)$ with the exception of x and y'. Moreover, it is a simple exercise to see that the ideal $I(x,y')$ is acyclic so that P_1 induces an isomorphism in homology.

This process has reduced the number of pairs of generators x, y of our algebra which are related by the formula $d_0x = y$ by one. We iterate it till no such pairs remain. The resulting algebra is $\mathcal{C}\ell_{min}$. □

Remark 4.4. We will always apply this proposition to functions f with a single local minimum and a single local maximum. These two generators are always d_0-cycles and they are not pertubed by the algebraic reduction decribed above.

As an application of this proposition we complete the proof of Corollary 1.15. Indeed, in the construction of the symmetric fine Floer homology we may very well use $\mathcal{C}\ell_{min}(L, f, J)$ instead of $\mathcal{C}\ell(L, f, J)$: it is simply a matter of changing coefficients by using the map P. The advantage is that if $H_*(L;\mathbb{Q})$ vanishes in even degrees except in dimension 0 and $\dim(L)$, then $\mathcal{C}\ell_{min}(L, f, J)$ can not have high free terms. Moreover, for any function f with a unique minimum m, if $d_{min}m \neq 0$, then $dm \neq 0$. The argument used to prove the Corollary in the special case when a perfect Morse function exists can therefore be applied also in this case and this concludes the proof of 1.15.

4.3. **Detection of disks of bounded area.** Recall that L is said to be displaceable if there exists a Hamiltonian $H : M \times S^1 \to \mathbb{R}$ whose associated Hamiltonian diffeomorphism ψ_H verifies $\psi^1_H(L) \cap L = \emptyset$. The displacement energy of L, $E(L)$, is then the infimum of the Hofer energies of the Hamiltonians with this property. The Hofer energy of a Hamiltonian H will be taken here simply as $\max_{x,t} H - \min_{x,t} H$.

The purpose of this subsection is to prove Corollary 1.18 from § 1.4.1. For the convenience of the reader we recall that this corollary claims that for a relatively spin, orientable Lagrangian submanifold L which is displaceable, any ω-tame almost complex structure J has the property that one of the following is true:

i. for any point $x \in L$ there exists a J-pseudo-holomorphic disk of symplectic area at most $E(L)$ whose boundary rests on L and which passes through x.
ii. there exists a J-disk of Maslov index at most

$$2 - \min\{2k \in \mathbb{N}^* \setminus \dim(L) : H_{2k}(L;\mathbb{Q}) \neq 0\}$$

and of symplectic area at most $E(L)$.

Proof. We start with a number of more general constructions.

Recall from Remark 1.9 that it is possible to define an action in the setting of the fine Floer complex. Of course, the definition given in that remark also applies to the symmetric fine complex and it is this version that will be of use here. Clearly, there is also an action in the complex $\hat{\mathcal{C}}(L, J, f, h)$: the action of a generator $\bar{x}e^{\lambda}$ in this complex is $h(x) - \omega(\lambda)$. Both functions f and h used here are assumed to have a single local minimum and a single local maximum. We fix a small disk D in L and we assume that both the minimum of f and that of h are included in this disk. We indend to show that either there is a J-disk passing through D and of area $\leq E(L) + \epsilon$ or there exists a J-disk of non-positive Maslov index with area again bounded by $E(L) + \epsilon$ where ϵ is an arbitrarily small constant and whose Maslov index satisfies the condition ii. above. As the disk D is arbitrary this implies the claim by Gromov compactness.

The proof is based on a comparison between three symmetric, fine Floer complexes. For all three of them we will use the same data $((f, g), J)$. The three

Hamiltonians will be $h : M \to \mathbb{R}$, $H : M \times S^1 \to \mathbb{R}$ and $h' : M \to \mathbb{R}$ so that: h is a Morse function with a single minimum $\bar{m}$, $\epsilon > \max(h) - \min(h)$, $\min_{x,t} H > 0$ and $K \geq \max(h) > \min(h) > \max_{x,t}(H)$, $h' = h - \max(h)$ where K is some positive constant and ϵ is small and h' is very C^2-small. The key remark that we will use is that, as in the usual Floer case, if we compare the different Hamiltonians by using monotone homotopies, then the action can not increase between the domain and the image of a map. In short, if one moduli spaces $\mathcal{W}^a_{\ldots;b}(\lambda)$ is not null, then $\mathcal{A}(a) - \mathcal{A}(be^{\lambda}) \geq 0$ and similarly for the moduli spaces of type $\mathcal{V}$ which are associated to the homotopies relating the different Hamiltonians in case these homotopies are monotone. Our choice of Hamiltonians imply the existence of such monotone homotopies and as a result we obtain morphisms:

$$\hat{I\!F}C(L, h) \xrightarrow{\Phi} \hat{I\!F}C(L, H) \xrightarrow{\Phi'} \hat{I\!F}C(L, h')$$

so that both preserve the action level.

As discussed in §4.1 we may replace $\hat{I\!F}C(L, h)$ with $\hat{\mathcal{C}}(L, h) = \hat{\mathcal{C}}(L, J, h, f, g)$ because it is easy to see that the comparison morphisms described there are also action preserving. We have a similar identification between $\hat{I\!F}C(L, h')$ and $\hat{\mathcal{C}}(L, h')$. The composition $\Phi' \circ \Phi$ is chain homotopic - see Remark 3.2 to a morphism induced by a trivial Morse cobordism G between h and h' (which has the form $G(x, t) = h(x) + k(t)$). Moreover, this chain homotopy is also action preserving.

For the complex $\hat{I\!F}C(L, H)$ we may define its action filtration as follows: $\hat{I\!F}C^s(L, H)$ is the (appropriate completion of the) rational vector space generated by the monomials $x_1 \ldots x_k a e^{\lambda}$ where $\mathcal{A}(\bar{a}e^{\lambda}) \leq s$ (here, as before, $x_i \in \mathrm{Crit}(f)$, a is a generator of the fine symmetric complex, and $\lambda \in \bar{\Lambda}$). We have similar filtrations also for $\hat{\mathcal{C}}(L, h)$ and $\hat{\mathcal{C}}(L, h')$ (clearly, the stages in these filtrations are not anymore $\bar{\Lambda}$-modules but this will not affect in any way the arguments that follow).

These filtrations are differential and for $s' < s$ we may define the chain complex:

$$\hat{I\!F}C^{s,s'}(L, h) = \hat{I\!F}C^s(L, h) / \hat{I\!F}C^{s'}(L, h)$$

and similarly for the two other complexes. Due to the fact that Φ and Φ' are action preserving the following composition is well defined:

$$\Psi = \Phi' \circ \Phi : \hat{\mathcal{C}}^{K,-\epsilon}(L, h) \longrightarrow \hat{I\!F}C^{K,-\epsilon}(L, H) \longrightarrow \hat{\mathcal{C}}^{K,-\epsilon}(L, h')$$

and as the chain homotopy between $\Phi' \circ \Phi$ is action preserving, Ψ is chain homotopic with the chain morphism Ψ' induced by the trivial Morse cobordims G. In the argument below we will only use the differential of these truncated complexes.

We now specialize our construction: we notice that we may pick the constant K, the Hamiltonian H and the Morse function h so that $\phi^H_1(L) \cap L = \emptyset$ and $K = E(L) + \epsilon$. In this case $\Psi = 0$.

We also need to fix the monotone homotopies $L : h \simeq H$ and $L' : H \simeq h'$ which induce respectively the ϕ and ϕ' as well as a monotone homotopy of homotopies $T : L\#L' \simeq G$ (here $L\#L'$ is the composed homotopy which joins h to h'). The chain homotopy η^T between Ψ and Ψ' is provided by this T - see Remark 3.2.

Let m' be the unique minimum of h'. We will add bars over the critical points of h or h' to denote the respective generators of the fine Floer complexes. Clearly, if we have $d\bar{m} \neq 0$ or $d\bar{m}' \neq 0$ by the same argument as in Corollary 1.14 we obtain that there is a J-disk passing through m whose area is at most $E(L) + \epsilon$ which consludes the proof. So from now on we shall assume that $d\bar{m} = 0 = d\bar{m}'$.

Notice also that we may assume that $\Psi'(\bar{m}) = \bar{m}'$. Indeed, we have that $\Psi'(\bar{m}) = \bar{m}' + \ldots$ because there is a unique flow line of G exiting $\bar{m}$ and this flow line ends in $\bar{m}'$. Geometrically, this flow line is constant equal to the critical point m. Therefore, if there is any cluster tree originating at $\bar{m}$ there will be a J-disk which crosses this line - but this means that this disk passes through m which again implies the claim.

In view of this, we have $d\eta^T(\bar{m}) = \bar{m}'$. Given Remark 1.1 the only possibility for $\bar{m}'$ to be a boundary is that there exists some $x \in Crit(f)$ so that $dx = a_0 e^{\lambda} + \ldots$ (we may assume $|x| \geq 0$ otherwise, again, our claim follows).

But this means precisely that the truncated cluster complex has high free terms which shows that there exist J-disks of area bounded by $E(L)+\epsilon$ and whose Maslov index is at most $2 - \min\{2k \in \mathbb{N}^* \backslash \dim(L) : \mathrm{Crit}_{2k}(f) \neq 0\}$. The last step in the proof is to note that, in this argument, instead of the cluster complex associated to f we could have used as well the minimal form of it, $\mathcal{C}\ell_{min}(L, f, J)$, as described in §4.2. This allows us to replace the condition $\mathrm{Crit}_{2k}(f) \neq 0$ by the one in the statement. $\square$

Remark 4.5. The method of proof used here is an adaptation of one used in [4] and Barraud-Cornea [5] to detect pseudoholomorphic strips of bounded area.

A particular case is worth stating separately.

Corollary 4.6. *If a relatively spin, orientable Lagrangian L verifies $H_{2k}(L;\mathbb{Q}) = 0$ if $2k \notin \{0, \dim(L)\}$ and is displaceable, then for any J, through each point of L passes a J-disk of symplectic area at most $E(L)$.*

For any Lagrangian submanifold $L \subset (M, \omega)$ define (as in [5], see also [4]) its real, or relative, Gromov radius $r(L)$ as the infimum of the positive constants r so that there is a symplectic embedding of the standard standard sphere $(B(r), \omega_0) \overset{e}{\hookrightarrow} (M, \omega)$ with the property that $e^{-1}(L) = \mathbb{R}^n \cap B(r)$.

Corollary 4.7. *If an orientable, relatively spin Lagrangian with $H_{2k}(L;\mathbb{Q}) = 0$ for $2k \notin \{0, \dim(L)\}$ is displaceable, then*

$$E(L) \geq \pi r(L)^2/2 \ .$$

This follows immediately from Corollaries 1.18 and 4.6 by a typical Gromov's "minimal surface" argument as in [4]. If $\mu_{min} \geq 2$, the result in Corollary 4.7 remains true without the homological, orientability or relatively spin conditions because we may apply the method in the Corollaries above directly to the complex $\mathcal{C}(L, J, f)$ described in §2.3.2 and this complex is defined over $\mathbb{Z}/2$. Corollary 4.7 is a particular case of a conjecture formulated in [5].

4.4. **Some calculations.** Our machinery easily leads to restrictions on the topology of Lagrangian submanifolds in $\mathbb{C}^n$. We will only discuss here one such immediate example. In this case, we recover results that agree with those obtained in [11] (see also [10]). We will pursue with other examples and applications elsewhere.

Example 4.8. We assume that our relatively spin Lagrangian submanifold $L \subset M$ is diffeomorphic to $S^1 \times S^{n-1}$ and $\hat{I\!F}H_*(L) = 0$ (for example, as happens in the case considered in [11] where $M = \mathbb{C}^n$).

We indend to deduce restrictions on the image of the Maslov index homomorphism

$$\mu : \pi_1(L) \approx \pi_2(\mathbb{C}^n, L) \to \mathbb{Z} \ .$$

We construct the cluster complex and the associated complex $\widetilde{\mathcal{C}\ell}(L,J;f)$ which computes the symmetric fine Floer homology (as in §4.1) by using a perfect Morse function f. We denote by m its minimum, by M its maximum, by a its critical point of index 1 and by b its critical point of index $n-1$. We denote by $\bar{x}$ the corresponding generators of $\widetilde{\mathcal{C}\ell}(L,J;f)$ (viewed as a module over $\mathcal{C}\ell(L,J,f)$).

The first key question in this type of calculation is whether the cluster complex has free terms or not (see Proposition 1.3). Suppose first that it has free terms. This means that some generator x verifies $dx = a_0 e^{\lambda} + \ldots$. As $\mu(\lambda)$ is even, it follows that x may be m (and, in this case $\mu(\lambda)=2$) but cannot be a. It might also be b in which case $n-1$ is even and $\mu(\lambda)=3-n$. In all cases, it cannot be M by Remark 1.4.

The other possibility to be considered is if the cluster differential does not have any free terms. In this case we will use the fact that the complex $\widetilde{\mathcal{C}\ell}(L,J;f)$ has a trivial homology and also has a differential that cannot decrease the word length.

The remark above concerning the maximum applies as well to $\bar{M}$ so that $d\bar{M} = 0$. This means that there has to be another generator y of $\widetilde{\mathcal{C}\ell}(L,J;f)$ verifying $dy = a_1\bar{M}e^{\lambda}+\ldots$, $a_1\in\mathbb{Q}^*$. First, $y\neq\bar{M}$ because $\mu(\lambda)$ is even. In case $y=\bar{b}$, then $\mu(\lambda)=2$. Another possible choice for y is $y=\bar{m}$ and then $\mu(\lambda)=n+1$ which has to be even. Similarly $y=\bar{a}$ implies $\mu(\lambda)=n$ which needs to be even.

We now summarize our results for n even:

- if n is even, then $\{2,n\}\cap Im(\mu)\neq\emptyset$.

Clearly, if $2\in Im(\mu)$, then $n\in Im(\mu)$ also but there exists an example due to Polterovich in which $2\notin Im(\mu)$.

We now pursue our discussion in case n is odd. As $2\in Im(\mu)$ implies $3-n\in Im(\mu)$, our discussion till now shows that if $3-n\notin Im(\mu)$, then the cluster complex has no free terms. In this case we see that $d\bar{m} = a_1\bar{M}e^{\lambda}+\ldots$ and so $\mu(\lambda)=n+1$. We now write $d\bar{m} = \bar{M}e^{\lambda}(a_1+a)+c$ where $a\in\mathcal{C}\ell(L,f)$ and is a term of word-area filtration greater than 0, $a_1\in\mathbb{Q}^*$ and c does not contain $\bar{M}$. We define a $\mathcal{C}\ell(L,f)$-differential module epimorphism

$$q:\widetilde{\mathcal{C}\ell}(L,f)\to(\mathcal{C}\ell(L,f)\otimes\mathbb{Q}<\bar{a},\bar{b}>,d')=\mathcal{K}$$

by sending $\bar{m}\to 0$, $\bar{M}\to -c(a_1+a)^{-1}e^{-\lambda}$ where the differential d' in the target module is the one induced by q (notice that (a_1+a) is invertible in $\mathcal{C}\ell(-)$ due to our completion). It is easy to see that this map q induces an isomorphism in homology and thus $H_*(\mathcal{K})$ is trivial. This means that either $d'\bar{a} = a_2 b e^{\lambda'}+\ldots$ and in this case $\mu(\lambda')=n-1$ and/or $d'\bar{b} = a_3 a e^{\lambda''}+\ldots$ ($a_i\in\mathbb{Q}^*$) which means $\mu(\lambda'')=3-n$. But as we also have $\mu(\lambda)=n+1$ both these possibilities contradict $3-n\notin Im(\mu)$. To conclude the argument in this case:

- if n is odd, then $\{2,3-n\}\cap Im(\mu)\neq\emptyset$.

4.5. **Detection of closed orbits of Hamiltonian systems.** We show here that, given any two compact oriented relatively spin Lagrangian submanifolds L, L' in a closed or geometrically bounded symplectic manifold (M,ω), there is a natural morphism of chain complexes

$$\mathrm{cyl}^{\Phi_0} : \mathcal{C}\ell\,(L) \to \mathcal{C}\ell\,(L) \otimes \mathcal{C}\ell\,(L') \otimes \Lambda_{\Phi_0}$$

defined using holomorphic cylinders (of varying conformal structure) with one boundary in L and the other in L' (here Λ_{Φ_0} is a coefficient ring defined below). When this morphism is not trivial in homology, and if some Hamiltonian in M *separates* L and L', one can easily deduce the existence of closed orbits of H with bounded period. Let us explain this more precisely.

4.5.1. *Reviewing the Maslov index.* Let L, L' be two oriented relatively spin Lagrangian submanifolds in (M, ω).

To define the Maslov index of a cylinder $u \in [S^1 \times [0,1], S^1 \times \{0\}, S^1 \times \{1\}; M, L, L']$, first choose a $SO(n)$-trivialization of $(u|_{S^1\times\{0\}})^*(T_*L)$, extend it in the natural way to a $U(n)$-trivialization of $(u|_{S^1\times\{0\}})^*(T_*M)$, and then to a well-defined, up to homotopy, $U(n)$-trivialization of $u^*(T_*M)$. The Maslov index of u is set to be equal to the Maslov index of the Lagrangian subbundle $(u|_{S^1\times\{1\}})^*(T_*L')$ with respect to this trivialization.

It is easy to see that this definition is independent of the choice of the $SO(n)$-trivialization of $(u|_{S^1\times\{0\}})^*(T_*L)$ and that it is a variant of the definition of the Maslov morphism as defined in §1.2. Similarly, one easily sees that, in the case when the homotopy class of $u|_{S^1\times\{0\}}$ vanishes, it is equal to the Maslov index of $(u|_{S^1\times\{0\}})^*(T_*L)$ with respect to the trivialization of any capping disc v, minus the Maslov index of $(u|_{S^1\times\{1\}})^*(T_*L')$ with respect to the trivialization of the capping disc $u\#v$. Note finally that the index of the space of parametrized holomorphic cylinders (with unprescribed conformal structure) in a class Φ is equal to $\mu(\Phi)+1$, so that the unparametrized index is $\mu(\Phi)$.

Now fix any homotopy class Φ_0 of continuous maps

$$S^1 \times [0,1] \to M$$

with the boundary component $S^1 \times \{0\}$ in L and the boundary component $S^1 \times \{1\}$ in L'. Consider the obvious left action of $\pi_2(M,L)$ on the space $[S^1 \times [0,1], S^1 \times \{0\}, S^1 \times \{1\}; M, L, L']$ and the right action of $\pi_2(M, L')$ on the same space. Obviously these two actions commute. Let us denote by K the subset of $[S^1 \times [0,1], S^1 \times \{0\}, S^1 \times \{1\}; M, L, L']$ equal to the orbit of these two actions on the element Φ_0.

Remark 4.9. We could as well take the full space $[S^1 \times [0,1], S^1 \times \{0\}, S^1 \times \{1\}; M, L, L']$ instead of the orbit K. Note that these two sets are in general distinct (for instance for Lagrangian submanifolds in a cotangent given as sections, there in no left and right action, so that K contain no multiple covering of Φ_0). The latter being smaller than the former, the applications of this theory will be simpler to compute.

As usual, denote by $\sim$ the quotient of the objects in $[S^1 \times [0,1], S^1 \times \{0\}, S^1 \times \{1\}; M, L, L']$, $\pi_2(M,L)$ and $\pi_2(M,L')$ by the relation requiring equality of both the Maslov index and the symplectic area. Now let $\bar{K}$ be the abelian subgroup of $\mathbb{Z} \times \mathbb{R}$ generated by the elements in $\pi_2(M,L)/\sim, \pi_2(M,L')/\sim$ and $K/\sim$, which of course is the same as taking the subgroup generated by the first two and by the singleton $(\mu(\Phi_0), \omega(\Phi_0))$. Let finally Λ_{Φ_0} be the usual completion of the group ring $\mathbb{Q} < \bar{K} >$.

4.5.2. *Appropriate moduli spaces.* We define the morphism cyl^{Φ_0} by considering the following moduli spaces. Fix Morse functions and metrics $(f, g), (f', g')$ on L and L', a compatible complex structure J and a generic perturbation ν (for simplicity, we will omit the ν in the symbols denoting our moduli spaces). For elements $x, x_1, \ldots, x_k \in \mathrm{Crit}(f)$, elements $y_1, \ldots, y_m \in \mathrm{Crit}(f')$, and a class $\bar{\lambda} \in \bar{K}$, denote by $\mathcal{C}^x_{x_1,\ldots,x_k,y_1,\ldots,y_m}(\bar{\lambda})$ the space of configurations (after quotient by the automorphism group) made from

1) a cluster on L rooted at x, with terminations $x_1, \ldots, x_k$ and where all vertices of the graph defining the cluster correspond to boundaries of (J, ν)-holomorphic disks in (M, L) except precisely one which is replaced by a loop γ_L in L;

2) a (J, ν)-holomorphic cylinder in class K

$$u : (S^1 \times [0, \sigma], S^1 \times \{\sigma\}, S^1 \times \{0\}) \to (M, L, L')$$

satisfying

$$u|_{S^1 \times \{\sigma\}} = \gamma_L,$$

where $\sigma > 0$ is the height of the cylinder defining its conformal structure (the cylinder is endowed with the standard complex structure coming from the quotient of the strip $[0, 1] \times [0, \sigma] \subset \mathbb{C}$ by the translation by 1 in the x-direction);

3) a cluster on L' rooted at a vertex replaced by the loop $\gamma_{L'} := u|_{S^1 \times \{0\}}$, with terminations $y_1, \ldots, y_m$ in which all other vertices of the graph defining the cluster correspond to boundaries of (J, ν)-holomorphic disks in (M, L').

We define the Maslov index of such a configuration C and its area by:

$$\sum_{\mathcal{D}} (\mu(D), \omega(D)) + (\mu(u), \omega(u))$$

where $\mathcal{D}$ is the set of (J, ν)-holomorphic discs appearing in the cluster elements of the configuration (in (M, L) as well as in (M, L')) and u is the above cylinder. We finally require that

4) $(\mu(C), \omega(C)) = \bar{\lambda}$.

Lemma 4.10. *a) The dimension of* $\mathcal{C}^x_{x_1,\ldots,x_k,y_1,\ldots,y_m}(\bar{\lambda})$ *is equal to*

$$|x| - \sum_i |x_i| - \sum_j |y_j| + \mu(\bar{\lambda}) - n + 2.$$

b) The cell of maximal dimension of the boundary of the compactification

$$\bar{\mathcal{C}}^x_{x_1,\ldots,x_k,y_1,\ldots,y_m}(\bar{\lambda}) - \mathcal{C}^x_{x_1,\ldots,x_k,y_1,\ldots,y_m}(\bar{\lambda})$$

is made of all configurations obtained from the ones described before by admitting one broken flowline of one of the flows $-\nabla_g f, -\nabla_{g'} f'$. *The break point belongs to* $\mathrm{Crit}(f) \cup \mathrm{Crit}(f')$. *The full boundary is obtained in a similar way by adding* d_1 *broken flowlines and* d_2 (J, ν)*-holomorphic* 2*-spheres attached to points belonging to discs or to the cylinder, where* $d_1 + 2d_2 \leq d$ *(here* d *is the dimension of the moduli space).*

The proof of this lemma is left to the reader. Two points are worth mentioning: the dimension formula in (a) is an easy consequence of the formula for the real dimension of parametrized pseudoholomorphic cylinders u with one boundary on L and the other on L' (of non-specified conformal structure): it is equal to $\mu(u) + 1$, see [12] (thus the unparametrized moduli space has dimension $\mu(u)$). The second remark concerns (b): a cylinder in class Φ may degenerate to a cylinder in class

$\Phi - \lambda - \lambda'$ where $\lambda \in \pi_2(M, L)/\sim$ and $\lambda' \in \pi_2(M, L')/\sim$ together with J-disks with boundaries in L and L' and of classes, respectively, λ and λ' . This shows that the correct Novikov ring to consider is indeed the larger ring Λ_{Φ_0} which contains both $\Lambda(L)$ and $\Lambda(L')$. $\square$

Set:

$$\mathrm{cyl}^{\Phi_0} : \mathcal{C}\ell\,(L) \to \mathcal{C}\ell\,(L) \otimes \mathcal{C}\ell\,(L') \otimes \Lambda_{\Phi_0}$$

defined by

$$\mathrm{cyl}^{\Phi_0}(x) = (-1)^{|x|} \sum_{x_1,\ldots,x_k,y_1,\ldots,y_m,\bar{\lambda}} \#\mathcal{C}^x_{x_1,\ldots,x_k,y_1,\ldots,y_m}(\bar{\lambda}) x_1,\ldots,x_k y_1,\ldots,y_m e^{\bar{\lambda}}$$

when the dimension of this configuration space is zero, and zero otherwise. Extend this definition by the Leibniz rule

$$\mathrm{cyl}^{\Phi_0}(xy) = \mathrm{cyl}^{\Phi_0}(x)\, y + (-1)^{|x|} x \,\mathrm{cyl}^{\Phi_0}(y)$$

and then by linearity. We define the degree of an element in $\mathcal{C}\ell\,(L) \otimes \mathcal{C}\ell\,(L') \otimes \Lambda_{\Phi_0}$ by the usual formula.

Lemma 4.11. cyl^{Φ_0} *is a morphism of chain complexes of degree* $2 - n$*.*

Proof. Let $x \in \mathrm{Crit}(f)$. We must show that $\mathrm{cyl}^{\Phi_0} \circ d(x) = d \circ \mathrm{cyl}^{\Phi_0}(x)$. A term on the left hand side corresponds to a configuration made from the one defining a term $a = x_1 \ldots x_k e^{\lambda}$ in the differential of x attached at one of its terminations x_i to the root x_i of a configuration defining a term $b = x'_1 \ldots x'_{k'} y_1 \ldots y_m e^{\bar{\lambda}}$ in $\mathrm{cyl}^{\Phi_0}(x_i)$. Smoothing the resulting configuration at x_i, one gets a real one-parameter family of configurations in $\mathcal{C}^x_{x_1,\ldots,\hat{x}_i,\ldots,x_k,x'_1,\ldots,x'_{k'},y_1,\ldots,y_m}(\lambda + \bar{\lambda})$. By the above compactification formula, the other end of this bordism must be obtained from an element of $\mathcal{C}^x_{x_1,\ldots,\hat{x}_i,\ldots,x_k,x'_1,\ldots,x'_{k'},y_1,\ldots,y_m}(\lambda + \bar{\lambda})$ by adding one broken flowline either on f or on f'. If the broken edge of the tree appears below the vertex corresponding to the cylinder u (on L or on L', it does not matter), then the resulting configuration belongs to $d \circ \mathrm{cyl}^{\Phi_0}(x)$. Otherwise, it belongs to $\mathrm{cyl}^{\Phi_0} \circ d(x)$. One checks easily that the same holds when one starts from a term on the right hand side of $\mathrm{cyl}^{\Phi_0} \circ d(x) = d \circ \mathrm{cyl}^{\Phi_0}(x)$. The statement concerning the degree is an immediate consequence of the formula for the dimension of the moduli spaces in the last lemma. $\square$

4.5.3. *Applications to the existence of closed orbits.* We will say that a morphism $\mathcal{C}\ell\, H_*(L) \to H_*(\mathcal{C}\ell\,(L) \otimes \mathcal{C}\ell\,(L') \otimes \Lambda_{\Phi_0})$ is *trivial* if it vanishes on all of the Λ-generators of $\mathcal{C}\ell\, H_*(L)$ (note that it need not vanish on the unit).

Corollary 4.12. *Let* L, L' *be oriented relatively spin submanifolds in some geometrically bounded manifold* (M, ω)*. Let* $H_{t\in[0,1]}$ *be a Hamiltonian on* M *that separates* L *and* L' *in the sense that the gap*

$$\int_0^1 (\min_L H_t - \max_{L'} H_t) dt$$

is strictly positive. Then, if the morphism cyl^{Φ_0} *is not trivial in homology, there must be a closed orbit* $\gamma(t)$ *of the system* TH_t *of period* 1*, for some* $T \leq A/gap$ *where* A *is the symplectic area of* Φ_0*. In particular, if* H *is autonomous, this gives the existence of a periodic orbit of* H *of period bounded above by* A/gap*.*

Proof. We proceed in three simple steps: the first one gives an a priori estimate, the second one is the core of the proof, and the third gives the definition of a homotopy operator used in the second step.

1) *First step.* From [12] (or Hofer-Viterbo [14] in a slightly different context), recall that there is an a priori estimate that gives an upper bound of the s-energy of the cylinder u in class K satisfying the equation $\bar{\partial}_J u = -T\nabla_{g_M} H_t$, where g_M is the metric $\omega(J\cdot,\cdot)$ on M and T is allowed to vary in the semi-infinite interval $[0,\infty)$ (here s is the vertical coordinate, whereas t will denote the S^1-horizontal coordinate). This estimate is

$$\int_{C_\sigma} \|\frac{\partial u}{\partial s}\|^2 dsdt \leq A - T\, gap.$$

To see this, calculate:

$$\begin{array}{lr} \int_{C_\sigma} \|\frac{\partial u}{\partial s}\|^2 dsdt & = \\ \int_{C_\sigma} < \frac{\partial u}{\partial s}, J\frac{\partial u}{\partial t} - T\nabla_{g_M} H_t > dsdt & = \\ \int_{C_\sigma} < \frac{\partial u}{\partial s}, J\frac{\partial u}{\partial t} > dsdt - \int_{C_\sigma} < \frac{\partial u}{\partial s}, T\nabla_{g_M} H_t > dsdt. & \end{array}$$

But the first term is the symplectic area of the cylinder in class K, bounded above by $A = \omega(\Phi_0)$. The second term is the the average over t of the difference of $H_t(t,\sigma) - H_t(t,0)$, which immedialty yields the above estimate.

2) *Second step.* Now assume that, in homology, cyl_{Φ_0} is not trivial and that TH_t has no closed orbit of period 1 for any $T \leq A/gap$. Then there is $\varepsilon > 0$ sufficiently small so that TH_t has no closed orbit of period 1 for any $T \leq (A/gap)+\varepsilon$. Consider the one-parameter family of morphisms $\mathrm{cyl}_T^{\Phi_0}$ in which the defining equation of the cylinder is replaced by

$$\bar{\partial}_J u = -T\nabla_{g_M} H_t.$$

For $A/gap \leq T \leq (A/gap)+\varepsilon$, the right hand side of our estimate becomes non-positive so that no solution exists and therefore $\mathrm{cyl}_T^{\Phi_0}$ obviously vanishes on all generators for these values of T. On the other hand, we prove in the third step below that cyl^{Φ_0} is chain homotopic to $\mathrm{cyl}_T^{\Phi_0}$ for any generic $T \leq (A/gap)+\varepsilon$. This is a contradiction.

3) *Third step.* There remains to construct the homotopy operator between the two morphisms cyl^{Φ_0} and $\mathrm{cyl}_T^{\Phi_0}$ where $T \in [A/gap, A/gap+\varepsilon]$ is generic. For this, consider the morphism of degree one more than the degree of cyl^{Φ_0}:

$$\Psi : \mathcal{C}\ell\,(L) \to \mathcal{C}\ell\,(L) \otimes \mathcal{C}\ell\,(L') \otimes \Lambda_{\Phi_0}$$

defined on a generator x by

$$\Psi(x) = \sum_{\tau\in[0,T]} \mathrm{cyl}_\tau^{\Phi_0}(x)$$

when, as usual, the corresponding obvious moduli space $\mathcal{C}^x_{x_1,\ldots,x_k,y_1,\ldots,y_m}(\bar{\lambda}, \tau \in [0,T])$ is of dimension 0. Extend it by the Leibniz rule and by linearity. In order to show the formula

$$\partial\Psi - \Psi\partial = \mathrm{cyl}_T^{\Phi_0} - \mathrm{cyl}^{\Phi_0}$$

i.e to see that this is indeed a homotopy operator between the two morphisms on the right hand side, we consider a one-parameter moduli space of the form $\mathcal{C}^x_{x'_1,\ldots,x'_k,y'_1,\ldots,y'_m}(\bar\lambda', \tau \in [0,T])$. This space can only degenerate to:

i) a configuration that reaches one of the two obvious boundaries corresponding to $\tau = 0, T$;

ii) a configuration that breaks in the ordinary Morse way, for an interior time $\tau \in (0,T)$, into two trees, at a critical point $\bar x \in \mathrm{Crit}(f)$ or $\bar y \in \mathrm{Crit}(f')$. In the second case, this belongs to $\partial\Psi(x)$ while in the first case, this belongs to either $\partial\Psi(x)$ or $\Psi\partial(x)$ according to whether the tree rooted at $\bar x$ contains or not the cylinder.

Note that the eventual bubbling off, from the cylinder, of a pseudoholomorphic disk with boundary on L or L' is not a boundary point of our 1-dimensional family since the cluster setting sees this phenomenon as an internal point.

Finally, there is no other case than the ones in (i) and (ii) above because obviously the degeneracy corresponding to the conformal parameter of the cylinder going to infinity cannot happen since there is, by assumption, no closed orbit of period 1 of TH_t for $\tau \le T$. □

There are many examples where the hypotheses of our theorem are satisfied.

Examples.

A. *In cotangent spaces*

Note first that if one considers sections of a cotangent bundle, there is no bubbling off and therefore one does not need to orient moduli spaces of holomorpic discs, so the above theorem applies as well without the hypothesis that L and L' be orientable or relatively spin. In general, if (V,g) is a Riemannian manifold and α is a closed 1-form whose g-dual X is g-parallel, the holomorphic cylinders, with respect to the natural almost complex structure on TV, from the zero section L to the graph of α, L', correspond to 1-jets of integral curves β of X which are also geodesics of V: the cylinder is then given by $\{(\beta(t), s\frac{d\beta}{dt}) : t \in [o,a], s \in [0,1]\}$ (actually, one reparametrizes the cylinder so that its circle-base is of length 1).

1) The simplest example is a generalisation from the case treated in [12] : one considers $V = K \times W$ where K is the Klein bottle and W any closed manifold admitting a perfect Morse function. Take L the zero section and L' the graph of the pull-back to $K \times W$ of the constant closed 1-form α on K which corresponds, through the euclidean paring, to the constant vector field $\partial/\partial x$ in $([0,1]\times[0,1]/R$ where R identifies the top and bottom boundaries, and reverse the orientation of the left and right boundaries. On K the geodesic whose image is $[0,1]\times\{1/2\}$ is an integral curve of the g-dual of α, and it is moreover unique in its homotopy class. Therefore, on $L = K \times W$, there is a W-family of such geodesics. One can consider a product Morse function on $L = K \times W$ and the configuration in $\mathcal{C}^x(L,L')(\bar\lambda)$ made of a flowline from a point x of index $\dim W$ to one element in the W-family of geodesics, with the corresponding holomorphic cylinder from L to L', and with no flowline in L'. These configurations are unique and give a map:

$$\mathrm{cyl}^{\Phi_0} : \mathcal{C}\ell\, H(L) \to \mathcal{C}\ell\, H(L) \otimes \mathcal{C}\ell\, H(L') \otimes \Lambda(\Phi_0)$$

which sends x to $1 \otimes e^{(\mu(\Phi_0),\omega(\Phi_0))}$. Because the Morse function on W is perfect, the ordinary flowlines in L do not contribute in the morphism. Thus cyl^{Φ_0} does not vanish since $1 \otimes e^{(\mu(\Phi_0,\omega(\Phi_0))}$ is not zero in

$$\mathcal{C}\ell\, H(L) \otimes \mathcal{C}\ell\, H(L') \otimes \Lambda(\Phi_0) \simeq (S(s^{-1}H_*(K \times W)) \otimes S(s^{-1}H_*(K \times W))) \otimes \Lambda(\Phi_0).$$

2) The same arguments apply modus vivendi to T^n. Indeed, if L is the zero section and L' the graph of a constant one-form in one of the S^1 direction, the geodesics in that same class foliate the torus. Thus once again, the map cyl^{Φ_0} sends the minimum x of a Morse function on T^n to a configuration made of x, sitting on a unique geodesic, and of the corresponding holomorphic cylinder. Hence, once again, $\mathrm{cyl}^{\Phi_0}(x) = 1 \otimes e^{(\mu(\Phi_0),\omega(\Phi_0))}$ and the rest of the argument is the same. One may here also generalise this to a product of $T^n \otimes W$ where W admits a perfect Morse function.

3) Let (V, g) be a hyperbolic manifold, consider a closed 1-form α and set as before L equal to the zero section and L' the graph of α. Clearly, if there is a geodesic which is an integral curve C of the g-dual of α, it is unique in its homotopy class. Assume that there is a Morse function on V with a critical point p of index $n-1$ such that its Morse differential vanishes and that its stable manifold has non-zero linking number with C. Then the map

$$\mathrm{cyl}^{\Phi_0} : \mathcal{C}\ell\, H(L) \to \mathcal{C}\ell\, H(L) \otimes \mathcal{C}\ell\, H(L') \otimes \Lambda(\Phi_0)$$

sends p to $1 \otimes e^{(\mu(\Phi_0),\omega(\Phi_0))}$ which is non-zero in homology.

B. *In manifolds*

Take any of the above examples imbedded as a Lagrangian submanifold of a geometrically bounded symplectic manifold (M, ω). Assume that neither L or L' has free terms (i.e their cluster homology does not vanish). Fix a class Φ_0 whose area is no greater than the smallest non-constant J-holomorphic disc in (M, L) or (M, L'). Then the same maps as in (A), (B) and (C) send some element to a cycle of the form

$$b = 1 \otimes e^{(\mu(\Phi_0,\omega(\Phi_0))} + \dots .$$

But by hypothesis, the class Φ_0 cannot degenerate so that, in the above expression, the only free term is $1 \otimes e^{(\mu(\Phi_0,\omega(\Phi_0))}$. Thus b does not vanish in homology.

Concluding remark. The morphism cyl is the first term of a series that defines a differential on the tensor product $\mathcal{C}\ell\,(L) \otimes \mathcal{C}\ell\,(L') \otimes \Lambda(\Phi_0)$. To explain this, first denote by S_{k+1} the standard sphere with $k+1$ disjoint open disks $D_0, D_1, \dots, D_k$ removed, and by $\mathcal{J}_{k+1}$ the space of conformal structures on S_{k+1}. For each integer $\ell > 0$, define:

$$d_\ell^{\ell\Phi_0} : \mathcal{C}\ell\,(L) \to \mathcal{C}\ell\,(L) \otimes \mathcal{C}\ell\,(L') \otimes \Lambda(\Phi_0)$$

by the formula

$$d_\ell^{\ell\Phi_0}(x) = \sum_{x_1,\dots,x_k,y_1,\dots,y_m,\bar{\lambda}} \#\mathcal{C}^x_{x_1,\dots,x_k,y_1,\dots,y_m}(\bar{\lambda}, \ell) x_1, \dots, x_k y_1, \dots, y_m e^{\bar{\lambda}}$$

when the dimension of this configuration space is zero, and zero otherwise. Extend this definition by the Leibniz rule. Here the moduli space of configurations $\mathcal{C}^x_{x_1,\dots,x_k,y_1,\dots,y_m}(\bar{\lambda}, \ell)$ is defined like $\mathcal{C}^x_{x_1,\dots,x_k,y_1,\dots,y_m}(\bar{\lambda})$ except that the unique cylinder in the definition of the latter moduli space is replaced by q surfaces $(\Sigma_1, j_1), \dots, (\Sigma_q, j_q)$ with each Σ_i equal to S_{k_i+1} and $j_i \in \mathcal{J}_{k_i+1}$ so that $\sum_i k_i = \ell$ and that each k_i is larger or equal to 1. More precisely, for elements $x, x_1, \dots, x_k \in$

$\mathrm{Crit}(f)$, elements $y_1, \ldots, y_m \in \mathrm{Crit}(f')$, and a class $\bar{\lambda} \in \bar{K}$, consider the space $\mathcal{C}^x_{x_1,\ldots,x_k,y_1,\ldots,y_m}(\bar{\lambda}, \ell)$ of configurations made from

1) a cluster on L rooted at x, with terminations $x_1, \ldots, x_k$ and where all vertices of the tree defining the cluster correspond to boundaries of (J, ν)-holomorphic disks in (M, L) except precisely q of them which are replaced by loops $\gamma_i, 1 \leq i \leq q$, in L;

2) q maps

$$u_i : (\Sigma_i, \partial D_0, \partial D_1 \cup \ldots \cup \partial D_k) \to (M, L, L'), \;\; 1 \leq i \leq q,$$

satisfying

$$\bar{\partial}_{(j_i, J)} u_i = \nu_i$$

and

$$(u_i)|_{\partial D_0} = \gamma_i;$$

3) ℓ clusters on L', each one being rooted at a vertex corresponding to a loop $\gamma'_i, 1 \leq i \leq \ell$, on L' so that these γ'_i's are equal to the restrictions of the u_i's to the various boundary circles $\partial D_j, 1 \leq j \leq k_i$ via a one-to-one correspondence. In each such cluster, all other vertices (i.e all but the root) of the tree defining the cluster correspond to boundaries of (J, ν)-holomorphic disks in (M, L'). The set of all terminations of these ℓ clusters consists of the ordered set $y_1, \ldots, y_m$.

We finally require that the total Malov index (whose computation can be reduced to the computation of a finite (connected) sum of cylinders) be equal to $\bar{\lambda}$, and that, topologically, the ℓ cylinders, be in class $\ell\Phi_0$ up to the left and right actions of $\pi_2(M, L)$ and $\pi_2(M, L')$.

Thus, for $\ell = 0$, this is simply the ordinary differential of the complex $\mathcal{C}\ell\,(L)$; for $\ell = 1$, it is, up to sign, equal to the morphism cyl^{Φ_0} above. For $\ell = 2$, the definition takes into account pairs of pants relating L to L'. Now extend each $d^{\Phi_0}_\ell$ to $\mathcal{C}\ell\,(L) \otimes \mathcal{C}\ell\,(L') \otimes \Lambda(\Phi_0)$ by defining it as the usual cluster differential on the $\mathcal{C}\ell\,(L')$-term, and denote the resulting map by the same symbol. Finally consider the *big differential*

$$d^{\Phi_0}_\infty = \sum_\ell d^{\Phi_0}_\ell : \mathcal{C}\ell\,(L) \otimes \mathcal{C}\ell\,(L') \otimes \Lambda(\Phi_0) \to \mathcal{C}\ell\,(L) \otimes \mathcal{C}\ell\,(L') \otimes \Lambda(\Phi_0).$$

At least up to a verification of signs, it is not difficult to see that d^2_∞ vanishes (note that this vanishing represents an infinite sequence of equations of which the first two are: $d^2 = 0$ where d is the usual cluster differential, and $\mathrm{cyl}^{\Phi_0} d - d\,\mathrm{cyl}^{\Phi_0} = 0$ which is another way of proving that cyl^{Φ_0} is a morphism of chain complexes). One could of course define the big differential d_∞ in which one would not limit to the class K the choices of the cylinders (but would instead consider all of them).

The computation of the corresponding big homology seems an interesting problem. By analogy with the differential of the rational model of the total space of a fibration, which is defined on the tensor product of the complexes of the base and fiber and is "twisted" only on the fiber, it seems that what this big differential computes is a quantum version of a fibration whose base would be L' and fiber L. But the geometric significance of this quantum fibration is not yet clear.

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