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LAGRANGIAN SHADOWS AND TRIANGULATED CATEGORIES

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***Abstract.* —** We introduce new metrics on spaces of Lagrangian submanifolds, not necessarily in a fixed Hamiltonian isotopy class. Our metrics arise from measurements involving Lagrangian cobordisms. We also show that splitting Lagrangians through cobordism has an energy cost and, from this cost being smaller than certain explicit bounds, we deduce some forms of rigidity of Lagrangian intersections. We also fit these constructions in the more general algebraic setting of triangulated categories, independent of Lagrangian cobordism. As a main technical tool, we develop aspects of the theory of (weakly) filtered A_∞-categories.

***Résumé* (*Ombres des sous-variétés Lagrangiennes et catégories triangulées*)**

Nous introduisons de nouvelles métriques sur les espaces des sous-variétés Lagrangiennes dont la classe d'isotopie Hamiltonienne n'est pas nécessairement fixée. Ces métriques proviennent de certaines quantités associées aux cobordismes Lagrangiens. Nous montrons également que la décomposition d'un Lagrangien à travers un cobordisme a un coût énergétique non-nul et, à partir d'une borne explicite de ce coût, nous déduisons des formes de rigidité des intersections Lagrangiennes. Ces constructions interviennent dans le cadre algébrique plus général des catégories triangulées, indépendament du cobordisme Lagrangien. Comme outil technique central, nous développons certains aspects de la théorie des catégories A_∞ (faiblement) filtrées.

CONTENTS

CHAPTER 1

INTRODUCTION AND MAIN RESULTS

One of the main objectives of this paper is to introduce new metrics and related measurements on certain classes of Lagrangian submanifolds of a given symplectic manifold. The (pseudo) metrics that we look for are supposed to have three features:

(i) Have significant symplectic content, in particular, be coherent with respect to Hofer's norm.

(ii) Be non-degenerate.

(iii) Be finite for a class of Lagrangians as large as possible.

Symplectic topology is characterized by an interplay of flexible and rigid phenomena, flexibility originating in the Gromov h-principle and rigidity being reflected through properties of J-holomorphic curves. This tension flexibility – rigidity renders non-trivial the definition of metrics with the three properties above: without restricting in an appropriate manner the class of Lagrangians considered, flexibility leads to pseudo-metrics that are degenerate. On the other hand, having finite distances between Lagrangians with different isotopy (and even homotopy) types is non-obvious.

Our measurements arise from the perspective of Lagrangian cobordism. The simplest non-trivial setting in which our metrics exist is the case when $(M, \omega = \mathrm{d}\lambda)$ is a Liouville manifold.

Denote by $\mathcal{L}ag^{\mathrm{ex}}(M)$ the collection of exact Lagrangian submanifolds in M which are compact without boundary. Given a Lagrangian cobordism $V \subset \mathbb{R}^2 \times M$ (see Section 1.1 for the definition), denote by $\mathcal{S}(V)$ the area of the projection of V to $\mathbb{R}^2$ together with all the bounded regions bounded by this projection.

We call this measurement the *shadow* of V. More precisely:

$$\mathcal{S}(V) = \mathrm{Area}(\mathbb{R}^2 \setminus \mathcal{U}), \tag{1.1}$$

where $\mathcal{U} \subset \mathbb{R}^2 \setminus \pi(V)$ is the union of all the *unbounded* connected components of $\mathbb{R}^2 \setminus \pi(V)$. Here $\pi : \mathbb{R}^2 \times M \to \mathbb{R}^2$ is the projection.

Fix a family of exact Lagrangians $\mathcal{F} \subset \mathcal{L}ag^{\mathrm{ex}}(M)$. For every $L, L' \in \mathcal{L}ag^{\mathrm{ex}}(M)$ define:

$$d^{\mathcal{F}}(L, L') := \inf_V \big\{ \mathcal{S}(V) \,;\, V : L \rightsquigarrow (F_1, \ldots, F_{i-1}, L', F_i, \ldots, F_k), k \geq 0,\, F_i \in \mathcal{F} \big\}, \tag{1.2}$$

where the infimum is taken over all (possibly disconnected) *exact* Lagrangian cobordisms V having L as its single positive end and whose negative ends consists of L' possibly together with other Lagrangians, all taken from the family $\mathcal{F}$. We use the convention that $\inf \emptyset = \infty$, so that $d^{\mathcal{F}}(L, L') = \infty$ if there is no exact cobordism V as in (1.2).

It is easy to see that $d^{\mathcal{F}}$ is a pseudo-metric (possibly with infinite values). However, $d^{\mathcal{F}}$ is generally degenerate (yet not identically zero). Fix a second family of Lagrangians $\mathcal{F}' \subset \mathcal{L}ag^{\mathrm{ex}}(M)$ and define

$$\widehat{d}^{\mathcal{F},\mathcal{F}'} := \max\{d^{\mathcal{F}}, d^{\mathcal{F}'}\}. \tag{1.3}$$

One of our main results is:

Theorem A. — *If $(\overline{\bigcup_{K\in\mathcal{F}} K}) \cap (\overline{\bigcup_{K'\in\mathcal{F}'} K'})$ is totally disconnected, then $\widehat{d}^{\mathcal{F},\mathcal{F}'}$ is non-degenerate, hence a metric, (possibly with infinite values) on $\mathcal{L}ag^{\mathrm{ex}}(M)$.*

We call $\widehat{d}^{\mathcal{F},\mathcal{F}'}$ the *shadow metric* associated to the pair of families $\mathcal{F}, \mathcal{F}'$. For example, one can take $\mathcal{F}$ to be a finite family of Lagrangians and for the family $\mathcal{F}'$ one can take a small and generic Hamiltonian perturbation of each of the elements in $\mathcal{F}$. Then $(\overline{\bigcup_{K\in\mathcal{F}} K}) \cap (\overline{\bigcup_{K'\in\mathcal{F}'} K'})$ is discrete and Theorem A applies.

The shadow metrics bear a simple relation to the well known Lagrangian Hofer metric [**Che00**] on the space of Lagrangian submanifolds in a given Hamiltonian isotopy class. Indeed, it is not hard to see that if L' is Hamiltonian isotopic to L then

$$\widehat{d}^{\mathcal{F},\mathcal{F}'}(L, L') \leq d_{\mathrm{Hofer}}(L, L')$$

and, in particular, shadow metrics satisfy property i from the beginning of the introduction. This is so because any Hamiltonian isotopy $\{\phi_t(L)\}$ between two exact Lagrangians L and L' gives rise to an exact Lagrangian cobordism $V : L \rightsquigarrow L'$ (called the Lagrangian suspension of the isotopy) with $\mathcal{S}(V) = \mathrm{length}_{\mathrm{Hofer}}\{\phi_t(L)\}$.

When $\mathcal{F} = \emptyset$ the pseudo-metric $d^{\emptyset}$ is already non-degenerate and coincides with the metric introduced in [**CS19**] which infimizes the shadow of cobordisms having only L and L' as ends (these are called simple cobordism). Of course $\widehat{d}^{\mathcal{F},\mathcal{F}'} \leq d^{\emptyset}$.

The use of multiple ended cobordisms and not of only simple ones in the definition of metrics such as $\widehat{d}^{\mathcal{F},\mathcal{F}'}$ is a crucial novelty brought forth in this paper. Three aspects of this construction are worth underlining at this point. Firstly, in the exact setting, it is conjectured that any simple cobordism is a Lagrangian suspension (progress on this question appears in [**Sua17**]). Therefore, $d^{\emptyset}$, at least conjecturally, coincides with the Lagrangian Hofer distance and, in particular, $d^{\emptyset}(L, L')$ is expected to be infinite as soon as L and L' are not Hamiltonian isotopic. However, for nonempty families $\mathcal{F}, \mathcal{F}'$ the associated distances $\widehat{d}^{\mathcal{F},\mathcal{F}'}(L, L')$ are often finite for pairs of Lagrangians L, L' that are not even smoothly isotopic or can even have different homotopy types. Secondly and more conceptually, the existence of the metrics $\widehat{d}^{\mathcal{F},\mathcal{F}'}$ for $\mathcal{F}, \mathcal{F}' \neq \emptyset$ is a reflection of the fact that the Lagrangian submanifolds in our setting can be organized in an A_∞-category which in turn, by a further algebraic process, gives rise to a triangulated category – the derived Fukaya category. As we will explain in detail below (see already Section 1.2), the metrics $\widehat{d}^{\mathcal{F},\mathcal{F}'}$ reflect the triangulated structure of this category in the sense that they can be understood as providing the infimum of an "energy" cost required for certain decompositions by iterated exact triangles in this category. Finally, the last point to mention is that, as a technical reflection of the second aspect mentioned just above, proving the non-degeneracy of the metrics $\widehat{d}^{\mathscr{F},\mathscr{F}'}$ requires, among other steps, a considerable development of A_∞-algebraic machinery in the filtered setting and this setup could potentially be of use elsewhere.

In Chapter 6 we will study further aspects of shadow metrics. In particular we will see that analogues of the shadow metric exist also for other classes of Lagrangian submanifolds, such as weakly exact Lagrangians and monotone ones and variants of Theorem A continue to hold in these settings.

1.1. Decomposition by Lagrangian cobordism

A *Lagrangian cobordism* [**Arn80**] (see [**BC13**] for the formalism in use here) is a Lagrangian submanifold $V \subset \mathbb{R}^2 \times M$ with the property that there exists a compact interval $[a_-, a_+] \subset \mathbb{R}$ such that

$$V \setminus \big([a_-, a_+] \times \mathbb{R} \times M\big) = \Big(\coprod_{i=1}^{k} \ell_- \times \{i\} \times L_i\Big) \coprod \Big(\coprod_{j=1}^{k'} \ell_+ \times \{j\} \times L'_j\Big),$$

where $\ell_- = (-\infty, a_-)$, $\ell_+ = (a_+, \infty)$ and the L_i's and L'_j's are Lagrangian submanifolds of M. The L_i's are called the *negative ends* of V and the L'_j's the *positive ends*. We write:

$$V : (L'_1, \dots, L'_{k'}) \rightsquigarrow (L_1, \dots, L_k).$$

We allow any of k' or k to be 0 in which case the positive or negative end of the cobordism is void.

Fix a collection $\mathscr{L}$ of Lagrangian submanifolds of M. Given a Lagrangian submanifold $L \subset M$ we are interested in the "splitting" (or decomposition) of L into Lagrangian submanifolds picked from the collection $\mathscr{L}$. The type of splitting that we focus on is through Lagrangian cobordisms V with a single positive end equal to L and multiple negative ends, $V : L \rightsquigarrow (L_1, \dots, L_k)$. This perspective on cobordism is natural not least because, as is known from previous work [**BC13**], [**BC14**] and under appropriate constraints on $\mathscr{L}$, such cobordisms induce genuine (iterated cone) decompositions of L with factors the negative ends L_i in the derived Fukaya category of M.

As already mentioned at the beginning of the introduction, the central point of view for this paper is to regard the shadow $\mathcal{S}(V)$ of a cobordism V as an energy cost for the splitting corresponding to V. We address two natural questions from this perspective:

1) Assuming L and $L_1, \dots, L_k$ fixed, find a lower bound for the minimal energy cost required to split L in the factors L_i (see Theorem B)?
2) Is there some form of Lagrangian intersections rigidity that is specific to low energy splittings (see Theorem C)?

For the following results we restrict to the class of Lagrangian submanifolds $L \subset M$ that are closed and *weakly exact* (*i.e.* $\omega_{|\pi_2(M,L)} = 0$). Similarly, cobordisms V are assumed to be weakly exact.

The next theorem shows that the shadow of cobordisms $V : L \rightsquigarrow (L_1, \ldots, L_k)$ between fixed L and $(L_1, \ldots, L_k)$ cannot become arbitrarily small unless these Lagrangians are placed in a very particular position.

Theorem B. — *Let $L, L_1, \ldots, L_k \subset M$ be weakly exact Lagrangian submanifolds. Assume that L is not contained in $L_1 \cup \cdots \cup L_k$. Then there exists $\delta = \delta(L; S) > 0$ which depends only on L and $S := L_1 \cup \cdots \cup L_k$, such that for every weakly exact Lagrangian cobordism $V : L \rightsquigarrow (L_1, \ldots, L_k)$ we have*

$$\mathcal{S}(V) \geq \tfrac{1}{2}\delta. \tag{1.4}$$

The proof is given Chapter 5. A non-technical outline of the proof is presented in next Section 1.3.

The next theorem establishes relations between L and $L_1, \ldots, L_k$ in case they are related by a Lagrangian cobordism with small shadow.

Theorem C. — *Let $L, L_1, \ldots, L_k \subset M$ be weakly exact Lagrangians and S as in Theorem B. Let $N \subset M$ be another weakly exact Lagrangian and assume that the Lagrangians $N, L, L_1, \ldots, L_k$ are in general position. There exists $\delta' = \delta'(N, S) > 0$ that depends on N and S (but not on L) such that for every weakly exact Lagrangian cobordism*

$$V : L \rightsquigarrow (L_1, \ldots, L_k)$$

with $\mathcal{S}(V) < \frac{1}{2}\delta'$ we have

$$\#(N \cap L) \geq \sum_{i=1}^{k} \#(N \cap L_i). \tag{1.5}$$

The numbers δ, δ' are variants of the Gromov width from [BC07]. Namely, δ is the Gromov width of L in the complement of S and δ' is a symplectic measure of the intersection $S \cap N$. The precise definitions are given in Chapter 5, where more precise versions of Theorems B and C are restated and proved as parts of a single, stronger statement, Theorem 5.1.

Analogues of Theorems B and C hold also in the monotone case, see Chapter 5.

1.2. Weighted fragmentation pseudo-metrics on triangulated categories

The construction of the shadow pseudo-metrics can be generalized to a more abstract setting, as discussed in §6.4.1. In summary, given a triangulated category $\mathcal{X}$ fix a family $\mathcal{F}$ of objects of $\mathcal{X}$. Assume that there is a way to associate a weight to each iterated cone decomposition in $\mathcal{X}$ which is well-behaved with respect to refinement of cone decompositions. For two objects K, K' in $\mathcal{X}$ we infimize the weight of the cone decompositions of K that express K as an iterated cone involving K' and elements of $\mathcal{F}$. By symmetrizing formally the resulting measurement we get a pseudo-metric on the objects of $\mathcal{X}$.

These pseudo-metrics are called *weighted fragmentation pseudo-metrics*. When the triangulated category in question is the derived Fukaya category an example of such pseudo-metrics are the shadow pseudo-metrics seen before. We also construct a more algebraic example, independent of cobordisms, that is based on the filtered chain level structures that appear in Floer theory.

1.2.1. ***Remark.*** — Since the submission of this paper, the machinery developed here has seen a few other applications beyond the Lagrangian intersection results included in the text. In one direction [**BCc**] the shadow fragmentation pseudo-metrics are used in the setting of certain cobordism categories of immersed Lagrangians. There is a class of such categories called *categories with surgery models*. A natural quotient of such a category is triangulated and carries shadow fragmentation pseudo-metrics as defined here. Under certain constraints, when the class of Lagrangians in study is unobstructed, these pseudo-metrics are non-degenerate, by an extension of Theorem A. Moreover, the respective triangulated category contains a subcategory isomorphic to the derived Fukaya category associated to the embedded Lagrangians. A second direction is related to a conjecture due to Viterbo [**Vit**, Conjecture 1] on the existence of a uniform bound on the spectral norm of a exact compact Lagrangian submanifold L in a fixed disk sub-bundle of a cotangent bundle T^*N of a closed manifold N. This conjecture was recently proved for a class of manifolds N including the original case of $N = T^n$ in the papers [**Sheb**], [**Shea**]. However, Viterbo's conjecture is still open for arbitrary closed N. In this general setting some of the filtration machinery developed here is used in [**BCa**] to deduce estimates for the spectral distance $\gamma(L, N)$ (where N is viewed as the zero section) in terms of the boundary depth of the Floer complex $\mathrm{CF}(L, T^*_xN)$, where T^*_xN is a fibre of the bundle. Finally, the paper [**KS**] that was partially inspired by the filtered Yoneda approach of the current paper has found numerous recent applications.

1.3. Outline of the proof of Theorem B

We focus here on the proof of Theorem B (the proof of Theorem C makes use of similar ideas). We consider a symplectic embedding of a standard ball $e : B(r) \longrightarrow M$ such that

$$e^{-1}(L) = B_{\mathbb{R}}(r), \quad e\big(B(r)\big) \cap (L_1 \cup \cdots \cup L_k) = \emptyset$$

and we put $P = e(0)$. The bulk of the proof is devoted to proving that for any almost complex structure J on M there exists a J-holomorphic polygon u in M with a boundary edge on L (and possibly on the other L_i's) going through P and with $\omega(u) \leq \mathcal{S}(V)$.

Once this is proved, the theorem follows by using a suitable choice of J, an application of the Lelong inequality, and the definition of $\delta = \delta(L; S)$ as in (5.1).

To control energy bounds in our arguments we set up in Chapter 2 the machinery of A_∞-categories and modules in the (weakly) filtered setting. Variants of this already appear in the literature, for instance in [**FOOO09a**], [**FOOO09b**] (in somewhat different form), but we give enough details so as to be able to extend – in Chapter 3 – the results from [BC14] to this setting. The wording *weakly* means that, to achieve regularity, we allow for small Hamiltonian perturbations in the definition of the various algebraic structures.

As a consequence, these structures are filtered only up to a system of small, controllable errors. We also prove in Section 2.6 a structural result, Theorem 2.14, concerning iterated cones $\mathcal{K}$ of (weakly) filtered A_∞-modules and, in particular, we show that each such cone admits a quasi-isomorphic model $\mathcal{K}'$ which is an iterated cone with the same factors as $\mathcal{K}$ and such that $\mathcal{K}'$ has a filtration that is well controlled with respect to that of $\mathcal{K}$ and the μ_1 operation of $\mathcal{K}'$ can be written explicitly in terms of higher μ_k's of the underlying A_∞-category – see (2.31).

This result is based on a (weakly) filtered version of the following property of the Yoneda embedding [Sei08]: for an A_∞-module $\mathcal{N}$ and an object Y there is a natural quasi-isomorphism $\mathcal{N}(Y) \cong \hom(\mathcal{Y}, \mathcal{N})$ (where $\mathcal{Y}$ is the Yoneda module of Y). We prove in Section 2.5 a weakly filtered version of this property which seems to be new (and somewhat delicate to prove).

With this preparation, the proof of the theorem is given in Chapter 5. By neglecting a number of technicalities, the argument can be sketched as follows. We consider a new cobordism $W : \emptyset \rightsquigarrow (L, L_1, \ldots, L_k)$ obtained from V by bending the end L of V clockwise half a turn, as in Figure 4. The main result in [BC14] implies that the Yoneda modules $\mathcal{L}, \mathcal{L}_i$ associated to the negative ends of the cobordism W fit into an iterated cone of A_∞-modules over the Fukaya category, $\mathcal{F}uk(M)$. The output of this iterated cone is a module $\mathcal{M}_W$ defined as:

$$\mathcal{M}_W = \mathcal{C}one\big(\mathcal{L}_k \xrightarrow{\varphi_k} \mathcal{C}one\big(\mathcal{L}_{k-1} \xrightarrow{\varphi_{k-1}} \cdots \to \mathcal{C}one(\mathcal{L}_1 \xrightarrow{\varphi_1} \mathcal{L}) \cdots\big)\big),$$

and, moreover, this module is acyclic. In view of our preparatory step all the modules and structures involved here are filtered.

By typical cobordism arguments, we show that there exists a null-homotopy ξ of the identity of $\mathcal{M}_W(L)$ (this is the chain complex given by applying the module $\mathcal{M}_W$ to the object L of $\mathcal{F}uk(M)$) that shifts filtrations by at most $\rho \leq \mathcal{S}(V) + \epsilon$ where we can take ϵ as small as desired. A cycle e_L in $\mathrm{CF}(L, L)$ representing the fundamental class $[L] \in \mathrm{HF}(L, L)$ still remains a cycle in $\mathcal{M}_W(L)$. We deduce that it has to be the boundary of some element in $\mathcal{M}_W(L)$ of filtration higher than that of e_L by not more than ρ or, in other words, the boundary depth (see [Ush11], and also Section 2.7) of e_L is at most ρ. By suitable choices, we may assume that e_L is the maximum point of a Morse function on L, which is achieved at P.

At this point it is crucial that $\mathcal{M}_W$ is an iterated cone of (weakly) filtered A_∞-*modules*. We now use the structural Theorem 2.14 to associate to $\mathcal{M}_W$ the quasi-isomorphic module $\mathcal{M}$ (provided by that theorem). Because the filtrations on $\mathcal{M}$ and $\mathcal{M}_W$ are tightly related, we deduce that the boundary depth of e_L in $\mathcal{M}(L)$ is at most $\rho + \epsilon'$ where $\epsilon' > 0$ can be taken arbitrarily small. From the special form of the differential of $\mathcal{M}(L)$ which involves the higher order A_∞-operations μ_d, we conclude that there is a pseudo-holomorphic polygon u in M with boundary on L and on some of the L_i's that appears in the differential of $\mathcal{M}(L)$ and that passes through P. Moreover, the area of this polygon is not more than $\rho + \epsilon'$. In essence, this concludes the argument by making $\epsilon, \epsilon' \to 0$.

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CHAPTER 2

WEAKLY FILTERED A_∞-THEORY

In this chapter we develop a general framework for weakly filtered A_∞-categories, with an emphasis on weakly filtered modules over such categories. In our context "weakly filtered" generally means that the morphisms in the category are filtered chain complexes but the higher A_∞-operations do not necessarily preserve these filtrations. Rather they preserve them up to prescribed errors which we call *discrepancies*. In the same vein one can consider also weakly filtered A_∞-functors and modules. Related notions of filtered A_∞-structures have been considered in the literature (*e.g.* [**FOOO09a**], [**FOOO09b**]), but the existing theory seems to differ from ours in its scope and applications.

Below we will cover only the most basic concepts of A_∞-theory in the weakly filtered setting. In particular we will not go into the topics of derived categories, split closure or generation in the weakly filtered framework. Our main goal is in fact much more modest: to provide an effective description of iterated cones of modules in the weakly filtered setting in terms of weakly filtered twisted complexes.

Some readers may find the details of the weakly filtered setting somewhat overwhelming, especially in what concerns keeping track of the discrepancies. If one assumes all the discrepancies to vanish, the theory becomes "genuinely filtered" and is easier to follow. However, the additional difficulty due to the weakly filtered setting is largely superficial. Indeed, significant parts of the theory developed in this chapter do not become easier if one works in the genuinely filtered setting, except in terms of notational convenience. We also remark that, as far as we know, a good part of the theory developed in this chapter, particularly the study of iterated cones, is new even in the genuinely filtered case. The reason for developing the theory in the weakly filtered setting (rather than filtered) has to do with the geometric applications we aim at which have to do with Fukaya categories of symplectic manifolds. For technical reasons, the weakly filtered framework fits better with the standard implementations of these categories.

2.1. Weakly filtered A_∞-categories

In the following we will often deal with sequences $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_d, \dots)$ of real numbers that we will refer to as discrepancies. We will use the following abbreviations and conventions:

- ▹ For two sequences $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}'$ we write $\boldsymbol{\epsilon} \leq \boldsymbol{\epsilon}'$ in order to say that $\epsilon_d \leq \epsilon'_d$ for all d.
- ▹ For $c \in \mathbb{R}$ we write $\boldsymbol{\epsilon} + c$ for the sequence $(\epsilon_1 + c, \dots, \epsilon_d + c, \dots)$.
- ▹ For a finite number of sequences $\boldsymbol{\epsilon}^{(1)}, \dots, \boldsymbol{\epsilon}^{(r)}$ we define $\max\{\boldsymbol{\epsilon}^{(1)}, \dots, \boldsymbol{\epsilon}^{(r)}\}$ to be the sequence $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_d, \dots)$ with $\epsilon_d := \max\{\epsilon_d^{(1)}, \dots, \epsilon_d^{(r)}\}$.

Fix a commutative ring R, which for simplicity we will henceforth assume to be of characteristic 2 (*i.e.* $2r = 0$ for all $r \in R$). Unless otherwise stated all tensor products will be taken over R.

The A_∞-theory developed below will be carried out in the ungraded framework. Also, in contrast to standard texts on the subject such as [**Sei08**], we will work in a homological (rather than cohomological) setting, following the conventions from [**BC14**].

Let $\mathscr{A}$ be an A_∞-category over R. To simplify notation, in what follows we will denote the morphisms between two objects $X, Y \in \mathrm{Ob}(\mathscr{A})$ by

$$C(X, Y) := \hom_{\mathscr{A}}(X, Y).$$

We denote the composition maps of $\mathscr{A}$ by $\mu_d^{\mathscr{A}}$, $d \geq 1$.

Let $\boldsymbol{\epsilon}^{\mathscr{A}} = (\epsilon_1^{\mathscr{A}}, \epsilon_2^{\mathscr{A}}, \dots, \epsilon_d^{\mathscr{A}}, \dots)$ be an infinite sequence of non-negative real numbers, with $\epsilon_1^{\mathscr{A}} = 0$. We call $\mathscr{A}$ a *weakly filtered* A_∞-category with *discrepancy* $\leq \boldsymbol{\epsilon}^{\mathscr{A}}$ if the following holds:

1) For every $X, Y \in \mathrm{Ob}(\mathscr{A})$, $C(X, Y)$ is endowed with an increasing filtration of R-modules indexed by the real numbers. We denote by

$$C^{\leq \alpha}(X, Y) \subset C(X, Y)$$

the part of the filtration corresponding to $\alpha \in \mathbb{R}$. By *increasing filtration* we mean that $C^{\leq \alpha'}(X, Y) \subset C^{\leq \alpha''}(X, Y)$ for every $\alpha' \leq \alpha''$.

2) The μ_d-operation preserves the filtration up to an "error" of $\epsilon_d^{\mathscr{A}}$. More precisely, for every $X_0, \dots, X_d \in \mathrm{Ob}(\mathscr{A})$ and $\alpha_1, \dots, \alpha_d \in \mathbb{R}$ we have

$$\mu_d\big(C^{\leq \alpha_1}(X_0, X_1) \otimes \cdots \otimes C^{\leq \alpha_d}(X_{d-1}, X_d)\big) \subset C^{\leq \alpha_1 + \cdots + \alpha_d + \epsilon_d^{\mathscr{A}}}(X_0, X_d).$$

Note that since $\epsilon_1^{\mathscr{A}} = 0$, $\mu_1^{\mathscr{A}}$ preserves the filtration, each $C^{\leq \alpha}(X, Y)$, $\alpha \in \mathbb{R}$, is a subcomplex of $C(X, Y)$. Note also that the discrepancy is not uniquely defined – in fact we can always increase it if needed. Namely, if $\boldsymbol{\epsilon}' = (\epsilon'_1 = 0, \epsilon'_2, \dots, \epsilon'_d, \dots)$ is another sequence like $\boldsymbol{\epsilon}^{\mathscr{A}}$ but with $\boldsymbol{\epsilon}^{\mathscr{A}} \leq \boldsymbol{\epsilon}'$ then $\mathscr{A}$ is also weakly filtered with discrepancy $\leq \boldsymbol{\epsilon}'$.

By analogy with symplectic topology we will often refer to the index of the filtration as an *action* and say that elements of $C^{\leq \alpha}(X, Y)$ have action $\leq \alpha$.

2.1.1. ***Unitality.*** — Let $\mathscr{A}$ be a weakly filtered A_∞-category and assume that $\mathscr{A}$ is homologically unital (h-unital for short). We say that $\mathscr{A}$ is *h-unital in the weakly filtered sense* if there exists $u^{\mathscr{A}} \in \mathbb{R}_{\geq 0}$ such that for every $X \in \mathrm{Ob}(\mathscr{A})$ we have a cycle e_X in $C^{\leq u^{\mathscr{A}}}(X, X)$ representing the homology unit

$$[e_X] \in H\big(C(X, X), \mu_1^{\mathscr{A}}\big).$$

We view the choices of e_X, $X \in \mathrm{Ob}(\mathscr{A})$ and $u^{\mathscr{A}}$ as part of the data of a weakly filtered h-unital A_∞-category. We call $u^{\mathscr{A}}$ the *discrepancy* of the units.

Occasionally we will have to impose the following additional assumption on $\mathscr{A}$.

Assumption U^e. — Let $\mathscr{A}$ be a weakly filtered A_∞-category which is h-unital in the weakly filtered sense. Let $2u^{\mathscr{A}} + \epsilon_2^{\mathscr{A}} \leq \zeta \in \mathbb{R}$. We say that $\mathscr{A}$ satisfies *Assumption* $U^e(\zeta)$ if for every $X \in \mathrm{Ob}(\mathscr{A})$ and for some $c \in C^{\leq \zeta}(X, X)$ we have

$$\mu_2^{\mathscr{A}}(e_X, e_X) = e_X + \mu_1^{\mathscr{A}}(c).$$

Put in different words, the assumption U^e says that

$$[e_X] \cdot [e_X] = [e_X]$$

in $H_*(C^{\leq \zeta}(X, X))$, where the dot '$\cdot$' stands for the product induced by $\mu_2^{\mathscr{A}}$ in homology. (The superscript e in U^e indicates that the assumption deals with the cycles e_X representing the units.) Below we will sometimes write $\mathscr{A} \in U^e(\zeta)$ to say that $\mathscr{A}$ satisfies Assumption $U^e(\zeta)$.

2.2. Typical classes of examples

Before we go on with the general algebraic theory of weakly filtered A_∞-structures, we make a short digression in order to exemplify what types of filtrations will actually occur in our applications. We resume with the general algebraic theory in Section 2.3 below.

The weakly filtered A_∞-categories that will appear in our applications are Fukaya categories associated to symplectic manifolds. They will mostly be of the following types, described in §§2.2.1–2.2.4 below.

2.2.1. ***Filtrations induced by an "action" functional on the generators.*** — In this class of weakly filtered A_∞-categories the collection of morphisms $C(X, Y)$ between any two objects is assumed to be a free R-module with a distinguished basis $B(X, Y)$, *i.e.*

$$C(X, Y) = \bigoplus_{b \in B(X,Y)} Rb.$$

We also have a function $\boldsymbol{A} : B(X, Y) \longrightarrow \mathbb{R}$, which (by analogy to symplectic topology) we call the *action function*, defined for every $X, Y \in \mathrm{Ob}(\mathscr{A})$, and this function induces the filtration, namely:

$$C^{\leq \alpha}(X, Y) = \bigoplus_{b \in B(X,Y),\ \boldsymbol{A}(b) \leq \alpha} Rb.$$

We will mostly assume that $C(X, Y)$ has finite rank and that R is a field.

2.2.2. ***Filtration coming from the Novikov ring.*** — Here we fix a commutative ring A and consider the (full) Novikov ring over A:

$$\Lambda = \Big\{ \sum_{k=0}^{\infty} a_k T^{\lambda_k} \ ; \ a_k \in A, \lim_{k \to \infty} \lambda_k = \infty \Big\}, \tag{2.1}$$

as well as the positive Novikov ring:

$$\Lambda_0 = \Big\{ \sum_{k=0}^{\infty} a_k T^{\lambda_k} \ ; \ a_k \in A, \lambda_k \geq 0, \lim_{k \to \infty} \lambda_k = \infty \Big\}. \tag{2.2}$$

The weakly filtered A_∞-categories $\mathscr{A}$ of the type discussed here are defined over Λ, but the weakly filtered structure is only over the ring $R = \Lambda_0$.

As in §2.2.1 above, we assume

$$C(X, Y) = \bigoplus_{b \in B(X,Y)} \Lambda b.$$

The filtration on $C(X, Y)$ is then defined by

$$C^{\leq \alpha}(X, Y) = \bigoplus_{b \in B(X,Y)} T^{-\alpha} \Lambda_0 b.$$

Note that $C^{\leq a}(X, Y)$ is not a Λ-module but rather a Λ_0-module.

We will mostly assume that $B(X, Y)$ are finite (hence $C(X, Y)$ have finite rank) and that A is a field (in which case Λ is a field too).

2.2.3. ***Mixed filtration.*** — In some situations the filtrations on our A_∞-categories occur as combination of §§2.2.1–2.2.2 above. More specifically, we have

$$C(X, Y) = \Lambda B(X, Y)$$

as in §2.2.2 and an action functional $\boldsymbol{A} : B(X, Y) \to \mathbb{R}$ as in §2.2.1. We then extend $\boldsymbol{A}$ to a functional

$$\boldsymbol{A} : C(X, Y) = \Lambda \cdot B(X, Y) \longrightarrow \mathbb{R} \cup \{-\infty\}$$

by first setting $\boldsymbol{A}(0) = -\infty$. Then for $P(T) \in \Lambda$ and $b \in B(X, Y)$ we define:

$$\boldsymbol{A}\big(P(T)b\big) := -\lambda_0 + \boldsymbol{A}(b),$$

where $\lambda_0 \in \mathbb{R}$ is the minimal exponent that appears in the formal power series of $P(T) \in \Lambda$, *i.e.* $P(T) = a_0 T^{\lambda_0} + \sum_{i=1}^{\infty} a_i T^{\lambda_i}$ with $a_0 \neq 0$ and $\lambda_i > \lambda_0$ for every $i \geq 1$. Finally, for a general non-trivial element

$$c = P_1(T)b_1 + \cdots + P_l(T)b_l \in C(X, Y),$$

define

$$\boldsymbol{A}(c) = \max \big\{ \boldsymbol{A}(P_k(T)b_k) \ ; \ 1 \leq k \leq l \big\}.$$

The filtration on $C(X, Y)$ is then induced by $\boldsymbol{A}$:

$$C^{\leq \alpha}(X, Y) = \big\{ c \in C(X, Y) \ ; \ \boldsymbol{A}(c) \leq \alpha \big\}.$$

It is easy to see that $C^{\leq \alpha}(X, Y)$ is a Λ_0-module.

2.2.4. ***Families of weakly filtered A_∞-categories.*** — The weakly filtered A_∞-categories in our applications will naturally occur in families $\{\mathcal{A}_s\}_{s\in\mathcal{P}}$ parametrized by choices of auxiliary structures s needed to define the A_∞-structure. The parameter s will typically vary over a subset $\mathcal{P} \subset E \setminus \{0\}$ where E is a neighborhood of 0 in a Banach (or Fréchet) space. The subset $\mathcal{P}$ will usually be residual (in the sense of Baire) so that 0 is in the closure of $\mathcal{P}$.

Typically all the members of the family $\{\mathcal{A}_s\}_{s\in\mathcal{P}}$ will be mutually quasi-equivalent (see [**Sei08**, Section 10] for several approaches to families of A_∞-categories). Of course, in the weakly filtered setting the quasi-equivalences between different $\mathcal{A}_s$'s are supposed to bear some compatibility with respect to the weakly filtered structures on the $\mathcal{A}_s$'s.

Apart from the above, in our applications the families $\{\mathcal{A}_s\}_{s\in\mathcal{P}}$ will enjoy the following additional property which will be crucial. The bounds $\boldsymbol{\epsilon}^{\mathcal{A}_s}$ for the discrepancies of the $\mathcal{A}_s$'s can be chosen such that

$$\lim_{s\to 0} \epsilon_d^{\mathcal{A}_s} = 0, \quad \text{for all } d.$$

Moreover, the categories $\mathcal{A}_s$ will mostly be h-unital with discrepancy of units $u^{\mathcal{A}_s}$ and satisfy Assumption $U^e(\zeta_s)$. The latter two quantities will satisfy

$$\lim_{s\to 0} u^{\mathcal{A}_s} = \lim_{s\to 0} \zeta_s = 0.$$

Below we will encounter further notions in the framework of weakly filtered A_∞-categories such as weakly filtered functors and modules. Each of these comes with its own discrepancy sequence $\boldsymbol{\epsilon}$. In our applications everything will occur in families and we will usually have $\lim_{s\to 0} \epsilon_d(s) = 0$ for each d.

While the algebraic theory below is developed without *a priori* assumptions on the size of discrepancies, it might be useful to view the discrepancies as quantities that can be made arbitrarily small.

2.2.5. ***The case of Fukaya categories.*** — The general description in §2.2.4 applies to the case of Fukaya categories which will be central in our applications. More specifically, in order to define the A_∞-structure of Fukaya categories one has to make choices of perturbation data (*e.g.* choices of almost complex structures as well as Hamiltonian perturbation – see *e.g.* [**Sei08**, Sections 8–9]). The space $\mathcal{P}$ will consist of those perturbation data that are regular (or admissible). This is normally a second category subset of the space of all perturbations E. The discrepancies occur as "error" curvature terms (associated to the perturbations) when defining the μ_d-operations. These discrepancies can be made arbitrarily small (for a fixed d) by choosing smaller and smaller perturbations. The same holds for the discrepancy of the units and the ζ_s's.

2.3. Weakly filtered A_∞-functors and modules

Let $\mathcal{A}$, $\mathcal{B}$ two weakly filtered A_∞-categories and $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ an A_∞-functor. Let $\boldsymbol{\epsilon}^{\mathcal{F}} = (\epsilon_1^{\mathcal{F}}, \epsilon_2^{\mathcal{F}}, \ldots, \epsilon_d^{\mathcal{F}}, \ldots)$ be a sequence of non-negative real numbers. In contrast to $\boldsymbol{\epsilon}^{\mathcal{A}}$ and $\boldsymbol{\epsilon}^{\mathcal{B}}$ we do allow here that $\epsilon_1^{\mathcal{F}} \neq 0$.

We say that $\mathcal{F}$ is a *weakly filtered* A_∞-functor with *discrepancy* $\leq \boldsymbol{\epsilon}^{\mathcal{F}}$ if for all $X_0, \dots, X_d \in \mathrm{Ob}(\mathcal{A})$ and $\alpha_1, \dots, \alpha_d \in \mathbb{R}$ we have

$$\mathcal{F}_d\big(C_{\mathcal{A}}^{\leq \alpha_1}(X_0, X_1) \otimes \cdots \otimes C_{\mathcal{A}}^{\leq \alpha_d}(X_{d-1}, X_d)\big) \subset C_{\mathcal{B}}^{\leq \alpha_1 + \cdots + \alpha_d + \epsilon_d^{\mathcal{F}}}(\mathcal{F}X_0, \mathcal{F}X_d). \tag{2.3}$$

Here we have denoted by $C_{\mathcal{A}}$ and $C_{\mathcal{B}}$ the hom's in $\mathcal{A}$ and $\mathcal{B}$ respectively and by $\mathcal{F}_d$ the higher order terms of the functor $\mathcal{F}$.

There is also a notion of weakly filtered natural transformations between weakly filtered functors but we will not go into this now as our main focus will be on a special case – weakly filtered modules and weakly filtered morphisms between them.

2.3.1. ***Weakly filtered modules.*** **—** Let $\mathcal{A}$ be a weakly filtered A_∞-category with discrepancy $\boldsymbol{\epsilon}^{\mathcal{A}}$. Let $\mathcal{M}$ be an $\mathcal{A}$-module with composition maps $\mu_d^{\mathcal{M}}$, $d \geq 1$. Let

$$\boldsymbol{\epsilon}^{\mathcal{M}} = (\epsilon_1^{\mathcal{M}}, \epsilon_2^{\mathcal{M}}, \dots, \epsilon_d^{\mathcal{M}}, \dots)$$

be an infinite sequence of non-negative real numbers with $\epsilon_1^{\mathcal{M}} = 0$. We say that $\mathcal{M}$ is *weakly filtered with discrepancy* $\leq \boldsymbol{\epsilon}^{\mathcal{M}}$ the following holds:

1) For every $X \in \mathrm{Ob}(\mathcal{A})$, $\mathcal{M}(X)$ is endowed with an increasing filtration $\mathcal{M}^{\leq \alpha}(X)$ indexed by $\alpha \in \mathbb{R}$.
2) The $\mu_d^{\mathcal{M}}$-operation respects the filtration up to an "error" of $\epsilon_d^{\mathcal{M}}$. Namely, for all $X_0, \dots, X_{d-1} \in \mathrm{Ob}(\mathcal{A})$ and $a_1, \dots, a_d \in \mathbb{R}$ we have

$$\mu_d^{\mathcal{M}}\big(C^{\leq \alpha_1}(X_0, X_1) \otimes \cdots \otimes C^{\leq \alpha_{d-1}}(X_{d-2}, X_{d-1}) \otimes \mathcal{M}^{\leq \alpha_d}(X_{d-1})\big) \subset \mathcal{M}^{\leq \alpha_1 + \cdots + \alpha_d + \epsilon_d^{\mathcal{M}}}(X_0).$$

Again, since $\epsilon_1^{\mathcal{M}} = 0$, every $(\mathcal{M}^{\leq \alpha}(X), \mu_1^{\mathcal{M}})$ is a sub-complex of $(\mathcal{M}(X), \mu_1^{\mathcal{M}})$.

2.3.2. ***Remark.*** **—** It is easy to see that weakly filtered $\mathcal{A}$-modules are the same as weakly filtered functors $\mathcal{F} : \mathcal{A} \longrightarrow \mathrm{Ch}_{\mathrm{f}}^{\mathrm{opp}}$ (having some discrepancy). Here Ch_{f} is the dg-category of filtered chain complexes (of R-modules) and $\mathrm{Ch}_{\mathrm{f}}^{\mathrm{opp}}$ stands for its opposite category. (Note that Ch_{f} and $\mathrm{Ch}_{\mathrm{f}}^{\mathrm{opp}}$ are in fact *filtered* dg-categories, *i.e.* they have discrepancies 0.) The correspondence between weakly filtered functors and weakly filtered modules is the same as in the "unfiltered" case [Sei08, Section (1j)]. Note that if $\mathcal{F} : \mathcal{A} \longrightarrow \mathrm{Ch}_{\mathrm{f}}^{\mathrm{opp}}$ has discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{F}}$ then the weakly filtered module $\mathcal{M}$ corresponding to it has discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{M}}$ with $\epsilon_d^{\mathcal{M}} = \epsilon_{d-1}^{\mathcal{F}}$ for every $d \geq 2$.

Next we define morphisms between weakly filtered $\mathcal{A}$-modules. Let $\mathcal{M}_0, \mathcal{M}_1$ be two weakly filtered $\mathcal{A}$-modules, both with discrepancy $\leq \boldsymbol{\epsilon}^m$. Let $f : \mathcal{M}_0 \longrightarrow \mathcal{M}_1$ be a pre-module homomorphism. We write $f = (f_1, \dots, f_d, \dots)$ where the f_d-component is an R-linear map

$$f_d : C(X_0, X_1) \otimes \cdots \otimes C(X_{d-2}, X_{d-1}) \otimes \mathcal{M}_0(X_{d-1}) \longrightarrow \mathcal{M}_1(X_0).$$

Let $\alpha \in \mathbb{R}$ and $\boldsymbol{\epsilon}^f = (\epsilon_1^f, \dots, \epsilon_d^f, \dots)$ be a vector of non-negative real numbers. In contrast to $\boldsymbol{\epsilon}^{\mathcal{A}}$ and $\boldsymbol{\epsilon}^m$ we *do allow* that $\epsilon_1^f \neq 0$.

We say that f *shifts action by* $\leq \rho$ and *has discrepancy* $\leq \boldsymbol{\epsilon}^f$ if for every d, the map f_d shifts action by not more than $\rho + \epsilon_d^f$, namely:

$$f_d\big(C^{\leq \alpha_1}(X_0, X_1) \otimes \cdots \otimes C^{\leq \alpha_{d-1}}(X_{d-2}, X_{d-1}) \otimes \mathcal{M}_0^{\leq \alpha_d}(X_{d-1})\big) \subset \mathcal{M}_1^{a_1+\cdots+a_d+\rho+\epsilon_d^f}(X_0).$$

We will generally refer to such f's as weakly filtered pre-module homomorphisms.

As before, if $\rho \leq \rho'$ and $\boldsymbol{\epsilon}^f \leq \boldsymbol{\epsilon}$ then f also shifts filtration by $\leq \rho'$ and has discrepancy $\leq \boldsymbol{\epsilon}$.

We will now define a filtration on the totality of pre-module homomorphisms. Denote

- $\hom(\mathcal{M}_0, \mathcal{M}_1)$ the pre-module homomorphisms $\mathcal{M}_0 \longrightarrow \mathcal{M}_1$ and
- $\hom^{\boldsymbol{\epsilon}^h}(\mathcal{M}_0, \mathcal{M}_1) \subset \hom(\mathcal{M}_0, \mathcal{M}_1)$ the weakly filtered pre-module homomorphisms of discrepancy $\leq \boldsymbol{\epsilon}^h$ (and arbitrary action shift).

The filtration will depend on an additional "discrepancy" parameter

$$\boldsymbol{\epsilon}^h = (\epsilon_1^h, \epsilon_2^h, \dots, \epsilon_d^h, \dots)$$

which is a sequence of non-negative real numbers (the superscript h stands for "homomorphisms"). Again, we *do not* assume here that ϵ_1^h is 0.

Our filtration is indexed by $\mathbb{R}$ and is defined as follows. The part of the filtration corresponding to $\rho \in \mathbb{R}$ is denoted by

$$\hom^{\leq \rho; \boldsymbol{\epsilon}^h}(\mathcal{M}_0, \mathcal{M}_1)$$

and consists of all pre-module homomorphisms $f : \mathcal{M}_0 \longrightarrow \mathcal{M}_1$ which shift action by not more than ρ and have discrepancy $\leq \boldsymbol{\epsilon}^h$. Clearly this yields an increasing filtration on $\hom^{\boldsymbol{\epsilon}^h}(\mathcal{M}_0, \mathcal{M}_1)$. Note however that, when viewed as a filtration on $\hom(\mathcal{M}_0, \mathcal{M}_1)$, this filtration might in general not be exhaustive since not every pre-module homomorphism must be weakly filtered.

Recall that $\mathcal{A}$-modules (and pre-module homomorphisms between them) form a dg-category $\mathrm{mod}_{\mathcal{A}}$ with differential μ_1^{mod} and composition μ_2^{mod} (see [**Sei08**, Section (1j)] for the definitions).

We now analyze these operations in the weakly filtered framework.

For the operation μ_1^{mod} one encounters the following problem. For general choices of $\boldsymbol{\epsilon}^{\mathcal{A}}$, $\boldsymbol{\epsilon}^m$ and $\boldsymbol{\epsilon}^h$ and two weakly filtered modules $\mathcal{M}_0$, $\mathcal{M}_1$ with discrepancy $\leq \boldsymbol{\epsilon}^m$ the differential μ_1^{mod} does not preserve $\hom^{\leq \rho; \boldsymbol{\epsilon}^h}(\mathcal{M}_0, \mathcal{M}_1)$. Nevertheless it is possible to correct this problem by restricting the choice of $\boldsymbol{\epsilon}^h$ as follows:

Assumption $\mathscr{E}$. — A sequence $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_d, \dots)$ is said to *satisfy Assumption* $\mathscr{E}$ if for every $d \geq 1$ we have

$$\varepsilon_d \geq \max\{\epsilon_i^m + \varepsilon_j, \epsilon_i^{\mathcal{A}} + \varepsilon_j \ ; \ i + j = d + 1\}.$$

Sometimes we will need to emphasize the dependence of Assumption $\mathscr{E}$ on the choices of $\boldsymbol{\epsilon}^{\mathcal{A}}$ and $\boldsymbol{\epsilon}^m$ in which case we will refer to it as Assumption $\mathscr{E}(\boldsymbol{\epsilon}^m, \boldsymbol{\epsilon}^{\mathcal{A}})$. Alternatively we will sometimes write $\boldsymbol{\varepsilon} \in \mathscr{E}(\boldsymbol{\epsilon}^m, \boldsymbol{\epsilon}^{\mathcal{A}})$.

An inspection of definition of μ_1^{mod} (see *e.g.* [Sei08, Section (1j)]) shows that if ϵ^h satisfies Assumption $\mathscr{E}$ then $\hom^{\leq \rho; \epsilon^h}(\mathcal{M}_0, \mathcal{M}_1)$ is preserved by μ_1^{mod} hence is a chain complex.

The following will be useful later on:

Lemma 2.1. — *For every $\epsilon^{\mathscr{A}}$ and ϵ^m there exists ε that satisfies Assumption $\mathscr{E}(\epsilon^m, \epsilon^{\mathscr{A}})$. Moreover, there exists a sequence of real numbers $\{A_d\}_{d \in \mathbb{N}}$ which is universal in the sense that it does not depend on $\epsilon^{\mathscr{A}}$ or ϵ^m and has the following property: for every sequence $\boldsymbol{\delta} = (\delta_1, \dots, \delta_d, \dots)$ of non-negative real numbers there exists an ε that satisfies Assumption $\mathscr{E}(\epsilon^m, \epsilon^{\mathscr{A}})$ and such that for all d:*

$$\delta_d \leq \varepsilon_d \leq A_d \sum_{j=1}^{d} (\epsilon_j^{\mathscr{A}} + \epsilon_j^m + \delta_j). \tag{2.4}$$

Proof of Lemma 2.1. — One can easily construct ε_d and A_d inductively: start with $\varepsilon_1 := \delta_1$ then set $\varepsilon_2 := \max\{\epsilon_2^m + \varepsilon_1, \epsilon_2^{\mathscr{A}} + \varepsilon_1, \delta_2\}$ and so on. (Note that $\epsilon_1^{\mathscr{A}} = \epsilon_1^m = 0$ so that the inequality in Assumption $\mathscr{E}$ is obviously satisfied for $i = 1, j = d$.) □

2.3.3. *Remarks*

1) If $\varepsilon \in \mathscr{E}(\epsilon^m, \epsilon^{\mathscr{A}})$ then the same holds for $\widetilde{\varepsilon} := \varepsilon + \boldsymbol{c}$, where $\boldsymbol{c} = (c, \dots, c, \dots)$ is a constant sequence.

2) When dealing with $\hom^{\leq \rho; \epsilon^h}$ we can always arrange that $\epsilon_1^h = 0$ by applying the following procedure. Suppose that $\epsilon^h \in \mathscr{E}(\epsilon^m, \epsilon^{\mathscr{A}})$. Put $\widetilde{\rho} := \rho + \epsilon_1^h$, $\widetilde{\epsilon}_d^h := \epsilon_d^h - \epsilon_1^h$. Note that $\widetilde{\epsilon}_1^h = 0$, $\widetilde{\epsilon}_d^h \geq 0$ and that $\widetilde{\epsilon}^h$ still satisfies Assumption $\mathscr{E}$. It is easy to see that

$$\hom^{\leq \widetilde{\rho}; \widetilde{\epsilon}^h}(\mathcal{M}_0, \mathcal{M}_1) = \hom^{\leq \rho; \epsilon^h}(\mathcal{M}_0, \mathcal{M}_1).$$

We now turn to the μ_2^{mod} operation. Let $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2$ be weakly filtered $\mathscr{A}$-modules with discrepancy $\leq \epsilon^m$. Let $f : \mathcal{M}_0 \longrightarrow \mathcal{M}_1$, $g : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ be two weakly filtered pre-module homomorphisms with $f \in \hom^{\leq \rho^f; \epsilon^f}(\mathcal{M}_0, \mathcal{M}_1)$, $g \in \hom^{\leq \rho^g; \epsilon^g}(\mathcal{M}_1, \mathcal{M}_2)$. Set $\varphi := \mu_2^{\mathrm{mod}}(f, g) : \mathcal{M}_0 \longrightarrow \mathcal{M}_2$. A simple calculation shows that φ is weakly filtered and that $\varphi \in \hom^{\leq \rho^f + \rho^g; \epsilon^f * \epsilon^g}(\mathcal{M}_0, \mathcal{M}_2)$, where the sequence of discrepancies $\epsilon^f * \epsilon^g$ is defined as:

$$(\epsilon^f * \epsilon^g)_d = \max\left\{\epsilon_i^f + \epsilon_j^g \,;\, i + j = d + 1\right\}. \tag{2.5}$$

Moreover, a simple calculation shows that if $\epsilon^f, \epsilon^g \in \mathscr{E}(\epsilon^m, \epsilon^{\mathscr{A}})$ then the same holds for $\epsilon^f * \epsilon^g$.

A few words are in order about the structure of the totality of weakly filtered $\mathscr{A}$-modules. Ideally one would like to view the weakly filtered modules (say with discrepancy $\leq \epsilon^m$, and with morphisms of discrepancy $\leq \epsilon^h$) as a sub-category of $\mathrm{mod}_{\mathscr{A}}$ and define a weakly filtered structure on it. As seen above, Assumption $\mathscr{E}$ assures that the $\hom^{\leq \rho; \epsilon^h}(\mathcal{M}_0, \mathcal{M}_1)$'s are closed under μ_1^{mod}. However without further restrictions on ϵ^h, the operation μ_2^{mod} does not map

$$\hom^{\leq \rho'; \epsilon^h}(\mathcal{M}_0, \mathcal{M}_1) \otimes \hom^{\leq \rho''; \epsilon^h}(\mathcal{M}_1, \mathcal{M}_2) \quad \text{to} \quad \hom^{\leq \rho' + \rho''; \epsilon^h}(\mathcal{M}_0, \mathcal{M}_2).$$

Thus for general $\epsilon^h \in \mathscr{E}(\epsilon^m, \epsilon^{\mathscr{A}})$ we still do not get a dg-category. We refer the reader to the expanded version of this paper [**BCS**] for possible solutions to this issue, as well as to further discussion on categorical aspects of weakly filtered modules such as the Yoneda embedding and triangulated structure in the weakly filtered framework. For the applications in this paper, we do not need the weakly filtered modules to form a dg-category, and therefore will generally not restrict ϵ^h beyond Assumption $\mathscr{E}$. We stress that Assumption $\mathscr{E}$ will continue to play an important role since it assures that the $\hom^{\leq\rho;\epsilon^h}$'s are preserved by μ_1^{mod}. Thus we will mostly continue to assume it.

2.3.4. ***Action-shifts.*** **—** Let $\mathscr{M}$ be a weakly filtered module over a weakly filtered A_∞-category $\mathscr{A}$. Let $\nu_0 \in \mathbb{R}$. Define a new weakly filtered $\mathscr{A}$-module $S^{\nu_0}\mathscr{M}$ to be the same module as $\mathscr{M}$ only that its filtration is shifted by ν_0, namely:

$$(S^{\nu_0}\mathscr{M})^{\leq\alpha}(N) := \mathscr{M}^{\leq\alpha+\nu_0}(N) \quad \text{for all } N \in \mathrm{Ob}(\mathscr{A}),\ \alpha \in \mathbb{R}.$$

Clearly $S^{\nu_0}\mathscr{M}$ has the same discrepancy as $\mathscr{M}$.

We call $S^{\nu_0}\mathscr{M}$ the *action-shift of $\mathscr{M}$ by ν_0*.

In what follows we will use the same notation S^{ν_0} also for action shifts of other filtered objects such as filtered chain complexes or more generally filtered R-modules.

2.3.5. ***Homologically unital $\mathscr{A}$-modules.*** **—** We have already discussed h-unital A_∞-categories in the weakly filtered sense on page 17. In what follows we will sometimes need an analogous, yet somewhat stronger, notion for modules.

Assumption U_m **.** — Let $\mathscr{A}$ be a weakly filtered A_∞-category with discrepancy $\leq \epsilon^{\mathscr{A}}$. Assume that $\mathscr{A}$ is h-unital in the weakly filtered sense as defined in Section 2.1, *i.e.* we have $u^{\mathscr{A}} \geq 0$ and choices of cycles $e_X \in C^{\leq u^{\mathscr{A}}}(X, X)$ for every $X \in \mathrm{Ob}(\mathscr{A})$ representing the units in homology. Let $\mathscr{M}$ be a weakly filtered $\mathscr{A}$-module with discrepancy $\leq \epsilon^m$, and let $u^{\mathscr{A}} + \epsilon_2^{\mathscr{M}} \leq \kappa \in \mathbb{R}$.

We say that $\mathscr{M}$ *satisfies Assumption $U_m(\kappa)$* (or $\mathscr{M} \in U_m(\kappa)$ for short) if for every X in $\mathrm{Ob}(\mathscr{A})$ and every $\alpha \in \mathbb{R}$ the map

$$\mathscr{M}^{\leq\alpha}(X) \longrightarrow \mathscr{M}^{\leq\alpha+\kappa}(X), \quad b \longmapsto \mu_2^{\mathscr{M}}(e_X, b) \tag{2.6}$$

induces in homology the same map as the one induced by $\mathscr{M}^{\leq\alpha}(X) \hookrightarrow \mathscr{M}^{\leq\alpha+\kappa}(X)$. Note that in particular, $\mathscr{M}$ is an h-unital module.

Sometimes the module $\mathscr{M}$ will be a Yoneda module $\mathscr{Y}$ associated to an object $Y \in \mathrm{Ob}(\mathscr{A})$. In that case we will sometimes write $Y \in U_m(\kappa)$ instead of $\mathscr{Y} \in U_m(\kappa)$. Note that in this case the map in (2.6) becomes

$$C^{\leq\alpha}(Y, X) \longrightarrow C^{\leq\alpha+\kappa}(Y, X), \quad b \longmapsto \mu_2^{\mathscr{A}}(e_X, b).$$

There is also a homotopical version of U_m (see [**BCS**, Section 2.3.4], where it is called U_s) but we will not need it here.

2.3.6. ***Pulling back weakly filtered modules.*** — Let $\mathcal{A}, \mathcal{B}$ be two weakly filtered A_∞-categories and $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ a weakly filtered A_∞-functor with discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{F}}$. Let $\mathcal{M}$ be a weakly filtered $\mathcal{B}$-module with discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{M}}$. Consider the $\mathcal{A}$-module $\mathcal{F}^*\mathcal{M}$ which is obtained by pulling back $\mathcal{M}$ via $\mathcal{F}$. We filter $\mathcal{F}^*\mathcal{M}$ by setting

$$(\mathcal{F}^*\mathcal{M})^{\leq \alpha}(N) = \mathcal{M}^{\leq \alpha}(\mathcal{F}N).$$

The following can be easily proved.

Lemma 2.2. — *The module $\mathcal{F}^*\mathcal{M}$ is weakly filtered with discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{F}^*\mathcal{M}}$, where for all $d \geq 2$*

$$\epsilon_d^{\mathcal{F}^*\mathcal{M}} = \max\left\{\epsilon_{s_1}^{\mathcal{F}} + \cdots + \epsilon_{s_k}^{\mathcal{F}} + \epsilon_{k+1}^{\mathcal{M}} \,;\, 1 \leq k \leq d-1, \;\; s_1 + \cdots + s_k = d-1\right\}.$$

In particular, if the higher order terms of $\mathcal{F}$ vanish, i.e. $\mathcal{F}_s = 0$ for all $s \geq 2$, then

$$\epsilon_d^{\mathcal{F}^*\mathcal{M}} = (d-1)\epsilon_1^{\mathcal{F}} + \epsilon_d^{\mathcal{M}}, \quad \textit{for all } d.$$

Let $\mathcal{M}_0, \mathcal{M}_1$ be two weakly filtered $\mathcal{B}$-modules and $f : \mathcal{M}_0 \to \mathcal{M}_1$ a weakly filtered module homomorphism that shifts action by $\leq \rho$ and has discrepancy $\leq \boldsymbol{\epsilon}^f$. Pulling back we obtain a homomorphism of $\mathcal{A}$-modules $\mathcal{F}^*f : \mathcal{F}^*\mathcal{M}_0 \to \mathcal{F}^*\mathcal{M}_1$. The following can be easily verified.

Lemma 2.3. — *The module homomorphism $\mathcal{F}^*f$ is weakly filtered with action shift $\leq \rho$ and discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{F}^*f}$, where $\epsilon_1^{\mathcal{F}^*f} = \epsilon_1^f$ and*

$$\epsilon_d^{\mathcal{F}^*f} = \max\left\{\epsilon_{s_1}^{\mathcal{F}} + \cdots + \epsilon_{s_k}^{\mathcal{F}} + \epsilon_{k+1}^{f} \,;\, 1 \leq k \leq d-1, \;\; s_1 + \cdots + s_k = d-1\right\}, \quad \textit{for all } d \geq 2.$$

In particular, if the higher order terms of $\mathcal{F}$ vanish, i.e. $\mathcal{F}_s = 0$ for all $s \geq 2$, then

$$\epsilon_d^{\mathcal{F}^*f} = (d-1)\epsilon_1^{\mathcal{F}} + \epsilon_d^{f}, \quad \textit{for all } d.$$

2.4. Weakly filtered mapping cones

Let $\mathcal{M}_0, \mathcal{M}_1$ be two weakly filtered $\mathcal{A}$-modules with discrepancies $\leq \boldsymbol{\epsilon}^{\mathcal{M}_0}$ and $\leq \boldsymbol{\epsilon}^{\mathcal{M}_1}$ respectively. Let $f : \mathcal{M}_0 \to \mathcal{M}_1$ be a *module homomorphism*, *i.e.* f is a pre-module homomorphism which is a cycle: $\mu_1^{\mathrm{mod}}(f) = 0$. Assume that f shifts action by $\leq \rho$ and has discrepancy $\leq \boldsymbol{\epsilon}^f$, or in other words $f \in \hom^{\leq \rho; \boldsymbol{\epsilon}^f}(\mathcal{M}_0, \mathcal{M}_1)$. We *generally do not* assume that $\boldsymbol{\epsilon}^f$ satisfies Assumption $\mathcal{E}(\boldsymbol{\epsilon}^m, \boldsymbol{\epsilon}^{\mathcal{A}})$ unless explicitly specified.

Consider the mapping cone $\mathcal{C} := \mathcal{C}one(f)$ viewed as an $\mathcal{A}$-module and endowed with its standard A_∞-composition maps $\mu_d^{\mathcal{C}}$. We endow $\mathcal{C}$ with a weakly filtered structure as follows. For $X \in \mathrm{Ob}(\mathcal{A})$ and $\alpha \in \mathbb{R}$, put

$$\mathcal{C}^{\leq \alpha}(X) := \mathcal{M}_0^{\leq \alpha - \rho - \epsilon_1^f}(X) \oplus \mathcal{M}_1^{\leq \alpha}(X). \tag{2.7}$$

Define (see page 16 for the precise meaning of this notation)

$$\boldsymbol{\epsilon}^{\mathcal{C}} := \max\left\{\boldsymbol{\epsilon}^{\mathcal{M}_0}, \boldsymbol{\epsilon}^{\mathcal{M}_1}, \boldsymbol{\epsilon}^f - \epsilon_1^f\right\}.$$

Then $\mathscr{C}$ is weakly filtered with discrepancy $\leq \boldsymbol{\epsilon}^{\mathscr{C}}$. This follows from (2.7) and the fact that

$$\mu_d^{\mathscr{C}}\big(a_1,\dots,a_{d-1},(b_0,b_1)\big) = \big(\mu_d^{\mathscr{M}_0}(a_1,\dots,a_{d-1},b_0),\\ f_d(a_1,\dots,a_{d-1},b_0) + \mu_d^{\mathscr{M}_1}(a_1,\dots,a_{d-1},b_1)\big).$$

2.4.1. ***Remark.*** **—** If we assume in addition that

$$\boldsymbol{\epsilon}^{\mathscr{M}_0}, \boldsymbol{\epsilon}^{\mathscr{M}_1} \leq \boldsymbol{\epsilon}^m \quad \text{and} \quad \boldsymbol{\epsilon}^f \in \mathscr{E}(\boldsymbol{\epsilon}^m, \boldsymbol{\epsilon}^{\mathscr{A}}),$$

then we have $\boldsymbol{\epsilon}^f - \epsilon_1^f \geq \boldsymbol{\epsilon}^m$, hence $\boldsymbol{\epsilon}^{\mathscr{C}} = \boldsymbol{\epsilon}^f - \epsilon_1^f$.

It is important to note that the filtration we have defined on $\mathscr{C}one(f)$ in (2.7) strictly depends on the choices of ρ and ϵ_1^f. Therefore, whenever these dependencies are relevant we will denote the weakly filtered cone of f by

$$\mathscr{C}one(f;\rho,\boldsymbol{\epsilon}^f) \quad \text{or by} \quad \mathscr{C}one(\mathscr{M}_0 \xrightarrow{(f;\rho,\boldsymbol{\epsilon}^f)} \mathscr{M}_1). \tag{2.8}$$

2.4.2. ***Remark.*** **—** We opted to define the filtration on the cone as in (2.7) so that the inclusion $\mathscr{M}_1 \to \mathscr{C}$ becomes a strictly filtered map.

We now discuss several elementary properties of weakly filtered mapping cones that will be useful later on. We begin with the effect of action-shifts (see §2.3.4) on mapping cones. The following follows immediately from the definitions.

Lemma 2.4. — *Let $f : \mathscr{M}_0 \to \mathscr{M}_1$ be a weakly filtered module homomorphism between two weakly filtered $\mathscr{A}$-modules. Assume that f shifts action by $\leq \rho$ and has discrepancy $\leq \boldsymbol{\epsilon}^f$. Let $\nu_0 \in \mathbb{R}$. Then we have the following equality of weakly filtered $\mathscr{A}$-modules:*

$$\begin{aligned} S^{\nu_0}\big(\mathscr{C}one(\mathscr{M}_0 \xrightarrow{(f;\rho,\boldsymbol{\epsilon}^f)} \mathscr{M}_1)\big) &= \mathscr{C}one\big(S^{\nu_0}\mathscr{M}_0 \xrightarrow{(f;\rho,\boldsymbol{\epsilon}^f)} S^{\nu_0}\mathscr{M}_1\big) \\ &= \mathscr{C}one\big(\mathscr{M}_0 \xrightarrow{(f;\rho,\boldsymbol{\epsilon}^f-\nu_0)} S^{\nu_0}\mathscr{M}_1\big) \\ &= \mathscr{C}one\big(\mathscr{M}_0 \xrightarrow{(f;\rho-\nu_0,\boldsymbol{\epsilon}^f)} S^{\nu_0}\mathscr{M}_1\big). \end{aligned}$$

Next, we analyze (a special case of) cones over a composition of module homomorphisms, from the weakly filtered perspective. Let $f : \mathscr{M}_0 \to \mathscr{M}_1$ be as at the beginning of the present chapter. Let $\mathscr{M}_1'$ be another weakly filtered $\mathscr{A}$-module with discrepancy $\leq \boldsymbol{\epsilon}^{\mathscr{M}'}$ and let $\xi : \mathscr{M}_1 \to \mathscr{M}_1'$ be a weakly filtered module homomorphism with $\xi \in \hom^{\leq s;\boldsymbol{\epsilon}^\xi}(\mathscr{M}_1, \mathscr{M}_1')$. Denote the composition of f and ξ by

$$f' = \xi \circ f := \mu_2^{\text{mod}}(f,\xi) : \mathscr{M}_0 \longrightarrow \mathscr{M}_1'.$$

We have $f' \in \hom^{\rho+s;\boldsymbol{\epsilon}^{f'}}(\mathscr{M}_0, \mathscr{M}_1')$, where

$$\epsilon_d^{f'} = (\boldsymbol{\epsilon}^f * \boldsymbol{\epsilon}^\xi)_d =: \max\big\{\epsilon_i^f + \epsilon_j^\xi \; ; \; i+j = d+1\big\}.$$

Lemma 2.5. — *There exists a weakly filtered module homomorphism*

$$\psi : \mathscr{C}one(f;\rho,\boldsymbol{\epsilon}^f) \longrightarrow \mathscr{C}one(f';\rho+s,\boldsymbol{\epsilon}^{f'})$$

that shifts action by $\leq s$ and has discrepancy $\leq \boldsymbol{\epsilon}^{\xi}$. The homomorphism ψ fits into the following (chain level) commutative diagram of $\mathscr{A}$-modules:

$$\begin{array}{ccccccc} \mathscr{M}_0 & \xrightarrow{\ f\ } & \mathscr{M}_1 & \longrightarrow & \mathscr{C}one(f) & \longrightarrow & T\mathscr{M}_0 \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ \mathscr{M}_0 & \xrightarrow{\ f'\ } & \mathscr{M}_1' & \longrightarrow & \mathscr{C}one(f') & \longrightarrow & T\mathscr{M}_0 \end{array} \tag{2.9}$$

where the horizontal unlabeled maps are the standard inclusion and projection maps (with zero higher order terms), and $T\mathscr{M}_0$ stands for the shift of $\mathscr{M}_0$ with respect to grading. Moreover, if ξ is a quasi-isomorphism then so is ψ.

As indicated earlier, in this paper we work in the ungraded setting, hence the equality $T\mathscr{M}_0 = \mathscr{M}_0$. Nevertheless we have written $T\mathscr{M}_0$ in (2.9) as a suggestion for how the statement should look like in the graded case.

Proof of Lemma 2.5. — Simply define $\psi_1(b_0, b_1) = \big(b_0, \xi_1(b_1)\big)$ and for $d \geq 2$:

$$\psi_d\big(a_1, \dots, a_{d-1}, (b_0, b_1)\big) := \big(0, \xi_d(a_1, \dots, a_{d-1}, b_1)\big).$$

All the statements asserted by the lemma can be verified by direct calculation. □

Next we discuss how the weakly filtered mapping cone changes if we alter the cycle f by a boundary. Assume now that $\mathscr{M}_0$ and $\mathscr{M}_1$ have both discrepancies $\leq \boldsymbol{\epsilon}^m$. Fix a sequence $\boldsymbol{\epsilon}^h$ that satisfies Assumption $\mathscr{E}(\boldsymbol{\epsilon}^m, \boldsymbol{\epsilon}^{\mathscr{A}})$. Let $f \in \hom^{\leq \rho; \boldsymbol{\epsilon}^h}(\mathscr{M}_0, \mathscr{M}_1)$ be a module homomorphism and $f' = f + \mu_1^{\mathrm{mod}}(\theta)$ for some $\theta \in \hom^{\leq \rho; \boldsymbol{\epsilon}^h}(\mathscr{M}_0, \mathscr{M}_1)$. Consider the two weakly filtered mapping cones $\mathscr{C}one(f;\rho,\boldsymbol{\epsilon}^h)$ and $\mathscr{C}one(f';\rho,\boldsymbol{\epsilon}^h)$.

Lemma 2.6. — *There exists a module homomorphism*

$$\vartheta : \mathscr{C}one(f;\rho,\boldsymbol{\epsilon}^h) \longrightarrow \mathscr{C}one(f';\rho,\boldsymbol{\epsilon}^h)$$

with the following properties:

(i) *ϑ is a quasi-isomorphism.*

(ii) *ϑ does not shift action (i.e. it shifts the action filtration by ≤ 0) and has discrepancy $\leq \boldsymbol{\epsilon}^{\vartheta} := \boldsymbol{\epsilon}^h - \epsilon_1^h$. In particular (since $\epsilon_1^{\vartheta} = 0$) the chain map*

$$\vartheta_1 : \mathscr{C}one(f;\rho,\boldsymbol{\epsilon}^h)(X) \longrightarrow \mathscr{C}one(f';\rho,\boldsymbol{\epsilon}^h)(X)$$

preserves the action filtration for every $X \in \mathrm{Ob}(\mathscr{A})$.

Proof. — Define $\vartheta_1(b_0, b_1) := \big((-1)^{|b_0|-1} b_0, (-1)^{|b_1|} b_1 + \theta_1(b_0)\big)$ and for $d \geq 2$ define:

$$\vartheta_d\big(a_1, \dots, a_{d-1}, (b_0, b_1)\big) = \big(0, \theta_d(a_1, \dots, a_{d-1}, b_0)\big).$$

Cf. [Sei08, Formula 3.7, p. 35].

Note that in this paper we work with a base ring R of characteristic 2, hence the signs in the preceding formula for ϑ_1 can actually be ignored. Nevertheless we included them, just as an indication for a possible extension to more general rings. □

The next lemma shows that weakly filtered cones are preserved under pulling back by weakly filtered functors.

Lemma 2.7. — *Let:*

- ▹ $\mathscr{A}, \mathscr{B}$ *be two weakly filtered* A_∞*-categories and* $\mathscr{F} : \mathscr{A} \longrightarrow \mathscr{B}$ *a weakly filtered* A_∞*-functor with discrepancy* $\leq \boldsymbol{\epsilon}^{\mathscr{F}}$*;*
- ▹ $\mathscr{M}_0, \mathscr{M}_1$ *weakly filtered* $\mathscr{B}$*-modules and* $f : \mathscr{M}_0 \longrightarrow \mathscr{M}_1$ *a weakly filtered module homomorphism which shifts action by* $\leq \rho$ *and has discrepancy* $\leq \boldsymbol{\epsilon}^f$*.*

Then we have the following equality of weakly filtered $\mathscr{A}$*-modules:*

$$\mathscr{F}^*\big(\mathscr{C}one(\mathscr{M}_0 \xrightarrow{(f;\rho,\boldsymbol{\epsilon}^f)} \mathscr{M}_1)\big) = \mathscr{C}one\big(\mathscr{F}^*\mathscr{M}_0 \xrightarrow{(\mathscr{F}^*f;\rho,\boldsymbol{\epsilon}^{\mathscr{F}^*f})} \mathscr{F}^*\mathscr{M}_1\big),$$

where $\boldsymbol{\epsilon}^{\mathscr{F}^*f}$ *is given in Lemma 2.3.*

The proof is straightforward, hence omitted.

We now return briefly to unitality of modules, more specifically to Assumption U_m. The following lemma shows that this assumption is preserved under certain quasi-isomorphisms of weakly filtered modules.

Lemma 2.8. — *Let:*

▹ $\mathscr{M}', \mathscr{M}''$ *be weakly filtered* $\mathscr{A}$*-modules with discrepancies* $\leq \boldsymbol{\epsilon}^{\mathscr{M}'}$ *and* $\leq \boldsymbol{\epsilon}^{\mathscr{M}''}$*.*

▹ $\phi' : \mathscr{M}' \longrightarrow \mathscr{M}''$ *be a weakly filtered module homomorphism,* $\phi' \in \hom^{\rho';\boldsymbol{\epsilon}^{\phi'}}(\mathscr{M}', \mathscr{M}'')$*.*

▹ *Let* ϕ'' *be a collection of chain maps* $\phi''_X : \mathscr{M}''(X) \longrightarrow \mathscr{M}'(X)$*, defined for all* X *in* $\mathrm{Ob}(\mathscr{A})$*, and assume for all* $X \in \mathrm{Ob}(\mathscr{A})$ *and* $\alpha \in \mathbb{R}$*,*

$$\phi''_X\big(\mathscr{M}''^{\leq\alpha}(X)\big) \subset \mathscr{M}'^{\leq \alpha+\rho''+\epsilon''}(X)$$

for some fixed $\rho'', \epsilon'' \in \mathbb{R}$*. (For example, if* $\phi'' : \mathscr{M}'' \longrightarrow \mathscr{M}'$ *is a weakly filtered module homomorphism with* $\phi'' \in \hom^{\rho'';\boldsymbol{\epsilon}^{\phi''}}(\mathscr{M}'', \mathscr{M}')$*, where* $\epsilon_1^{\phi''} \leq \epsilon''$*, then the assumptions on* ϕ'' *are clearly satisfied.)*

▹ *Let* $\nu, \kappa'' \in \mathbb{R}$ *and assume further that*

α) *For every* $X \in \mathrm{Ob}(\mathscr{A})$ *and every* $\alpha \in \mathbb{R}$ *the composition of chain maps*

$$\mathscr{M}'^{\leq\alpha}(X) \xrightarrow{\phi''_X\circ\phi'_1} \mathscr{M}'^{\leq\alpha+\rho'+\rho''+\epsilon_1^{\phi'}+\epsilon''}(X) \xrightarrow{\mathrm{inc}} \mathscr{M}'^{\leq\alpha+\rho'+\rho''+\epsilon_1^{\phi'}+\epsilon''+\nu}(X)$$

induces in homology the same map as the one induced by the inclusion

$$\mathscr{M}'^{\leq\alpha}(X) \longrightarrow \mathscr{M}'^{\leq\alpha+\rho'+\rho''+\epsilon_1^{\phi'}+\epsilon''+\nu}(X).$$

β) $\mathscr{M}'' \in U_m(\kappa'')$*.*

Then $\mathscr{M}'$ *belongs to* $U_m(\kappa')$*, where*

$$\kappa' = \rho' + \rho'' + \epsilon'' + \max\big\{\epsilon_1^{\phi'} + u^{\mathscr{A}} + \epsilon_2^{\mathscr{M}'} + \nu, \epsilon_1^{\phi'} + \kappa'', \epsilon_2^{\phi'} + u^{\mathscr{A}}\big\}. \tag{2.10}$$

Proof. — Fix $X \in \mathrm{Ob}(\mathscr{A})$, $\alpha \in \mathbb{R}$ and let $b \in \mathscr{M}'^{\leq \alpha}(X)$ be a *cycle*. Since ϕ' is a module homomorphism (*i.e.* $\mu_1^{\mathrm{mod}}(\phi') = 0$) we have

$$\phi'_1 \mu_2^{\mathscr{M}'}(e_X, b) = \mu_2^{\mathscr{M}''}\big(e_X, \phi'_1(b)\big) \pm \mu_1^{\mathscr{M}''}\phi'_2(e_X, b).$$

Applying ϕ''_X to both sides of this identity we obtain

$$\phi''_X \phi'_1 \mu_2^{\mathscr{M}'}(e_X, b) = \phi''_X \mu_2^{\mathscr{M}''}\big(e_X, \phi'_1(b)\big) \pm \mu_1^{\mathscr{M}'}\phi''_X \phi'_2(e_X, b). \tag{2.11}$$

Since $\mu_2^{\mathscr{M}'}(e_X, b) \in \mathscr{M}'^{\leq \alpha + u^{\mathscr{A}} + \epsilon_2^{\mathscr{M}'}}(X)$ our assumption on $\phi''_X \circ \phi'_1$ implies that there exists $x \in \mathscr{M}'^{\leq \alpha + u^{\mathscr{A}} + \epsilon_2^{\mathscr{M}'} + \rho' + \rho'' + \epsilon_1^{\phi'} + \epsilon'' + \nu}(X)$ such that

$$\phi''_X \phi'_1 \mu_2^{\mathscr{M}'}(e_X, b) = \mu_2^{\mathscr{M}'}(e_X, b) - \mu_1^{\mathscr{M}'}(x). \tag{2.12}$$

Since $\mathscr{M}'' \in U_m(\kappa'')$ there exists an element $y \in \mathscr{M}''^{\leq \alpha + \rho' + \epsilon_1^{\phi'} + \kappa''}(X)$ such that

$$\mu_2^{\mathscr{M}''}\big(e_X, \phi'_1(b)\big) = \phi'_1(b) + \mu_1^{\mathscr{M}''}(y).$$

Substituting the last identity together with (2.12) into (2.11) yields:

$$\mu_2^{\mathscr{M}'}(e_X, b) = \mu_1^{\mathscr{M}'}(x) + \phi''_X \phi'_1(b) + \mu_1^{\mathscr{M}'}\big(\phi''_X(y)\big) + \mu_1^{\mathscr{M}'}\big(\phi''_X \phi'_2(e_X, b)\big). \tag{2.13}$$

Using our assumption on $\phi''_X \circ \phi'_1$, we can write the second term of (2.13) as $\phi''_X \phi'_1(b) = b + \mu_1^{\mathscr{M}'}(z)$ for some $z \in \mathscr{M}'^{\leq \alpha + \rho' + \rho'' + \epsilon_1^{\phi'} + \epsilon'' + \nu}(X)$. Substituting this in (2.13) we obtain

$$\mu_2^{\mathscr{M}'}(e_X, b) = b + \mu_1^{\mathscr{M}'}(x) + \mu_1^{\mathscr{M}'}(z) + \mu_1^{\mathscr{M}'}\big(\phi''_X(y)\big) + \mu_1^{\mathscr{M}'}\big(\phi''_X \phi'_2(e_X, b)\big), \tag{2.14}$$

where

$$x \in \mathscr{M}'^{\leq \alpha + u^{\mathscr{A}} + \epsilon_2^{\mathscr{M}'} + \rho' + \rho'' + \epsilon_1^{\phi'} + \epsilon'' + \nu}(X), \quad \phi''_X(y) \in \mathscr{M}''^{\leq \alpha + \rho' + \rho'' + \epsilon_1^{\phi'} + \epsilon'' + \kappa''}(X),$$
$$z \in \mathscr{M}'^{\leq \alpha + \rho' + \rho'' + \epsilon_1^{\phi'} + \epsilon'' + \nu}(X), \quad \phi''_X \phi'_2(e_X, b) \in \mathscr{M}'^{\leq \alpha + u^{\mathscr{A}} + \rho' + \rho'' + \epsilon'' + \epsilon_2^{\phi'}}.$$

The estimate (2.10) for κ' readily follows. □

It is known that h-unitality is preserved under mapping cones [Sei08, Section 3e]. The following Lemma is a weakly filtered analogue, concerning Assumption U_m.

Lemma 2.9. — *Assume that $\mathscr{A}$ satisfies Assumption $U^e(\zeta)$ (see page 17). Let:*

- *$\mathscr{M}_0$, $\mathscr{M}_1$ be weakly filtered $\mathscr{A}$-modules with discrepancies $\leq \boldsymbol{\epsilon}^{\mathscr{M}_0}$ and $\leq \boldsymbol{\epsilon}^{\mathscr{M}_1}$ respectively and assume that $\mathscr{M}_0 \in U_m(\kappa^{\mathscr{M}_0})$ and $\mathscr{M}_1 \in U_m(\kappa^{\mathscr{M}_1})$.*
- *$f \in \hom^{\leq \rho; \boldsymbol{\epsilon}^f}(\mathscr{M}_0, \mathscr{M}_1)$ be a module homomorphism.*

Then the weakly filtered module $\mathscr{C}one(f; \rho, \boldsymbol{\epsilon}^f)$ satisfies Assumption $U_m(\kappa)$, where

$$\kappa = \max\left\{2\kappa^{\mathscr{M}_0}, 2\kappa^{\mathscr{M}_1}, 2u^{\mathscr{A}} + \epsilon_3^{\mathscr{C}}, 2u^{\mathscr{A}} + 2\epsilon_2^{\mathscr{C}}, \zeta + \epsilon_2^{\mathscr{C}}\right\}, \tag{2.15}$$

and $\boldsymbol{\epsilon}^{\mathscr{C}} := \max\{\boldsymbol{\epsilon}^{\mathscr{M}_0}, \boldsymbol{\epsilon}^{\mathscr{M}_1}, \boldsymbol{\epsilon}^f - \epsilon_1^f\}$. (Recall that $\boldsymbol{\epsilon}^{\mathscr{C}}$ is the standard bound on the discrepancy of $\mathscr{C} = \mathscr{C}one(f; \rho, \boldsymbol{\epsilon}^f)$ – see page 24.)

To show this lemma we will make use of the following proposition that is of independent interest.

Proposition 2.10. — *Assume that* $\mathscr{A} \in U^e(\zeta)$. *Let* $\mathscr{M}$ *be a weakly filtered* $\mathscr{A}$*-module with discrepancy* $\leq \boldsymbol{\epsilon}^{\mathscr{M}}$ *and let* $X \in \mathrm{Ob}(\mathscr{A})$. *Then the chain maps*

$$v : \mathscr{M}(X) \longrightarrow \mathscr{M}(X), \quad v(b) := \mu_2^{\mathscr{M}}(e_X, b)$$

and $v \circ v : \mathscr{M}(X) \to \mathscr{M}(X)$ *are chain homotopic via a chain homotopy that shifts action by not more than* $\max\{2u^{\mathscr{A}} + \epsilon_3^{\mathscr{M}}, \zeta + \epsilon_2^{\mathscr{M}}\}$.

Proof. — The A_∞-identities for $\mathscr{M}$ (+ the fact that e_X is a cycle) imply that for every $b \in \mathscr{M}(X)$ we have

$$\mu_1^{\mathscr{M}}\mu_3^{\mathscr{M}}(e_X, e_X, b) - \mu_2^{\mathscr{M}}\big(e_X, \mu_2^{\mathscr{M}}(e_X, b)\big) + \mu_3^{\mathscr{M}}\big(e_X, e_X, \mu_1^{\mathscr{M}}(b)\big) + \mu_2^{\mathscr{M}}\big(\mu_2^{\mathscr{A}}(e_X, e_X), b\big) = 0. \tag{2.16}$$

Since $\mathscr{A} \in U^e(\zeta)$ we have $\mu_2^{\mathscr{A}}(e_X, e_X) = e_X + \mu_1^{\mathscr{A}}(c)$, for some $c \in C^{\leq\zeta}(X, X)$. Substituting this in (2.16) together with $\mu_2^{\mathscr{M}}(\mu_1^{\mathscr{A}}(c), b) + \mu_2^{\mathscr{M}}(c, \mu_1^{\mathscr{M}}(b)) + \mu_1^{\mathscr{M}}\mu_2^{\mathscr{M}}(c, b) = 0$ yields:

$$\mu_2^{\mathscr{M}}\big(e_X, \mu_2^{\mathscr{M}}(e_X, b)\big) - \mu_2^{\mathscr{M}}(e_X, b) = \mu_1^{\mathscr{M}} h(b) + h\mu_1^{\mathscr{M}}(b),$$

where $h(b) = \mu_3^{\mathscr{M}}(e_X, e_X, b) - \mu_2^{\mathscr{M}}(c, b)$. Clearly the chain homotopy h shifts action by not more than $\max\{2u^{\mathscr{A}} + \epsilon_3^{\mathscr{M}}, \zeta + \epsilon_2^{\mathscr{M}}\}$. □

We now return to the proof of the lemma.

Proof of Lemma 2.9. — Denote $\mathscr{C} = \mathscr{C}one(f; \rho, \boldsymbol{\epsilon}^f)$. Recall that this module has discrepancy $\leq \boldsymbol{\epsilon}^{\mathscr{C}} := \max\{\boldsymbol{\epsilon}^{\mathscr{M}_0}, \boldsymbol{\epsilon}^{\mathscr{M}_1}, \boldsymbol{\epsilon}^f - \epsilon_1^f\}$. Put

$$\delta := \max\{u^{\mathscr{A}} + \epsilon_2^{\mathscr{C}}, \kappa^{\mathscr{M}_0}, \kappa^{\mathscr{M}_1}\}, \quad \kappa := \max\{2\delta, 2u^{\mathscr{A}} + \epsilon_3^{\mathscr{C}}, \zeta + \epsilon_2^{\mathscr{C}}\}.$$

It is easy to see that the latter expression for κ coincides with (2.15).

For an A_∞-module $\mathscr{M}$ and $X \in \mathrm{Ob}(\mathscr{A})$ we will typically denote by

$$V_{\mathscr{M}} : H_*\big(\mathscr{M}^{\leq\alpha}(X)\big) \longrightarrow H_*\big(\mathscr{M}^{\leq\alpha+r}(X)\big)$$

the map induced in homology by the chain map

$$v_{\mathscr{M}} : \mathscr{M}^{\leq\alpha}(X) \to \mathscr{M}^{\leq\alpha+r}(X), \quad b \longmapsto \mu_2^{\mathscr{M}}(e_X, b).$$

Here $r \in \mathbb{R}$ is chosen such that $u^{\mathscr{A}} + \epsilon_2^{\mathscr{M}} \leq r$ so that $v_{\mathscr{M}}$ is well defined with the above given target. We will need to consider such maps for different values of r, and whenever a need to distinguish between them arises we will use additional "decorations" such as $V'_{\mathscr{M}}, V''_{\mathscr{M}}$, *etc.*

Fix $\alpha \in \mathbb{R}$, $X \in \mathrm{Ob}(\mathscr{A})$. Since

$$\mathscr{C}^{\leq\alpha}(X) = \mathscr{C}one\big(\mathscr{M}_0^{\leq\alpha-\rho-\epsilon_1^f}(X) \xrightarrow{f_1} \mathscr{M}_1^{\leq\alpha}(X)\big)$$

we have a long exact sequence in homology:

$$\cdots \to H_k\big(\mathscr{M}_1^{\leq\alpha}(X)\big) \xrightarrow{\iota} H_k\big(\mathscr{C}^{\leq\alpha}(X)\big) \xrightarrow{\pi} H_k\big(\mathscr{M}_0^{\leq\alpha-\rho-\epsilon_1^f}(X)\big) \to \cdots,$$

where ι and π are the maps in homology induced by the inclusion $\mathscr{M}_1(X) \to \mathscr{C}(X)$ and the projection $\mathscr{C}(X) \to \mathscr{M}_0(X)$ respectively.

Replacing α by $\alpha+\delta$ and by $\alpha+\kappa$ we obtain two similar long exact sequences. These three sequences are mapped one to the other via maps induced from the inclusions coming from raising the action level from α to $\alpha+\delta$ and then to $\alpha+\kappa$. In particular, the degree-k chunks of these exact sequences gives the following commutative diagram with exact rows:

$$\begin{array}{ccccc} H_k\big(\mathcal{M}_1^{\leq\alpha}(X)\big) & \longrightarrow & H_k\big(\mathcal{C}^{\leq\alpha}(X)\big) & \longrightarrow & H_k\big(\mathcal{M}_0^{\leq\alpha-\rho-\epsilon_1^f}(X)\big) \\ \Big\downarrow{\scriptstyle V'_{\mathcal{M}_1}=i'_{\mathcal{M}_1}} & & {\scriptstyle V'_{\mathcal{C}}}\Big\downarrow\Big\downarrow{\scriptstyle i'_{\mathcal{C}}} & & \Big\downarrow{\scriptstyle V'_{\mathcal{M}_0}=i'_{\mathcal{M}_0}} \\ H_k\big(\mathcal{M}_1^{\leq\alpha+\delta}(X)\big) & \xrightarrow{\ \iota\ } & H_k\big(\mathcal{C}^{\leq\alpha+\delta}(X)\big) & \xrightarrow{\ \pi\ } & H_k\big(\mathcal{M}_0^{\leq\alpha-\rho-\epsilon_1^f+\delta}(X)\big) \\ \Big\downarrow{\scriptstyle V''_{\mathcal{M}_1}=i''_{\mathcal{M}_1}} & & {\scriptstyle V''_{\mathcal{C}}}\Big\downarrow\Big\downarrow{\scriptstyle i''_{\mathcal{C}}} & & \Big\downarrow{\scriptstyle V''_{\mathcal{M}_0}=i''_{\mathcal{M}_0}} \\ H_k\big(\mathcal{M}_1^{\leq\alpha+\kappa}(X)\big) & \longrightarrow & H_k\big(\mathcal{C}^{\leq\alpha+\kappa}(X)\big) & \longrightarrow & H_k\big(\mathcal{M}_0^{\leq\alpha-\rho-\epsilon_1^f+\kappa}(X)\big) \end{array} \tag{2.17}$$

The maps $i'_{\mathcal{C}}$, $i''_{\mathcal{C}}$ are induced by the corresponding inclusions and similarly for $i'_{\mathcal{M}_0}, i''_{\mathcal{M}_0}, i'_{\mathcal{M}_1}, i''_{\mathcal{M}_1}$. By assumption (and by the choices of δ and κ) we have

$$V'_{\mathcal{M}_j} = i'_{\mathcal{M}_j}, \quad V''_{\mathcal{M}_j} = i''_{\mathcal{M}_j}, \quad \text{for} \quad j = 0, 1.$$

Note also that each of the maps $V'_{\mathcal{C}}$ and $i'_{\mathcal{C}}$ makes the above diagram commutative, and similarly for $V''_{\mathcal{C}}$ and $i''_{\mathcal{C}}$. Denote by

$$V'''_{\mathcal{C}} : H_k\big(\mathcal{C}^{\leq\alpha}(X)\big) \longrightarrow H_k\big(\mathcal{C}^{\leq\alpha+\kappa}(X)\big)$$

the map induced in homology by $v_{\mathcal{C}} : \mathcal{C}^{\leq\alpha}(X) \to \mathcal{C}^{\leq\alpha+\kappa}(X)$, and by

$$i'''_{\mathcal{C}} : H_k\big(\mathcal{C}^{\leq\alpha}(X)\big) \longrightarrow H_k\big(\mathcal{C}^{\leq\alpha+\kappa}(X)\big)$$

the map induced by the inclusion. Clearly we have

$$V'''_{\mathcal{C}} = i''_{\mathcal{C}} \circ V'_{\mathcal{C}} = V''_{\mathcal{C}} \circ i'_{\mathcal{C}}.$$

To prove the lemma, we need to show that $V'''_{\mathcal{C}}(x) = i'''_{\mathcal{C}}(x)$ for all $x \in H_k(\mathcal{C}^{\leq\alpha}(X))$. To prove the latter equality, we first note that since both $V'_{\mathcal{C}}$ and $i'_{\mathcal{C}}$ make diagram (2.17) commutative, we have

$$V'_{\mathcal{C}}(x) - i'_{\mathcal{C}}(x) \in \ker\pi = \operatorname{image}\iota.$$

Now write $V'_{\mathcal{C}}(x) - i'_{\mathcal{C}}(x) = \iota(y)$ for some $y \in H_k(\mathcal{M}_1^{\leq\alpha+\delta}(X))$. As both $V''_{\mathcal{C}}$ and $i''_{\mathcal{C}}$ make diagram (2.17) commutative we also have $V''_{\mathcal{C}} \circ \iota(y) = i''_{\mathcal{C}} \circ \iota(y)$. It follows that

$$V''_{\mathcal{C}}\big(V'_{\mathcal{C}}(x) - i'_{\mathcal{C}}(x)\big) = i''_{\mathcal{C}}\big(V'_{\mathcal{C}}(x) - i'_{\mathcal{C}}(x)\big).$$

Applying Proposition 2.10 with $\mathcal{M} = \mathcal{C}$ we obtain

$$V'''_{\mathcal{C}}(x) - V''_{\mathcal{C}} \circ i'_{\mathcal{C}}(x) = i''_{\mathcal{C}} \circ V'_{\mathcal{C}}(x) - i'''_{\mathcal{C}}(x).$$

Since $V'''_{\mathcal{C}} = V''_{\mathcal{C}} \circ i'_{\mathcal{C}} = i''_{\mathcal{C}} \circ V'_{\mathcal{C}}$ the lemma follows. □

2.5. The λ-map

Let $\mathscr{A}$ be an A_∞-category and $\mathscr{M}$ an $\mathscr{A}$-module. Let $Y \in \mathrm{Ob}(\mathscr{A})$ and denote by $\mathscr{Y}$ the Yoneda module corresponding to Y. Consider the map:

$$\lambda : \mathscr{M}(Y) \longrightarrow \hom(\mathscr{Y}, \mathscr{M}), \quad c \longmapsto \lambda(c) = \big(\lambda(c)_1, \lambda(c)_2, \dots, \lambda(c)_d, \dots\big), \tag{2.18}$$

where $\lambda(c)_d(a_1, \dots, a_{d-1}, b) = \mu^{\mathscr{M}}_{d+1}(a_1, \dots, a_{d-1}, b, c)$.

This map was defined by Seidel [Sei08, Section (1l)] in the context of the Yoneda embedding of A_∞-categories. A straightforward calculation shows that it is a chain map. We will refer to it from now on as the λ-*map*.

Seidel [Sei08, Lemma 2.12] proves that, under the additional assumptions that $\mathscr{A}$ and $\mathscr{M}$ are h-unital, the λ-map is a quasi-isomorphism. Our goal is to establish a weakly filtered analogue of this result.

We begin with a technical assumption on a given object $Y \in \mathrm{Ob}(\mathscr{A})$.

Assumption $U^{R,e}$. — Let $\kappa \geq \epsilon^{\mathscr{A}}_2 + u^{\mathscr{A}}$ be a real number (recall that $\epsilon^{\mathscr{A}}_2$ and $u^{\mathscr{A}}$ are the discrepancies associated respectively to the μ_2-operation and units in $\mathscr{A}$, see §2.1.1). We say that Y *satisfies Assumption* $U^{R,e}(\kappa)$ (or $Y \in U^{R,e}(\kappa)$ for short) if for every $X \in \mathrm{Ob}(\mathscr{A})$ the map

$$C(X,Y) \longrightarrow C(X,Y), \quad b \longmapsto \mu_2(b, e_Y)$$

is chain homotopic to the identity via a chain homotopy h_X that shifts action by $\leq \kappa$. The superscript R, e stand for "Right"-multiplication with e_Y.

We now define the right setting for the λ-map in the weakly filtered case. Assume that $\mathscr{A}$ and $\mathscr{M}$ are both weakly filtered with discrepancies $\leq \boldsymbol{\epsilon}^{\mathscr{A}}$ and $\leq \boldsymbol{\epsilon}^{\mathscr{M}}$ respectively. Clearly $\mathscr{Y}$ is also a weakly filtered module with discrepancy $\leq \boldsymbol{\epsilon}^{\mathscr{A}}$.

Without loss of generality we assume from now on that $\boldsymbol{\epsilon}^{\mathscr{M}} \geq \boldsymbol{\epsilon}^{\mathscr{A}}$ so that $\mathscr{Y}$ can be regarded also as a weakly filtered module with discrepancy $\leq \boldsymbol{\epsilon}^{\mathscr{M}}$. (If needed, we can always increase $\boldsymbol{\epsilon}^{\mathscr{M}}$ and $\mathscr{M}$ will continue being weakly filtered with discrepancy less than the increased $\boldsymbol{\epsilon}^{\mathscr{M}}$.)

Let $\boldsymbol{\epsilon}^h$ be any sequence that satisfies Assumption $\mathscr{E}$ and assume in addition that

$$\epsilon^h_d \geq \epsilon^{\mathscr{M}}_{d+1} \quad \text{for all} \quad d. \tag{2.19}$$

Under these assumptions, the λ-map restricts to maps:

$$\lambda^\alpha : \mathscr{M}^{\leq \alpha}(Y) \longrightarrow \hom^{\leq \alpha; \boldsymbol{\epsilon}^h}(\mathscr{Y}, \mathscr{M}), \tag{2.20}$$

defined for all $\alpha \in \mathbb{R}$. Since $\boldsymbol{\epsilon}^h$ satisfies Assumption $\mathscr{E}$, the right-hand side of (2.20) is a chain complex with respect to μ^{mod}_1 and the λ-map from (2.20) is a chain map.

Let $\mathscr{A}$, $\mathscr{M}$, Y and $\mathscr{Y}$ be as at the beginning of Section 2.5. Fix also $\boldsymbol{\epsilon}^{\mathscr{M}}$, $\boldsymbol{\epsilon}^h$ as above. For every $\alpha \in \mathbb{R}$ set

$$\mathscr{H}^{\leq \alpha} := \hom^{\leq \alpha; \boldsymbol{\epsilon}^h}(\mathscr{Y}, \mathscr{M})$$

and for every $k \geq 1$:

$$Q^{\leq \alpha}_{(k)} := \big\{ t \in \mathscr{H}^{\leq \alpha} \, ; \, t_1 = \cdots = t_k = 0 \big\}, \quad \mathscr{H}^{\leq \alpha}_{(k)} := \mathscr{H}^{\leq \alpha} / Q^{\leq \alpha}_{(k)}.$$

As explained above, the λ-map restricts to maps $\lambda^\alpha : \mathcal{M}^{\leq\alpha}(Y) \to \mathcal{H}^{\leq\alpha}$ for every $\alpha \in \mathbb{R}$ and we also have the induced maps:

$$\lambda^\alpha_{(k)} : \mathcal{M}^{\leq\alpha}(Y) \longrightarrow \mathcal{H}^{\leq\alpha}_{(k)}$$

defined by composing λ^α with the quotient map $\pi_{(k)} : \mathcal{H}^{\leq\alpha} \to \mathcal{H}^{\leq\alpha}_{(k)}$.

Proposition 2.11. — *Suppose that $\mathcal{A}$ is h-unital in the weakly filtered sense with discrepancy of units $\leq u^{\mathcal{A}}$. Let $\kappa \in \mathbb{R}$ such that $\kappa \geq u^{\mathcal{A}} + \epsilon_2^{\mathcal{M}}, u^{\mathcal{A}} + \epsilon_2^{\mathcal{A}}$ and assume that $\mathcal{M} \in U_m(\kappa)$ and $Y \in U^{R,e}(\kappa)$. Let $\alpha \in \mathbb{R}$. Fix $1 \leq \ell \in \mathbb{Z}$ and put $\alpha' := \alpha + \ell\kappa$. Consider the commutative diagram in cohomology:*

$$\begin{array}{ccc} H_*\big(\mathcal{M}^{\leq\alpha}(Y)\big) & \xrightarrow{\lambda^\alpha_*} & H_*\big(\mathcal{H}^{\leq\alpha}\big) \\ \downarrow{\scriptstyle i^H} & & \downarrow{\scriptstyle i^H} \\ H_*\big(\mathcal{M}^{\leq\alpha'}(Y)\big) & \xrightarrow{\lambda^{\alpha'}_*} & H_*\big(\mathcal{H}^{\leq\alpha'}\big) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi^H_{(\ell)}} \\ H_*\big(\mathcal{M}^{\leq\alpha'}(Y)\big) & \xrightarrow{\lambda^{\alpha'}_{(\ell)*}} & H_*\big(\mathcal{H}^{\leq\alpha'}_{(\ell)}\big) \end{array} \tag{2.21}$$

where the i^H maps are induced by the inclusions $\mathcal{M}^{\leq\alpha}(Y) \to \mathcal{M}^{\leq\alpha'}(Y)$ and $\mathcal{H}^{\leq\alpha} \to \mathcal{H}^{\leq\alpha'}$ and $\pi^H_{(\ell)}$ is induced by the projection $\pi_{(\ell)} : \mathcal{H}^{\leq\alpha'} \to \mathcal{H}^{\leq\alpha'}_{(\ell)}$. Then for every $b \in H_*(\mathcal{H}^{\leq\alpha})$ there exists $c \in H_*(\mathcal{M}^{\leq\alpha'}(Y))$ such that

$$\pi^H_{(\ell)} \circ i^H(b) = \lambda^{\alpha'}_{(\ell)*}(c).$$

In other words, for every cycle $\beta \in \mathcal{H}^{\leq\alpha}$ there exists a cycle $\gamma \in \mathcal{M}^{\leq\alpha'}(Y)$ such that

$$\beta = \lambda(\gamma) + \mu_1^{\mathrm{mod}}(\theta) + \tau, \tag{2.22}$$

for some $\theta \in \mathcal{H}^{\leq\alpha'}$ and some cycle $\tau = (\tau_1, \tau_2, \dots) \in \mathcal{H}^{\leq\alpha'}$ with $\tau_1 = \cdots = \tau_\ell = 0$.

Proof. — The proof below follows the general scheme of the proof of Lemma 2.12 from of [Sei08], however the weakly filtered setting entails significant adjustments with respect to [Sei08].

Before we go on, two quick remarks on grading. The first is that in this paper we generally work in an ungraded framework. Nevertheless, the proof below works also in the graded case, hence we have written it in this setting. [1] The second remark is that, in order to keep compatibility with the proof of Lemma 2.12 from of [Sei08], we will work in this proof with cohomological grading (although we generally use homological conventions). We will therefore denote by H^i the homology in "cohomological degree" i.

1. The ungraded case can be viewed as a special case of the graded one, by replacing each chain complex in the statement of Proposition 2.11 by a graded one which equals in all degrees to the original chain complex. Note that, in the graded case, none of the chain complexes in the statement of Proposition 2.11 is assumed to be bounded.

We begin with some preparations regarding the weakly filtered version of the λ-map. Let $\rho \in \mathbb{R}$, $d \in \mathbb{N}$. Recall that we have the chain map $\lambda^{\rho} : \mathcal{M}^{\leq \rho}(Y) \to \mathcal{H}^{\leq \rho}_{(d)}$ and consider its mapping cone:

$$\mathcal{K}^{\rho}_{(d)} = \mathcal{C}one\big(\mathcal{M}^{\leq \rho}(Y) \xrightarrow{\lambda^{\rho}} \mathcal{H}^{\leq \rho}_{(d)}\big).$$

Define a decreasing filtration $F^r \mathcal{K}^{\rho}_{(d)}$, $r \in \mathbb{Z}_{\geq 0}$ on this chain complex by setting

$$F^r \mathcal{K}^{\rho}_{(d)} = \begin{cases} \mathcal{K}^{\rho}_{(d)} & \text{if } r = 0, \\ \mathcal{H}^{\leq \rho}_{(d)} & \text{if } r = 1, \\ \big\{ f \in \mathcal{H}^{\leq \rho}_{(d)} \ ; \ f_1 = \cdots = f_{r-1} = 0 \big\} & \text{if } 2 \leq r. \end{cases} \tag{2.23}$$

Note that this is a bounded filtration and we actually have $F^r \mathcal{K}^{\rho}_{(d)} = 0$ for $r \geq d+1$.

Consider now the cohomological spectral sequence $\{E^{p,q}_r(\rho), \partial_r\}_{r \in \mathbb{Z}_{\geq 0}}$ associated to the filtration $F^{\bullet}$. Since the filtration is bounded the spectral sequence converges to $H^*(\mathcal{K}^{\rho}_{(d)})$. Note also that for $\rho \leq \rho'$ we have an obvious inclusion of chain complexes $i : \mathcal{K}^{\rho}_{(d)} \to \mathcal{K}^{\rho'}_{(d)}$. Moreover, this inclusion preserves the filtrations $F^{\bullet}$ on the corresponding chain complexes. Therefore, i induces a map of spectral sequences

$$i^E_r : E^{p,q}_r(\rho) \longrightarrow E^{p,q}_{r+1}(\rho'), \qquad \text{for all } r \geq 0 \text{ and } p, q \ .$$

We now describe more explicitly the first two pages of $E^{p,q}_r(\rho)$. A simple calculation gives the following description of the E_0-page of this spectral sequence. We have $E^{p,\bullet}_0(\rho) = 0$ for $p > d$ and for $p < 0$. Next we have $E^{0,\bullet}_0(\rho) = \mathcal{M}^{\leq \rho}(Y)^{\bullet}$, where the superscript $^{\bullet}$ stands here for the (cohomological) grading of the chain complex $\mathcal{M}^{\leq \rho}(Y)$. The differential $\partial_0 : E^{0,q}_0(\rho) \to E^{0,q+1}_0(\rho)$ is simply $\mu^{\mathcal{M}}_1$.

The rest of the columns, $1 \leq p \leq d$, are

$$E^{p,\bullet}_0(\rho) = \prod \hom_R^{\leq \rho + \epsilon^h_p ; \bullet}\big(C(X_0, X_1) \otimes \cdots \otimes C(X_{p-2}, X_{p-1}) \otimes C(X_{p-1}, Y), \mathcal{M}(X_0)\big), \tag{2.24}$$

where the product is taken for $X_0, \ldots, X_{p-1} \in \mathrm{Ob}(\mathcal{A})$, the superscript '$\bullet$' stands again for (cohomological) grading and $\hom_R^{\leq \rho + \epsilon^h_p}$ stands for R-linear homomorphisms that shift action by not more than $\rho + \epsilon^h_p$. (Recall that $\boldsymbol{\epsilon}^h$ has been fixed at the beginning of Section 2.5 and is used in the definitions of $\mathcal{H}^{\leq \rho}$ and $\mathcal{H}^{\leq \rho}_{(d)}$.) For $1 \leq p \leq d$, the differentials $\partial_0 : E^{p,q}_0(\rho) \to E^{p,q+1}_0(\rho)$ are induced in a standard way from $\mu^{\mathcal{A}}_1$ and $\mu^{\mathcal{M}}_1$.

The E_1-page is consequently the following: $E^{p,\bullet}_1(\rho) = 0$ for all $p > d$ and for $p < 0$. For $p = 0$ we have $E^{0,q}_1(\rho) = H^q(\mathcal{M}^{\leq \rho}(Y))$ for all q. And for $1 \leq p \leq d$ we have

$$E^{p,q}_1(\rho) = \prod_{X_0, \ldots, X_{p-1} \in \mathrm{Ob}(\mathcal{A})} H^q\big(\hom_R^{\leq \rho + \epsilon^h_p}\big(C(X_0, X_1) \otimes \cdots \cdots \otimes C(X_{p-2}, X_{p-1}) \otimes C(X_{p-1}, Y), \mathcal{M}(X_0)\big)\big). \tag{2.25}$$

We now describe the differentials $\partial_1 : E^{p,q}_1 \to E^{p+1,q}_1$ on the E_1-page.

We start with $p = 0$. Let $[c] \in E_1^{0,q} = H^q(\mathcal{M}^{\leq \rho}(Y))$, where c is a cycle. Then

$$\partial_1[c] \in E_1^{1,q} = \prod_{X \in \mathrm{Ob}(\mathcal{A})} H^q\big(\hom_R^{\leq \rho + \epsilon_1^h}\big(C(X,Y), \mathcal{M}(X)\big)\big)$$

is the cycle represented by the homomorphism

$$C(X,Y) \longrightarrow \mathcal{M}(X), \quad b \longmapsto \mu_2^{\mathcal{M}}(b,c).$$

It is easy to check that this homomorphism is a cycle and that it shifts action by not more than $\rho + \epsilon_1^h$. (The latter hold because $\epsilon_1^h \geq \epsilon_2^{\mathcal{M}}$ by (2.19).)

The formula for ∂_1 for $1 \leq p \leq d-1$ is the following. Let f be an element in the RHS of (2.24) which is a cycle. Then

$$\partial_1[f] = [g] \in E_1^{p+1,q},$$

where g is a collection of R-linear homomorphism

$$g : C(X_0,X_1) \otimes \cdots \otimes C(X_{p-1},X_p) \otimes C(X_p,Y) \to \mathcal{M}(X_0),$$

defined for all objects $X_0, \ldots, X_p \in \mathrm{Ob}(\mathcal{A})$ and is given by the formula:

$$\begin{aligned} g(a_1,\ldots,a_p,b) \longmapsto \pm\mu_2^{\mathcal{M}}\big(a_1, f(a_2,\ldots,a_p,b)\big) \pm f\big(a_1,\ldots,a_{p-1},\mu_2^{\mathcal{A}}(a_p,b)\big) \\ + \sum_{n=0}^{p-2} \pm f\big(a_1,\ldots,\mu_2^{\mathcal{A}}(a_{n+1},a_{n+2}),\ldots,b\big). \end{aligned} \tag{2.26}$$

This follows from a direct calculation. See the proof of Lemma 2.12 in [Sei08] for the precise signs in formula (2.26). Note also that g shifts action by

$$\leq \max\big\{\rho + \epsilon_{p-1}^h + \epsilon_2^{\mathcal{M}}, \rho + \epsilon_{p-1}^{\mathcal{A}}\big\} \leq \rho + \epsilon_p^h,$$

where the latter inequality follows from Assumption $\mathscr{E}(\epsilon^m, \epsilon^{\mathcal{A}})$.

Consider now the inclusion $i : \mathcal{K}_{(d)}^{\rho} \to \mathcal{K}_{(d)}^{\rho+\kappa}$. As indicated earlier this induces a map of spectral sequences $i^E : E(\rho) \to E(\rho+\kappa)$, namely

$$i_r^E : E_r^{p,q}(\rho) \longrightarrow E_r^{p,q}(\rho+\kappa), \quad \text{for all } r \geq 0.$$

Claim 2.12. — *For every q the chain map $i_1^E : E_1^{\bullet,q}(\rho) \to E_1^{\bullet,q}(\rho+\kappa)$ is chain homotopic to 0 in the degree range $0 \leq \bullet \leq d-1$. In other words, for every q there exist homomorphisms*

$$S^{p,q} : E_1^{p,q}(\rho) \longrightarrow E_1^{p-1,q}(\rho+\kappa),$$

defined for all p, such that

$$i_1^E|_{E_1^{p,q}(\rho)} = \partial_1 \circ S^{p,q} + S^{p+1,q} \circ \partial_1, \quad \textit{for all } 0 \leq p \leq d-1. \tag{2.27}$$

We postpone the proof of this claim till later in this section and continue now with the proof of Proposition 2.11.

Claim 2.12 implies that $i_2^E : E_2^{p,q}(\rho) \to E_2^{p,q}(\rho+\kappa)$ is the 0 map for every $0 \leq p \leq d-1$ and every q. It follows that the same holds for the maps $i_r^E : E_r^{p,q}(\rho) \to E_r^{p,q}(\rho+\kappa)$ for every $r \geq 2$.

Since both the spectral sequences converge after a finite number of pages (in fact they collapse at page $r = d+1$) we conclude that $i^E_\infty : E^{p,q}_\infty(\rho) \to E^{p,q}_\infty(\rho+\kappa)$ is 0 for all $0 \le p \le d-1$ and all q. Denote by $F^\bullet H^*(\mathcal{K}^\rho_{(d)})$ the filtration on $H^*(\mathcal{K}^\rho_{(d)})$ induced by $F^\bullet \mathcal{K}^\rho_{(d)}$. Since

$$E^{p,q}_\infty(\rho) = F^p H^{p+q}(\mathcal{K}^\rho_{(d)})/F^{p+1}H^{p+q}(\mathcal{K}^\rho_{(d)}),$$

and similarly for $E^{p,q}_\infty(\rho+\kappa)$, we have proved the following auxiliary statement:

Lemma 2.13. — *The inclusion $i : \mathcal{K}^\rho_{(d)} \to \mathcal{K}^{\rho+\kappa}_{(d)}$ induces in homology the map*

$$i^H : H^n(\mathcal{K}^\rho_{(d)}) \longrightarrow H^n(\mathcal{K}^{\rho+\kappa}_{(d)})$$

which sends $F^p H^n(\mathcal{K}^\rho_{(d)})$ to $F^{p+1}H^n(\mathcal{K}^{\rho+\kappa}_{(d)})$ for every n and $0 \le p \le d-1$.

We are now in position to conclude the proof of Proposition 2.11.

Fix α, ℓ and α' as in the statement of the proposition.

Choose $d \gg \ell$ and apply what we have proved above to $\mathcal{K}^\alpha_{(d)}$ (*i.e.* take $\rho = \alpha$).

Lemma 2.13, applied with $p = 0$, implies that i^H maps $H^n(\mathcal{K}^\alpha_{(d)})$ to $F^1 H^n(\mathcal{K}^{\alpha+\kappa}_{(d)})$ for all n.

Apply Lemma 2.13 this time with $p = 1$, $\rho = \alpha+\kappa$ and $\mathcal{K}^{\alpha+\kappa}_{(d)} \to \mathcal{K}^{\alpha+2\kappa}_{(d)}$. Together with the previous conclusion we infer that[2] i^H maps $H^n(\mathcal{K}^\alpha_{(d)})$ to $F^2 H^n(\mathcal{K}^{\alpha+2\kappa}_{(d)})$ for all n.

Applying the same argument over and over again, ℓ times, we conclude that the map $i^H : H^n(\mathcal{K}^\alpha_{(d)}) \to H^n(\mathcal{K}^{\alpha+\ell\kappa}_{(d)})$ induced by the inclusion $\mathcal{K}^\alpha_{(d)} \to \mathcal{K}^{\alpha+\ell\kappa}_{(d)}$ maps $H^n(\mathcal{K}^\alpha_{(d)})$ to $F^\ell H^n(\mathcal{K}^{\alpha+\ell\kappa}_{(d)})$.

Let now $\beta \in \mathcal{H}^{\le\alpha}$ be a cycle and denote by $\bar\beta$ its image in $\mathcal{H}^{\le\alpha'}_{(d)}$, where $\alpha' = \alpha + \ell\kappa$. Consider the cycle $(0,\bar\beta) \in \mathcal{K}^{\alpha'}_{(d)}$. By what we have proved before we know that $[(0,\bar\beta)]$ belongs to $F^\ell H^*(\mathcal{K}^{\alpha'}_{(d)})$. It follows that there exists $\tau' \in \hom^{\le\alpha';\epsilon^h}(\mathcal{Y},\mathcal{M})$ such that

$$\tau'_1 = \cdots = \tau'_\ell = 0 \quad \text{and} \quad \big[(0,\bar\beta)\big] = \big[(0,\tau')\big] \quad \text{in} \quad H^*(\mathcal{K}^{\alpha'}_{(d)}).$$

Therefore, there exist $\gamma \in \mathcal{M}^{\le\alpha'}(Y)$ and $\theta \in \hom^{\le\alpha';\epsilon^h}(\mathcal{Y},\mathcal{M})$ such that

$$(0,\bar\beta) = (0,\tau') + \big(\mu_1^{\mathcal{M}}(\gamma), \lambda^{\alpha'}_{(d)}(\gamma) + \mu_1^{\hom}(\theta)\big)$$

in $\mathcal{K}^{\alpha'}_{(d)}$. In order to lift the last equation from $\mathcal{K}^{\alpha'}_{(d)}$ to

$$\mathcal{C}one\big(\mathcal{M}^{\le\alpha'}(Y) \xrightarrow{\lambda^{\alpha'}} \mathcal{H}^{\le\alpha'}\big)$$

we can correct if necessary the terms beyond order d by replacing τ' with a suitable τ that coincides with τ' up to order d (recall that $d \gg \ell$).

2. We have denoted here by the same symbol, i^H, the maps induced in homology by the different inclusions: $\mathcal{K}^\rho_{(d)} \to \mathcal{K}^{\rho+\kappa}_{(d)}$, $\mathcal{K}^{\rho+\kappa}_{(d)} \to \mathcal{K}^{\rho+2\kappa}_{(d)}$ and $\mathcal{K}^\rho_{(d)} \to \mathcal{K}^{\rho+2\kappa}_{(d)}$. Below we will continue with this notation.

Summing up, we have proved that there exists a cycle $\gamma \in \mathcal{M}^{\leq \alpha'}(Y)$, and a pre-module homomorphism $\theta \in \hom^{\leq \alpha';\epsilon^h}(\mathcal{Y}, \mathcal{M})$ such that

$$\beta = \lambda(\gamma) + \mu_1^{\mathrm{mod}}(\theta) + \tau,$$

where $\tau \in \hom^{\leq \alpha';\epsilon^h}(\mathcal{Y}, \mathcal{M})$ is a cycle with $\tau_1 = \cdots = \tau_\ell = 0$, as claimed by the proposition [3].

This concludes the proof of Proposition 2.11, modulo the proof of Claim 2.12. □

Proof of Claim 2.12. — Fix q. We define the chain homotopy $S^{\bullet,q}$ as follows. Define $S^{0,q} = 0$ (note that $E_1^{-1,q}(\rho) = 0$). Next, to define $S^{1,q}$, let

$$f \in E_0^{1,q}(\rho) = \prod_{X \in \mathrm{Ob}(\mathcal{A})} \hom_R^{\leq \rho + \epsilon_1^h}\big(C(X, Y), \mathcal{M}(X)\big)$$

be a ∂_0-cycle. We define

$$S^{1,q}[f] := \big[f(e_Y)\big] \in E_1^{0,q}(\rho + \kappa) = H^q\big(\mathcal{M}^{\leq \rho + \epsilon_1^h + \kappa}(X)\big).$$

Since $\kappa \geq u^{\mathcal{A}}$, $f(e_Y)$ indeed belongs to $E_0^{0,q}(\rho + \kappa)$. Moreover, a straightforward calculation shows that $f(e_Y)$ is a ∂_0-cycle and that its homology class $[f(e_Y)]$ depends only on the homology class $[f] \in E_1^{1,q}(\rho)$.

For the range of degrees $2 \leq p \leq d$ we define $S^{p,q}$ by a similar formula: let $f \in E_0^{p,q}(\rho)$, *i.e.* a collection of R-linear homomorphism as in (2.24). Assume that f is a ∂_0-cycle. Define $S^{p,q}[f]$ to be the homology class $[g] \in E_1^{p-1,q}(\rho + \kappa)$ of the element $g \in E_0^{p-1,q}(\rho + \kappa)$ given by

$$g(a_1, \dots, a_{p-2}, b) = f(a_1, \dots, a_{p-2}, b, e_Y),$$

for all $a_i \in C(X_{i-1}, X_i)$, $i = 1, \dots, p-2$, and $b \in C(X_{p-2}, Y)$. Since $\kappa \geq u^{\mathcal{A}}$, we have $g \in E_0^{p-1,q}(\rho + \kappa)$. A straightforward calculation shows that g is a ∂_0-cycle and moreover its homology class, $[g] \in E_1^{p-1,q}(\rho + \kappa)$ depends only on the homology class $[f]$ of f. This concludes the definition [4] of the maps $S^{p,q}$.

We verify now the identity (2.27). We begin with $2 \leq p \leq d-1$. Let $f \in E_0^{p,q}(\rho)$ be a cycle. A straightforward calculation shows that

$$(\partial_1 S^{p,q} + S^{p+1,q}\partial_1)[f] = [\widetilde{f}],$$

where $\widetilde{f}(a_1, \dots, a_{p-1}, b) = f\big(a_1, \dots, a_{p-1}, \mu_2^{\mathcal{A}}(b, e_Y)\big)$. We claim that

$$[\widetilde{f}] = [f] \text{ in } E_1^{p,q}(\rho + \kappa).$$

Indeed, since Y belongs to $U^{R,e}(\kappa)$ there exists a chain homotopy $h_{X_{p-1}} : C(X_{p-1}, Y) \longrightarrow C(X_{p-1}, Y)$ that shifts action by $\leq \kappa$ such that

$$\mu_2^{\mathcal{A}}(b, e_Y) = b + h_{X_{p-1}}\mu_1^{\mathcal{A}}(b) + \mu_1^{\mathcal{A}} h_{X_{p-1}}(b), \quad \text{for all } b \in C(X_{p-1}, Y).$$

3. The statement concerning diagram (2.21) is a rephrasing of what we have just proved.
4. For p's outside of the range $0, \dots, d$ we can define $S^{p,q}$ in an arbitrary way.

Define $\psi \in E_0^{p,q-1}(\rho + \kappa)$ by

$$\psi(a_1, \dots, a_{p-1}, b) := f\big(a_1, \dots, a_{p-1}, h_{X_{p-1}}(b)\big).$$

A straightforward calculation shows that $\widetilde{f} - f = \partial_0 \psi$, hence $[\,\widetilde{f}\,] = [\,f\,]$ in $E_1^{p,q}(\rho+\kappa)$. This proves (2.27) for $2 \leq p \leq d$.

A similar argument shows that (2.27) holds also for $p = 1$.

It remains to verify (2.27) the case $p = 0$. Let $m \in \mathcal{M}^{\leq \rho}(Y)$ be a cycle. We have

$$(\partial_1 S^{0,q} + S^{1,q}\partial_1)[m] = S^{1,q}\partial_1[m] = \big(\partial_1[m]\big)(e_Y) = \big[\mu_2^{\mathcal{M}}(e_Y, m)\big].$$

By assumption $\mathcal{M} \in U_m(\kappa)$, hence $[\mu_2^{\mathcal{M}}(e_Y, m)] = [m]$ in $H^q(\mathcal{M}^{\leq \rho+\kappa}(Y))$.

This proves (2.27) for $p = 0$ and concludes the proof of Claim 2.12. □

2.6. Structure theorem for weakly filtered iterated cones

Let $\mathcal{A}$ be an h-unital weakly filtered A_∞-category with discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{A}}$ and discrepancy of units $u^{\mathcal{A}}$. Let $L_0, \dots, L_r \in \mathrm{Ob}(\mathcal{A})$ and for every i denote by $\mathcal{L}_i$ the Yoneda module associated to L_i, viewed as a weakly filtered module. In this section we analyze iterated cones in the weakly filtered framework.

By *iterated cones* we mean modules of the type

$$\mathcal{C}one(\mathcal{L}_r \xrightarrow{\phi_r} \mathcal{C}one\big(\mathcal{L}_{r-1} \xrightarrow{\phi_{r-1}} \mathcal{C}one\big(\cdots \mathcal{C}one\big(\mathcal{L}_2 \xrightarrow{\phi_2} \mathcal{C}one\big(\mathcal{L}_1 \xrightarrow{\phi_1} \mathcal{L}_0\big)\big)\big)\cdots\big)\big). \tag{2.28}$$

The weakly filtered structure is defined by iterating the construction from Section 2.4. More precisely, we define a sequence of weakly filtered $\mathcal{A}$-modules $\mathcal{K}_0, \dots, \mathcal{K}_r$ as follows. We start by setting $\mathcal{K}_0 := \mathcal{L}_0$ which is a weakly filtered module with discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{K}_0} := \boldsymbol{\epsilon}^{\mathcal{A}}$. Note that all the modules $\mathcal{L}_i$ have discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{A}}$ too. Suppose that $\phi_1 \in \hom^{\leq \rho_1; \boldsymbol{\delta}^{\phi_1}}(\mathcal{L}_1, \mathcal{K}_0)$ is a module homomorphism, where $\rho_1 \in \mathbb{R}$ and $\boldsymbol{\delta}^{\phi_1}$ is some sequence. We *do not* assume that $\boldsymbol{\delta}^{\phi_1}$ satisfies anything like Assumption $\mathcal{E}$. We define

$$\mathcal{K}_1 = \mathcal{C}one\big(\mathcal{L}_1 \xrightarrow{(\phi_1; \rho_1, \boldsymbol{\delta}^{\phi_1})} \mathcal{K}_0\big).$$

Since $\boldsymbol{\epsilon}^{\mathcal{K}_0} = \boldsymbol{\epsilon}^{\mathcal{A}}$, the discrepancy of $\mathcal{K}_1$ is $\leq \boldsymbol{\epsilon}^{\mathcal{K}_1} := \max\{\boldsymbol{\epsilon}^{\mathcal{A}}, \boldsymbol{\delta}^{\phi_1} - \delta_1^{\phi_1}\}$.

Let $i \geq 1$ and suppose that we have already defined the weakly filtered modules $\mathcal{K}_0, \dots, \mathcal{K}_i$. Let $\phi_{i+1} : \mathcal{L}_{i+1} \longrightarrow \mathcal{K}_i$ be a module homomorphism that shifts action by $\leq \rho_{i+1}$ and has discrepancy $\leq \boldsymbol{\delta}^{\phi_{i+1}}$. Again, we *do not* assume that $\boldsymbol{\delta}^{\phi_{i+1}}$ satisfies any assumption of the type $\mathcal{E}$. We define

$$\mathcal{K}_{i+1} = \mathcal{C}one\big(\mathcal{L}_{i+1} \xrightarrow{(\phi_{i+1}; \rho_{i+1}, \boldsymbol{\delta}^{\phi_{i+1}})} \mathcal{K}_i\big).$$

The $\mathcal{A}$-module $\mathcal{K}_{i+1}$ has discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{K}_{i+1}} := \max\{\boldsymbol{\epsilon}^{\mathcal{K}_i}, \boldsymbol{\delta}^{\phi_{i+1}} - \delta_1^{\phi_{i+1}}\}$ because (by induction) $\boldsymbol{\epsilon}^{\mathcal{K}_i} \geq \boldsymbol{\epsilon}^{\mathcal{A}}$. The final $\mathcal{A}$-module $\mathcal{K}_r$ is precisely the one described by (2.28) and moreover now it also has the structure of a weakly filtered module.

The following expressions will be used frequently in what follows:

$$\chi_{m,d} := \sum_{j=1}^{m}\sum_{i=1}^{d+m} \delta_i^{\phi_j} + \sum_{i=1}^{d+m} \epsilon_i^{\mathcal{A}}, \quad \xi_q := \kappa + \sum_{i=1}^{q+3} \epsilon_i^{\mathcal{A}} + \sum_{j=1}^{q}\sum_{i=1}^{q+2} \delta_i^{\phi_j}. \tag{2.29}$$

Theorem 2.14. — *Let $\mathcal{K}_i$, $0 \leq i \leq r$ be as above. Assume that $\mathcal{A}$ is h-unital in the weakly filtered sense with discrepancy of units $\leq u^{\mathcal{A}}$. Let $\kappa \geq 2u^{\mathcal{A}} + \epsilon_2^{\mathcal{A}}$ be a real number and assume that $\mathcal{A}$ and the objects L_i satisfy the following two conditions:*

- *$\mathcal{A} \in U^e(\kappa)$.*
- *For every $0 \leq i \leq r$, $L_i \in U^{R,e}(\kappa)$, and $L_i \in U_m(\kappa)$.*

Then there exists a weakly filtered $\mathcal{A}$-module $\mathcal{M}$ with the following properties:

(i) *For every $X \in \mathrm{Ob}(\mathcal{A})$, we have $\mathcal{M}(X) = \mathcal{K}_r(X)$ as R-modules, namely the R-module $\mathcal{M}(X)$ is a direct sum:*

$$\mathcal{M}(X) = C(X, L_0) \oplus C(X, L_1) \oplus \cdots \oplus C(X, L_r). \tag{2.30}$$

(ii) *Denote by $\mu_1^{\mathcal{M}}$ the differential of the chain complex $\mathcal{M}(X)$. Then the matrix of $\mu_1^{\mathcal{M}}$ with respect to the splitting* (2.30) *has the following shape:*

$$\mu_1^{\mathcal{M}} = (a_{ij})_{0\leq i,j\leq r} \ \text{ with } \ a_{i,j} : C(X, L_j) \longrightarrow C(X, L_i), \text{ where}$$

α) $a_{i,j} = 0$ for every $i > j$. In other words, the matrix of $\mu_1^{\mathcal{M}}$ is upper triangular.

β) $a_{i,i} = \mu_1^{\mathcal{A}} : C(X, L_i) \to C(X, L_i)$.

γ) There exist elements $c_{q,p} \in C(L_q, L_p)$ for all $0 \leq p < q \leq r$, such that for every $i < j$ the (i, j)-th entry of the matrix of $\mu_1^{\mathcal{M}}$ is given by

$$a_{i,j}(\bullet) = \sum_{2\leq d,\, \underline{k}} \mu_d^{\mathcal{A}}(\bullet, c_{k_d,k_{d-1}}, \ldots, c_{k_2,k_1}), \tag{2.31}$$

where $\underline{k} = (k_1, \ldots, k_d)$ runs over all partitions $i = k_1 < k_2 < \cdots < k_{d-1} < k_d = j$ (the sum in (2.31) *is finite because $d \leq j - i \leq r$).*

δ) $c_{q,p} \in C^{\leq \alpha_{q,p}}(L_q, L_p)$, where

$$\alpha_{q,p} = \rho_q - \rho_p + B_q \xi_q, \tag{2.32}$$

where B_q is a universal constant in the sense that it depends only on q, but not on $\mathcal{A}$, the modules $\mathcal{K}_i$ or their discrepancy data. (In (ii.δ) *and in what follows we use the convention that $\rho_0 = 0$.)*

(iii) *There exists a quasi-isomorphism of $\mathcal{A}$-modules $\sigma : \mathcal{K}_r \longrightarrow \mathcal{M}$ which shifts action by $\leq \rho^\sigma$ and has discrepancy $\leq \boldsymbol{\epsilon}^\sigma$. The latter quantities admit the estimates*

$$\rho^\sigma \leq C_r \xi_r, \quad \epsilon_d^\sigma \leq D_{r,d} \chi_{r,d}, \tag{2.33}$$

where the constants C_r and $\{D_{r,d}\}_{d\in\mathbb{N}}$ are universal in the sense mentioned at point (ii.δ) *above.*

(iv) *The first order part* $\sigma_1 : \mathcal{K}_r(X) \to \mathcal{M}(X)$ *of the quasi-isomorphism* σ *is an isomorphism of chain complexes for all* $X \in \mathrm{Ob}(\mathcal{A})$*, and the matrix corresponding to* σ_1 *with respect to the splitting* (2.30) (*taken both for* $\mathcal{K}_r(X)$ *and* $\mathcal{M}(X)$) *is upper triangular with* id*-maps along its diagonal.*

(v) *The inverse* $\sigma_1^{-1} : \mathcal{M}(X) \to \mathcal{K}_r(X)$ *of* σ_1 *is action preserving* (*i.e. it is filtered and shifts action by* ≤ 0).

(vi) *For every* $0 \leq j \leq r$ *the diagonal element*

$$\Delta_j = \mathrm{pr}_{C(X,L_j)} \circ \sigma_1|_{C(X,L_j)} : C(X, L_j) \longrightarrow C(X, L_j)$$

is the identity map (*as follows from point* (2.14) *above*)*. However, when the domain inherits filtration from* $\mathcal{K}_r(X)$ *and the target from* $\mathcal{M}(X)$ *this map shifts action by* $\leq \rho^\sigma$*.* (*Note that for* $j \geq 1$*,* $C(X, L_j)$ *is in general not a subcomplex of either* $\mathcal{K}_r(X)$ *or of* $\mathcal{M}(X)$)*. For* $j = 0$*,* $C(X, L_0)$ *is a subcomplex of both* $\mathcal{K}_r(X)$ *and of* $\mathcal{M}(X)$ *and the two inherited filtrations on* $C(X, L_0)$ *coincide, hence* $\Delta_0 = \mathrm{id}$ *preserves filtration* (*i.e. shifts action by* ≤ 0).

Proof of Theorem 2.14. — We will construct inductively a sequence of weakly filtered modules $\mathcal{M}_i$, $i = 1, \ldots, r$ such that $\mathcal{M}_i$ is quasi-isomorphic to $\mathcal{K}_i$ and whose differential $\mu_1^{\mathcal{M}_i}$ has a matrix of the type describe by (2.31). The desired module $\mathcal{M}$ will then be $\mathcal{M}_r$. In the course of the construction we will successively apply Proposition 2.11, Lemma 2.6 and Lemma 2.5.

Fix once and for all $\ell := r + 2$.

We begin the construction with $i = 1$. Put $\mathcal{M}_0 = \mathcal{K}_0 = \mathcal{L}_0$, $\mathcal{K}_1' = \mathcal{K}_1$. Set also $\kappa_0 = \kappa$, so that $\mathcal{L}_1, \mathcal{K}_0 \in U_m(\kappa_0)$. Define an auxiliary weakly filtered module

$$\mathcal{K}_1'' := \mathcal{C}one(\mathcal{L}_1 \xrightarrow{(\phi_1;\rho_1+\ell\kappa_0,\boldsymbol{\epsilon}^{(1)})} \mathcal{K}_0),$$

where $\boldsymbol{\epsilon}^{(1)}$ is chosen such that

$$\boldsymbol{\epsilon}^{(1)} \geq \boldsymbol{\delta}^{\phi_1}, \quad \boldsymbol{\epsilon}^{(1)} \in \mathcal{E}(\boldsymbol{\epsilon}^{\mathcal{A}}, \boldsymbol{\epsilon}^{\mathcal{K}_0}), \quad \epsilon_d^{(1)} \geq \epsilon_{d+1}^{\mathcal{K}_0} \quad \text{for all } d.$$

By Proposition 2.11 there exists a cycle $c_1 \in \mathcal{K}_0^{\leq \rho_1 + \ell\kappa_0}(L_1) = C^{\leq \rho_1 + \ell\kappa_0}(L_1, L_0)$ as well as $\theta_1, \tau_1 \in \hom^{\leq \rho_1 + \ell\kappa_0; \boldsymbol{\epsilon}^{(1)}}(\mathcal{L}_1, \mathcal{K}_0)$ with τ_1 a cycle and $(\tau_1)_1 = \cdots = (\tau_1)_\ell = 0$, such that

$$\phi_1 = \lambda(c_1) + \mu_1^{\mathrm{mod}}(\theta) + \tau_1$$

in $\hom^{\leq \rho_1 + \ell\kappa_0; \boldsymbol{\epsilon}^{(1)}}(\mathcal{L}_1, \mathcal{K}_0)$. Define now

$$\mathcal{M}_1 := \mathcal{C}one(\mathcal{L}_1 \xrightarrow{(\phi_1 - \mu_1^{\mathrm{mod}}(\theta_1);\rho_1+\ell\kappa_0,\boldsymbol{\epsilon}^{(1)})} \mathcal{K}_0).$$

Note that $\boldsymbol{\epsilon}^{\mathcal{M}_1} = \max\{\boldsymbol{\epsilon}^{\mathcal{A}}, \boldsymbol{\epsilon}^{\mathcal{K}_0}, \boldsymbol{\epsilon}^{(1)} - \epsilon_1^{(1)}\} = \boldsymbol{\epsilon}^{(1)} - \epsilon_1^{(1)}$ because $\boldsymbol{\epsilon}^{(1)} \in \mathcal{E}(\boldsymbol{\epsilon}^{\mathcal{A}}, \boldsymbol{\epsilon}^{\mathcal{K}_0})$.

For later use we will need to address Assumption U_m for the module $\mathcal{M}_1$. Indeed, by Lemma 2.9 we have $\mathcal{M}_1 \in U_m(\kappa_1)$, where

$$\kappa_1 := \max\left\{2\kappa_0, 2u^{\mathcal{A}} + \epsilon_3^{(1)} - \epsilon_1^{(1)}, 2u^{\mathcal{A}} + 2\epsilon_2^{(1)} - 2\epsilon_1^{(1)}, \kappa_0 + \epsilon_2^{(1)} - \epsilon_1^{(1)}\right\}.$$

The modules $\mathcal{K}_1' = \mathcal{K}_1$, $\mathcal{K}_1''$ and $\mathcal{M}_1$ are related by weakly filtered quasi-isomorphisms as follows. The identity homomorphism can be viewed as a weakly filtered quasi-isomorphism $I_1 : \mathcal{K}_1 \to \mathcal{K}_1''$ which shifts action by $\leq \ell\kappa_0$ and has discrepancy $\leq (\epsilon_1^{(1)} - \delta_1^{\phi_1}, 0, \dots, 0, \dots)$. Lemma 2.6 provides a quasi-isomorphism

$$\vartheta_1 : \mathcal{K}_1'' \longrightarrow \mathcal{M}_1$$

which shifts action by ≤ 0 and has discrepancy $\leq \boldsymbol{\epsilon}^{(1)} - \epsilon_1^{(1)}$. Consider the quasi-isomorphism $\eta_1 : \mathcal{K}_1 \to \mathcal{M}_1$ given by the composition $\eta_1 := \vartheta_1 \circ I_1$ which shifts action by $\leq \ell\kappa_0$ and has discrepancy $\leq \boldsymbol{\epsilon}^{(1)} - \delta_1^{\phi_1}$.

The first order part $(\eta_1)_1 : \mathcal{K}_1(X) \to \mathcal{M}_1(X)$ of the module homomorphism η_1 is an isomorphism of chain complexes for all X and its matrix (with respect to the splitting $C(X, L_0) \oplus C(X, L_1)$ of $\mathcal{K}_1(X)$ and $\mathcal{M}_1(X)$ as R-modules) is upper triangular with id's along the diagonal. This follows from the explicit formula of $(\vartheta_1)_1$ from the proof of Lemma 2.6.

The same formula also shows that $(\vartheta_1)_1^{-1}$ shift action by ≤ 0 and the same holds for $(I_1)_1^{-1}$. It follows that the inverse $(\eta_1)_1^{-1}$ of $(\eta_1)_1$ shifts action by ≤ 0.

Next, consider the composition $\eta_1 \circ \phi_2 : \mathcal{L}_2 \to \mathcal{M}_1$. This is a module homomorphism that shifts action by $\leq \rho_2 + \ell\kappa_0$ and has discrepancy $\leq \boldsymbol{\epsilon}^{\eta_1 \circ \phi_2} = \boldsymbol{\epsilon}^{\eta_1} * \boldsymbol{\delta}^{\phi_2}$. Define now

$$\mathcal{K}_2' = \mathcal{C}one\big(\mathcal{L}_2 \xrightarrow{(\eta_1 \circ \phi_2; \rho_2 + \ell\kappa_0, \boldsymbol{\epsilon}^{\eta_1} * \boldsymbol{\delta}^{\phi_2})} \mathcal{M}_1\big),$$
$$\mathcal{K}_2'' = \mathcal{C}one\big(\mathcal{L}_2 \xrightarrow{(\eta_1 \circ \phi_2; \rho_2 + \ell\kappa_1 + \ell\kappa_0, \boldsymbol{\epsilon}^{(2)})} \mathcal{M}_1\big),$$

where $\boldsymbol{\epsilon}^{(2)}$ is chosen such that

$$\boldsymbol{\epsilon}^{(2)} \geq \boldsymbol{\epsilon}^{\eta_1} * \boldsymbol{\delta}^{\phi_2}, \quad \boldsymbol{\epsilon}^{(2)} \in \mathcal{E}(\boldsymbol{\epsilon}^{\mathcal{A}}, \boldsymbol{\epsilon}^{\mathcal{M}_1}), \quad \epsilon_d^{(2)} \geq \epsilon_{d+1}^{\mathcal{M}_1} \quad \text{for all } d.$$

Applying Proposition 2.11 we can write:

$$\eta_1 \circ \phi_2 = \lambda(c_2) + \mu_1^{\mathrm{mod}}(\theta_2) + \tau_2,$$

where $c_2 \in \mathcal{M}_1^{\leq \rho_2 + \ell\kappa_1 + \ell\kappa_0}(L_2)$ is a cycle and $\theta_2, \tau_2 \in \hom^{\leq \rho_2 + \ell\kappa_1 + \ell\kappa_0; \boldsymbol{\epsilon}^{(2)}}(\mathcal{L}_2, \mathcal{M}_1)$ with τ_2 being a cycle such that $(\tau_2)_1 = \cdots = (\tau_2)_\ell = 0$.

We define now

$$\mathcal{M}_2 := \mathcal{C}one\big(\mathcal{L}_2 \xrightarrow{(\eta_1 \circ \phi_2 - \mu_1^{\mathrm{mod}}(\theta_2); \rho_2 + \ell\kappa_1 + \ell\kappa_0, \boldsymbol{\epsilon}^{(2)})} \mathcal{M}_1\big).$$

The discrepancy of $\mathcal{M}_2$ is $\leq \boldsymbol{\epsilon}^{\mathcal{M}_2} := \max\{\boldsymbol{\epsilon}^{\mathcal{A}}, \boldsymbol{\epsilon}^{\mathcal{M}_1}, \boldsymbol{\epsilon}^{(2)} - \epsilon_1^{(2)}\} = \max\{\boldsymbol{\epsilon}^{(1)} - \epsilon_1^{(1)}, \boldsymbol{\epsilon}^{(2)} - \epsilon_1^{(2)}\}$. By Lemma 2.9 we have $\mathcal{M}_2 \in U_m(\kappa_2)$, where

$$\kappa_2 := \max\big\{2\kappa_1, 2u^{\mathcal{A}} + \epsilon_3^{(1)} - \epsilon_1^{(1)}, 2u^{\mathcal{A}} + \epsilon_3^{(2)} - \epsilon_1^{(2)}, 2u^{\mathcal{A}} + 2\epsilon_2^{(1)} - 2\epsilon_1^{(1)},$$
$$2u^{\mathcal{A}} + 2\epsilon_2^{(2)} - 2\epsilon_1^{(2)}, \kappa_0 + \epsilon_2^{(1)} - \epsilon_1^{(1)}, \kappa_0 + \epsilon_2^{(2)} - \epsilon_1^{(2)}\big\}.$$

The modules $\mathcal{K}_2, \mathcal{K}_2', \mathcal{K}_2''$ and $\mathcal{M}_2$ are related by weakly filtered quasi-isomorphisms

$$\mathcal{K}_2 \xrightarrow[\simeq]{\psi_2} \mathcal{K}_2' \xrightarrow[\simeq]{I_2} \mathcal{K}_2'' \xrightarrow[\simeq]{\vartheta_2} \mathcal{M}_2,$$

where the shifts in action and discrepancies of these maps are given by

$$\begin{aligned}
&\mathrm{shift}(\psi_2) \leq \ell\kappa_0, \quad \boldsymbol{\epsilon}^{\psi_2} \leq \boldsymbol{\epsilon}^{(1)} - \delta_1^{\phi_1},\\
&\mathrm{shift}(I_2) \leq \ell\kappa_1, \quad \boldsymbol{\epsilon}^{I_2} \leq (\epsilon_1^{(2)} - \delta_1^{\phi_2} - \epsilon_1^{(1)} + \delta_1^{\phi_1}, 0, \ldots, 0, \ldots),\\
&\mathrm{shift}(\vartheta_2) \leq 0, \quad \boldsymbol{\epsilon}^{\vartheta_2} \leq \boldsymbol{\epsilon}^{(2)} - \epsilon_1^{(2)}.
\end{aligned}$$

The quasi-isomorphism ψ_2 is obtained from Lemma 2.5 and ϑ_2 from Lemma 2.6. The quasi-isomorphism I_2 is basically the identity map, relating the same module with two (slightly) different structures of weakly filtered module.

Consider now the composition $\eta_2 = \vartheta_2 \circ I_2 \circ \psi_2 : \mathcal{K}_2 \longrightarrow \mathcal{M}_2$. This quasi-isomorphism has the following action shift and discrepancy:

$$\mathrm{shift}(\eta_2) \leq \ell(\kappa_1 + \kappa_0), \quad \boldsymbol{\epsilon}^{\eta_2} \leq \boldsymbol{\epsilon}^{(2)} * \boldsymbol{\epsilon}^{(1)} - (\delta_1^{\phi_2} + \epsilon_1^{(1)}).$$

As in the previous step, the first order part $(\eta_2)_1 : \mathcal{K}_2(X) \longrightarrow \mathcal{M}_2(X)$ of η_2 is an isomorphism of chain complexes and its matrix (with respect to the splitting $C(X, L_0) \oplus C(X, L_1) \oplus C(X, L_2)$ of $\mathcal{K}_2(X)$ and $\mathcal{M}_2(X)$ as R-modules) is upper triangular with id's along the diagonal. Moreover, the inverse $(\eta_2)_1^{-1}$ of $(\eta_2)_1$ shifts action by ≤ 0.

These assertions easily follows from the explicit formulas of $(\psi_2)_1$ and $(\vartheta_2)_1$ given in the proofs of Lemmas 2.5 and 2.6 respectively and the fact, already shown in the previous step, that $(\eta_1)_1$ is a chain isomorphism represented by an upper triangular matrix with id's along the diagonal. Recall also from the previous step that $(\eta_1)_1^{-1}$ shifts action by ≤ 0. An examination of the action shifts shows that each of the maps $(I_2)_1^{-1}$, $(\psi_2)_1^{-1}$ and $(\vartheta_2)_1^{-1}$ shifts action by ≤ 0, hence the same holds for $(\eta_2)_1^{-1}$.

Continuing as above by induction we obtain, for every $1 \leq j \leq r$:

1) A weakly filtered module $\mathcal{M}_j$.
2) Two sequences of non-negative real numbers $\boldsymbol{\epsilon}^{(j)}$ and $\boldsymbol{\epsilon}^{\eta_j}$ that satisfy:
   (a) $\boldsymbol{\epsilon}^{(j)} \geq \boldsymbol{\epsilon}^{\eta_{j-1}} * \boldsymbol{\delta}^{\phi_j}, \quad \boldsymbol{\epsilon}^{(j)} \in \mathcal{E}(\boldsymbol{\epsilon}^{\mathcal{A}}, \boldsymbol{\epsilon}^{\mathcal{M}_{j-1}}), \quad \epsilon_d^{(j)} \geq \epsilon_{d+1}^{\mathcal{M}_{j-1}}$ for all d.
   (b) $\boldsymbol{\epsilon}^{\eta_j} \leq \boldsymbol{\epsilon}^{(j)} * \cdots * \boldsymbol{\epsilon}^{(1)} - (\delta_1^{\phi_j} + \sum_{i=1}^{j-1} \epsilon_1^{(i)})$.
   We use the convention that $\boldsymbol{\epsilon}^{\eta_0} = (0, \ldots, 0, \ldots)$.
3) A positive real number κ_j defined (inductively) by

$$\kappa_j := \max\left\{2\kappa_{j-1}, 2u^{\mathcal{A}} + \epsilon_3^{(i)} - \epsilon_1^{(i)}, 2u^{\mathcal{A}} + 2\epsilon_2^{(i)} - 2\epsilon_1^{(i)}, \kappa_0 + \epsilon_2^{(i)} - \epsilon_1^{(i)} \; ; \; 1 \leq i \leq j\right\}.$$

   (Recall that $\kappa_0 = \kappa$.)
4) A cycle $c_j \in \mathcal{M}_{j-1}^{\leq \rho_j + \sum_{i=0}^{j-1} \kappa_i}(L_i)$.
5) The module $\mathcal{M}_j$ is related to $\mathcal{M}_{j-1}$ by

$$\mathcal{M}_j = \mathcal{C}one\big(\mathcal{L}_j \xrightarrow{(\lambda(c_j)+\tau_j; \rho_j + \ell \sum_{i=0}^{j-1} \kappa_i, \boldsymbol{\epsilon}^{(j)})} \mathcal{M}_{j-1}\big), \tag{2.34}$$

   where $\tau_j \in \hom^{\leq \rho_j + \ell(\sum_{i=0}^{j-1} \kappa_i); \boldsymbol{\epsilon}^{(j)}}(\mathcal{L}_j, \mathcal{M}_{j-1})$ is a cycle with $(\tau_j)_1 = \cdots = (\tau_j)_\ell = 0$.
6) The discrepancy of $\mathcal{M}_j$ is $\boldsymbol{\epsilon}^{\mathcal{M}_j} \leq \max\{\boldsymbol{\epsilon}^{(i)} - \epsilon_1^{(i)} \; ; \; 1 \leq i \leq j\}$.

7) $\mathcal{M}_j \in U_m(\kappa_j)$.
8) A weakly filtered quasi-isomorphism $\eta_j : \mathcal{K}_j \longrightarrow \mathcal{M}_j$ which shifts action by $\leq \ell(\kappa_0 + \cdots + \kappa_{j-1})$ and with discrepancy $\leq \boldsymbol{\epsilon}^{\eta_j}$, where the sequences $\boldsymbol{\epsilon}^{\eta_j}$ is the one from point 2) above. Moreover, the first order part $(\eta_j)_1$ is a chain isomorphism represented by an upper triangular matrix with id's along the diagonal (with respect to the splitting $\mathrm{CF}(X, L_0) \oplus \cdots \oplus C(X, L_r)$) and its inverse $(\eta_j)_1^{-1}$ shifts action by ≤ 0.

The module $\mathcal{M}$ claimed in the statement of the theorem is the module $\mathcal{M}_r$, and the quasi-isomorphism of $\mathcal{A}$-modules is $\sigma : \mathcal{K}_r \longrightarrow \mathcal{M}$ is η_r.

Next, we analyze the differential $\mu_1^{\mathcal{M}_j}$ on the modules $\mathcal{M}_j$. We begin with the module

$$\mathcal{M}_1 = \mathcal{C}one\big(\mathcal{L}_1 \xrightarrow{(\lambda(c_1)+\tau_1;\rho_1+\ell\kappa_0,\boldsymbol{\epsilon}^{(1)})} \mathcal{K}_0\big). \tag{2.35}$$

Recall that $c_1 \in \mathcal{K}_0^{\leq \rho_1+\ell\kappa_0}(L_1) = C^{\leq \rho_1+\ell\kappa_0}(L_1, L_0)$. For further use, we will write

$$c_{1,0} := c_1.$$

Let $X \in \mathrm{Ob}(\mathcal{A})$. Write

$$\mathcal{M}_1(X) = C(X, L_1) \oplus C(X, L_0)$$

as R-modules. By the definition of the map λ we have according to this splitting:

$$\mu_1^{\mathcal{M}_1}(b_1, b_0) = \big(\mu_1^{\mathcal{A}}(b_1), \mu_1^{\mathcal{A}}(b_0) + \mu_2^{\mathcal{A}}(b_1, c_{1,0})\big), \quad \text{for all } b_1 \in C(X, L_1), b_0 \in C(X, L_0).$$

More generally, the higher operations $\mu_d^{\mathcal{M}_1}$ have the following form. Let $1 \leq d \leq \ell - 1$ and $X_0, \ldots, X_{d-1} \in \mathrm{Ob}(\mathcal{A})$. One has , for all $a_i \in C(X_{i-1}, X_i)$, $i = 1, \ldots, d$ and for all $(b_1, b_0) \in C(X_d, L_1) \oplus C(X_d, L_0)$:

$$\begin{aligned} \mu_d^{\mathcal{M}_1}&\big(a_1, \ldots, a_{d-1}, (b_1, b_0)\big) \\ &= \big(\mu_d^{\mathcal{A}}(a_1, \ldots, a_{d-1}, b_1), \mu_d^{\mathcal{A}}(a_1, \ldots, a_{d-1}, b_0) + \mu_{d+1}^{\mathcal{A}}(a_1, \ldots, a_{d-1}, b_1, c_{1,0})\big). \end{aligned} \tag{2.36}$$

Note that the term τ_1 in (2.35) does not play any role in the expression for $\mu_d^{\mathcal{M}_1}$ as long as $d \leq \ell - 1$, since $(\tau_1)_1 = \cdots = (\tau_1)_\ell = 0$. Recall also that $\ell = r + 2$.

We now analyze $\mathcal{M}_2$. Recall that

$$\mathcal{M}_2 := \mathcal{C}one\big(\mathcal{L}_2 \xrightarrow{(\lambda(c_2)+\tau_2;\rho_2+\ell\kappa_1+\ell\kappa_0,\boldsymbol{\epsilon}^{(2)})} \mathcal{M}_1\big), \tag{2.37}$$

where $c_2 \in \mathcal{M}_1^{\leq \rho_2+\ell(\kappa_1+\kappa_0)}(L_2)$. Recall that, as R-modules,

$$\mathcal{M}_1^{\leq \rho_2+\ell(\kappa_1+\kappa_0)}(L_2) = C^{\leq \rho_2-\rho_1+\ell\kappa_1-\epsilon_1^{(1)}}(L_2, L_1) \oplus C^{\leq \rho_2+\ell(\kappa_1+\kappa_0)}(L_2, L_0).$$

Write $c_2 = (c_{2,1}, c_{2,0})$ with respect to this splitting.

Let $X \in \mathrm{Ob}(\mathcal{A})$ and write, as R-modules,

$$\mathcal{M}_2(X) = C(X, L_2) \oplus \mathcal{M}_1(X) = C(X, L_2) \oplus C(X, L_1) \oplus C(X, L_0). \tag{2.38}$$

By the definition of λ together with (2.36) we have

$$\begin{aligned}(2.39)\qquad \mu_1^{\mathscr{M}_2}(b_2, b_1, b_0) &= \big(\mu_1^{\mathscr{A}}(b_2), \mu_1^{\mathscr{M}_1}(b_1, b_0) + \mu_2^{\mathscr{M}_1}(b_2, c_2)\big)\\ &= \big(\mu_1^{\mathscr{A}}(b_2), \mu_1^{\mathscr{A}}(b_1) + \mu_2^{\mathscr{A}}(b_2, c_{2,1}),\\ &\qquad \mu_1^{\mathscr{A}}(b_0) + \mu_2^{\mathscr{A}}(b_1, c_{1,0}) + \mu_2^{\mathscr{A}}(b_2, c_{2,0}) + \mu_3^{\mathscr{A}}(b_2, c_{2,1}, c_{1,0})\big).\end{aligned}$$

In other words, the matrix of $\mu_1^{\mathscr{M}_2}$ has the following shape:

$$(2.40)\qquad \mu_1^{\mathscr{M}_2} = \begin{pmatrix} \mu_1^{\mathscr{A}}(\bullet) & \mu_2^{\mathscr{A}}(\bullet, c_{1,0}) & \mu_2^{\mathscr{A}}(\bullet, c_{2,0}) + \mu_3^{\mathscr{A}}(\bullet, c_{2,1}, c_{1,0}) \\ 0 & \mu_1^{\mathscr{A}}(\bullet) & \mu_2^{\mathscr{A}}(\bullet, c_{2,1}) \\ 0 & 0 & \mu_1^{\mathscr{A}}(\bullet) \end{pmatrix}$$

Here the matrix has been calculated with respect to the splitting

$$\mathscr{M}_2(X) = C(X, L_0) \oplus C(X, L_1) \oplus C(X, L_2)$$

(in contrast to (2.38) and (2.39)) in order to be compatible with (2.30).

A similar formula holds also for the higher operations $\mu_d^{\mathscr{M}_2}$. Let $1 \le d \le \ell - 2$ and $X_0, \dots, X_{d-1} \in \mathrm{Ob}(\mathscr{A})$. Then, for all $\underline{a} \in C(X_0, X_1) \otimes \cdots \otimes C(X_{d-2}, X_{d-1})$:

$$\begin{aligned}(2.41)\qquad \mu_d^{\mathscr{M}_2}(\underline{a}, b_2, b_1, b_0) = \Big(&\mu_d^{\mathscr{A}}(\underline{a}, b_2),\ \mu_d^{\mathscr{A}}(\underline{a}, b_1) + \mu_{d+1}^{\mathscr{A}}(\underline{a}, b_2, c_{2,1}),\\ &\mu_d^{\mathscr{A}}(\underline{a}, b_0) + \mu_{d+1}^{\mathscr{A}}(\underline{a}, b_1, c_{1,0})\\ &\quad + \mu_{d+1}^{\mathscr{A}}(\underline{a}, b_2, c_{2,0}) + \mu_{d+2}^{\mathscr{A}}(\underline{a}, b_2, c_{2,1}, c_{1,0})\Big).\end{aligned}$$

Continuing by induction, we obtain the $c_{q,p} \in C(L_q, L_p)$ for all $0 \le q < p \le r$ and the operators $a_{i,j}$, $i > j$, as described in (2.31), which form the matrix of the differentials $\mu_1^{\mathscr{M}}$ for the module $\mathscr{M} = \mathscr{M}_r$.

Note that the $\mu_k^{\mathscr{M}_j}$-operation of the intermediate module $\mathscr{M}_j$ involves expressions containing $\mu_d^{\mathscr{A}}$ for $d \le j + k$ but no higher order μ's. It is also important to remark that at every step of the construction, the operations $\mu_d^{\mathscr{M}_j}$ for $d \le r + 1 - j$ will depend on the cycles $c_{q,p}$ with $0 \le p < q \le j$ but *not* on the elements τ_i that appear in (2.34). The reason is that $(\tau_i)_1 = \cdots = (\tau_i)_\ell = 0$ and we have chosen in advance $\ell = r + 2$.

Next, we estimate the action levels $\alpha_{q,p}$ of $c_{q,p}$ from (ii.δ) and the action shift and discrepancy of the quasi-isomorphism $\sigma = \eta_r$ as claimed in (2.33).

An inspection of the previous steps in the proof shows that

$$c_{q,p} \in C^{\le \rho_q - \rho_p + \ell(\kappa_p + \cdots + \kappa_{q-1}) - \epsilon_1^{(p)}}(L_q, L_p).$$

Thus we need to estimate the κ_j's. This, in turn, would require to estimate the $\boldsymbol{\epsilon}^{(i)}$'s. Note that we can choose at every step of the previous inductive construction the sequence $\boldsymbol{\epsilon}^{(j)}$ at 2.a) page 41 to satisfy

$$\epsilon_d^{(j)} \le \epsilon_{d+1}^{\mathscr{M}_{j-1}} + \sum_{i=1}^{d} \big(\epsilon_i^{\eta_{j-1}} + \delta_i^{\phi_j} + \epsilon_i^{\mathscr{A}} + \epsilon_i^{\mathscr{M}_{j-1}}\big).$$

A simple inductive argument now implies the desired estimates for the $\epsilon_d^{(j)}$'s the κ_j's as well as for the action shift of η_j and its discrepancy.

Finally, the first statement at point (2.14) follows easily from the induction process defining the maps η_i, $i = 1, \ldots, r$, by examining the filtrations induced on $\mathrm{CF}(X, L_j)$ by each of $\mathcal{K}_i(X)$ and $\mathcal{M}_i(X)$ for $j \leq i \leq r$. That $C(X, L_0)$ is a subcomplex of both $\mathcal{K}_i(X)$ and $\mathcal{M}_i(X)$ follows from the fact that $\mathcal{K}_i$ and $\mathcal{M}_i$ are both iterated cones starting with the object $\mathcal{L}_0$. □

2.7. Invariants and measurements for filtered chain complexes

As a supplement to the previous material we describe here a number of numerical invariants of filtered chain complexes that will be useful in Chapter 5 when we prove our main geometric results. More details and further results can be found in the expanded version of this paper [**BCS**].

We begin with basic definitions. Fix a commutative ring $\mathcal{R}$ with unity.

- By a *filtered chain complex* we mean a chain complex (C, d^C) of $\mathcal{R}$-modules endowed with an increasing filtration by sub-chain complexes $C^{\leq \alpha} \subset C$, indexed by the real numbers $\alpha \in \mathbb{R}$.
- An $\mathcal{R}$-linear map $f : C \to D$ between the filtered chain complexes (C, d^C), (D, d^D) is called *filtered* if there exists $\rho \in \mathbb{R}$ such that $f(C^{\leq \alpha}) \subset D^{\leq \alpha + \rho}$ for every α. In that case we also say that f *shifts action by* $\leq \rho$. In case f preserves the filtrations (*i.e.* it shifts filtration by ≤ 0) we say that f is *strictly filtered*.
- Let (C, d^C) be a filtered chain complex, and $x \in C$. Define $A(x) \in \mathbb{R} \cup \{-\infty, \infty\}$ to be the infimal filtration level of C which contains x, *i.e.*

  $$A(x) := \inf \left\{ \alpha \in \mathbb{R} \ ; \ x \in C^{\leq \alpha} \right\}.$$

  We call $A(x)$ the *action level* of x. Sometimes we will write $A(x; C)$ instead of $A(x)$ in order to keep track of the chain complex C that x belongs to.

By our conventions we have $A(0) = -\infty$ and if $\bigcap_{\alpha \in \mathbb{R}} C^{\leq \alpha} = \{0\}$ then $A(x) = -\infty$ iff $x = 0$. Also, if the filtration on C is exhaustive, *i.e.* $\bigcup_{\alpha \in \mathbb{R}} C^{\leq \alpha} = C$, then $A(x) < \infty$ for every $x \in C$.

- Another measurement relevant to our considerations is the following. Define the "*action drop*" of the differential d^C of the filtered chain complex (C, d^C) as

$$\delta_{d^C} = \sup \left\{ r \in [0, \infty) \ ; \ \forall a \in \mathbb{R}, \ d^C(C^{\leq a}) \subset C^{\leq a - r} \right\}. \tag{2.42}$$

2.7.1. ***Boundary depth and related algebraic notions.*** — Boundary depth was introduced and studied extensively in symplectic topology (in a slightly different formulation than below) by Usher [**Ush11**], [**Ush13**]. Here we introduce variants of this measurement, such as boundary level and homotopical boundary level and explain their relation to boundary depth.

Let (C, d^C) be a filtered chain complex and $c \in C$ a boundary.

Define the *boundary level of c* by

$$B(c;C) = \inf\big\{\alpha \in \mathbb{R}\ ;\ \exists\, b \in C^{\leq \alpha} \text{ such that } c = d^C b\big\}. \tag{2.43}$$

A central measurement in our framework is the following special case. Let (C, d^C) and (D, d^D) be filtered chain complexes. Let $\psi : C \to D$ be a filtered chain map and assume that ψ is null-homotopic.

Define the *homotopical boundary level* $B_h(\psi)$ of ψ to be the infimal action shift needed for a chain homotopy between ψ and 0. More precisely:

$$B_h(\psi) = \inf\big\{\rho \in \mathbb{R}\ ;\ \exists \text{ an } \mathcal{R}\text{-linear map } h : C \to D \text{ which shifts action by} \leq \rho \text{ and such that } \psi = hd^C + d^D h\big\}. \tag{2.44}$$

Note that $B_h(\psi) = B(\psi; \hom_{\mathcal{R}}(C, D))$, where we view ψ as a boundary in the chain complex $\hom_{\mathcal{R}}(C, D)$. The latter chain complex is filtered as follows: for $\gamma \in \mathbb{R}$, $\hom_{\mathcal{R}}^{\leq\gamma}(C, D)$ is the subcomplex consisting of all $\mathcal{R}$-linear maps $C \to D$ that shift action by $\leq \gamma$.

The notion of boundary level is closely related to the boundary depth measurement introduced by Usher [**Ush11**], [**Ush13**]. The relation is the following. Let (C, d^C) be a filtered chain complex and $c \in C$ a boundary.

The *boundary depth* $\beta(c; C)$ of c is defined by the equality

$$B(c;C) = A(c;C) + \beta(c;C), \tag{2.45}$$

where $A(c;C)$ is the action level of c. It is easy to see that

$$\beta(c;C) = \inf\big\{r \geq 0\ ;\ \forall \alpha \text{ such that } c \in C^{\leq\alpha}, \exists\, b \in C^{\leq \alpha + r} \text{ such that } d^C b = c\big\}.$$

Now let (C, d^C) be a filtered chain complex which is acyclic. Its boundary depth is

$$\beta(C) := \inf\big\{r \geq 0\ ;\ \forall \alpha \text{ and } \forall c \in C^{\leq \alpha} \text{ with } d^C(c) = 0,\ \exists\, b \in C^{\leq \alpha + r} \text{ such that } d^C b = c\big\}.$$

If we assume that (C, d^C) is acyclic (*i.e.* homotopy equivalent to the trivial chain complex), then we have the inequality

$$\beta(C) \leq B_h(\mathrm{id}_C). \tag{2.46}$$

Let (C, d^C) and (D, d^D) be filtered chain complexes and $\psi : C \to D$ a filtered chain map which is null-homotopic. Similarly to the homotopical boundary level we also have the *homotopical boundary depth* of ψ:

$$\beta_h(\psi) := \beta\big(\psi; \hom_{\mathcal{R}}(C, D)\big). \tag{2.47}$$

More explicitly, $\beta_h(\psi)$ is the infimal $r \geq 0$ for which ψ is null-homotopic via a chain homotopy that shifts filtration by $\leq A(\psi) + r$. As in (2.45) we have

$$B_h(\psi) = A\big(\psi; \hom_{\mathcal{R}}(C, D)\big) + \beta_h(\psi).$$

Finally, here is another variant of the above measurements.

Let $\mathcal{A}$ be a weakly filtered A_∞-category with discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{A}}$ (see Chapter 2).

Let $\boldsymbol{\epsilon}^m = (\epsilon_1^m = 0, \epsilon_2^m, \ldots, \epsilon_d^m, \ldots)$ be a sequence of non-negative real numbers, and let $\mathcal{M}_0, \mathcal{M}_1$ be two weakly filtered $\mathcal{A}$-modules with discrepancy $\leq \boldsymbol{\epsilon}^m$.

Let ϵ^h be another sequence of non-negative real numbers, and assume that $\epsilon^h \in \mathscr{E}(\epsilon^m, \epsilon^{\mathscr{A}})$ (see page 21).

Let $\hom^{\epsilon^h}(\mathscr{M}_0, \mathscr{M}_1)$ be the weakly filtered pre-module homomorphisms $\mathscr{M}_0 \to \mathscr{M}_1$ with discrepancy $\leq \epsilon^h$ (and arbitrary action shift). As explained in §2.3.1, it is a chain complex when endowed with the differential μ_1^{mod} of the dg-category of $\mathscr{A}$-modules. Moreover, this chain complex is filtered by $\hom^{\leq \rho; \epsilon^h}(\mathscr{M}_0, \mathscr{M}_1)$, $\rho \in \mathbb{R}$.

Now let $\psi : \mathscr{M}_0 \to \mathscr{M}_1$ be a weakly filtered module homomorphism with discrepancy $\leq \epsilon^h$, and assume that ψ is a boundary in $\hom^{\epsilon^h}(\mathscr{M}_0, \mathscr{M}_1)$ (*i.e.* ψ is chain homotopic to 0 via a chain homotopy of pre-module maps with discrepancy $\leq \epsilon^h$). Then we can define

$$\beta_h(\psi; \epsilon^h) := \beta\big(\psi; \hom^{\epsilon^h}(\mathscr{M}_0, \mathscr{M}_1)\big)$$

and similarly define $B_h(\psi; \epsilon^h)$.

Further variants of the boundary level/depth measurements and their properties can be found in the expanded version [**BCS**].

2.7.2. ***Homotopies of chain isomorphisms.*** — Here we prove a simple algebraic approximation lemma which says that altering a chain isomorphism by a chain homotopy yields an injective map provided that the chain homotopy shifts action by a small enough amount.

Lemma 2.15. — *Let (C, d^C) and (D, d^D) be filtered chain complexes, and assume that the filtration on C is exhaustive (i.e. $\bigcup_{\alpha \in \mathbb{R}} C^{\leq \alpha} = C$) and separated (i.e. $\bigcap_{\alpha \in \mathbb{R}} C^{\leq \alpha} = 0$). Let $f, g : C \to D$ be chain maps with the following properties:*

- *g is an isomorphism.*
- *g and g^{-1} are strictly filtered.*
- *$f - g$ is null-homotopic and $B_h(f - g) < \min\{\delta_{d^C}, \delta_{d^D}\}$.*

Then f is strictly filtered and moreover f is injective.

In our geometric applications $D = C$, g will be the identity, and f will be the composition of two chain morphisms $C \xrightarrow{f_1} C' \xrightarrow{f_2} C$ that are constructed geometrically. The lemma shows in this case that the middle complex C' contains C as a retract. Results of this sort are familiar in symplectic topology since [**CR03**].

Proof of Lemma 2.15. — Since the filtration on C is both exhaustive and separated, we have $-\infty < A(x) < \infty$ for every $x \neq 0$, and $A(0) = -\infty$. Set $\rho := B_h(f - g) + \epsilon$, where $\epsilon > 0$ is small enough such that $\rho < \min\{\delta_{d^C}, \delta_{d^D}\}$. Write

$$f = g + \eta d^C + d^D \eta,$$

where $\eta : C \to D$ is $\mathscr{R}$-linear and shifts action by $\leq \rho$. Since $\rho < \min\{\delta_{d^C}, \delta_{d^D}\}$ and g is strictly filtered we have

$$A\big(f(x)\big) = A\big(g(x) + \eta d^C(x) + d^D \eta(x)\big) \leq A(x), \quad \text{for all } x \in C,$$

hence f is strictly filtered.

For the injectivity of f, assume that $f(x) = 0$ for some $x \neq 0$. Then

$$g(x) = -\big(\eta d^C(x) + d^D \eta(x)\big),$$

and using again inequality $\rho < \min\{\delta_{d^C}, \delta_{d^D}\}$ we obtain that

$$A\big(g(x)\big) = A\big(\eta d^C(x) + d^D \eta(x)\big) < A(x).$$

The last inequality together with the assumption that g^{-1} is strictly filtered imply

$$A(x) = A\big(g^{-1} g(x)\big) \leq A\big(g(x)\big) < A(x).$$

A contradiction. □

Under additional assumptions we can obtain a somewhat stronger result. Before we state it, here are a couple of relevant notions. The filtration $C^{\leq \alpha} \subset C$, $\alpha \in \mathbb{R}$ induces a topology on C which is generated by the cosets of $C^{\leq \alpha}$, $\alpha \in \mathbb{R}$, as basic open subsets. The assumption that the filtration is *separated* (*i.e.* $\bigcap_{\alpha \in \mathbb{R}} C^{\leq \alpha} = 0$) which implies that C is Hausdorff in this topology.

The filtration on C is called *complete* if the obvious map

$$C \longrightarrow \varprojlim_{\alpha} (C/C^{\leq \alpha})$$

is surjective. This assumption implies that the previously mentioned topology on C turns C into a complete topological space (in the sense that every Cauchy sequence converges).

Lemma 2.16. — *Let (C, d^C), (D, d^D), f, g be as in Lemma 2.15 and assume in addition that the filtration on C is complete. Then f is a strictly filtered isomorphism and moreover f^{-1} is also strictly filtered.*

Proof. — In view of Lemma 2.15 we only need to show that f is an isomorphism and that f^{-1} is strictly filtered.

We will use a well-known inversion trick, that has already been used in a similar setting in [**Ush11**], [**Ush13**]. Fix

$$0 < \epsilon < \tfrac{1}{2}\big(\min\{\delta_{d^C}, \delta_{d^D}\} - B_h(f-g)\big).$$

By the definition of B_h there is an $\mathcal{R}$-linear map $\eta : C \to D$ that shifts actions by $\leq B_h(f-g) + \epsilon$ such that $f - g = d^C \eta + \eta d^C$. Note that $f - g$ *decreases* action by at least ϵ. Now write

$$f = g + (f-g) = g\big(\mathrm{id} + g^{-1}(f-g)\big) = g(\mathrm{id} - k),$$

where $k : C \to C$ is defined by $k = -g^{-1}(f-g)$. Since g^{-1} is strictly filtered and $f - g$ decreases filtration by at least ϵ, the same is true for k. As the filtration on C is complete, the series $a = \mathrm{id} + \sum_{n \geq 1} k^n$ converges, and satisfies $(\mathrm{id} - k)a = a(\mathrm{id} - k) = \mathrm{id}$. Therefore f is invertible with inverse ag^{-1}, a strictly filtered chain-morphism of $\mathcal{R}$-modules. □

For the next result we will assume that $\mathcal{R} = \Lambda_0$ (the positive Novikov ring over any field R). Recall that the Novikov ring Λ is the field of fractions of Λ_0. Denote by $\nu : \Lambda \to \mathbb{R} \cup \{\infty\}$ the standard valuation defined by

$$\nu\Big(a_0 T^{\lambda_0} + \sum_{i=1}^{\infty} a_i T^{\lambda_i}\Big) = \lambda_0,$$

where $a_0 \neq 0$ and $\lambda_i > \lambda_0$ for every $i \geq 1$. As usual we set $\nu(0) = \infty$.

Let (C, d^C) be a finite dimensional chain complex over Λ. Fix a basis $\mathcal{G}$ of C over Λ and let $A : \mathcal{G} \to \mathbb{R}$ be a function. Similarly to §2.2.3 we will use A to define a filtration on C by Λ_0-modules. Extend A to a function $A : C \to \mathbb{R} \cup \{-\infty\}$, by

$$A\Big(\sum \lambda_j e_j\Big) = \max\big\{ -\nu(\lambda_j) + A(e_j)\big\},$$

where e_j are the elements of the basis $\mathcal{G}$, $0 \neq \lambda_j \in \Lambda$, $A(e_j)$ is the pre-determined value of A on the generator e_j, and ν is the preceding valuation. Define now

$$C^{\leq \alpha} := \big\{x \in C \, ; \, A(x) \leq \alpha\big\}.$$

It is easy to see that $C^{\leq\alpha} \subset C$, $\alpha \in \mathbb{R}$, is an increasing filtration of C by Λ_0-modules (though not by vector spaces over Λ). Since $A(x) = -\infty$ iff $x = 0$, this filtration is separated. Moreover, it is exhaustive and complete.

From now on we will make the following *standing assumption:* $A(d^C x) \leq A(x)$, for all $x \in C$. In other words, we assume that each $C^{\leq\alpha} \subset C$, $\alpha \in \mathbb{R}$, is a subcomplex of C (over Λ_0).

It is important to note that the function A, as defined above, coincides with the action level of the preceding filtration on C, as defined at the beginning of Section 2.7. Thus no confusion should arise by denoting them both by A.

We will make use of the following definition from [**UZ16**].

Definition 2.17. — A subspace $V \subset \mathrm{Ker}(d^C) \subset C$ is called δ*-robust* if for all $v \in V$ and $w \in C$ such that $v = d^C(w)$, we have $A(w) \geq A(v) + \delta$.

2.7.3. ***Remark.*** — According to the above definition, a complement W in $\mathrm{Ker}(d^C)$ to $\mathrm{Im}(d^C)$ is a δ-robust subspace for all $\delta > 0$. Hence if $V \subset \mathrm{Im}(d^C)$ is δ-robust then $V \oplus W$ is also δ-robust. We will call a δ-robust subspace $V \subset \mathrm{Im}(d^C)$ a *proper* δ*-robust* subspace.

Proposition 2.18. — *Let (C, d^C) be a chain complex as above, and let $f : C \to C$ be a chain map. Assume that d^C splits as a sum $d^C = d_0 + d_1$ such that d_0 is a Λ-linear differential which (like d^C) also preserves the given filtration on C. Furthermore, assume that*

$$\dim_\Lambda \big(H_*(C, d_0)\big) \geq \dim_\Lambda \big(H_*(C, d^C)\big).$$

If $B_h(f - \mathrm{id}_C) < \delta_{d_1}$, then

$$\dim_\Lambda \big(\mathrm{Im}(f)\big) \geq \dim_\Lambda \big(H_*(C, d_0)\big).$$

The proposition follows directly from the following two lemmas.

Lemma 2.19. — *Let (C, d^C) be a chain complex as above, and assume that its differential splits as $d^C = d_0 + d_1$ with d_0 satisfying the same assumptions as in Proposition 2.18. Then*

$$\dim_\Lambda \big(H_*(C, d_0)\big) - \dim_\Lambda \big(H_*(C, d^C)\big) \quad \text{is even} .$$

Furthermore, denote the latter number by $2k$ and assume that $k \geq 0$. Then (C, d_C) admits a proper δ_{d_1}-robust subspace of dimension at least k.

Lemma 2.20. — *Let (C, d^C) be a chain complex as in Lemma 2.19 and $f : C \to C$ be a chain map. Let $0 < \epsilon < \delta$ and suppose that $B_h(f - \mathrm{id}_C) = \delta - \epsilon$. Then f is injective on each (resp. proper) δ-robust subspace, and maps it to a (resp. proper) ϵ-robust subspace.*

Proof of Proposition 2.18. — By Lemma 2.19, there exists a proper δ_{d_1}-robust subspace V in (C, d^C) of dimension k (where k is given by that lemma). By Lemma 2.20, $f(V)$ will be a proper ϵ-robust subspace of dimension k. Consider a subspace $V' \subset C$ of dimension k such that $d^C(V') = V$, and a complement W in $\mathrm{Ker}(d^C)$ to $\mathrm{Im}(d^C)$. Then $d^C(f(V')) = f(V)$, showing that $\dim d^C(f(V')) = k$, and $f(W)$ will again be a complement in $\mathrm{Ker}(d^C)$ to $\mathrm{Im}(d^C)$. (Note that $f(W) \cap d^C(C) = 0$ because, by assumption, $f - \mathrm{id}_C$ is null-homotopic, so f induces an isomorphism in homology.) Now, by Lemma 2.20 again, $f(W)$ will have the correct dimension. Finally the three subspaces $f(V), f(V'), f(W)$ are direct summands of C whence

$$\dim_\Lambda \big(\mathrm{Im}(f)\big) \geq \dim_\Lambda \big(H_*(C, d^C)\big) + 2k,$$

finishing the proof. □

Proof of Lemma 2.19. — The identities

$$\dim_\Lambda(C) = \dim_\Lambda \big(H_*(C, d^C)\big) + 2 \dim_\Lambda \big(\mathrm{Im}(d_C)\big),$$
$$\dim_\Lambda(C) = \dim_\Lambda \big(H_*(C, d_0)\big) + 2 \dim_\Lambda \big(\mathrm{Im}(d_0)\big),$$

show that $\dim_\Lambda(H_*(C, d_0)) - \dim_\Lambda(H_*(C, d^C))$ is even. Moreover we obtain

$$\dim_\Lambda \big(\mathrm{Im}(d^C)\big) = \dim_\Lambda \big(\mathrm{Im}(d_0)\big) + k. \tag{2.48}$$

From [**UZ16**, Proposition 7.4], it is immediate to construct a projection $\pi : C \to \mathrm{Im}(d_0)$, that restricts to the identity on $\mathrm{Im}(d_0)$ and satisfies $A(\pi(x)) \leq A(x)$ for all $x \in C$.

From (2.48) we now have that $\dim(\mathrm{Ker}(\pi_{|\mathrm{Im}(d^C)})) \geq k$. We claim that

$$V = \mathrm{Ker}\left(\pi_{|\mathrm{Im}(d^C)}\right)$$

is δ_{d_1}-robust. Indeed, if $v \in V, w \in C$, and $v = d^C w$, then writing $d^C w = d_0 w + d_1 w$, and using $\pi(v) = 0$ we obtain $d_0 w = \pi(d_0 w) = -\pi(d_1 w)$, whence $v = (\mathrm{id} - \pi)(d_1 w)$. Therefore

$$A(v) = A\big((\mathrm{id} - \pi)(d_1 w)\big) \leq A(d_1 w) \leq A(w) - \delta_{d_1}.$$

This implies $A(w) \geq A(v) + \delta_{d_1}$, concluding the proof. □

Proof of Lemma 2.20. — Let $V \subset C$ be a δ-robust subspace. We write

$$f = \mathrm{id}_C + d^C h - h d^C,$$

where $A(h(x)) \leq A(x) + (\delta - \epsilon)$, for all $x \in C$.

If $v \in V$ is such that $f(v) = 0$, we would have $v + d^C(h(v)) = 0$, which would yield $w = -h(v)$, with $v = dw$ and $A(w) \leq A(v) + \delta - \epsilon$. On the other hand δ-robustness implies $A(w) \geq A(v) + \delta$. A contradiction.

If $f(v) = d^C z$, we would have

$$v + d^C\big(h(v)\big) = d^C z,$$

which would yield $w = z - h(v)$, with $v = dw$. Therefore by δ-robustness we obtain

$$A(v) + \delta \leq A\big(z - h(v)\big) \leq \max\big\{A(h(v)), A(z)\big\}.$$

Since $A(h(v)) \leq A(v) + \delta - \epsilon$, we get

$$A(v) + \delta \leq A(z) \quad \text{and} \quad A\big(f(v)\big) \leq \max\big\{A(v), A(h(v))\big\} \leq A(v) + \delta - \epsilon \leq A(z) - \epsilon.$$

We conclude that $A(z) \geq A(f(v)) + \epsilon$, which finishes the proof. □

CHAPTER 3

FLOER THEORY AND FUKAYA CATEGORIES

We set up the variant of Floer theory that will be used in this book. In particular, we discuss how to choose the auxiliary parameters of this theory so that the Fukaya category becomes a *weakly filtered* A_∞-category.

Let (M, ω) be a symplectic manifold, either closed or convex at infinity. We always assume M to be connected.

Denote by $\mathscr{L}ag^{\mathrm{we}}(M)$ the collection of all closed connected Lagrangian submanifolds $L \subset M$ that are *weakly exact*. Recall that $L \subset M$ is weakly exact if for every $A \in H_2^D(M, L)$ we have $\int_A \omega = 0$. [5]

Let $\mathscr{C} \subset \mathscr{L}ag^{\mathrm{we}}$ be a collection of weakly exact Lagrangians. Unless explicitly stated otherwise, we henceforth make the following mild assumption on $\mathscr{C}$, whenever M is not compact. There exists an open domain $U_0 \subset M$ with compact closure, such that all Lagrangians $L \subset \mathscr{C}$ lie inside U_0. For further use, also fix another open domain with compact closure $U_1 \supset \overline{U}_0$ as well as an ω-compatible almost complex structure J_{conv} which is compatible with the convexity of M outside of $\overline{U}_1$.

Fix a base ring R of characteristic 2 (*e.g.* $R = \mathbb{Z}_2$) and let Λ be the Novikov ring over R as defined in (2.1). Denote by $\mathscr{F}uk(\mathscr{C})$ the Fukaya category, with coefficients in Λ, whose objects are $L \in \mathscr{C}$. We mostly follow here the implementation of the Fukaya category due to Seidel [**Sei08**] with several modifications that will be explained shortly.

As in [**Sei08**], for every pair of Lagrangians $L_0, L_1 \in \mathscr{C}$ we choose a Floer datum

$$\mathscr{D}_{L_0,L_1} = (H^{L_0,L_1}, J^{L_0,L_1})$$

consisting of a Hamiltonian function $H^{L_0,L_1} : [0,1] \times M \to \mathbb{R}$ and a time-dependent ω-compatible almost complex structure $J^{L_0,L_1} = \{J_t^{L_0,L_1}\}_{t\in[0,1]}$. In case M is not compact we require that outside of U_1 we have $H^{L_0,L_1} \equiv 0$ and $J_t^{L_0,L_1} \equiv J_{\mathrm{conv}}$.

Denote by $\mathcal{O}(H^{L_0,L_1})$ the set of orbits $\gamma : [0,1] \to M$ of the Hamiltonian flow $\phi_t^{H^{L_0,L_1}}$ generated by H^{L_0,L_1} such that $\gamma(0) \in L_0$ and $\gamma(1) \in L_1$. The Floer complex

5. The group $H_2^D(M, L) \subset H_2(M, L)$ is by definition the image of the Hurewicz homomorphism $\pi_2(M, L) \to H_2(M, L)$.

$\mathrm{CF}(L_0, L_1; \mathscr{D}_{L_0,L_1})$ is a free Λ-module generated by the set $\mathcal{O}(H^{L_0,L_1})$:

$$\mathrm{CF}(L_0, L_1; \mathscr{D}_{L_0,L_1}) = \bigoplus_{\gamma \in \mathcal{O}(H^{L_0,L_1})} \Lambda\gamma. \tag{3.1}$$

We work here in an ungraded setting. The differential μ_1 on the Floer complex is defined by counting solutions u of the Floer equation:

$$\begin{aligned} & u : \mathbb{R} \times [0,1] \to M, \quad u(\mathbb{R} \times 0) \subset L_0, \ \ u(\mathbb{R} \times 1) \subset L_1, \\ & \partial_s u + J_t^{L_0,L_1}(u)\partial_t u = -\nabla H_t^{L_0,L_1}(u), \\ & E(u) := \int_{-\infty}^{\infty}\int_0^1 |\partial_s u|^2 \mathrm{d}t\, \mathrm{d}s < \infty. \end{aligned} \tag{3.2}$$

where $(s,t) \in \mathbb{R} \times [0,1]$. Here,

$$H_t^{L_0,L_1}(x) := H^{L_0,L_1}(t,x)$$

and $\nabla H_t^{L_0,L_1}$ is the gradient of the function $H_t^{L_0,L_1} : M \to \mathbb{R}$ with respect to the Riemannian metric $g_t(\bullet, \bullet) = \omega(\bullet, J_t^{L_0,L_1} \bullet)$ associated to ω and $J_t^{L_0,L_1}$. Quantity $E(u)$ in the last line of (3.2) is the energy of a solution u and we consider only finite energy solutions. (Note also that the norm $|\partial_s u|$ in the definition of $E(u)$ is calculated with respect to the metric g_t.) Solutions u of (3.2) are also called Floer trajectories.

For $\gamma_-, \gamma_+ \in \mathcal{O}(H^{L_0,L_1})$ consider the space of *parametrized* Floer trajectories u connecting γ_- to γ_+:

$$\mathcal{M}(\gamma_-, \gamma_+; \mathscr{D}_{L_0,L_1}) = \left\{u \ ; \ u \text{ solves (3.2) and } \lim_{s \to \pm\infty} u(s,t) = \gamma_\pm(t)\right\}. \tag{3.3}$$

Note that $\mathbb{R}$ acts on this space by translations along the s-coordinate. This action is generally free, with the only exception being $\gamma_- = \gamma_+$ and the stationary solution $u(s,t) = \gamma_-(t)$ at γ_-.

Whenever $\gamma_- \neq \gamma_+$, we denote by

$$\mathcal{M}^*(\gamma_-, \gamma_+; \mathscr{D}_{L_0,L_1}) := \mathcal{M}(\gamma_-, \gamma_+; \mathscr{D}_{L_0,L_1})/\mathbb{R} \tag{3.4}$$

the quotient space (*i.e.* the space of non-parametrized solutions).

In the case $\gamma_- = \gamma_+$ we define $\mathcal{M}^*(\gamma_-, \gamma_-; \mathscr{D}_{L_0,L_1})$ in the same way except that we omit the stationary solution at γ_-.

For a generic choice of Floer datum $\mathscr{D}_{L_0,L_1}$ the space $\mathcal{M}^*(\gamma_-, \gamma_+; \mathscr{D}_{L_0,L_1})$ is a smooth manifold (possibly with several components having different dimensions). Moreover, its 0-dimensional component $\mathcal{M}_0^*(\gamma_-, \gamma_+; \mathscr{D}_{L_0,L_1})$ is compact hence a finite set.

Define now $\mu_1 : \mathrm{CF}(L_0, L_1; \mathscr{D}_{L_0,L_1}) \to \mathrm{CF}(L_0, L_1; \mathscr{D}_{L_0,L_1})$ by

$$\mu_1(\gamma_-) := \sum_{\gamma_+}\sum_{u} T^{\omega(u)}\gamma_+, \quad \text{for all } \ \gamma_- \in \mathcal{O}(H^{L_0,L_1}), \tag{3.5}$$

and extending linearly over Λ. Here, the outer sum runs over all $\gamma_+ \in \mathcal{O}(H^{L_0,L_1})$ and the inner sum over all solutions $u \in \mathcal{M}_0^*(\gamma_-, \gamma_+; \mathscr{D}_{L_0,L_1})$. The term $\omega(u)$ is a shorthand notation for the symplectic area of a Floer trajectory u, namely $\omega(u) := \int_{\mathbb{R}\times[0,1]} u^*\omega$.

It is well known that μ_1 is a differential and we denote the homology of $\mathrm{CF}(L_0, L_1; \mathscr{D}_{L_0,L_1})$ by $\mathrm{HF}(L_0, L_1; \mathscr{D}_{L_0,L_1})$ – the Floer homology of (L_0, L_1). This homology is independent of the choice of the Floer datum in the sense that for every two regular choices of Floer data $\mathscr{D}_{L_0,L_1}$, $\mathscr{D}'_{L_0,L_1}$ there is a canonical isomorphism

$$\psi_{\mathscr{D},\mathscr{D}'} : \mathrm{HF}(L_0, L_1; \mathscr{D}_{L_0,L_1}) \longrightarrow \mathrm{HF}(L_0, L_1; \mathscr{D}'_{L_0,L_1})$$

which form a directed system. Therefore we can regard this collection of Λ-modules as one and denote it by $\mathrm{HF}(L_0, L_1)$. The canonical isomorphisms $\psi_{\mathscr{D},\mathscr{D}'}$ do not preserve action-filtrations in general, hence there is no meaning to $H(\mathrm{CF}^{\leq\alpha}(L_0, L_1))$ without specifying the Floer datum.

The higher operations μ_d, $d \geq 2$, follow the same scheme as in [**Sei08**], with the main difference being that we work over the Novikov ring Λ.

More precisely, we first make a choice of strip-like ends along the compactification of the moduli-spaces $\mathscr{R}^{d+1}$, $d \geq 2$, of disks with $(d+1)$-boundary punctures. For every $r \in \mathscr{R}^{d+1}$ denote by S_r the punctured disk corresponding to r (thus S_r is the actual punctured Riemann surface corresponding to the parameter $r \in \mathscr{R}^{d+1}$). Denote the punctures by ζ_i, $i = 0, \ldots, d$, going in *clockwise* direction. The puncture ζ_0 will be called the exit and $\zeta_1, \ldots, \zeta_d$ the entry punctures. We denote the arc along ∂S_r connecting ζ_i to ζ_{i+1} by C_i, with the convention that $\zeta_{d+1} := \zeta_0$. [6]

Next we make a choice of perturbation data

$$\mathscr{D}_{L_0,\ldots,L_d} = (K^{L_0,\ldots,L_d}, J^{L_0,\ldots,L_d})$$

for every tuple of $d+1$ Lagrangians $L_0, \ldots, L_d \in \mathscr{C}$. The first item

$$K^{L_0,\ldots,L_d} = \{K_r^{L_0,\ldots,L_r}\}_{r\in\mathscr{R}^{d+1}}$$

is a family of 1-forms parametrized by $r \in \mathscr{R}^{d+1}$, with values in the space of Hamiltonian functions $M \to \mathbb{R}$. The second one is a family

$$J^{L_0,\ldots,L_d} = \{J_r^{L_0,\ldots,L_d}\}_{r\in\mathscr{R}^{d+1}}$$

of ω-compatible domain-dependent almost complex structures on M, parametrized by $r \in \mathscr{R}^{d+1}$. In other words for every $r \in \mathscr{R}^{d+1}$, $J_r^{L_0,\ldots,L_d}$ is itself a family $\{J_{r,z}^{L_0,\ldots,L_d}\}_{z\in S_r}$ of ω-compatible almost complex structure on M, parametrized by $z \in S_r$.

The perturbation data are required to satisfy several additional conditions. The first one is that along each of the strip-like ends the perturbation data coincides with the Floer data associated to the pair of Lagrangians corresponding to that end. More precisely, along the strip-like end corresponding to the puncture ζ_i of S_r we have

$$K_r^{L_0,\ldots,L_r} = H_t^{L_{i-1},L_i}\,dt, \quad J^{L_0,\ldots,L_d} = J_t^{L_{i-1},L_i}, \quad i = 1, \ldots, d+1, \tag{3.6}$$

where we have used here the convention that $L_{d+1} = L_0$. Here (s,t) are the conformal coordinates corresponding to the strip-like ends.

The second condition is that along the arc C_i we have

$$K^{L_0,\ldots,L_d}(\xi)_{|L_i} = 0, \quad \text{for all } \xi \in T(C_i),\ i = 0, \ldots, d. \tag{3.7}$$

6. Of course, ζ_i and C_i all depend on r but we suppress this from the notation.

The choices of strip-like ends and perturbation data along $\mathcal{R}^{d+1}$ are required to be compatible with gluing and splitting, or in the language of [**Sei08**] "consistent". This means essentially that these choices extend smoothly over the compactification $\overline{\mathcal{R}}^{d+1}$ of the space of boundary-punctured disks. In turn, this requires that for every d, the choices of strip-like ends and perturbation data done over $\mathcal{R}^{d+1}$ are compatible with those that appear on all the strata of the boundary $\partial\overline{\mathcal{R}}^{d+1}$ of the compactification $\overline{\mathcal{R}}^{d+1}$ of $\mathcal{R}^{d+1}$. We refer the reader to [**Sei08**, Chapter 9] for the precise definitions and implementation.

In case M is not compact we add the following conditions on the perturbation data. For every $r \in \mathcal{R}^{d+1}$ and $\xi \in T(S_r)$ the Hamiltonian function $K_r^{L_0,\dots,L_d}(\xi)$ is required to vanish outside of U_1 and $J_r^{L_0,\dots,L_d} \equiv J_{\text{conv}}$ outside of U_1.

Once we have made consistent choices of strip-like ends and perturbation data we define the higher operations μ_d for $L_0, \dots, L_d \in \mathcal{C}$ as follows.

For $r \in \mathcal{R}^{d+1}$, $z \in S_r$ and $\xi \in T_z(S_r)$ define $Y_{r,z}(\xi)$ to be the Hamiltonian vector field of the function $K_{r,z}^{L_0,\dots,L_d}(\xi) : M \to \mathbb{R}$. Consider now the following Floer equation:

$$
\begin{aligned}
&u : S_r \to M, \quad u(C_i) \subset L_i, \ \ i = 0, \dots, d, \\
&Du_z + J_{r,z}^{L_0,\dots,L_d}(u) \circ Du_z \circ j_r = Y_{r,z}(u) + J_{r,z}^{L_0,\dots,L_d} \circ Y_{r,z}(u) \circ j_r, \\
&E(u) := \int_{S_r} |Du - Y_r|_J^2 \sigma < \infty.
\end{aligned}
\tag{3.8}
$$

Here j_r stands for the complex structure on S_r. The last quantity in (3.8) is the energy of a solution u and we consider only solutions of finite energy. The definition of $E(u)$ involves an area form σ on S_r and the norm $|\bullet|_J$ on the space of linear maps $T_z(S_r) \to T_{u(z)}(M)$ which is induced by j_r, $J := J_r^{L_0,\dots,L_d}$ and σ. See [**MS12**, Section 2.2, page 20] for the definition. Note that $E(u)$ does not depend on σ.

Given orbits $\gamma_-^1, \dots, \gamma_-^d, \gamma_+$ with $\gamma_-^i \in \mathcal{O}(H^{L_{i-1},L_i})$ and $\gamma_+ \in \mathcal{O}(H^{L_0,L_d})$ define the space of so called *Floer polygons* connecting $\gamma_-^1, \dots, \gamma_-^d$ to γ_+ to be the space of all pairs (r, u) with $r \in \mathcal{R}^{d+1}$ and $u : S_r \to M$ such that

1) u is a solution of (3.8);
2) $\lim_{s\to\infty} u(s,t) = \gamma_-^i(t)$ for $1 \le i \le d$ on the strip-like end corresponding to puncture ζ_i, where $(s,t) \in (-\infty, 0] \times [0,1]$ are the conformal coordinates on the strip-like end of ζ_i;
3) $\lim_{s\to\infty} u(s,t) = \gamma_+(t)$ for $1 \le i \le d$ on the strip-like end corresponding to puncture ζ_0, where $(s,t) \in [0,\infty) \times [0,1]$ are the conformal coordinates on the strip-like end of ζ_0.

We denote this space by $\mathcal{M}(\gamma_-^1, \dots, \gamma_-^d, \gamma_+; \mathcal{D}_{L_0,\dots,L_d})$. For generic choices of Floer and perturbation data this space is a smooth manifold and its 0-dimensional component

$$\mathcal{M}_0(\gamma_-^1, \dots, \gamma_-^d, \gamma_+; \mathcal{D}_{L_0,\dots,L_d})$$

is compact hence a finite set. [7] Define now

$$\mu_d(\gamma_-^1,\ldots,\gamma_-^d) = \sum_{\gamma_+}\sum_{(r,u)} T^{\omega(u)}\gamma_+ \in \mathrm{CF}(L_0,L_d;\mathscr{D}_{L_0,L_d}), \tag{3.9}$$

where the first sum goes over all $\gamma_+ \in \mathscr{O}(H^{L_0,L_d})$ and the second sum goes over all pairs $(r,u) \in \mathscr{M}_0(\gamma_-^1,\ldots,\gamma_-^d,\gamma_+;\mathscr{D}_{L_0,\ldots,L_d})$. The term $\omega(u)$ stands for the symplectic area of u,

$$\omega(u) := \int_{S_r} u^*\omega.$$

Extending μ_d multi-linearly over Λ we obtain an operation:

$$\mu_d : \mathrm{CF}(L_0,L_1;\mathscr{D}_{L_0,L_1}) \otimes \cdots \otimes \mathrm{CF}(L_{d-1},L_d;\mathscr{D}_{L_{d-1},L_d}) \longrightarrow \mathrm{CF}(L_0,L_d;\mathscr{D}_{L_0,L_d}).$$

With all the operations above $\mathscr{F}uk(\mathscr{C})$ becomes an A_∞-category. The proof of this is essentially the same as the one in [**Sei08**], the only difference is that one needs to keep track of the areas appearing as exponents in the variable T of the Novikov ring.

3.1. Units

We now explain briefly how to construct homology units in $\mathscr{F}uk(\mathscr{C})$. More details can be found in [**Sei08**, Chapter 8]. Denote by $S = D \setminus \zeta_0$ the unit disk punctured at one boundary point $\zeta_0 \in \partial D$. Fix a strip-like end around ζ_0 making ζ_0 an exit puncture and let (s,t) be the conformal coordinates associated to this strip-like end. Let $L \in \mathscr{C}$ and $\mathscr{D}^{L,L}$ be a regular Floer datum for the pair (L,L). Pick a regular perturbation datum $\mathscr{D}_S = (K,J)$, as described earlier with the only difference that K and J are are defined only on S (*i.e.* there is no dependence on any space like $\mathscr{R}^{d+1}$). As before, we require that D_S coincides with the Floer datum $\mathscr{D}_{L,L}$ along the strip-like ends in the sense of (3.6). For $z \in S$, $\xi \in T_z(S)$ define $Y_z(\xi)$ as before.

Given $\gamma \in \mathscr{O}(H^{L,L})$ consider the space $\mathscr{M}(\gamma;\mathscr{D}_S)$ of solutions $u : (S,\partial S) \to (M,L)$ of the last two lines of equation (3.8), with S_r, $Y_{r,z}$, $J_{r,z}^{L_0,\ldots,L_d}$, j_r replaced by S, Y_z, J_z and i respectively, and such that along the strip-like end at ξ_0 we have $\lim_{s\to\infty} u(s,t) = \gamma(t)$.

Define now an element $e_L \in \mathrm{CF}(L,L;\mathscr{D}_{L,L})$ by

$$e_L := \sum_{\gamma\in\mathscr{O}(H^{L,L})}\sum_u T^{\omega(u)}\gamma, \tag{3.10}$$

where the second sum runs over all solutions u in the 0-dimensional component $\mathscr{M}_0(\gamma;\mathscr{D}_S)$ of $\mathscr{M}(\gamma;\mathscr{D}_S)$. By standard theory e_L is a cycle and its homology class in $\mathrm{HF}(L,L)$ is independent of the choice of the Floer and perturbation data. Moreover, $[e_L] \in \mathrm{HF}(L,L)$ is a unit for the product induced by μ_2.

7. Recall that $d \geq 2$ hence we do not divide here by any reparametrization group.

3.2. Families of Fukaya categories

The Fukaya category $\mathcal{F}uk(\mathcal{C})$ depends on all the choices made – strip-like ends, Floer and perturbation data. We fix once and for all a consistent choice of strip-like ends and denote by E the space of all consistent choices of perturbation data (compatible with the fixed choice of strip-like ends). The space E can be endowed with a natural topology (and a structure of a Fréchet manifold), induced from a larger space in which one allows perturbation data in appropriate Sobolev spaces (see [Sei08, Chapter 9]). The subspace $E_{\text{reg}} \subset E$ of regular perturbation data is residual hence a dense subset.

The space E contains a distinguished subspace $\mathcal{N} \subset E$ consisting of all consistent choices of perturbation data $\mathcal{D} = (K, J)$ with $K \equiv 0$. Fix a subset $E'_{\text{reg}} \subset E_{\text{reg}}$ whose closure $\overline{E}'_{\text{reg}}$ contains $\mathcal{N}$.

For $p \in E'_{\text{reg}}$ we denote by $\mathcal{F}uk(\mathcal{C}; p)$ the associated Fukaya category with choice of perturbation data p. We thus obtain a family of A_∞-categories $\{\mathcal{F}uk(\mathcal{C}; p)\}_{p \in E'_{\text{reg}}}$, parametrized by $p \in E'_{\text{reg}}$. It is well known that this is a *coherent* system of A_∞-categories (see [Sei08, Chapter 10]), in particular they are all mutually quasi-equivalent.

In what follows we will sometimes use the following notation. Given $L_0, L_1 \in \mathcal{C}$ and $p \in E'_{\text{reg}}$ we write $\mathrm{CF}(L_0, L_1; p)$ for $\mathrm{CF}(L_0, L_1; \mathcal{D}_{L_0,L_1})$, where $\mathcal{D}_{L_0,L_1}$ is the Floer datum prescribed by the choice $p \in E'_{\text{reg}}$.

3.3. Weakly filtered structure on Fukaya categories

We start by defining filtrations on the Floer complexes of pairs of Lagrangians in $\mathcal{C}$. We follow here the general recipe from §2.2.3.

Denote by $\nu : \Lambda \to \mathbb{R} \cup \{\infty\}$ the standard valuation defined by

$$\nu\Big(a_0 T^{\lambda_0} + \sum_{i=1}^{\infty} a_i T^{\lambda_i}\Big) = \lambda_0, \tag{3.11}$$

where $a_0 \neq 0$ and $\lambda_i > \lambda_0$ for every $i \geq 1$. As usual we set $\nu(0) = \infty$.

Let $L_0, L_1 \in \mathcal{C}$ be two Lagrangians and $\mathcal{D}_{L_0,L_1} = (H^{L_0,L_1}, J^{L_0,L_1})$ a Floer datum. We define an "action functional"

$$\boldsymbol{A} : \mathrm{CF}(L_0, L_1; \mathcal{D}_{L_0,L_1}) \longrightarrow \mathbb{R} \cup \{-\infty\}$$

as follows. Let $P(T) = \sum_{i=0}^{\infty} a_i T^{\lambda_i} \in \Lambda$ with $\lambda_0 < \lambda_i$ for all $i \geq 1$, and $a_0 \neq 0$. Let $\gamma \in \mathcal{O}(H^{L_0,L_1})$ be a Hamiltonian orbit. We first define:

$$\boldsymbol{A}\big(P(T)\gamma\big) := -\nu(P(T)) + \int_0^1 H_t^{L_0,L_1}\big(\gamma(t)\big)\,\mathrm{d}t = -\lambda_0 + \int_0^1 H_t^{L_0,L_1}\big(\gamma(t)\big)\,\mathrm{d}t.$$

Now let $\sum_{k=1}^{l} P_k(T)\gamma_k \in \mathrm{CF}(L_0, L_1; \mathcal{D}_{L_0,L_1})$ be a general non-trivial element, where the γ_k's are mutually distinct. We extend the definition of $\boldsymbol{A}$ to such an element by

$$\boldsymbol{A}\big(P_1(T)\gamma_1 + \cdots + P_l(T)\gamma_l\big) := \max\big\{\boldsymbol{A}\big(P_k(T)\gamma_k\big)\ ;\ k = 1, \ldots, l\big\}.$$

Finally, we put $\boldsymbol{A}(0) = -\infty$.

We now define a filtration on $\mathrm{CF}(L_0, L_1; \mathscr{D}_{L_0,L_1})$ by

$$CF^{\leq\alpha}(L_0, L_1; \mathscr{D}_{L_0,L_1}) := \{x \in \mathrm{CF}(L_0, L_1; \mathscr{D}_{L_0,L_1})\ ;\ \boldsymbol{A}(x) \leq \alpha\}. \tag{3.12}$$

Before we go on, a quick remark regarding the Hamiltonian functions H^{L_0,L_1} in the Floer data is in order. We do *not* assume that these functions are normalized (*e.g.* by requiring them to have zero mean when M is closed, or to be compactly supported when M is open). This means that if we replace H^{L_0,L_1} by $H^{L_0,L_1} + c(t)$ for some family of constants $c(t)$, we get the same chain complex as $\mathrm{CF}(L_0, L_1; \mathscr{D}_{L_0,L_1})$ but with a shifted action-filtration.

Returning to (3.12), it is easy to see that $CF^{\leq\alpha}(L_0, L_1; \mathscr{D}_{L_0,L_1})$ is a Λ_0-module (though not a Λ-module). The fact that this filtration is preserved by μ_1 and moreover, that it provides $\mathscr{F}uk(\mathscr{C})$ with a structure of a weakly filtered A_∞-category are the subject of the following proposition.

Proposition 3.1. — *There exists a choice $E'_{\mathrm{reg}} \subset E_{\mathrm{reg}} \setminus \mathcal{N}$ with $\overline{E}'_{\mathrm{reg}} \supset \mathcal{N}$ and such that the following holds. Let $p \in E'_{\mathrm{reg}}$ and $\mathscr{F}uk(\mathscr{C}; p)$ be the corresponding Fukaya category. Then there exist a sequence of non-negative real numbers $\boldsymbol{\epsilon}(p) = (\epsilon_1(p) = 0, \epsilon_2(p), \ldots, \epsilon_d(p), \ldots)$ and $u(p), \zeta(p), \kappa(p) \in \mathbb{R}_+$, depending on p, such that:*

(i) *With the filtrations described above on the Floer complexes, $\mathscr{F}uk(\mathscr{C}; p)$ becomes a weakly filtered A_∞-category with discrepancy $\leq \boldsymbol{\epsilon}(p)$.*

(ii) *$\mathscr{F}uk(\mathscr{C}; p)$ is h-unital in the weakly filtered sense and there is a choice of homology units with discrepancy $\leq u(p)$.*

(iii) *$\mathscr{F}uk(\mathscr{C}; p) \in U^e(\zeta(p))$.*

(iv) *Let $L \in \mathscr{C}$ and denote by $\mathscr{L}$ its Yoneda module. Then $\mathscr{L} \in U_m(\kappa(p))$.*

(v) *For every $p_0 \in \mathcal{N} \subset E$ (see page 56) we have*

$$\lim_{p \to p_0} \epsilon_d(p) = 0, \ \textit{for all}\ d \geq 2, \qquad \lim_{p \to p_0} u(p) = \lim_{p \to p_0} \zeta(p) = \lim_{p \to p_0} \kappa(p) = 0.$$

Proof. — We will only give a sketch of the proof, as most of the ingredients are standard in the theory (see *e.g.* [Sei08]).

The precise definition of the set of choices of perturbation data E'_{reg} will be given in the course of the proof.

We begin by showing that the filtration (3.12) is preserved by μ_1. Let $L_0, L_1 \in \mathscr{C}$ and $\mathscr{D}_{L_0,L_1} = (H^{L_0,L_1}, J^{L_0,L_1})$ be a Floer datum. Let $\gamma_-, \gamma_+ \in \mathscr{O}(H^{L_0,L_1})$ be two generators of $\mathrm{CF}(L_0, L_1; \mathscr{D}_{L_0,L_1})$ and let $u \in \mathscr{M}_0(\gamma_-, \gamma_+; \mathscr{D}_{L_0,L_1})$ be an element of the 0-dimensional component of Floer trajectories connecting γ_- to γ_+. By (3.5), the contribution of u to $\mu_1(\gamma_-)$ is $T^{\omega(u)}\gamma_+$. We now have the following standard energy-area identity for solutions $u \in \mathscr{M}(\gamma_-, \gamma_+; \mathscr{D}_{L_0,L_1})$ of the Floer equation

$$E(u) = \omega(u) + \int_0^1 H_t^{L_0,L_1}(\gamma_-(t))\,\mathrm{d}t - \int_0^1 H_t^{L_0,L_1}(\gamma_+(t))\,\mathrm{d}t. \tag{3.13}$$

It immediately follows that

$$\boldsymbol{A}(T^{\omega(u)}\gamma_+) = -\omega(u) + \int_0^1 H_t^{L_0,L_1}\big(\gamma_+(t)\big)\,\mathrm{d}t \leq \int_0^1 H_t^{L_0,L_1}\big(\gamma_-(t)\big)\,\mathrm{d}t = \boldsymbol{A}(\gamma_-).$$

This shows that μ_1 preserves the filtration (3.12) on $\mathrm{CF}(L_0, L_1; \mathscr{D}_{L_0,L_1})$.

The next step is to analyze the behavior of the higher operations μ_d, $d \geq 2$, with respect to our filtration.

Let $L_0, \ldots, L_d \in \mathscr{C}$ and $\mathscr{D}_{L_0,\ldots,L_d}$ be the corresponding perturbation data.

Let $\gamma_-^i \in \mathcal{O}(H^{L_{i-1},L_i})$, $\gamma_+ \in \mathcal{O}(H^{L_0,L_d})$, and $(r, u) \in \mathscr{M}_0(\gamma_-^1, \ldots, \gamma_-^d, \gamma_+; \mathscr{D}_{L_0,\ldots,L_d})$.

The contribution of u to $\mu_d(\gamma_-^1, \ldots, \gamma_-^d)$ is $T^{\omega(u)}\gamma_+$.

Similarly to (3.13) we have the following energy-area identity

$$E(u) = \omega(u) - \int_0^1 H_t^{L_0,L_d}\big(\gamma_+(t)\big)\,\mathrm{d}t + \sum_{j=1}^{d}\int_0^1 H_t^{L_{j-1},L_j}\big(\gamma_-^j(t)\big)v + \int_{S_r} R^{K^{L_0,\ldots,L_d}}(u), \tag{3.14}$$

for solutions u of (3.8), where $R^{K^{L_0,\ldots,L_d}}$ is the curvature 2-form on S_r associated to the perturbation form $K^{L_0,\ldots,L_d}$. In local conformal coordinates $(s, t) \in S_r$ it can be written as follows. Write

$$K^{L_0,\ldots,L_d} = F_{s,t}\,\mathrm{d}s + G_{s,t}\,\mathrm{d}t$$

for some functions $F_{s,t}, G_{s,t} : M \to \mathbb{R}$. Then

$$R_{s,t}^{K^{L_0,\ldots,L_d}} = \Big(-\frac{\partial F_{s,t}}{\partial t} + \frac{\partial G_{s,t}}{\partial s} - \{F_{s,t}, G_{s,t}\}\Big)\,\mathrm{d}s \wedge \mathrm{d}t, \tag{3.15}$$

where $\{F_{s,t}, G_{s,t}\} := -\omega(X^{F_{s,t}}, X^{G_{s,t}})$ is the Poisson bracket of the functions $F_{s,t}, G_{s,t}$.

We now need to bound the term $\int_{S_r} R^{K^{L_0,\ldots,L_d}}(u)$ from (3.14) independently of (r, u). To this end, first note that for any given $r \in \mathscr{R}^{d+1}$, the curvature $R^{K^{L_0,\ldots,L_d}}$ vanishes identically along the strip-like ends of S_r by assumption on the perturbation 1-form. Next, let $\mathcal{S}^{d+1}$ be the universal family of disks with $d+1$ boundary punctures (see [**Sei08**, Chapter 9], see also [**BC14**, Section 3.1]). This is a fiber bundle over $\mathscr{R}^{d+1}$ whose fiber over $r \in \mathscr{R}^{d+1}$ is the surface S_r. The space $\mathcal{S}^{d+1}$ admits a partial compactification $\bar{\mathcal{S}}^{d+1}$ over $\overline{\mathscr{R}}^{d+1}$ and can be endowed with a smooth structure. Since the perturbation data $\mathscr{D}_{L_0,\ldots,L_d}$ was chosen consistently, the forms $K^{L_0,\ldots,L_d}$ extend to the partial compactification $\bar{\mathcal{S}}^{d+1}$ over $\overline{\mathscr{R}}^{d+1}$. Now let $\mathscr{W} \subset \bar{\mathcal{S}}^{d+1}$ be the union of all the strip-like ends corresponding to all the surfaces parametrized by $r \in \overline{\mathscr{R}}^{d+1}$. Then $\bar{\mathcal{S}}^{d+1} \setminus \operatorname{Int}\mathscr{W}$ is compact. It follows that for all $(r, u) \in \mathscr{M}_0(\gamma_-^1, \ldots, \gamma_-^d, \gamma_+; \mathscr{D}_{L_0,\ldots,L_d})$ we have

$$\Big|\int_{S_r} R^{K^{L_0,\ldots,L_d}}(u)\Big| \leq \epsilon_d(K^{L_0,\ldots,L_d}), \tag{3.16}$$

where $\epsilon_d(K^{L_0,\ldots,L_d})$ depends only on the C^1-norm of $K^{L_0,\ldots,L_d}$ (defined in the $\mathcal{S}^{d+1}$ as well as M directions). Moreover, we have $\epsilon_d(K^{L_0,\ldots,L_d}) \to 0$ as $K^{L_0,\ldots,L_d} \to 0$ in the C^1-topology (along $\bar{\mathcal{S}}^{d+1} \setminus \operatorname{Int}\mathscr{W}$ and M).

A few words are in order for the case when M is not compact. In that case the arguments above continue to work due to our choice of perturbation data. More precisely, recall that we had two open domains $U_0, U_1 \subset M$ with compact closure, with $\overline{U}_0 \subset U_1$, and with the following properties: all Lagrangians $L \in \mathcal{L}$ lie in U_0 and outside of U_1 we have $\mathscr{D}^{L_0,\dots,L_d} = (0, J_{\text{conv}})$ for all $r \in \mathscr{R}^{d+1}$. This implies that the Floer equations (3.2) and (3.8) become homogeneous at the points where $u(z) \in M \setminus U_1$. Since $(M, \omega, J_{\text{conv}})$ is convex at infinity, the maximum principle implies that all solutions u lie within one compact domain of M. Thus the estimate (3.16) follows by bounding the C^1-norm of $K^{L_0,\dots,L_d}$ only along that compact domain.

Coming back to the estimate (3.16), it follows from (3.14) that

$$\boldsymbol{A}(T^{\omega(u)}\gamma_+) \leq \boldsymbol{A}(\gamma_-^1) + \cdots + \boldsymbol{A}(\gamma_-^d) + \epsilon_d(K^{L_0,\dots,L_d}). \tag{3.17}$$

In order to obtain a weakly filtered structure on $\mathcal{F}uk(\mathscr{C}; p)$ we need to bound from above $\epsilon_d(K^{L_0,\dots,L_d})$ uniformly in $L_0, \dots, L_d \in \mathscr{C}$, so that the ultimate discrepancy $\epsilon_d(p)$ depends only on the choice of $p \in E'_{\text{reg}}$. This is easily done by restricting the set E'_{reg} to choices of perturbation data $p = \{\mathscr{D}_{L_0,\dots,L_d}\}_{L_0,\dots,L_d \in \mathscr{C}}$ for which the C^1-norms of the forms $K^{L_0,\dots,L_d}$ are uniformly bounded (in $L_0, \dots, L_d$). Since $E_{\text{reg}} \subset E$ is residual it follows that the restricted set of choices E'_{reg} still has $\mathcal{N}$ in its closure.

This concludes the proof that $\mathcal{F}uk(\mathscr{C}; p)_{p \in E'_{\text{reg}}}$ is a family of weakly filtered A_∞-categories, and that the bounds on their discrepancies $\boldsymbol{\epsilon}(p)$ have the property that for all $p_0 \in \mathcal{N}$ we have $\lim_{p \to p_0} \epsilon_d(p) = 0$ for every $d \geq 2$.

We now turn to the statements about the unitality of the categories $\mathcal{F}uk(\mathscr{C}; p)$ and their Yoneda modules. Let $p \in E'_{\text{reg}}$. Fix $L \in \mathscr{C}$ and let $\mathscr{D}_{L,L} = (H^{L,L}, J^{L,L})$ be the Floer datum of (L, L) prescribed by p. Recall that a homology unit $e_L \in \text{CF}(L, L; \mathscr{D}_{L,L})$ can be defined by (3.10). Let $S = D \setminus \{\zeta_0\}$ and $\mathscr{D}_S = (K, J)$ as in Section 3.1. Let $\gamma \in \mathcal{O}(H^{L,L})$ and $u \in \mathcal{M}_0(\gamma; \mathscr{D}_S)$. According to (3.10) the contribution of γ and u to e_L is $T^{\omega(u)}\gamma$. The energy-area identity for u gives

$$E(u) = \omega(u) - \int_0^1 H_t^{L,L}\big(\gamma(t)\big)\,\mathrm{d}t - \int_S R^K(u),$$

where $R^K(u)$ is the curvature associated to the 1-form K from the perturbation datum $\mathscr{D}_S$ and is defined in a similar way as in (3.15). Note that we can choose the perturbation datum $\mathscr{D}_S = (K, J)$ such that the C^1-norm of the 1-form K is of the same order size as the C^1-norm of $H^{L,L}$ (*i.e.* $\|K\|_{C^1} \leq C\|H^{L,L}\|_{C^1}$ for some constant C). By doing that we obtain $|\int_S R^K(u)| \leq C'\|H^{L,L}\|_{C^1}$ for some constant C'. It follows that

$$\boldsymbol{A}(e_L) \leq C'\|H^{L,L}\|_{C^1}.$$

By restricting all the Hamiltonians $H^{L,L}$, for all $L \in \mathscr{C}$, to have a uniformly bounded C^1-norm we obtain one constant $u(p)$ (that depends on the choice p) such that for all every $L \in \mathscr{C}$ we have $\boldsymbol{A}(e_L) \leq u(p)$. Moreover, $u(p) \to 0$ as $p \to p_0 \in \mathcal{N}$ in the C^1-topology. This proves the statement about the discrepancy of the units in $\mathcal{F}uk(\mathscr{C}; p)$.

We now turn to proving statements (iii) and (iv) of Proposition 3.1 and the corresponding claims on $\zeta(p)$ and $\kappa(p)$ from statement (v).

Let $L, L' \in \mathscr{C}$. Choose $S = D \setminus \{\zeta_0\}$, $\mathscr{D}_S$ and define e_L as explained above. Denote by $\mathscr{D}_{L,L,L'}$ be the perturbation datum of the triple (L, L, L') as prescribed by p. Consider also a disk $S' = D \setminus \{\zeta'_0, \zeta'_1, \zeta'_2\}$ with three boundary punctures, ordered clockwise along ∂D. We fix strip-like ends near these three punctures such that ζ'_0, ζ'_1 are entries and ζ'_2 is an exit. Consider a 1-parametric family $(\{S''_\tau\}_{\tau\in(0,1]}, j_\tau)$ of surfaces (endowed with complex structures) obtained by performing gluing S and S' at the points ζ_0, ζ'_0 respectively. We construct this family so that $S''_\tau \longrightarrow S \coprod S'$ as $\tau \longrightarrow 0$ and $S''_1 = \mathbb{R}\times[0, 1]$ is the standard strip. Next, we choose a generic family $\{\mathscr{D}_\tau\}_{\tau\in(0,1]}$ of perturbation data over the family $\{S''_\tau\}_{\tau\in(0,1]}$ such that

1) for $\tau \longrightarrow 0$, $\mathscr{D}_\tau$ converges to $\mathscr{D}_S$ on the S component and $\mathscr{D}_{L,L,L'}$ on the S'' component.
2) $\mathscr{D}_1 = \mathscr{D}_{L,L}$.

As the family $\{\mathscr{D}_\tau\}_{\tau\in(0,1]}$ is generic, none of the elements in the $\mathscr{D}_\tau$, $\tau < 1$ is invariant under reparametrization by any non-trivial automorphism $\sigma \in \mathrm{Aut}(S_\tau)$.

Let $\gamma, \lambda \in \mathscr{O}(H^{L,L'})$ and consider the space $\mathscr{M}(\gamma, \lambda; \{\mathscr{D}_\tau\})$ of all pairs (τ, u), with $\tau \in (0, 1]$ and $u : S_\tau \longrightarrow M$ a solution of the Floer equation (3.8) with the obvious modifications: namely, the lower part of ∂S_τ is mapped by u to L and the upper one to L', u converges to γ at the entry ζ'_1 and to λ at the exit ζ'_2, (S_r, j_r) is replaced by (S_τ, j_τ), and $J_{r,z}$ and $Y_{r,z}$ are replaced by the corresponding structures from $\mathscr{D}_\tau$.

Assume that $\gamma \neq \lambda$, and consider the 0-dimensional component $\mathscr{M}_0(\gamma, \lambda; \{\mathscr{D}_\tau\})$. This is compact 0-dimensional manifold hence a finite set. It gives rise to a map

$$\Phi : \mathrm{CF}(L, L'; \mathscr{D}_{L,L'}) \longrightarrow \mathrm{CF}(L, L'; \mathscr{D}_{L,L'}),$$
$$\Phi(\gamma) := \sum_{\lambda} \sum_{(\tau,u)} T^{\omega(u)}\lambda, \quad \text{for all } \gamma \in \mathscr{O}(H^{L,L'}), \tag{3.18}$$

where the outer sum is over all $\lambda \in \mathscr{O}(H^{L,L'})$ with $\lambda \neq \gamma$ and the second sum is over all $(\tau, u) \in \mathscr{M}_0(\gamma, \lambda; \{\mathscr{D}_\tau\})$. We extend the formula in the second line of (3.18) linearly over Λ. We claim that the following formula holds:

$$\mu_2(e_L, x) = x + \mu_1 \circ \Phi(x) + \Phi \circ \mu_1(x), \quad \text{for all } x \in \mathrm{CF}(L, L'; \mathscr{D}_{L,L'}), \tag{3.19}$$

i.e. Φ is a chain homotopy between the map $\mu_2(e_L, \cdot)$ and the identity.

To prove this, let $\gamma, \gamma_+ \in \mathscr{O}(H^{L,L'})$ and consider the 1-dimensional component $\mathscr{M}_1(\gamma, \gamma_+; \{\mathscr{D}_\tau\})$ of the space $\mathscr{M}(\gamma, \gamma_+; \{\mathscr{D}_\tau\})$. This space admits a compactification $\overline{\mathscr{M}}_1(\gamma, \gamma_+; \{\mathscr{D}_\tau\})$ which is a 1-dimensional manifold with boundary. The elements in the boundary of this space fall into four types:

1) Elements corresponding to the splitting of S'' into S and S' at $\tau = 0$. These can be written as pairs $(u_S, u_{S'})$ with $u_S \in \mathscr{M}_0(\gamma'; \mathscr{D}_S)$ for some $\gamma' \in \mathscr{O}(H^{L,L})$ and $u_{S'} \in \mathscr{M}_0(\gamma', \gamma, \gamma_+; \mathscr{D}_{L,L,L'})$.
2) Elements corresponding to splitting of S_τ at some $0 < \tau_0 < 1$ into a Floer strip u_0 followed by a solution $u_1 : S_{\tau_0} \longrightarrow M$ of the Floer equation for the perturbation datum $\mathscr{D}_{\tau_0}$. More precisely, these can be written as (τ_0, u_0, u_1) with $0 < \tau_0 < 1$, $u_0 \in \mathscr{M}^*(\gamma, \gamma'; \mathscr{D}_{L,L'})$ and $u_1 \in \mathscr{M}(\gamma', \gamma_+; \mathscr{D}_{\tau_0})$ for some $\gamma' \in \mathscr{O}(H^{L,L'})$.

3) The same as 2) only that the splitting occurs in reverse order, namely first an element of $\mathcal{M}(\gamma, \gamma'; \mathcal{D}_{\tau_0})$ followed by an element of $\mathcal{M}^*(\gamma', \gamma_+; \mathcal{D}_{L,L'})$.

4) Elements corresponding to $\tau = 1$. These are $u : \mathbb{R}\times[0,1] \to M$ that belong to the space $\mathcal{M}_0(\gamma, \gamma_+; \mathcal{D}_{L,L'})$ or in other words elements of the 0-dimensional component of the space $\mathcal{M}(\gamma, \gamma_+; \mathcal{D}_{L,L'})$ of *parametrized* Floer trajectories connecting γ to γ_+. The latter space has a 0-dimensional component if and only if $\gamma = \gamma_+$ in which case that component contains only the stationary trajectory at γ. Summing up, this type of boundary point occurs if and only if $\gamma = \gamma_+$ and u is the stationary solution at γ.

Summing up over all these four possibilities (for every given area of solutions u) yields formula (3.19). Note that the first term (*i.e.* the summand x) on the right-hand side of (3.19) comes exactly from the boundary points of type 4).

To conclude the proof we only need to estimate the shift in action (or filtration) of the chain homotopy Φ. This is done in a similar way to the argument used above to estimate $\epsilon(p)$, namely by using an energy-area identity as in (3.14). Indeed we can choose the perturbation data $\mathcal{D}_S$ and $\mathcal{D}_\tau$, $0 < \tau < 1$, to be of the same size order (in the C^1-norm) as the Hamiltonian $H^{L,L'}$, hence the curvature term in the energy-area identity can be bounded by a constant $C(H^{L,L'})$ that depends on $H^{L,L'}$ and such that $C(H^{L,L'}) \to 0$ as $H^{L,L'} \to 0$ in the C^1-topology. By taking all the Hamiltonians $H^{L,L'}$ for all $L, L' \in \mathcal{C}$ to be uniformly bounded in the C^1-topology we obtain a uniform bound $\kappa(p)$ on the action shift of the chain homotopy Φ that holds for all pairs $L, L' \in \mathcal{C}$ and such that $\kappa(p) \to 0$ as $p \to p_0 \in \mathcal{N}$. This shows that the Yoneda module $\mathcal{L}$ satisfies Assumption $U_m(\kappa(p))$. By taking $L' = L$ it also follows immediately that $\mu_2(e_L, e_L) = e_L + \mu_1(c)$ for some chain c with $\boldsymbol{A}(c) \leq \kappa(p)$, hence $\mathcal{F}uk(\mathcal{C}; p) \in U^e(\kappa(p))$ (so we can actually take $\zeta(p) = \kappa(p)$). □

3.3.1. ***Remark.*** — In some variants of Floer theory it is common to normalize the Hamiltonian functions involved in the definition of the Floer complexes. For example, when the ambient manifold is closed one often normalizes the Hamiltonian functions to have zero mean, and for open manifolds one requires the Hamiltonian functions to have compact support. This solves the ambiguity of adding constants to the Hamiltonian functions and consequently provides a "canonical" way to define action filtrations. This especially makes sense when one aims to construct invariants of Hamiltonian diffeomorphisms (or flows) by means of filtered Floer homology. See *e.g.* the theory of spectral numbers [**Sch00**], [**Oh06**], [**Oh05a**], [**Oh05b**], [**EP03**], see also [**Vit92**] for an earlier approach via generating functions.

While we could have normalized the Hamiltonian functions in the Floer and perturbation data, we have opted not to do so. At first glance, this might seem to have odd implications. For example, suppose that $p_1, p_2 \in E'_{\text{reg}}$ are two choices of perturbation data such that p_2 is obtained by adding a (different) constant to each of the Hamiltonian functions (or forms) in the perturbation data from p_1. Clearly, the Fukaya categories $\mathcal{F}uk(\mathcal{C}; p_1)$ and $\mathcal{F}uk(\mathcal{C}; p_2)$ are precisely the same, but they have completely different (and generally unrelated) weakly filtered structures.

Our justification for not imposing any normalization on the Hamiltonian functions is that their role is purely auxiliary, and moreover, ideally we would like to make them arbitrarily small. More specifically, the focus of our study is the collections of Lagrangians $\mathscr{C}$ and its Fukaya category, whereas the Hamiltonian functions in the Floer data serve only as perturbations whose sole purpose is technical, namely to set up the Floer theory so that it fits into a (infinite dimensional) Morse theoretic framework. In reality, we view the Hamiltonian perturbations as quantities that can be taken arbitrarily small and consider families of Fukaya categories parametrized by choices of perturbations that tend to 0. (See *e.g.* Proposition 3.1.)

In fact, our theory would become simpler and cleaner if we could set up the Fukaya category without appealing to any perturbations at all. If this were possible (which means that all the Floer trajectories and polygons are unperturbed pseudo-holomorphic curves) our Fukaya categories would be genuinely filtered rather than only "weakly filtered". (See [**FOOO09a**], [**FOOO09b**] for a "perturbation-less" construction of an A_∞-algebra associated to a single Lagrangian.)

Another point related to the matter of normalization is that when extending our theory to Lagrangian cobordisms (see Section 3.4) we are forced to work with non-compactly supported Hamiltonian perturbations. While one could have attempted a different sort of normalization in that case (suited for the class of non-compactly supported perturbations used for cobordisms), we will not do that for the very same reasons as those for not doing it for $\mathscr{F}uk(\mathscr{C}; p)$.

3.4. Extending the theory to Lagrangian cobordisms

Most of the theory developed in the previous subsections of Chapter 3 extends to Lagrangian cobordisms. We will briefly go over the main points here and refer the reader to [**BC14**], [**BC13**] for more details.

Let (M, ω) be a symplectic manifold as at the beginning of Chapter 3. We fix a collection $\mathscr{C}$ of Lagrangians in M as in Chapter 3. Consider $\widetilde{M} := \mathbb{R}^2 \times M$ endowed with the split symplectic structure $\widetilde{\omega} := \omega_{\mathbb{R}^2} \oplus \omega$, where $\omega_{\mathbb{R}^2}$ is the standard symplectic structure of $\mathbb{R}^2$.

Fix a strip $B = [a, b] \times \mathbb{R} \subset \mathbb{R}^2$ in the plane. Consider the collection $\widetilde{\mathscr{C}}$ of all Lagrangian cobordisms $V : (L'_1, \ldots, L'_s) \rightsquigarrow (L_1, \ldots, L_r)$ in $\widetilde{M}$ that have the following additional properties. We assume that V is cylindrical (with horizontal ends) outside of $B \times M$ and that all of its ends L'_i, L_j are Lagrangian submanifolds from the collection $\mathscr{C}$. Moreover, we assume that the ends of V are all located along horizontal ends whose y-coordinates are in $\mathbb{Z}$. Finally, we further assume that V is weakly exact as a Lagrangian submanifold of $\widetilde{M}$.

The Lagrangians $(L'_1, \ldots, L'_s)$ are referred to as the positive ends and $(L_1, \ldots, L_r)$ are the negative ends. Note that the values of r and s are allowed to vary arbitrarily. We also allow s or r to be 0 in which case V is a null cobordism, *i.e.* a cobordism with only negative ends (if $s = 0$) or only positive ends (if $r = 0$). The case $r = s = 0$ means that V is a closed Lagrangian submanifold of $\widetilde{M}$.

One can associate a Fukaya category $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}})$ to the collection $\widetilde{\mathscr{C}}$. This is an A_∞-category (or rather a family of such categories, depending on auxiliary choices) whose objects are the elements of $\widetilde{\mathscr{C}}$. The precise construction is detailed in [**BC14**]. The main ingredients in the construction are completely analogous to the case $\mathcal{F}uk(\mathscr{C})$, the main differences being the following. The Floer datum $\mathscr{D}_{V,V'} = (\widetilde{H}^{V,V'}, \widetilde{J}^{V,V'})$ of a pair of cobordisms V, V' has a special form at infinity. Namely, there is a compact subset $C_{V,V'} \subset B \times M$ such that outside of $C_{V,V'}$ we have

$$\widetilde{H}^{V,V'}(t, z, p) = h(z) + H^{V,V'}(t, p),$$

where $z \in \mathbb{R}^2$, $p \in M$, $H^{V,V'} : [0, 1] \times M \longrightarrow \mathbb{R}$ is a Hamiltonian function on M and $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the so called *profile function* whose purpose is to generate a Hamiltonian perturbation at infinity which disjoins V' from V at infinity while keeping both of them cylindrical and horizontal at infinity.

Note that the profile function h is not (and in fact cannot be) compactly supported. We use the same function h for the perturbation data of all pairs of Lagrangians $V, V' \in \widetilde{\mathscr{C}}$. We remark also that h can be taken to be arbitrarily small in the C^1-topology. Precise details on the construction of h can be found in [**BC14**, Section 3].

The almost complex structures $\widetilde{J}^{V,V'}$ appearing in the Floer data have also a special form whose purpose is to retain compactness of the space of Floer trajectories. We will not repeat its definition here, since its particular form does not have any significance to the weakly filtered structure on $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}})$ that we want to achieve. The only relevant thing is that with this choice of Floer data, there is a compact subset $C'_{V,V'} \subset B \times M$ such that all orbits $\mathcal{O}(\widetilde{H}^{V,V'})$ lie inside $C_{V,V'}$ and moreover all Floer trajectories for the pair (V, V') lie inside $C_{V,V'}$.

The perturbation data used for the definition of $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}})$ are analogous to those used for $\mathcal{F}uk(\mathscr{C})$ with the following differences. For a given tuple $\mathscr{V} = (V_0, \ldots, V_d)$ with $V_j \in \widetilde{\mathscr{C}}$ the perturbation data $\mathscr{D}_{\mathscr{V}} = (\widetilde{K}^{\mathscr{V}}, \widetilde{J}^{\mathscr{V}})$ is chosen so that

$$\widetilde{K}^{\mathscr{V}}|_{S_r} = h \cdot da_r + \widetilde{K}_0^{\mathscr{V}}, \tag{3.20}$$

where $a_r : S_r \longrightarrow [0, 1]$ are the so called *transition functions* which depend smoothly on $r \in \mathscr{R}^{d+1}$. See [**BC14**, Section 3.1] for their precise definition. The 1-form

$$\widetilde{K}_0^{\mathscr{V}} \in \Omega^1\big(S_r, C^\infty(\widetilde{M})\big)$$

is chosen so that it has the following two additional properties.

The first one is that $\widetilde{K}_0^{\mathscr{V}}$ satisfies condition (3.7).

The second one is that there is a compact subset $C_{\mathscr{V}} \subset B \times M$ that contains all the subsets C'_{V_i,V_j} (mentioned earlier) such that the Hamiltonian vector fields $X^{\widetilde{K}_0^{\mathscr{V}}}(\xi)$, generated by the function $\widetilde{K}_0^{\mathscr{V}}(\xi) : \widetilde{M} \longrightarrow \mathbb{R}$ are vertical for all $r \in \mathscr{R}^{d+1}$ and $\xi \in T(S_r)$ outside of $C_{\mathscr{V}}$. By "vertical" we mean that

$$D\pi(X^{\widetilde{K}_0^{\mathscr{V}}(\xi)}) = 0,$$

where $\pi : \widetilde{M} \to \mathbb{R}^2$ is the projection. Note that due to the $h \cdot da_r$ term in the perturbation form $\widetilde{K}^{\mathcal{V}}$ this form does *not* satisfy condition (3.7). However, this will not play any role for the purposes of establishing a weakly filtered A_∞-category.

The almost complex structures $\widetilde{J}^{\mathcal{V}}$ from $\mathcal{D}_{\mathcal{V}}$ are also chosen to have restricted form, similarly to the ones appearing in the Floer data. We refer the reader to [**BC14**, Section 3.2] for the details. With these choices made it can be proved that there exists a compact subset $C_{\mathcal{V}} \subset \widetilde{M}$ such that for all $(r,u) \in \mathcal{M}(\widetilde{\gamma}^1, \ldots, \widetilde{\gamma}_d; \mathcal{D}_{\mathcal{V}})$ we have $\operatorname{image}(u) \subset C'_{\mathcal{V}}$. See [**BC14**, Section 3.3] and in particular Lemma 3.3.2 there.

Of course, apart from the above the perturbation data $\mathcal{D}_{\mathcal{V}}$ are assumed to be consistent and also compatible with the Floer data along the strip-like ends of the S_r's.

With these choices made we can define the A_∞-category $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}})$ by the same recipe as in the previous sections of Chapter 3, in particular by formula (3.9). This A_∞-category is h-unital, and a choice of homology units $e_V \in CF(V, V; \mathcal{D}_{V,V})$ can be constructed by the same recipe as in Section 3.1 (see also [**BC14**, Remark 3.5.1] for an alternative approach).

Similarly to $\mathcal{F}uk(\mathcal{C})$ our category $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}})$ depends on the various choices made, namely a choice of strip-like ends and perturbation data. Note that part of the choices made for the perturbation data is the choice of a profile function and the choice of transition functions.

We now fix the same choice of strip-like ends as for $\mathcal{F}uk(\mathcal{C})$ and denote the space of choices of perturbation data by $\widetilde{E}$. We denote the subspace of regular choices of perturbation data by $\widetilde{E}_{\mathrm{reg}}$. For $\widetilde{p} \in \widetilde{E}_{\mathrm{reg}}$ we denote by $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}; \widetilde{p})$ the category corresponding to $\widetilde{p}$.

Next we endow $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}; \widetilde{p})$ with a weakly filtered structure. This is done in precisely the same way as for $\mathcal{F}uk(\mathcal{C}; p)$. More precisely we define the action filtration on the Floer complexes $CF(V_0, V_1; \mathcal{D}_{V_0,V_1})$ by the same recipe as in Section 3.3.

With these filtrations fixed, we now have the following:

Proposition 3.2. — *The statement of Proposition 3.1 holds for the A_∞-categories*

$$\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}; \widetilde{p}), \quad \widetilde{p} \in \widetilde{E}'_{\mathrm{reg}},$$

where $\widetilde{E}'_{\mathrm{reg}} \subset \widetilde{E}_{\mathrm{reg}}$ is defined in an analogous way as $E'_{\mathrm{reg}} \subset E_{\mathrm{reg}}$ (see page 56).

The proof of this Proposition is essentially the same as that of Proposition 3.2 with straightforward modifications related to the special form of the perturbation data $\widetilde{p}$.

For technical reasons we will need in the following also enlargements of the categories $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}; \widetilde{p})$ which will be denoted $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}_{1/2}; \widetilde{p})$. These are defined in the same was as $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}; \widetilde{p})$ only that the collection of objects $\widetilde{\mathcal{C}}$ is extended to allow Lagrangian cobordisms V with ends from $\mathcal{C}$ but these ends are now allowed to lie over horizontal rays with y-coordinate in $\frac{1}{2}\mathbb{Z}$ (rather than only $\mathbb{Z}$). This larger collection of objects is denoted by $\widetilde{\mathcal{C}}_{1/2}$. The perturbation data, the A_∞-operations as well as the weakly filtered structures are defined in an analogous way as for $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}; \widetilde{p})$. We denote the space of choices of perturbation data for these categories by $\widetilde{E}_{1/2}$

and the space of regular such choices by $\widetilde{E}_{\text{reg},1/2}$. Similarly to E'_{reg} and $\widetilde{E}'_{\text{reg}}$ we also have the space $\widetilde{E}'_{\text{reg},1/2} \subset \widetilde{E}_{\text{reg},1/2}$. An obvious analogue of Proposition 3.2 continues to hold for the family of categories $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}}_{1/2};\widetilde{p}\,)$, $\widetilde{p} \in \widetilde{E}'_{\text{reg},1/2}$.

The relation between $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}})$ and $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}}_{1/2})$ is simple. Any regular choice of perturbation data for $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}}_{1/2})$ can be used, by restriction to smaller class of objects, for $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}})$. Thus, with the right choices of perturbation data we obtain a full and faithful embedding $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}}) \to \mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}}_{1/2})$. We shall give now a more precise description of this.

There is an obvious restriction map $r : \widetilde{E}_{1/2} \to \widetilde{E}$ with

$$r(\widetilde{E}_{\text{reg},1/2}) \subset \widetilde{E}_{\text{reg}} \quad \text{and} \quad r(\widetilde{E}'_{\text{reg},1/2}) \subset \widetilde{E}'_{\text{reg}}$$

and such that the closure of $r(\widetilde{E}'_{\text{reg},1/2})$ contains $\widetilde{\mathcal{N}}$ (the space of perturbations with perturbation form 0, similarly to $\mathcal{N}$ on page 56). We will replace from now on $\widetilde{E}'_{\text{reg}}$ with $r(\widetilde{E}'_{\text{reg},1/2})$ and continue to denote the latter by $\widetilde{E}'_{\text{reg}}$.

There is also a (non-unique) right inverse to r which is an extension map

$$j : r(\widetilde{E}_{1/2}) \longrightarrow \widetilde{E}_{1/2}$$

with $j(\widetilde{N}) \subset \widetilde{N}_{1/2}$ and such that $j(\widetilde{E}'_{\text{reg}}) \subset \widetilde{E}'_{\text{reg},1/2}$. The map j induces an obvious family of extension functors

$$\mathcal{J} : \mathcal{F}uk_{\text{cob}}(\mathcal{C};\widetilde{p}\,) \longrightarrow \mathcal{F}uk_{\text{cob}}\big(\widetilde{\mathcal{C}}_{1/2}; j(\widetilde{p}\,)\big), \quad \widetilde{p} \in \widetilde{E}'_{\text{reg}}. \tag{3.21}$$

These are A_∞-functors which are full and faithful (on the chain level). Note also that these functors $\mathcal{J}$ are filtered, *i.e.* they have discrepancy $\leq \mathbf{0}$.

From now on we replace $\widetilde{E}'_{\text{reg},1/2}$ with $j(\widetilde{E}'_{\text{reg}})$ and continue to denote the latter by $\widetilde{E}'_{\text{reg},1/2}$. With these conventions made, the maps

$$r_{|\widetilde{E}'_{\text{reg},1/2}} : \widetilde{E}'_{\text{reg},1/2} \to \widetilde{E}'_{\text{reg}} \quad \text{and} \quad j_{|\widetilde{E}'_{\text{reg}}} : \widetilde{E}'_{\text{reg}} \to \widetilde{E}'_{\text{reg}}$$

become bijections, inverse one to the other. Therefore, whenever no confusion arises we omit j and r from the notation and denote $j(\widetilde{p}\,)$ by $\widetilde{p}$ keeping in mind that $\widetilde{p}$ is a regular choice of perturbation data for $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}})$ which admits an extension, still denoted by $\widetilde{p}$, to a regular choice of perturbation data for $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}}_{1/2})$.

An important property of the extension map j is the following. For every $\widetilde{p}_0 \in \widetilde{\mathcal{N}}$ we have

$$\lim_{\widetilde{p} \to \widetilde{p}_0} \epsilon_d^{\mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}}_{1/2}; j(\widetilde{p}\,))} = 0, \quad \text{for all } d. \tag{3.22}$$

This follows easily from Proposition 3.2 together with the fact that j is continuous, that $j(\widetilde{\mathcal{N}}) \subset \widetilde{\mathcal{N}}_{1/2}$ and that the closure of $j(\widetilde{E}'_{\text{reg}})$ contains $\widetilde{\mathcal{N}}_{1/2}$.

3.5. The monotone case

The theory developed earlier in the paper continues to work in the more general setting of monotone Lagrangian submanifolds. We will assume henceforth all symplectic manifolds as well as Lagrangian submanifolds to be connected.

Let (M, ω) be a symplectic manifold and $L \subset M$ a Lagrangian submanifold. Recall that L is called *monotone* if the following two conditions are satisfied:

1) There exists a constant $\rho > 0$ such that

$$\omega(A) = \rho \cdot \mu(A), \quad \text{for all } A \in H_2^D(M, L).$$

Here $H_2^D(M, L) \subset H_2(M, L)$ is the image of the Hurewicz homomorphism $\pi_2(M, L) \to H_2(M, L)$ and μ is the Maslov index of L.

2) The minimal Maslov number N_L of L, defined by

$$N_L := \min \left\{ \mu(A) \, ; \, A \in H_2^D(M, L), \mu(A) > 0 \right\}$$

satisfies $N_L \geq 2$. (We use the convention that $\min \varnothing = \infty$.)

A basic invariant of monotone Lagrangians L is the *Maslov*-2 *disk count*, $\mathbf{d}_L \in \Lambda_0$. This element is defined as

$$\mathbf{d}_L := dT^a,$$

where $d \in \mathbb{Z}_2$ is the number of J-holomorphic disks (for generic J) of Maslov index 2 whose boundaries go through a given point in L, and $a = 2\rho > 0$ is the area of each of these disks. Note that if there are no J-holomorphic disks of Maslov 2 at all then $\mathbf{d}_L = 0$ by definition.

It is well known that $\mathbf{d}_L$ is independent of the choices made in the definition (the almost complex structure J and the point on L through which we count the disks - recall that L is assumed to be connected). We refer the reader to [**BC12**, Section 2.5.1] for the precise definition of the coefficient d in $\mathbf{d}_L$ and its properties. [8] In different forms this invariant has appeared in [**Oh93**], [**Oh95**], [**Che97**], [**FOOO09a**], [**Aur07**]. Under additional assumptions on L, one can define a version of this invariant also over other base rings (such as $\mathbb{Z}$ and $\mathbb{C}$) sometimes taking additional structures (like local systems) into account (see *e.g.* [**Aur07**], [**BC12**]), but we will not need that in the sequel.

Fix an element $\mathbf{d} \in \Lambda_0$ of the form $\mathbf{d} = dT^a$, $d \in \mathbb{Z}_2$, $a > 0$. Denote by $\mathcal{L}ag^{\mathrm{mon},\mathbf{d}}(M)$ the class of closed monotone Lagrangians $L \subset M$ with $\mathbf{d}_L = \mathbf{d}$. Let $\mathcal{C} \subset \mathcal{L}ag^{\mathrm{mon},\mathbf{d}}(M)$ be a collection of Lagrangians. Then one can define the Fukaya categories $\mathcal{F}uk(\mathcal{C}; p)$ in the same way as described earlier and the theory developed above in Chapter 3 carries over without any modifications. (The main difference in the monotone case is that $\mathrm{HF}(L, L)$ might not be isomorphic to $H_*(L)$, and in fact may even vanish. This however will not affect any of our considerations. Apart from that, the monotone case poses some grading issues for Floer complexes, but in this paper we work in an ungraded framework.)

8. Note that the definition in that paper is done over $\mathbb{Z}$ so the d above is obtained by reducing mod 2.

Before we go on, we mention another basic measurement for monotone Lagrangians that will be relevant in the sequel. Given a monotone Lagrangian $L \subset M$ define its *minimal disk area* A_L by

$$A_L = \min\left\{\omega(A)\,;\, A \in H_2^D(M,L), \omega(A) > 0\right\}. \tag{3.23}$$

Turning to cobordisms, the theory continues to work if we restrict to monotone Lagrangian cobordisms $V \subset \mathbb{R}^2 \times M$. Note that if $V : (L'_1, \dots, L'_s) \rightsquigarrow (L_1, \dots, L_r)$ is monotone then its ends L'_i and L'_j are automatically monotone too. Moreover, as observed by Chekanov [**Che97**], if V is a monotone Lagrangian cobordism then one can define the Maslov-2 disk count $\mathbf{d}_V$ in the same way as above (*i.e.* for closed Lagrangian submanifolds) and $\mathbf{d}_V$ continues to be invariant of the choices made in its definition. Furthermore, if V is connected then

$$\mathbf{d}_V = \mathbf{d}_{L'_i} = \mathbf{d}_{L_j}, \quad \text{for all } i, j.$$

Given $\mathbf{d} = dT^a \in \Lambda_0$ and a collection $\mathscr{C} \subset \mathscr{L}ag^{\text{mon},\mathbf{d}}(M)$, denote by $\widetilde{\mathscr{C}}$ the collection of connected monotone Lagrangian cobordisms V all of whose ends are in $\mathscr{C}$. Note that by the preceding discussion each $V \in \widetilde{\mathscr{C}}$ must have $\mathbf{d}_V = \mathbf{d}$. Therefore we omit $\mathbf{d}$ from the notation of $\mathscr{C}$ and $\widetilde{\mathscr{C}}$. This also keeps the notation consistent with the weakly exact case.

From now, unless explicitly indicated, we treat uniformly both the weakly exact case as well as the monotone one. In particular the class of admissible Lagrangians will be denoted by $\mathscr{L}ag^*(M)$, where $* =$ we in the weakly exact case, and $* = (\text{mon}, \mathbf{d})$ in the monotone case. We will use similar notation $\mathscr{L}ag^*(\mathbb{R}^2 \times M)$ for the admissible classes of cobordisms.

3.6. Inclusion functors

Let $\gamma \subset \mathbb{R}^2$ be an embedded curve with horizontal ends, *i.e.* γ is the image of a proper embedding $\mathbb{R} \hookrightarrow \mathbb{R}^2$ whose image outside of a compact set coincides with two horizontal rays having y-coordinates in $\frac{1}{2}\mathbb{Z}$.

In [**BC14**, Section 4.2] we associated to γ a family of mutually quasi-isomorphic A_∞-functors

$$\mathscr{I}_\gamma : \mathscr{F}uk(\mathscr{C}) \longrightarrow \mathscr{F}uk_{\text{cob}}(\widetilde{\mathscr{C}}_{1/2})$$

which we called *inclusion functors*. They all have the same action on objects which is given by $\mathscr{I}_\gamma(L) = \gamma \times L$ for every $L \in \mathscr{C}$.

Here is a more precise description of this family of functors. Denote by $\mathscr{H}_{\text{prof}}$ the space of profile functions (see Section 3.4, see also [**BC14**, Section 3] for the precise definition). The construction of the inclusion functors from [**BC14**] involves the following ingredients. First, we restrict to a special subset $\mathscr{H}'_{\text{prof}}(\gamma) \subset \mathscr{H}_{\text{prof}}$ which contains arbitrarily C^1-small profile functions.

Apart from being profile functions, these functions $h : \mathbb{R}^2 \to \mathbb{R}$ have the following additional properties:

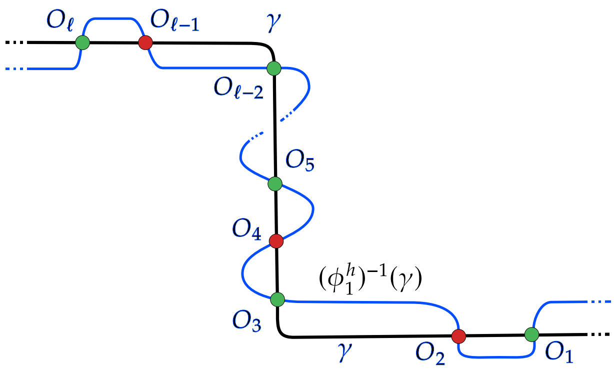

FIGURE 1. The curves γ and $(\phi_1^h)^{-1}(\gamma)$.

1) $h_{|\gamma}$ is a Morse function with an odd number of critical points $O_1, \dots, O_l \in \gamma$, where $5 \leq l =$ odd. Moreover, in a small Darboux-Weinstein neighborhood of γ, h is constant along each cotangent fiber. Thus $\phi_t^h(\gamma) \cap \gamma = \{O_1, \dots, O_l\}$ for every t.
2) The image, $(\phi_1^h)^{-1}(\gamma)$, of γ under the inverse of the time-1 map of the Hamiltonian diffeomorphism generated by h is as depicted in Figure 1.

We refer the reader to [**BC14**, Section 4] for more details. In that paper such functions were called *extended profile functions*. The word "extended" indicates that these functions are adapted to cobordisms with ends along rays having y-coordinates in $\frac{1}{2}\mathbb{Z}$ rather than just $\mathbb{Z}$.

Next, there is a map

$$\iota_\gamma : E'_{\text{reg}} \times \mathcal{H}'_{\text{prof}}(\gamma) \longrightarrow \widetilde{E}'_{\text{reg},1/2}, \tag{3.24}$$

and an A_∞-functor

$$\mathcal{I}_{\gamma;p,h} : \mathcal{F}uk(\mathcal{C}; p) \to \mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}}_{1/2}; \iota_\gamma(p,h)), \tag{3.25}$$

defined for every $(p,h) \in E'_{\text{reg}} \times \mathcal{H}'_{\text{prof}}(\gamma)$ such that for all (p,h) the following holds:

1) For $L_0, L_1 \in \mathcal{C}$, let
- ▹ $\mathcal{D}_{L_0,L_1} = (H^{L_0,L_1}, J^{L_0,L_1})$ be the Floer datum of (L_0, L_1) prescribed by p and
- ▹ $\mathcal{D}_{\gamma\times L_0,\gamma\times L_1} = (H^{\gamma\times L_0,\gamma\times L_1}, J^{\gamma\times L_0,\gamma\times L_1})$ the one prescribed by $\iota_\gamma(p,h)$.

Let $1 \leq j =$ odd $\leq l$. Then for a small neighborhood $\mathcal{U}_j$ of O_j we have

$$H^{\gamma\times L_0,\gamma\times L_1}(z,m) = h(z) + H^{L_0,L_1}(m)$$

for all $(z,m) \in \mathcal{U}_j \times M$. Moreover, we have $O_j \times x \in \mathcal{O}(H^{\gamma\times L_0,\gamma\times L_1})$ for every orbit $x \in \mathcal{O}(H^{L_0,L_1})$ and $1 \leq j =$ odd $\leq l$. In the following we will denote

$$x^{(j)} := O_j \times x.$$

Furthermore, we may assume that these are all the orbits in $\mathcal{O}(H^{\gamma\times L_0,\gamma\times L_1})$, *i.e.*

$$\mathcal{O}(H^{\gamma\times L_0,\gamma\times L_1}) = \bigcup_j \big(O_j \times \mathcal{O}(H^{\gamma\times L_0,\gamma\times L_1})\big),$$

where the union runs over all $1 \leq j = \text{odd} \leq l$.

2) $\mathcal{I}_{\gamma;p,h}(L) = \gamma \times L$ for every $L \in \mathcal{C}$.

3) The first order term $(\mathcal{I}_{\gamma;p,h})_1$ is the chain map

$$(\mathcal{I}_{\gamma;p,h})_1 : CF(L_0, L_1; \mathcal{D}_{L_0,L_1}) \longrightarrow CF(\gamma \times L_0, \gamma \times L_1; \mathcal{D}_{\gamma\times L_0,\gamma\times L_1}), \tag{3.26}$$

defined by the formula $(\mathcal{I}_{\gamma;p,h})_1(x) = x^{(1)} + x^{(3)} + \cdots + x^{(l)}$, for all $x \in \mathcal{O}(H_{L_0,L_1})$.

4) The higher terms of $\mathcal{I}_{\gamma;p,h}$ vanish: $(\mathcal{I}_{\gamma;p,h})_d = 0$ for every $d \geq 2$.

5) The homological functor associated to $\mathcal{I}_{\gamma;p,h}$ is full and faithful.

6) For every $p_0 \in \mathcal{N}$ we have $\lim \iota_\gamma(p,h) \in \widetilde{\mathcal{N}}_{1/2}$ as $h \to 0$, $p \to p_0$. (The limits here are in the C^1-topology.)

7) Let $p_0 \in \mathcal{N}$. The weakly filtered A_∞-categories $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}_{1/2}; \iota(p,h))$ have discrepancy $\leq \boldsymbol{\epsilon}^{\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}_{1/2};\iota(p,h))}$, where for every d, $\lim \epsilon_d^{\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}_{1/2};\iota(p,h))} = 0$ as $p \to p_0$ and $h \to 0$. (The limits here are in the C^1-topology.)

8) In case the ends of γ are along rays with y-coordinates in $\mathbb{Z}$ the map ι_γ and functors $\mathcal{I}_{\gamma;p,h}$ can be assumed to have values in $\widetilde{E}'_{\mathrm{reg}}$ and $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}; \widetilde{p})$ respectively. More precisely, the map ι_γ factors as a composition

$$E'_{\mathrm{reg}} \times \mathcal{H}'_{\mathrm{prof}}(\gamma) \xrightarrow{\iota'_\gamma} \widetilde{E}'_{\mathrm{reg}} \xrightarrow{j} \widetilde{E}'_{\mathrm{reg},1/2}$$

and the functors $\mathcal{I}_{\gamma;p,h}$ factor as the composition of the following two A_∞-functors:

$$\mathcal{F}uk(\mathcal{C}; p) \xrightarrow{\mathcal{I}'_{\gamma;p,h}} \mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}; \iota'_\gamma(p,h)) \xrightarrow{\mathcal{J}} \mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}_{1/2}; \iota_\gamma(p,h)). \tag{3.27}$$

The map ι'_γ and the A_∞-functor $\mathcal{I}'_{\gamma;p,h}$ have the same properties as described above for ι_γ and $\mathcal{I}_{\gamma;p,h}$ respectively, with obvious modifications). The map j and functor $\mathcal{J}$ are the ones introduced in (3.21).

We refer the reader to [**BC14**, Section 4.2] for a more detailed construction of these functors.

3.6.1. ***Additional properties relative to a given cobordism.*** — Suppose we fix in advance a Lagrangian cobordism $W \in \widetilde{\mathcal{C}}$ with the following properties. Let $K_1, \ldots, K_r \in \mathcal{C}$ be the negative ends of W. Let $\pi : \mathbb{R}^2 \times M \to \mathbb{R}^2$ be the projection. Assume that γ intersects $\pi(W)$ only along the projection of the horizontal cylindrical negative part of W (corresponding to its negative ends) with one intersection point corresponding to each end. Assume further that the intersection of γ and $\pi(W)$ is transverse and denote the intersection points by $Q_1, \ldots, Q_r \in \mathbb{R}^2$, where Q_j corresponds to the j'th negative end of W. Then we can restrict to profile functions h that have $O_{2j+1} = Q_j$ for every $1 \leq j \leq k$ and redefine the spaces $\mathcal{H}_{\mathrm{prof}}$ and $\mathcal{H}'_{\mathrm{prof}}$ by adding this restriction to their definitions. For simplicity, we will continue to denote these spaces by $\mathcal{H}'_{\mathrm{prof}}$ and $\mathcal{H}_{\mathrm{prof}}$.

Now, *in addition* to the previous list of properties, the map ι_γ can be assumed to have also the following property: let $L \in \mathscr{C}$ be a Lagrangian, and denote by

- $\mathscr{D}_{L,K_j} = (H^{L,K_j}, J^{L,K_j})$ the Floer datum of (L, K_j) prescribed by p,
- $\mathscr{D}_{\gamma\times L,V} = (H^{\gamma\times L,V}, J^{\gamma\times L,V})$ the Floer datum of $(\gamma \times L, V)$ prescribed by $\iota_\gamma(p, h)$.

Then we may assume that for small neighborhoods $\mathcal{U}_j$ of Q_j we have

$$H^{\gamma\times L,V}(z, m) = h(z) + H^{L,K_j}(m)$$

for every $(z, m) \in \mathcal{U}_j \times M$. Moreover, we may assume that

$$\mathcal{O}(H^{\gamma\times L,V}) = \textstyle\bigcup_{j=1}^{k}\big(Q_j \times \mathcal{O}(H^{L,K_j})\big).$$

3.6.2. ***The weakly filtered structure of the inclusion functors.*** — The next proposition shows that the inclusion functors are weakly filtered and gives more information on their discrepancies.

Proposition 3.3. — *The family of A_∞-functors $\mathcal{I}_{\gamma;p,h}$, $(p, h) \in E'_{\mathrm{reg}} \times \mathcal{H}'_{prof}(\gamma)$, has the following properties:*

(i) *$\mathcal{I}_{\gamma;p,h}$ is weakly filtered (see Section 2.3 for the definition).*

(ii) *$\mathcal{I}_{\gamma;p,h}$ has discrepancy $\leq \epsilon^{\mathcal{I}_{\gamma;p,h}}$, where $\epsilon_d^{\mathcal{I}_{\gamma;p,h}} = 0$ for every $d \geq 2$ and*

$$\epsilon_1^{\mathcal{I}_{\gamma;p,h}} \leq \max\big\{h(O_k)\ ;\ 1 \leq k = \mathrm{odd} \leq l\big\}.$$

Note that $\epsilon_1^{\mathcal{I}_{\gamma;p,h}} \longrightarrow 0$ as $h \longrightarrow 0$ in the C^0-topology.

(iii) *$\mathcal{I}_{\gamma;p,h}$ is homologically unital.*

(iv) *For every $L \in \mathscr{C}$ denote by $e'_{\gamma\times L} = (\mathcal{I}_{\gamma;p,h})_1(e_L) \in \mathrm{CF}(\gamma \times L, \gamma \times L; \mathscr{D}_{\gamma\times L,\gamma\times L})$ the image of the homology unit $e_L \in \mathrm{CF}(L, L; \mathscr{D}_{L,L})$ under the functor $\mathcal{I}_{\gamma;p,h}$. The collection of elements $\{e'_{\gamma\times L}\}_{L\in\mathscr{C}}$ can be extended to a collection of homology units*

$$\widetilde{\mathscr{E}} = \{e'_V\}_{V\in\widetilde{\mathscr{C}}}$$

for $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}}_{1/2}, \iota_\gamma(p, h))$ with discrepancy $\leq \widetilde{u}'(p, h)$, where $\widetilde{u}'(p, h) \longrightarrow 0$ as $p \longrightarrow p_0 \in \mathcal{N}$ and $h \longrightarrow 0$ in the C^1-topologies.

(v) *With respect to the collection of homology units $\widetilde{\mathscr{E}}$ above, $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}}_{1/2}; \iota_\gamma(p, h))$ belongs to $U^e(\widetilde{\zeta}(p, h))$, where $\widetilde{\zeta}(p, h) \longrightarrow 0$ as $p \longrightarrow p_0 \in \mathcal{N}$ and $h \longrightarrow 0$ in the C^1-topologies.*

(vi) *Let $\mathcal{V}$ be the Yoneda module of $V \in \widetilde{\mathscr{C}}_{1/2}$. Then, with respect to the collection of homology units $\widetilde{\mathscr{E}}$ above we have*

$$\mathcal{V} \in U_m\big(\widetilde{\kappa}(p, h)\big),$$

where $\widetilde{\kappa}(p, h) \longrightarrow 0$ as $p \longrightarrow p_0 \in \mathcal{N}$ and $h \longrightarrow 0$ in the C^1-topologies.

In case the ends of γ have y-coordinates in $\mathbb{Z}$ an obvious analogue holds for the family of functors $\mathcal{I}'_{\gamma;p,h}$ from (3.27).

The proof of this proposition is straightforward, given the precise definition of the functors $\mathcal{I}_{\gamma;p,h}$ which is described in detail in [**BC14**, Section 4.2].

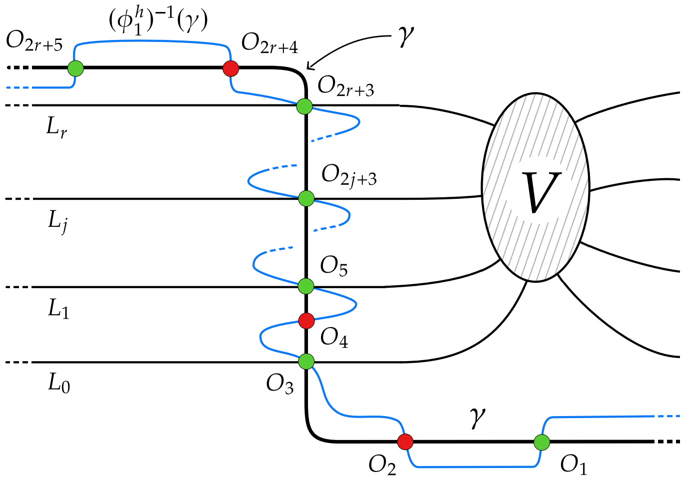

FIGURE 2. The curves γ, $(\phi_1^h)^{-1}(\gamma)$ and the cobordism V.

3.7. Weakly filtered iterated cones coming from cobordisms

Let $V \in \widetilde{\mathscr{C}}$ be a Lagrangian cobordism and denote by $L_0, \dots, L_r \in \mathscr{C}$ its negative ends. (In contrast to Section 3.4 as well as [BC14], in this section we index the negative ends from 0 to r rather than from 1 to r.)

Let $\gamma \subset \mathbb{R}^2$ be the curve depicted in Figure 2. Let $p \in E'_{\mathrm{reg}}$ and $h \in \mathscr{H}'_{\mathrm{prof}}(\gamma)$ be such that $l := \#(\phi_1^h)^{-1}(\gamma) \cap \gamma = 2r + 5$.

Denote by $\mathscr{V}$ the Yoneda module of V, which we view here as an A_∞-module over the category $\mathscr{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}}_{1/2}; \iota_\gamma(p, h))$. Consider now the pullback module

$$\mathscr{M}_{V;\gamma,p,h} := \mathscr{I}^*_{\gamma;p,h}\mathscr{V}, \tag{3.28}$$

which is a $\mathscr{F}uk(\mathscr{C}; p)$-module. Since $\mathscr{I}_{\gamma;p,h}$ is a weakly filtered functor the module $\mathscr{M}_{V;\gamma,p,h}$ is weakly filtered.

PROPOSITION 3.4. — *The weakly filtered module $\mathscr{M}_{V;\gamma,p,h}$ has the following properties.*

(i) *For every $N \in \mathscr{C}$ and $\alpha \in \mathbb{R}$ we have*

$$\mathscr{M}^{\leq \alpha}_{V;\gamma,p,h}(N) = \mathrm{CF}^{\leq \alpha - h(O_3)}(N, L_0; p) \oplus \mathrm{CF}^{\leq \alpha - h(O_5)}(N, L_1; p) \oplus \cdots$$
$$\cdots \oplus \mathrm{CF}^{\leq \alpha - h(O_{2r+3})}(N, L_r; p),$$

where the last equality is of Λ_0-modules (but not necessarily of chain complexes). Here $\mathrm{CF}(N, L_i; p)$ stands for $\mathrm{CF}(N, L_i; \mathscr{D}_{N,L_i})$, where $\mathscr{D}_{N,L_i}$ is the Floer datum prescribed by $p \in E'_{\mathrm{reg}}$.

(ii) *$\mathscr{M}_{V;\gamma,p,h}$ has discrepancy $\leq \epsilon^{\mathscr{M}_{V;\gamma,p,h}}$, where*

$$\epsilon_d^{\mathscr{M}_{V;\gamma,p,h}} \leq (d-1)\max\left\{h(O_k)\ ;\ 1 \leq k = \mathrm{odd} \leq 2r+5\right\} + \epsilon_d^{\mathscr{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}}_{1/2};\iota_\gamma(p,h))}. \tag{3.29}$$

Proof. — The second statement follows immediately from Proposition 3.3 and Lemma 2.2 together with the fact that the higher terms of $\mathcal{I}_{\gamma;p,h}$ vanish. The first statement can be verified by a straightforward calculation. ◻

3.7.1. ***Remark.*** — An inspection of the arguments from [**BC14**, Section 4.4] shows that the estimate for the discrepancy $\epsilon_d^{\mathcal{M}_{V;\gamma,p,h}}$ in (3.29) can be slightly improved by replacing the "max" term from (3.29) with $\max\{h(O_k) \ ; \ 3 \leq k = \text{odd} \leq 2r+3\}$. We will not go into details on that since this improvement will not play any role in our applications.

Recall from [**BC14**, Section 4.4] that the module $\mathcal{M}_{V;\gamma,p,h}$ is naturally isomorphic to an iterated cone with attaching objects corresponding to the ends $L_0, \ldots, L_r$ of V. More precisely, denote by $\mathcal{L}_j$ the Yoneda module corresponding to L_j. Then

$$\mathcal{M}_{V;\gamma,p,h} \cong \mathcal{C}one\big(\mathcal{L}_r \xrightarrow{\phi_r} \mathcal{C}one\big(\mathcal{L}_{r-1} \xrightarrow{\phi_{r-1}} \mathcal{C}one\big(\cdots \\ \cdots \mathcal{C}one\big(\mathcal{L}_2 \xrightarrow{\phi_2} \mathcal{C}one\big(\mathcal{L}_1 \xrightarrow{\phi_1} \mathcal{L}_0\big)\big)\cdots\big)\big),$$

where ϕ_j is a module homomorphism between $\mathcal{L}_j$ and the intermediate iterated cone involving the attachment of only the first $j+1$ objects $\mathcal{L}_0, \ldots, \mathcal{L}_j$.

As we will see shortly, the module homomorphisms ϕ_j are weakly filtered (and obviously the $\mathcal{L}_i$'s too) and consequently the iterated cone $\mathcal{M}_{V;\gamma,p,h}$ can be endowed with a weakly filtered structure by the algebraic recipe of Sections 2.4 and 2.6. At the same time, we have just seen that $\mathcal{M}_{V;\gamma,p,h}$ has another weakly filtered structure as it is the pull back module by an inclusion functor, as described in Proposition 3.4. Our goal now is to compare these two weakly filtered structures and show that they are essentially the same.

Consider the following collection of curves $\gamma_1, \ldots, \gamma_r \subset \mathbb{R}^2$ with horizontal ends, as depicted in Figure 3 next page. We assume that $\gamma_r = \gamma$, the curve involved in the definition of $\mathcal{M}_{V;\gamma,p,h}$.

We also choose profile functions $h_1, \ldots, h_r : \mathbb{R}^2 \to \mathbb{R}$ with $h_j \in \mathcal{H}'_{\text{prof}}(\gamma_j)$ and such that the following holds (see Figure 3):

1) $h_r = h$.
2) $(\phi_1^h)^{-1}(\gamma) \cap \gamma = \{O_1, \ldots, O_{2r+5}\}$
3) $(\phi_1^{h_j})^{-1}(\gamma_j) \cap \gamma_j = \{O_1^j, \ldots, O_{2j+5}^j\}$, where $O_k^j = O_k$ for all $1 \leq k \leq 2j+3$. Thus only the last two intersection points O_{2j+4}^j, O_{2j+5}^j do not belong to the γ_l's for $l > j$.
4) h_j coincides with h over the half-plane $\{y \leq y_{2j+3} + \frac{1}{100}\}$, where y_{2j+3} is the y-coordinate of O_{2j+3}.

We denote the space of all tuples of profile functions $(h_1, \ldots, h_r)$ satisfying these conditions by $\mathcal{H}'_{\text{prof}}(\gamma_1, \ldots, \gamma_r)$ and denote elements of this space by $v = (h_1, \ldots, h_r)$.

With this notation, it is possible as in (3.24) to choose maps

$$\iota_{\gamma_j} : E'_{\text{reg}} \times \mathcal{H}'_{\text{prof}}(\gamma_j) \longrightarrow \widetilde{E}'_{\text{reg},1/2}, \quad j = 1, \ldots, r,$$

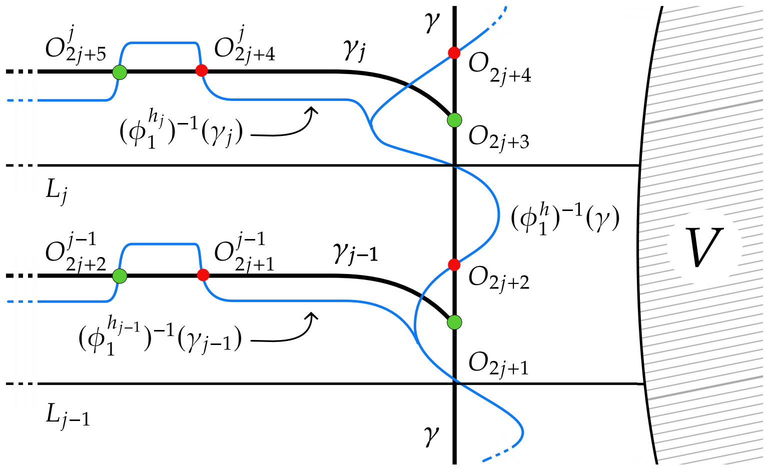

FIGURE 3. A closer look at the curves γ, γ_{j-1}, and γ_j near the $(j-1)$-th and j-th ends of V.

satisfying the following. For every $(p, v) \in E'_{\text{reg}} \times \mathcal{H}'_{\text{prof}}(\gamma_1, \ldots, \gamma_r)$ the choice of data $\iota_{\gamma_j}(p, h_j) \in \widetilde{E}'_{\text{reg},1/2}$ has the properties listed for $\iota_\gamma(p, h)$ on page 69 but with γ replaced by γ_j and h by h_j. (Consequently, for every $\widetilde{p}_0 \in \mathcal{N}$ we have $\lim \iota_{\gamma_j}(p, h_j) \in \widetilde{\mathcal{N}}_{1/2}$ as $p \to p_0$ and $v = (h_1, \ldots, h_r) \to (0, \ldots, 0)$.) Moreover, we require that for every j and $p \in E'_{\text{reg}}$, $v = (h_1, \ldots, h_r) \in \mathcal{H}'_{\text{prof}}(\gamma_1, \ldots, \gamma_r)$ the data prescribed by $\iota_{\gamma_j}(p, h_j)$ is compatible with that prescribed by $\iota_{\gamma_{j-1}}(p, h_{j-1})$ (in the obvious sense, similar to h_j being compatible with h_{j-1}). By Section 3.6, the curves γ_j and the maps ι_{γ_j} induce a family of inclusion functors

$$\mathcal{I}_{\gamma_j;p,h_j} : \mathcal{F}uk(\mathcal{C}; p) \longrightarrow \mathcal{F}uk\big(\widetilde{\mathcal{C}}_{1/2}; \iota_{\gamma_j}(p, h_j)\big),$$

parametrized by $p \in E'_{\text{reg}}$, $v = (h_1, \ldots, h_r) \in \mathcal{H}'_{\text{prof}}(\gamma_1, \ldots, \gamma_r)$. We will use below the notation

$$\mathcal{I}_{\gamma_j;p,v} := \mathcal{I}_{\gamma_j;p,h_j},$$

where h_j is the j-th entry in the tuple v since it reflects better the parameters (p, v) parametrizing this family of functors. We will also write $\iota_{\gamma_j}(p, v)$ for $\iota_{\gamma_j}(p, h_j)$ sometimes.

Consider now the pullback $\mathcal{F}uk(\mathcal{C}; p)$-modules

$$\mathcal{M}_{V;\gamma_j,p,v} = \mathcal{I}^*_{\gamma_j;p,v}\mathcal{V}, \quad j = 1, \ldots, r. \tag{3.30}$$

We endow each of these modules with its weakly filtered structure as defined at the beginning of Section 3.7 and further described by Proposition 3.4 (where $l = 2j + 5$, and γ should be replaced by γ_j and h by h_j). Next, for every $0 \leq j \leq r$ denote by $\mathcal{L}_j$ the Yoneda module associated to L_j, endowed with its weakly filtered structure induced from $\mathcal{F}uk(\mathcal{C}; p)$. Finally, recall that for a weakly filtered module $\mathcal{M}$ and $\nu \in \mathbb{R}$, $S^\nu\mathcal{M}$ stands for the weakly filtered module obtained from $\mathcal{M}$ by an action-shift of ν (see §2.3.4).

Proposition 3.5. — *For every* $(p,v) \in E'_{\mathrm{reg}} \times \mathcal{H}'_{\mathrm{prof}}(\gamma_1,\ldots,\gamma_r)$ *there exist weakly filtered module homomorphisms*

$$\phi_1 : \mathcal{L}_1 \longrightarrow \mathcal{L}_0 \quad \textit{and} \quad \phi_j : \mathcal{L}_j \longrightarrow S^{h(O_3)}\mathcal{M}_{V;\gamma_{j-1},p,v} \quad (j = 2,\ldots,r)$$

such that the following holds for every $1 \leq j \leq r$*:*

(i) ϕ_j *shifts action by* ≤ 0.

(ii) *The discrepancy of* ϕ_j *is* $\leq \boldsymbol{\delta}^{\phi_j}$*, where*

$$\delta_d^{\phi_j} := (d-1) \max_{\substack{1 \leq k \leq 2j+3 \\ k \text{ odd}}} h(O_k) + \epsilon_d^{\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}_{1/2};\iota_{\gamma_j}(p,v))} + h(O_{2j+3}) - h(O_3). \tag{3.31}$$

(iii) *For every* $1 \leq j \leq r$, $S^{h(O_3)}\mathcal{M}_{V;\gamma_j,p,v} = \mathcal{C}one(\phi_j; 0, \boldsymbol{\delta}^{\phi_j})$ *as weakly filtered module. (See Section 2.4 for our conventions for weakly filtered cones.) In other words, the weakly filtered module* $S^{h(O_3)}\mathcal{M}_{V;\gamma_j,p,v}$ *coincides with the weakly filtered mapping cone over* ϕ_j.

Recalling that $\mathcal{M}_{V;\gamma,p,h} = \mathcal{M}_{V;\gamma_r,p,v}$, the above proposition implies that

$$S^{h(O_3)}\mathcal{M}_{V;\gamma,p,h} = \mathcal{C}one(\mathcal{L}_r \xrightarrow{\overline{\phi}_r} \mathcal{C}one(\mathcal{L}_{r-1} \xrightarrow{\overline{\phi}_{r-1}} \mathcal{C}one(\cdots \cdots \mathcal{C}one(\mathcal{L}_2 \xrightarrow{\overline{\phi}_2} \mathcal{C}one(\mathcal{L}_1 \xrightarrow{\overline{\phi}_1} \mathcal{L}_0))\cdots))), \tag{3.32}$$

where $\overline{\phi}_j := (\phi_j; 0, \boldsymbol{\delta}^{\phi_j})$ and the cones in (3.32) are endowed with the filtrations as defined in Section 2.4. In other words, up to a small action-shift, $\mathcal{M}_{V;\gamma,p,h}$ can be viewed as a weakly filtered iterated cone by the very same recipe described at the beginning of Section 2.6 (with $\rho_j = 0$ and $\mathcal{K}_j = S^{h(O_3)}\mathcal{M}_{V;\gamma_j,p,v}$). Consequently, in our geometric applications we can use Theorem 2.14 for $\mathcal{K}_r = S^{h(O_3))}\mathcal{M}_{V;\gamma,p,h}$.

Proof of Proposition 3.5. — The proof is based on two main ingredients. The first one is the theory developed in [**BC14**, Sections 4.2, 4.4] from which it follows that, ignoring action-filtrations, we have $\mathcal{M}_{V;\gamma_j,p,v} = \mathcal{C}one(\mathcal{L}_j \to \mathcal{M}_{V;\gamma_{j-1},p,v})$. The second one comprises direct action-filtration calculations for the modules $\mathcal{M}_{V;\gamma_j,p,v}$ and the homomorphisms ϕ_j.

Before we go on, we should remark a notational difference between [**BC14**] and the present paper. In [**BC14**] the negative ends of the cobordism V are indexed from 1 to r, whereas in the present text the indexing runs between 0 and r. This results in several other indexing differences between the two texts. For example, the curves γ_j in the present text are the same as γ_{j+1} in [**BC14**]. In the present text, the number of intersection points between $\phi_1^{h_j}(\gamma_j)$ and γ_j is $2j+5$, whereas in [**BC14**] this number is $2j+3$, *etc.*

We start by adding to the collection of curves $\gamma_1,\ldots,\gamma_r$ another curve γ_0, defined in the same way as the γ_j's only that it is adapted to the L_0-end of V in the sense that the negative end of γ_0 goes above the L_0-end and below the L_1 end. We also choose $h_0 \in \mathcal{H}'_{\mathrm{prof}}(\gamma_0)$ satisfying the same conditions as the h_j's (see page 72) only for $j = 0$.

We write

$$(\phi_1^* h_0)^{-1}(\gamma_0) \cap \gamma_0 = \{O_1, O_2, O_3, O_4^0, O_5^0\}.$$

To simplify the notation we also extend the tuple $v = (h_1, \dots, h_r)$ to contain also h_0 and write $v = (h_0, \dots, h_r)$. As before we have an inclusion functor associated to γ_0, p, h_0 and we consider the pullback module

$$\mathcal{M}_{V;\gamma_0,p,h_0} := \mathcal{I}^*_{\gamma_0;p,h_0} \mathcal{V}.$$

We will denote this module also by $\mathcal{M}_{V;\gamma_0,p,v}$ to be consistent with the previous notation.

We first claim that there exist module homomorphisms $\phi_j : \mathcal{L}_j \to \mathcal{M}_{V;\gamma_{j-1},p,h_{j-1}}$ for all $1 \leq j \leq r$, such that

$$\mathcal{M}_{V;\gamma_j,p,h_j} = \mathcal{C}one\big(\mathcal{L}_j \xrightarrow{\phi_j} \mathcal{M}_{V;\gamma_{j-1},p,h_{j-1}}\big), \tag{3.33}$$

where at the moment we ignore the action filtrations. This statement is not explicitly stated in [**BC14**, Section 4.4.2], but it follows easily from the arguments in that paper. More specifically, what is stated explicitly in [**BC14**, Section 4.4.2] is that there exists an exact triangle – in the derived category $D\mathcal{F}uk(\mathcal{C}; p)$ – of the form

$$\mathcal{L}_j \longrightarrow \mathcal{M}_{V;\gamma_{j-1},p,h_{j-1}} \longrightarrow \mathcal{M}_{V;\gamma_j,p,h_j}.$$

Here however, we claim a stronger statement, namely that (3.33) holds at the chain level. We will now explain how to deduce (3.33) from the theory developed in [**BC14**]. In doing that we will mostly follow the notation from that paper.

By [**BC14**, Proposition 4.4.1] for every $0 \leq j \leq r$ we have the following:

1) A_∞-categories $\mathcal{B}_j$ and $\mathcal{B}'_j$ (depending on γ_j, p and h_j).

2) Quasi-isomorphisms of A_∞-categories: $e_j : \mathcal{F}uk(\mathcal{C}; p) \to \mathcal{B}_j$, $p_j : \mathcal{B}_j \to \mathcal{B}'_j$, $\sigma_j : \mathcal{B}'_j \to \mathcal{F}uk(\mathcal{C}; p)$ and $q_j : \mathcal{B}'_j \to \mathcal{B}'_{j-1}$, for $j \geq 1$, all with vanishing higher order terms. Moreover, they satisfy:

$$\sigma_j \circ p_j \circ e_j = \mathrm{id}, \quad \text{for all } j \geq 0, \quad \text{and} \quad q_j \circ p_j \circ e_j = p_{j-1} \circ e_{j-1} \quad \text{for all } j \geq 1. \tag{3.34}$$

3) A $\mathcal{B}_j$-module $\overline{\mathcal{M}}_j$ and a $\mathcal{B}'_j$-module $\mathcal{M}'_j$ such that

$$\begin{aligned} &\mathcal{M}_{V;\gamma_j,p,h_j} = e_j^* \overline{\mathcal{M}}_j, \quad p_j^* \mathcal{M}'_j = \overline{\mathcal{M}}_j, \quad \forall j \geq 0, \\ &\mathcal{M}'_j = \mathcal{C}one\big(\sigma_j^* \mathcal{L}_j \xrightarrow{\varphi_j} q_j^* \mathcal{M}'_{j-1}\big), \quad \forall j \geq 1, \end{aligned} \tag{3.35}$$

for some module homomorphism φ_j. (This homomorphism was denoted by ϕ_j in [**BC14**, Proposition 4.4.1]. We have denoted it here by φ_j since ϕ_j is already used for a slightly different homomorphism.)

4) For $j = 0$ we have $\mathcal{M}'_0 = \sigma_0^* \mathcal{L}_0$.

We now pull back the second line of (3.35) by the functor $p_j \circ e_j$. The desired equality (3.33) now follows by using (3.34) together with the fact that A_∞-functors pull back mapping cones to mapping cones (at the chain level). Note that for $j = 0$, pulling back the equality from point (3.7.1) above yields: $\mathcal{M}_{V;\gamma_0,p,h_0} = \mathcal{L}_0$.

We now turn to the weakly filtered setting. Throughout the rest of the proof it is useful to keep in mind that $h_j(O_k) = h(O_k)$ for every $0 \leq j \leq r$ and $1 \leq k \leq 2j+3$.

We claim that in the weakly filtered setting the correct version of (3.33) is

$$(3.36)\quad \begin{aligned} & S^{h(O_3)}\mathcal{M}_{V;\gamma_j,p,v} = \mathcal{C}one\big(\mathcal{L}_j \xrightarrow{(\phi_j;0,\boldsymbol{\delta}^{\phi_j})} S^{h(O_3)}\mathcal{M}_{V;\gamma_{j-1},p,v}\big), \quad \text{for all } 1 \leq j \leq r, \\ & S^{h(O_3)}\mathcal{M}_{V;\gamma_0,p,v} = \mathcal{L}_0. \end{aligned}$$

Of course, by Lemma 2.4, the first line of (3.36) is equivalent to:

$$(3.37)\quad \mathcal{M}_{V;\gamma_j,p,v} = \mathcal{C}one\big(\mathcal{L}_j \xrightarrow{(\phi_j;0,\boldsymbol{\delta}^{\phi_j}+h(O_3))} \mathcal{M}_{V;\gamma_{j-1},p,v}\big), \quad \text{for all } 1 \leq j \leq r.$$

To prove (3.37) one needs to go over the arguments in the proof of [**BC14**, Proposition 4.4.1] and take action-filtrations into consideration. An inspection of these arguments shows that the categories $\mathcal{B}_j$, $\mathcal{B}'_j$ and functors e_j, p_j, σ_j, q_j are all weakly filtered, and so are the modules $\mathcal{M}'_j$ and $\overline{\mathcal{M}}_j$. Moreover, we have:

1) The discrepancies of both $\mathcal{B}_j$ and $\mathcal{B}'_j$ are $\leq \boldsymbol{\epsilon}^{\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}_{1/2};\iota_{\gamma_j}(p,h_j))}$.
2) Both functors p_j and q_j are filtered, *i.e.* have discrepancies $\leq \mathbf{0}$.
3) e_j has discrepancy $\leq \boldsymbol{\epsilon}^{e_j}$, where $\epsilon_1^{e_j} = \max\{h_j(O_k^j)\ ;\ 1 \leq k = \text{odd} \leq 2j+5\}$ and $\epsilon_d^{e_j} = 0$ for all $d \geq 2$.
4) $p_j \circ e_j$ has discrepancy $\leq \boldsymbol{\epsilon}^{p_j\circ e_j}$, where
$$\epsilon_1^{p_j\circ e_j} = \max\big\{h(O_k)\ ;\ 1 \leq k = \text{odd} \leq 2j+3\big\}$$
and $\epsilon_d^{p_j\circ e_j} = 0$ for all $d \geq 2$.
5) σ_j has discrepancy $\leq \boldsymbol{\epsilon}^{\sigma_j}$, where $\epsilon_1^{\sigma_j} = -h(O_{2j+3})$ and $\epsilon_d^{\sigma_j} = 0$ for all $d \geq 2$.
6) The module homomorphism $\varphi_j : \sigma_j^*\mathcal{L}_j \longrightarrow q_j^*\mathcal{M}'_{j-1}$ shifts action by ≤ 0 and has discrepancy $\leq \boldsymbol{\epsilon}^{\varphi_j}$, where
$$\epsilon_d^{\varphi_j} = \epsilon_d^{\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}_{1/2};\iota_{\gamma_j}(p,v))} + h(O_{2j+3}).$$
7) The modules $\mathcal{M}'_j$ and $\overline{\mathcal{M}}'_j$ have discrepancies $\leq \boldsymbol{\epsilon}^{\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathcal{C}}_{1/2};\iota_{\gamma_j}(p,h_j))}$.
8) The equalities (or identifications) from (3.35) hold also in the weakly filtered sense, where the cone over φ_j on the second line of (3.35) is now taken over $(\varphi_j; 0, \boldsymbol{\epsilon}^{\varphi_j})$.
9) $\mathcal{M}'_0 = S^{-h(O_3)}\sigma_0^*\mathcal{L}_0$ as weakly filtered modules.

To conclude the proof of (3.37) we pull back the weakly filtered version of the second line of (3.35) by $p_j \circ e_j$ and use Lemmas 2.7, 2.2 and 2.3 (recall that p_j, e_j do not have higher order terms). The assertion that $S^{h(O_3)}\mathcal{M}_{V;\gamma_0,p,v} = \mathcal{L}_0$ follows in a similar way. □

CHAPTER 4

QUASI-EXACT AND QUASI-MONOTONE COBORDISMS

For reasons that will become apparent when we introduce shadow metrics in Chapter 6 we need to extend some of the theory from Chapter 3, especially from Section 3.7, to the cases of quasi-exact and quasi-monotone cobordisms. Quasi-exact cobordisms form a larger class than the usual weakly-exact cobordisms considered earlier in the paper but, from the point of view of J-holomorphic machinery, they behave in the the same way except that only for particular classes of almost complex structures J. The same applies to quasi-monotone cobordisms versus monotone ones.

4.1. Quasi-exact cobordisms

Fix a symplectic manifold (M, ω), as at the beginning of Section 3 and denote by

$$\mathcal{L}ag^{\mathrm{we}}(M)$$

the class of weakly-exact Lagrangian submanifolds $L \subset M$. As before, we write

$$(\widetilde{M}, \widetilde{\omega}) = (\mathbb{R}^2 \times M, \omega_{\mathbb{R}^2} \oplus \omega)$$

and denote by $\pi : \widetilde{M} \to \mathbb{R}^2$ the projection.

We begin with a simple definition that will be useful in the following.

Definition 4.1. — Let $V \subset \mathbb{R}^2 \times M$ be a Lagrangian cobordism and let $K_V \subset \mathbb{R}^2$ be a subset with compact closure. We say that V is *cylindrical* over $\mathbb{R}^2 \setminus K_V$ if over $\mathbb{R}^2 \setminus K_V$ the cobordism V is equal to a disjoint union with terms $\gamma_k \times L_k$ where γ_k are pairwise disjoint, unbounded, connected, and embedded curves in the plane, horizontal at infinity, and $L_k \subset M$ are Lagrangians.

Next, we introduce *quasi-exact* Lagrangian cobordisms (with weakly exact ends).

Definition 4.2. — Let $V \subset \widetilde{M}$ be a Lagrangian cobordism with ends in $\mathcal{L}ag^{\mathrm{we}}(M)$. We say that V is *quasi-exact* if there is a compact subset $K_V \subset \mathbb{R}^2$ and an $\widetilde{\omega}$-compatible almost complex structure J_V such that:

1) V is cylindrical over $\mathbb{R}^2 \setminus \mathrm{Int}\,(K_V)$.

2) π is (J_V, i)-holomorphic over $\mathbb{R}^2 \setminus \mathrm{Int}(K_V)$.

3) There are no non-constant J_V-holomorphic disks $u : (D, \partial D) \to (\widetilde{M}, V)$.

Sometimes we will say that (V, J_V, K_V) is *quasi-exact*.

A pair (J_V, K_V) as above will be called *quasi-exact admissible for* V. Sometimes the focus will be on the subset K_V, and we will say that K_V is *quasi-exact admissible for* V if there exists J_V such that (V, J_V, K_V) is quasi-exact.

We denote by $\mathcal{L}ag^{q,\mathrm{we}}(\mathbb{R}^2 \times M)$ the collection of quasi-exact Lagrangian cobordisms $V \subset \mathbb{R}^2 \times M$.

4.1.1. *Remarks*

1) If $V \subset \widetilde{M}$ is quasi-exact then V must have at least one (non-void) end. Indeed, if V has no ends at all, then V is a closed Lagrangian submanifold of $\mathbb{R}^2 \times M$ and so it can be displaced by a (compactly supported) Hamiltonian diffeomorphism. By standard results, for every $\widetilde{\omega}$-compatible almost complex structure $\widetilde{J}$ there exists a non-constant J_V-holomorphic disk with boundary on V, contradicting the quasi-exactness of V.

2) If $V \subset \widetilde{M}$ is quasi-exact, then necessarily M is weakly-exact in the sense that $\int_S \omega = 0$ for every $A \in H_2^S(M)$, where $H_2^S(M) \subset H_2(M)$ is the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$. Obviously the same holds also for $\widetilde{M}$. In particular neither M nor $\widetilde{M}$ has non-constant pseudo-holomorphic spheres, for any compatible almost complex structure.

Indeed, by point (4.1.1) above, V has at least one (non-void) end, say L. By assumption $L \subset M$ is weakly-exact, hence so is M.

3) The condition that $\pi : \widetilde{M} \to \mathbb{R}^2$ is $(\widetilde{J}, i)$-holomorphic over a subset $S \subset \mathbb{R}^2$ is equivalent to $\widetilde{J}$ being fiberwise split over S. The space of $\widetilde{\omega}$-compatible almost complex structures $\widetilde{J}$ that are fiberwise split over $S \subset \mathbb{R}^2$ is path-connected (and, in fact, contractible).

4) One can also define quasi-exact cobordisms with ends being quasi-exact Lagrangians (not just weakly-exact). We will not pursue this degree of generality here.

4.1.2. ***Examples.*** **—** Here are several examples of quasi-exact cobordisms.

1) Weakly-exact cobordisms.

2) Cobordisms $V \subset \mathbb{R}^2 \times M$, where $\dim_{\mathbb{R}} M = 2$ and $\mu = 0$ on $\pi_2(\widetilde{M}, V)$.

3) More generally, cobordisms $V \subset \mathbb{R}^2 \times M$ with $\mu(A) \leq 1 - \frac{1}{2}\dim_{\mathbb{R}}(M)$, for all $A \in \pi_2(\widetilde{M}, V)$ with $\widetilde{\omega}(A) > 0$.

4) As will be seen in Proposition 6.2 in Section 6.1, compositions of quasi-exact cobordisms (along a pair of matching ends) are quasi-exact.

4.2. Extending the results from Section 3.7 to quasi-exact cobordisms

Let $V \subset \widetilde{M}$ be a quasi-exact Lagrangian cobordism with ends in $\mathcal{L}ag^{\mathrm{we}}(M)$. Fix a quasi-exact admissible pair (J_V, K_V). Let $\gamma \subset \mathbb{R}^2$ be a plane curve with horizontal ends, *e.g.* as depicted in Figure 2, page 71. Assume in addition that:

1) $\gamma \subset \mathbb{R}^2 \setminus K_V$.
2) γ intersects $\pi(V)$ only along the horizontal rays associated to the ends of V (be they on the negative or positive side of V) and γ intersects each such ray at most once. Moreover these intersections are transverse.

Fix p and h as at the beginning of Section 3.7. Denote by $\mathscr{C}$ the collection of weakly-exact Lagrangians in M and by $\widetilde{\mathscr{C}}$ the collection of weakly-exact Lagrangian cobordisms in $\mathbb{R}^2 \times M$. As in Section 3.7 we have the Fukaya categories $\mathcal{F}uk(\mathscr{C}; p)$ and $\mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}}_{1/2}; \iota_\gamma(p, h))$. Note that, unless V is weakly-exact, V is not an object of the latter category.

Consider now the (full) subcategory $\mathcal{F}uk_{\mathrm{cob},\mathscr{C},\gamma} \subset \mathcal{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}}_{1/2}; \iota_\gamma(p, h))$ whose objects are $\gamma \times N$ with $N \in \mathscr{C}$.

We will define now a $\mathcal{F}uk_{\mathrm{cob},\mathscr{C},\gamma}$-module $\mathcal{V}_{\mathrm{qe}}$ associated to V, constructed in an analogous way to the Yoneda module $\mathcal{V}$ from Section 3.7. More precisely, we set $\mathcal{V}_{\mathrm{qe}}(\gamma \times N) = \mathrm{CF}(\gamma \times N, V; \mathscr{D}_{\gamma \times N, V})$ and define the higher A_∞-module operations $\mu_d^{\mathcal{V}_{\mathrm{qe}}}$ as for a Yoneda module (associated to V) but with the following modifications for the Floer and perturbations data $\mathscr{D} = \mathscr{D}_{\gamma \times N_1, \ldots, \gamma \times N_d, V} = (\widetilde{K}, \widetilde{J})$:

(P1) We force the transition functions $a_r : S_r \to [0, 1]$ to be identically 0 on the arc $\partial_V S_r$ corresponding to V. See (3.20) on page 63 for how the transition functions are incorporated into the perturbation data. See also [**BC14**, pp. 1757–1759 and 1762–1764], for more details.

(P2) The almost complex structures $\widetilde{J}$ in the perturbation data $\mathscr{D} = (\widetilde{K}, \widetilde{J})$ are such that $\widetilde{J}|_{\partial_V S_r} = J_V$.

As usual we make the preceding choices of perturbation data to be consistent with the compactification of the spaces $\mathcal{R}^{d+1}$, $d \geq 2$, of punctured disks (in other words, the perturbation data can be chosen to be consistent with breaking and gluing).

Before we proceed, here are a few important remarks explaining why this type of perturbation data makes sense at all, and why it does not collide with other aspects of the construction coming from [**BC14**]. First note that as we are only aiming at defining a module $\mathcal{V}_{\mathrm{qe}}$ over $\mathcal{F}uk_{\mathrm{cob},\mathscr{C},\gamma}$, the Floer polygons $u : S_r \to \widetilde{M}$ involved in the definition of $\mu_d^{\mathcal{V}_{\mathrm{qe}}}$ map the last arc along the boundary of S_r to V, and all other arcs to Lagrangians of the type $\gamma \times N_i$. Moreover, as γ is transverse to the rays of $\pi(V)$, then when defining the $\mu_d^{\mathcal{V}_{\mathrm{qe}}}$-operations we do not need to perform any horizontal perturbation (in the $\mathbb{R}^2$-direction) for strip-like ends corresponding to $(\gamma \times N_d, V)$ and $(\gamma \times N_1, V)$. Thus vertical perturbations (in the M-direction) are enough. Therefore we can force the transition functions a_r to be 0 along $\partial_V S_r$.

Recall also that apart from (P2) above, the almost complex structures $\widetilde{J}$ in the perturbation data also have a restricted form, as described in [**BC14**] page 1764, namely they should satisfy that the projection π is $(\widetilde{J}_z, (\phi^h_{a_r(z)})_*(i))$-holomorphic for every $r \in \mathcal{R}^{d+1}$, $z \in S_r$, over the complement of some compact subset in $\mathbb{R}^2$. We note that the latter condition is compatible with (P2) above because $a_r = 0$ along $\partial_V S_r$ and because π is (J_V, i)-holomorphic over the complement of a compact subset $K_V \subset \mathbb{R}^2$.

Finally, it is straightforward to see that a consistent choice of perturbation data as described above indeed exists.

We now claim that with these modification the $\mu_d^{\mathcal{V}_{\mathrm{qe}}}$-operations are well defined and satisfy the A_∞-module identities. To see this we need to address the following points: compactness and transversality of the relevant spaces of Floer polygons (defined using the preceding perturbation data), and finally, that the A_∞-module identities indeed hold.

Assuming compactness and transversality, the last point easily follows from the fact that the perturbation data can be chosen in a consistent way.

For transversality, the arguments used in [**BC14**, Sections 3.4 and 4.3] (see also [**BCb**, Section 4.3.2 and Remark 4.3.5]) can be easily adapted to the present setting. The point is that imposing conditions (P1) and (P2) has no effect on transversality for the spaces of Floer trajectories, since these conditions affect the values of a_r and $\widetilde{J}$ only along $\partial_V S_r$ while in the interior of S_r we can perform arbitrary perturbations (subject to [**BC14**, pp. 1762–1764]).

We now address compactness. Here there are two separate issues to take care of. The first one is to verify that all Floer polygons (with fixed input and output chords) lie in a compact region of $\widetilde{M}$. The second issue is to control bubbling of holomorphic disks and spheres (recall that V is not assumed to be weakly-exact anymore but only quasi-exact).

The first point can be dealt with by the same arguments as in [**BC14**, Section 3.3]. Indeed, since γ is assumed to be transverse to the rays of $\pi(V)$ corresponding to the ends, condition (P1) does not interfere with the arguments from [**BC14**, Section 3.3]. Condition (P2) works well with the the arguments from [**BC14**, Section 3.3] since π is (J_V, i) holomorphic over $\mathbb{R}^2 \setminus K_V$. This concludes the argument showing that all Floer polygons lie within a compact region of $\widetilde{M}$.

Finally, we claim that in our setting no bubbling of holomorphic disks or spheres can occur. Indeed, by condition (P2) $\widetilde{J}|_{\partial_V S_r} = J_V$ and by assumption there are no non-constant J_V-holomorphic disks with boundary on V. Therefore, bubbling of disks cannot occur at the $\partial_V S_r$ arc. As all the Lagrangians corresponding to the other arcs of S_r are weakly-exact (they are of the type $\gamma \times N$ with $N \in \mathcal{C}$) bubbling of disks cannot occur at these arcs too. Bubbling of holomorphic disks is also impossible since by point (4.1.1) of Remark 4.1.1, $\widetilde{M}$ is a weakly-exact symplectic manifold.

This concludes the definition of the $\mathcal{F}uk_{\mathrm{cob},\mathcal{C},\gamma}$-module $\mathcal{V}_{\mathrm{qe}}$.

4.2.1. ***Remark***. — The module $\mathcal{V}_{\mathrm{qe}}$ is, strictly speaking, not a Yoneda module (although it is defined in an analogous way to Yoneda modules). The reason is that a quasi-exact (but not weakly-exact) cobordism V is not an object of $\mathcal{F}uk_{\mathrm{cob},\mathscr{C},\gamma}$, nor of any other A_∞-category we are considering in this paper. It is possible to set up an A_∞-category whose objects are quasi-exact cobordisms, by further modifications of the construction above. But this is not needed for the applications in this paper and so we will not pursue this direction here.

We continue with extending the constructions from Sections 3.6 and 3.7 to the quasi-exact setting.

Let $V \subset \mathbb{R}^2 \times M$ be a cobordism as at the beginning of Section 3.7, only that now we assume that V is only quasi-exact. Let γ, p, h be as in Section 3.7. Consider also the module $\mathscr{V}_{\mathrm{qe}}$ as constructed above. Note that the inclusion functor $\mathcal{I}_{\gamma;p,h}$ has its image in $\mathcal{F}uk_{\mathrm{cob},\mathscr{C},\gamma}$ hence can be viewed as a functor

$$\mathcal{I}_{\gamma;p,h} : \mathcal{F}uk(\mathscr{C};p) \longrightarrow \mathcal{F}uk_{\mathrm{cob},\mathscr{C},\gamma}.$$

By analogy to (3.28) we define a $\mathcal{F}uk(\mathscr{C};p)$-module:

$$\mathscr{M}^{\mathrm{qe}}_{V;\gamma,p,h} := \mathcal{I}^*_{\gamma;p,h}\mathscr{V}_{\mathrm{qe}}. \tag{4.1}$$

We define also the modules $\mathscr{M}^{\mathrm{qe}}_{V;\gamma_j,p,v}$, $j = 1, \ldots, r$, in the same way as in (3.30), only that we now use the module $\mathscr{V}_{\mathrm{qe}}$ instead of the Yoneda module $\mathscr{V}$.

Proposition 4.3. — *The statements of Propositions 3.4 and 3.5 continue to hold for the modules $\mathscr{M}^{\mathrm{qe}}_{V;\gamma,p,h}$ and $\mathscr{M}^{\mathrm{qe}}_{V;\gamma_j,p,v}$ that have just been defined.*

The proof is exactly the same as the proofs of Propositions 3.4 and 3.5.

4.3. Quasi-monotone cobordisms

By analogy to the "quasi-exact vs. weakly-exact" case, there is also a similar notion of quasi-monotone cobordisms generalizing monotone ones.

Fix $\mathbf{d} := dT^a \in \Lambda_0$, where $d \in \mathbb{Z}_2$, $a > 0$. As in Section 3.5 denote by $\mathscr{L}ag^{\mathrm{mon},\mathbf{d}}(M)$ the class of closed monotone Lagrangians $L \subset M$ with $\mathbf{d}_L = \mathbf{d}$. Note that existence of a monotone Lagrangian in M implies that the ambient manifold M is monotone too. In particular, for every $A \in \pi_2(M)$ with $\omega(A) > 0$ we have $c_1(A) > 0$.

Definition 4.4. — Let $V \subset \mathbb{R}^2 \times M$ be a Lagrangian cobordism with ends in $\mathscr{L}ag^{\mathrm{mon},\mathbf{d}}(M)$, *not all void*. We say that V is quasi-monotone if there is a compact subset $K_V \subset \mathbb{R}^2$ and an $\widetilde{\omega}$-compatible almost complex structure J_V such that:

1) V is cylindrical over $\mathbb{R}^2 \setminus \mathrm{Int}(K_V)$.
2) π is (J_V, i)-holomorphic over $\mathbb{R}^2 \setminus \mathrm{Int}(K_V)$.
3) For all J_V-holomorphic disks $u : (D, \partial D) \to (\widetilde{M}, V)$ we have $\mu(u) \geq 2$.

As in the quasi-exact case we will call (J_V, K_V) *quasi-monotone* admissible for V, and sometimes say that (V, J_V, K_V) is quasi-monotone.

We denote the class of quasi-monotone cobordisms V as above by

$$\mathscr{L}ag^{\mathrm{qm},\mathbf{d}}(\mathbb{R}^2 \times M).$$

The parameter $\mathbf{d}$ indicates the value of $\mathbf{d}_{L_i}$ for the ends L_i of $V \in \mathscr{L}ag^{\mathrm{qm},\mathbf{d}}(\mathbb{R}^2 \times M)$.

In the following we will need the following lemma, which is valid both in the quasi-exact and quasi-monotone cases.

Lemma 4.5. — *Let (V, J_V, K_V) be quasi-exact (resp. quasi-monotone). Let J'_V be another $\widetilde{\omega}$-compatible almost complex structure such that $J'_V = J_V$ over K_V and J'_V is fiberwise split over $\mathbb{R}^2 \setminus \mathrm{Int}(K_V)$. Then (V, J'_V, K_V) is also quasi-exact (resp. quasi-monotone).*

Proof. — This is an immediate application of the open mapping theorem, combined with the weak exactness (resp. monotonicity) of the ends of V. Indeed, by the open mapping theorem we deduce that any J'_V-holomorphic disk with boundary on V has to have its image inside $\pi^{-1}(K_V)$. As (V, J_V, K_V) is quasi-exact (resp. quasi-monotone) and $J_V = J'_V$ over K_V this implies the claim. □

4.4. Extending the results from Section 3.7 to quasi-monotone cobordisms

This is similar to Section 4.2 only that now we have to take care of bubbling of holomorphic disks. The goal is to construct the modules $\mathcal{V}_{qm}$ and $\mathcal{M}^{qm}_{V;\gamma,p,h}$ analogous to $\mathcal{V}^{\mathrm{qe}}$ and $\mathcal{M}^{\mathrm{qe}}_{V;\gamma,p,h}$. For brevity, denote by

$$\mathcal{C} = \mathcal{L}ag^{\mathrm{mon},\mathbf{d}}(M).$$

Let $V \subset \widetilde{M}$ be a quasi-monotone Lagrangian cobordism with ends in $\mathcal{C}$. Fix a quasi-monotone admissible pair (J_V, K_V).

Let γ and p, h be as in Section 4.2. Recall that in the monotone Fukaya category of M the choices of the Floer data prescribed by p are assumed to satisfy the following additional conditions. Let $K_0, K_1 \in \mathcal{C}$ and $\mathcal{D}_{K_0,K_1} = (H^{K_0,K_1}, \{J_t^{K_0,K_1}\})$ be the Floer datum of (K_0, K_1) prescribed by p. Let $\eta \in \mathcal{O}(H^{K_0,K_1})$ be a Hamiltonian chord. Then for both $\nu = 0$ and $\nu = 1$, the almost complex structure $J_\nu^{K_0,K_1}$ is regular for all $J_\nu^{K_0,K_1}$-holomorphic disks with boundary on K_ν that have Maslov index 2 and moreover $\eta(\nu) \in K_\nu$ is a regular value of the evaluation maps $\mathrm{ev}_{K_\nu,A} : (\mathcal{M}(A, J_\nu^{K_0,K_1}) \times \partial D)/\mathrm{Aut}(D) \longrightarrow K_\nu$, for all $A \in \pi_2(M, K_\nu)$ with $\mu(A) = 2$. (And of course, by assumption the $\sum_A \deg_{\mathbb{Z}_2} \mathrm{ev}_{K_\nu,A} = d$, where the sum is over all $A \in \pi_2(M, K_\nu)$ with $\mu(A) = 2$. Here $d \in \mathbb{Z}_2$ is the coefficient of T^a in $\mathbf{d}$, *i.e.* $\mathbf{d} = dT^a$.)

Fix $N \in \mathcal{C}$. Let $J^{\gamma\times N,V} = \{J_t^{\gamma\times N,V}\}$ be a time-dependent $\widetilde{\omega}$-compatible almost complex satisfying the following properties:

1) For each intersection point $x \in \gamma \cap \pi(V)$, denote by $L_x \subset M$ the Lagrangian corresponding to the end of V over x. We require that $J_t^{\gamma\times N,V} = i \oplus J_t^{N,L_x}$ in $U_x \times M$ for some small neighborhood U_x of x which is contained in $\mathbb{R}^2 \setminus K_V$. Here, $\{J_t^{N,L_x}\}$ is the choice prescribed by p for the pair (N, L_x).
2) $J_1^{\gamma\times N,V}$ is fiberwise split over $\mathbb{R}^2 \setminus K_V$.
3) $J_1^{\gamma\times N,V}$ coincides with J_V over K_V.

As will be seen soon, $J^{\gamma\times N,V}$ will be used as the almost complex structure for the Floer datum $\mathcal{D}_{\gamma\times N,V}$ of the pair $(\gamma \times N, V)$. As such, $J^{\gamma\times N,V}$ needs to satisfy the usual additional conditions we impose on Floer data for Lagrangian cobordisms, as described in [BC14, Section 3.2, p. 1764]. It is easy to see that almost complex structures $\{J_t^{\gamma \times N, V}\}$ as described above exist (recall that the space of $\widetilde{\omega}$-compatible fiberwise split almost complex structures is connected). Note that by Lemma 4.5 $(J_1^{\gamma \times N, V}, K_V)$ continues to be quasi-monotone admissible for V.

The A_∞-category $\mathcal{F}uk_{\mathrm{cob}, \mathscr{C}, \gamma}$ is constructed in a similar way to what we have done in Section 4.2, only that we work in the monotone framework.

To define the module $\mathcal{V}_{qm}$ we take Floer data of the type

$$\mathscr{D}_{\gamma \times N, V} = \left(H^{\gamma \times N, V}, \{J^{\gamma \times N, V}\}\right),$$

where the almost complex structure $\{J^{\gamma \times N, V}\}$ is as described above. The Hamiltonian term $H^{\gamma \times N, V}$ is assumed to have the following form: for every $x \in \gamma \cap \pi(V)$ we have $H^{\gamma \times N, V}(z, u) = \sigma_{(x)}(z) H^{N, L_x}(u)$, for $z \in U_x$, $u \in M$. Here, H^{N, L_x} is the Hamiltonian term in the Floer datum of (N, L_x) and $\sigma_{(x)} : U_x \longrightarrow [0, 1]$ is a smooth function with compact support in U_x and such that $\sigma_{(x)} \equiv 1$ near x. Outside of the union of the subsets U_x, $x \in \gamma \cap \pi(V)$, we set $H^{\gamma \times N, V}$ to be 0.

Next, we define in a similar way to Section 4.2 perturbation data $\mathscr{D} = (\widetilde{K}, \widetilde{J})$ for tuples of the type $(\gamma \times N_1, \ldots, \gamma \times N_d, V)$ with the difference that we require now that $\widetilde{J}_{|\partial_V S_r}$ coincides with J_V over K_V. It is straightforward to see that consistent choices of perturbation data with these additional properties exist. Moreover, there exist such consistent choices which are regular. The latter does not require any new arguments beyond those remarked in the quasi-exact case.

The definition of the module $\mathcal{V}_{qm}$ is now done in the same way as for the module $\mathcal{V}^{\mathrm{qe}}$ in the quasi-exact case. Beyond the arguments for the weakly-exact and quasi-exact cases, there is only one point that needs to be analyzed – bubbling of disks and spheres within spaces of Floer polygons of dimensions ≤ 1 and its effect on the A_∞-module identities for the $\mu_d^{\mathcal{V}_{qm}}$ operations.

To this ends, suppose that bubbling of a holomorphic disk or sphere occurs in a sequence of Floer polygons whose index is ≤ 1 (*i.e.* the dimension of the space of these polygons is ≤ 1). We claim that this can happen only if the Floer polygons are in fact Floer strips (*i.e.* the polygons are 2-gons with boundaries on two Lagrangians), the incoming and exit chords coincide and moreover, after removing the bubbles we are left with a *"constant"* Floer strip, namely a degenerate Floer strip whose image is that common Hamiltonian chord.

Indeed, if bubbling of disks occurs along V then by quasi-monotonicity each such bubble has Maslov index ≥ 2. If bubbling of a holomorphic disk occurs along one of the $\gamma \times N_i$'s, then by the monotonicity of N_i we again have that the Maslov index of each such bubble is ≥ 2. Finally, if bubbling of a holomorphic sphere occurs, then the Chern number of such bubbles is ≥ 1 because M is a monotone symplectic manifold (see the beginning of Section 4.3). Thus, in all cases the total index of the Floer polygon that remains after removing the bubbles is negative. By transversality this cannot happen unless that polygon is "constant at a chord". Moreover, if the perturbation data are chosen generically, such a limit can occur only if the polygons are strips.

We are thus left only with the case when bubbling occurs for Floer strips (in a 1-dimensional space) connecting a chord η to itself, and after bubbling of a holomorphic disk the remaining Floer strip is "constant" at η. The holomorphic disk bubble has boundary on one of the Lagrangians involved and passes through $\eta(0)$ or $\eta(1)$.

Now, the only effect of the last phenomenon on the $\mu_d^{\mathcal{V}_{qm}}$ operation is for $d = 1$, namely when trying to show that $\mu_1^{\mathcal{V}_{qm}} \circ \mu_1^{\mathcal{V}_{qm}}$ is 0. Note that the two pairs of Lagrangians involved in this operation are of the type $\gamma \times N$ with $N \in \mathcal{C}$ and V. By our choices of Floer data, the only Hamiltonian chords η between these two Lagrangians are of the type $x \times \eta'$, where $x \in \gamma \cap \pi(V)$ and $\eta' \in \mathcal{O}(H^{N,L_x})$. Here $L_x \subset M$ is the Lagrangian corresponding to the end of V over x. By our choice of almost complex structures in the Floer data and by applying the open mapping theorem all the holomorphic disks with boundary on either $\gamma \times N$ or on V that pass through $\eta(0)$ or $\eta(1)$ must have constant projection to $\mathbb{R}^2$. Thus these disks lie in $x \times M$ and are in fact either J_0^{N,L_x}-holomorphic with boundary on N and pass through $\eta'(0)$ or are J_1^{N,L_x}-holomorphic with boundary on L_x and pass through $\eta'(1)$. By gluing results the outcome of this is that

$$\mu_1^{\mathcal{V}_{qm}} \circ \mu_1^{\mathcal{V}_{qm}}(x \times \eta') = \sum_{\xi' \in \mathcal{O}(H^{N,L_x})} (\mathbf{d}_N - \mathbf{d}_{L_x})(x \times \xi'). \tag{4.2}$$

Since both N and all the ends of V belong to $\mathcal{C} = \mathcal{L}ag^{\mathrm{mon},\mathbf{d}}(M)$ we have $\mathbf{d}_N = \mathbf{d}_{L_x}$, hence (4.2) vanishes. This concludes the construction of the $\mathcal{F}uk_{\mathrm{cob},\mathcal{C},\gamma}$-module $\mathcal{V}_{qm}$.

The construction of the $\mathcal{F}uk(\mathcal{C};p)$-modules $\mathcal{M}^{qm}_{V;\gamma,p,h}$ and $\mathcal{M}^{qm}_{V;\gamma_j,p,h}$ is done as in (4.1) with $\mathcal{V}_{qe}$ replaced by $\mathcal{V}_{qm}$.

As earlier, Propositions 3.4 and 3.5 continue to hold in the quasi-monotone case (with the same proofs):

Proposition 4.6. — *The statements of Propositions 3.4 and 3.5 continue to hold for the module* $\mathcal{M}^{qm}_{V;\gamma,p,h}$ *and* $\mathcal{M}^{qm}_{V;\gamma_j,p,v}$ *that has just been defined.*

CHAPTER 5

PROOF OF THE MAIN GEOMETRIC STATEMENTS

We prove here the main geometric results.

We will make use of the following variants of the notion of Gromov width. Let (M^{2n}, ω) be a symplectic manifold, $L \subset M$ a Lagrangian submanifold and $Q \subset M$ a subset. Following [BC07], [BC06] we define the *Gromov width* $\delta(L; Q)$ of L relative to Q as follows.

Assume first that $L \not\subset Q$. Define:

$$(5.1) \quad \delta(L; Q) = \sup\left\{\pi r^2 \in (0, \infty] \; ; \; \exists \text{a symplectic embedding } e : B(r) \to M \right. \\ \left. \text{such that } e^{-1}(L) = B_{\mathbb{R}}(r) \text{ and } e(B(r)) \cap Q = \varnothing \right\}.$$

Here $B(r) \subset \mathbb{R}^{2n}$ is the standard $2n$-dimensional closed ball of radius r, endowed with the standard symplectic structure from $\mathbb{R}^{2n}$, and $B_{\mathbb{R}}(r) := B(r) \cap (\mathbb{R}^n \times \{0\})$ is the real part of $B(r)$.

In case $L \subset Q$ we set $\delta(L; Q) := 0$.

Another variant of the Gromov width is associated to an immersed Lagrangian. Let $\widehat{\mathbb{L}}$ be a smooth closed manifold (possibly disconnected) and let $\iota : \widehat{\mathbb{L}} \to M$ be a Lagrangian immersion with image $\mathbb{L} := \iota(\widehat{\mathbb{L}})$. We will measure the "size" of a subset of the double points of $\mathbb{L}$ relative to a given subset $Q \subset M$. Denote by $\Sigma(\iota) \subset \mathbb{L}$ the set of points that have more than one preimage under the immersion ι. Let $\Sigma' \subset \Sigma(\iota)$ be a non-empty subset such that each point in Σ' is a transverse intersection of two branches of the immersion. As before, let $Q \subset M$ be a subset.

Assume first that $\Sigma' \not\subset Q$. We define the *Gromov width* $\delta^{\Sigma'}(\mathbb{L}; Q)$ of the self-intersection set Σ' relative to Q by

$$\delta^{\Sigma'}(\mathbb{L}; Q) = \sup\left\{\pi r^2 \in (0, \infty] \; ; \forall x \in \Sigma', \exists \text{ a symplectic embedding } e_x : B(r) \to M \text{ with} \right. \\ e_x(0) = x,\ e_x^{-1}(\mathbb{L}) = B_{\mathbb{R}}(r) \cup iB_{\mathbb{R}}(r),\ e_x(B(r)) \cap Q = \varnothing, \\ \left. \text{and } e_{x'}(B(r)) \cap e_{x''}(B(r)) = \varnothing \text{ whenever } x' \neq x''\right\}.$$

Here $iB_{\mathbb{R}}(r)$ stands for the imaginary part of the ball, $iB_{\mathbb{R}}(r) := B(r) \cap (\{0\} \times \mathbb{R}^n)$.

In case $\varnothing \neq \Sigma' \subset Q$ we set $\delta^{\Sigma'}(\mathbb{L}; Q) = 0$. In case $\Sigma' = \varnothing$ we set $\delta^{\varnothing}(\mathbb{L}; Q) = \infty$.

In what follows, if $Q = \varnothing$, then we omit the set Q from the notation in both $\delta(L;Q)$ and $\delta^{\Sigma'}(\mathbb{L};Q)$.

The next important geometric measurement is the *shadow* of a cobordism, as defined in [CS19] and already mentioned in the introduction. Let $V \subset \mathbb{R}^2 \times M$ be a Lagrangian cobordism. Denote by $\pi : \mathbb{R}^2 \times M \to \mathbb{R}^2$ the projection. The shadow $\mathcal{S}(V)$ of V is defined as

$$\mathcal{S}(V) = \mathrm{Area}\big(\mathbb{R}^2 \setminus \mathcal{U}\big), \tag{5.2}$$

where $\mathcal{U} \subset \mathbb{R}^2 \setminus \pi(V)$ is the union of all the *unbounded* connected components of $\mathbb{R}^2 \setminus \pi(V)$.

We now restate here the main geometric results for the convenience of the reader. Recall that $\mathcal{L}ag^*(M)$ denotes the collection of closed Lagrangian submanifolds of M of class $*$, where $*$ stands either for the weakly exact Lagrangians ($* =$ we in short), or for the monotone Lagrangians with given Maslov-2 disk count $\mathbf{d} \in \Lambda_0$ ($* = (\mathrm{mon}, \mathbf{d})$ in short) as introduced in Section 3.5. Similarly, we have the collection $\mathcal{L}ag^*(\mathbb{R}^2 \times M)$ of Lagrangian cobordisms $V \subset \mathbb{R}^2 \times M$ of class $*$, where $*$ is as above.

Theorem 5.1. — *Let $L, L_1, \ldots, L_k \in \mathcal{L}ag^{\mathrm{we}}(M)$ and $V : L \rightsquigarrow (L_1, \ldots, L_k)$ a weakly exact Lagrangian cobordism. Denote $S := \bigcup_{i=1}^k L_i$ the union of the Lagrangians corresponding to the negative ends of V. Then*

$$\mathcal{S}(V) \geq \tfrac{1}{2}\delta(L;S). \tag{5.3}$$

For the next two points of the theorem we will use the following. Let $N \in \mathcal{L}ag^{\mathrm{we}}(M)$ be another weakly exact Lagrangian submanifold and consider $S = \bigcup_{i=1}^k L_i \subset M$ and $N \cup S \subset M$ as immersed Lagrangians (parametrized by $\coprod_{i=1}^k L_i$ and $N \coprod (\coprod_{i=1}^k L_i)$ respectively).

(i) *Assume that N intersects each of the Lagrangians $L_1, \ldots, L_k$ transversely and that $N \cap L_i \cap L_j = \varnothing$ for all $i \neq j$. Denote $\Sigma' := N \cap S$. If $\mathcal{S}(V) < \frac{1}{2}\,\delta^{\Sigma'}(N \cup S)$ then*

$$\#\,(N \cap L) \geq \sum_{i=1}^k \#(N \cap L_i). \tag{5.4}$$

(ii) *Assume that the Lagrangians $L_1, \ldots, L_k$ intersect pairwise transversely and that no three of them have a common intersection point (i.e. $L_i \cap L_j \cap L_r = \varnothing$ for all distinct indices i, j, r). Let Σ'' be the set of all double points of S, i.e. $\Sigma'' := \bigcup_{1 \leq i < j \leq k} L_i \cap L_j$. If $\mathcal{S}(V) < \frac{1}{4}\,\delta^{\Sigma''}(S;N)$ then*

$$\#\,(N \cap L) \geq \sum_{i=1}^k \dim_\Lambda \big(\mathrm{HF}(N, L_i)\big). \tag{5.5}$$

The proof is given in Section 5.1 below.

5.0.1. ***Remark.*** **—** Note that the inequality (5.4) in Theorem 5.1 implies variants of both inequalities (5.3) and (5.5), but with slightly different assumptions and for different constants δ. This is obvious concerning (5.5) and for inequality (5.3) it is seen by applying (5.4) to the cobordism $W : \varnothing \rightsquigarrow (L, L_1, \ldots, L_k)$ obtained by bending the positive end of V half way clockwise – as in Figure 4 – and taking N to be a suitable Hamiltonian perturbation of L. □

Theorem 5.1 has an analogue in the monotone case too. Recall from Section 3.5 the Maslov-2 disk count $\mathbf{d} \in \Lambda_0$ associated to a monotone Lagrangian L and also its minimal disk area A_L defined by (3.23) in Section 3.5.

Theorem 5.2. — *Let $L, L_1, \ldots, L_k \subset M$ be monotone Lagrangians and*

$$V : L \rightsquigarrow (L_1, \ldots, L_k)$$

a connected monotone cobordism. Let S be the same as in Theorem 5.1. Denote by $\mathbf{d} \in \Lambda_0$ the Maslov-2 disk count of L (hence by Section 3.5 also of the L_i's) and let $N \subset M$ be another monotone Lagrangian with $\mathbf{d}_N = \mathbf{d}$. Then:

$$\mathcal{S}(V) \geq \min\left\{\tfrac{1}{2}\delta(L; S), A_L\right\}. \tag{5.6}$$

Moreover, under the above assumptions inequalities (5.4) *and* (5.5) *continue to hold as stated in Theorem 5.1.*

The proof is given in Section 5.2.

Before the proofs of Theorems 5.1 and 5.2, here is bit of notation that will be used throughout. Denote by Λ the Novikov ring (with coefficients in $\mathbb{Z}_2$) and by $\Lambda_0 \subset \Lambda$ the positive Novikov ring, as defined in (2.1) and (2.2). Recall from (3.11) the standard valuation $\nu : \Lambda \to \mathbb{R} \cup \{\infty\}$ defined by

$$\nu\Big(a_0 T^{\lambda_0} + \sum_{i=1}^{\infty} a_i T^{\lambda_i}\Big) = \lambda_0,$$

where $a_0 \neq 0$ and $\lambda_i > \lambda_0$ for every $i \geq 1$. As usual we set $\nu(0) = \infty$.

All Floer complexes will be taken with coefficients in Λ as in Section 3 and the filtrations on them will be defined by action, according to the recipe from Section 3.3. Given such a Floer complex, say C, we will denote by $A : C \to \mathbb{R} \cup \{-\infty\}$ the action level, as defined in Section 2.7. (Recall that for $x \in C$ we write $A(x)$ and $A(x; C)$ interchangeably.) Note that A coincides with $\boldsymbol{A}$ from Section 3.3, and below we will continue to denote it by A (rather than $\boldsymbol{A}$) to keep compatibility with our general algebraic conventions.

5.1. Proof of Theorem 5.1

We begin with the proof of inequality (5.3). We first assume that the Lagrangians $L, L_1, \ldots, L_k$ intersect pairwise transversely, and treat the general case afterwards.

We start by bending the positive end of V by 180° clockwise in such a way as to get a cobordism W without positive ends, and whose negative ends are $(L_0, L_1, \ldots, L_k)$, where $L_0 := L$ (see Figure 4). Clearly $\mathcal{S}(W) = \mathcal{S}(V)$.

Fix $\epsilon > 0$. Let γ, γ' be two curves, as depicted in Figure 5, and such that there exists a (not compactly supported) Hamiltonian isotopy, horizontal at infinity, $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$,

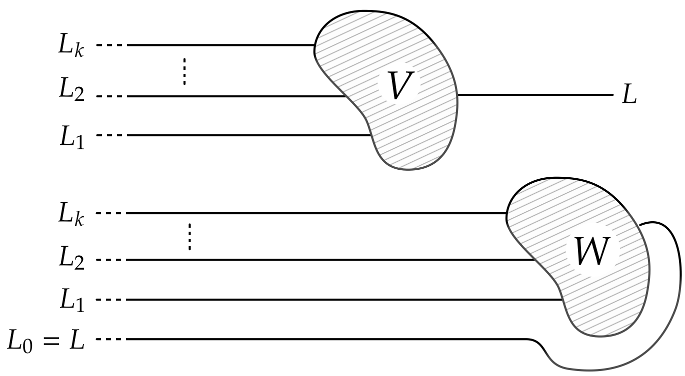

FIGURE 4. The cobordisms W obtained from V by bending the positive end.

$t \in [0, 1]$, with $\varphi_0 = \mathrm{id}$, $\varphi_1(\gamma) = \gamma'$ and with

$$\text{length}\{\varphi_t\} \leq \mathcal{S}(W) + \tfrac{1}{2}\epsilon, \tag{5.7}$$

where length$\{\varphi_t\}$ stands for the Hofer length of the isotopy $\{\varphi_t\}$.

Put $S = \bigcup_{i=1}^{k} L_i$ and let $e : B(r) \to M \setminus S$ be a symplectic embedding as in the definition of $\delta(L_0; S)$ in (5.1), with

$$\delta(L_0, S) - \epsilon \leq \pi r^2 \leq \delta(L_0, S).$$

Next, let

$$B := \text{image}\,(e), \quad q := e(0) \in L_0, \quad J^B := e_*(J_{\text{std}}), \tag{5.8}$$

where the latter is the complex structure on B corresponding to the standard complex structure J_{std} of $B^{2n}(r)$ via the embedding e.

Next, we fix a symplectic identification between a small open neighborhood U of L_0 in M and a neighborhood U' of the zero-section in $T^*(L_0)$. Let $f : L_0 \to \mathbb{R}$ be a C^1-small Morse function with exactly one local maximum at the point $q \in L_0$. We extend f to a function $\widetilde{f} : U' \to \mathbb{R}$ by setting it to be constant along the fibers of the cotangent bundle. Finally, let $H_f^{L_0,L_0} : M \to \mathbb{R}$ be a smooth function such that $H_f^{L_0,L_0}|_U$ coincides with $\widetilde{f}$ via the identification between U and U' that we have just fixed.

We now turn to the Fukaya categories relevant for this proof. Let $\mathscr{C}$ be the collection of Lagrangians $L_0, \ldots, L_k$. We will use the Fukaya categories $\mathcal{F}uk(\mathscr{C})$ and $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathscr{C}})$ associated to $\mathscr{C}$. More specifically, we consider regular perturbation data $p \in E'_{\text{reg}}$ and C^1-small profile functions $h \in \mathscr{H}'_{\text{prof}}(\gamma)$ as in Section 3.6.

We impose two additional restrictions on the admissible choices of perturbation data p as follows. The first one is that the datum $\mathscr{D}_{L_0,L_0}$ of the pair (L_0, L_0) should have the function $H_f^{L_0,L_0}$ as its Hamiltonian function, defined using any choice of a C^1-small Morse function f as described above. Furthermore, we allow only for functions f that are sufficiently C^1-small such that $\mathcal{O}(H_f^{L_0,L_0}) = \text{Crit}(f)$. Note that for every $y \in \mathcal{O}(H_f^{L_0,L_0})$ we have $A(y) = f(y)$.

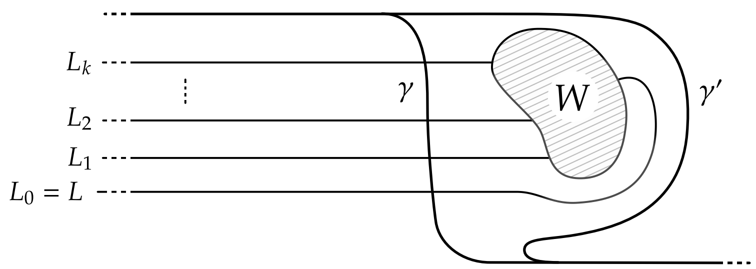

FIGURE 5. The curves γ and γ' and the cobordism W.

The second restriction is that the Hamiltonian functions H^{L_i,L_j} in the Floer data $\mathscr{D}_{L_i,L_j}$, $i \neq j$, are all 0. It is possible to impose these additional restriction and still maintain regularity since we have assumed that the Lagrangians $L_0, L_1, \dots, L_k$ intersect pairwise transversely. With these choices we have for every $i \neq j$:

$$\mathcal{O}(H^{L_i,L_j}) = L_i \cap L_j, \quad A(z) = 0, \quad \text{for all } z \in \mathcal{O}(H^{L_i,L_j}).$$

We denote the space of all such regular choices of perturbation data by $E''_{\text{reg}} \subset E'_{\text{reg}}$. We remark that the Morse function f is not fixed over E''_{reg} and each choice $p \in E''_{\text{reg}}$ comes with its own function f. Finally, note that $\mathcal{N}$ is still in the closure of E''_{reg}.

We now appeal to the theory developed in Section 3. Consider the Fukaya category $\mathcal{F}uk(\mathscr{C}; p)$ (see Section 3.2) as well as the Fukaya category of cobordisms $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathscr{C}}_{1/2}; \iota_\gamma(p, h))$ (see Section 3.4 and (3.25) in Section 3.6). Recall that we have an "inclusion" functor

$$\mathcal{I}_{\gamma;p,h} : \mathcal{F}uk(\mathscr{C}; p) \longrightarrow \mathcal{F}uk_{\text{cob}}\big(\widetilde{\mathscr{C}}_{1/2}; \iota_\gamma(p, h)\big).$$

Denote by $\mathscr{W}$ the Yoneda module corresponding to the object

$$W \in \text{Ob}\,\big(\mathcal{F}uk_{\text{cob}}(\widetilde{\mathscr{C}}_{1/2}; \iota_\gamma(p, h))\big)$$

and consider its pull-back by the functor $\mathcal{I}_{\gamma;p,h}$:

$$\mathscr{M}_{W;\gamma,p,h} := \mathcal{I}^*_{\gamma;p,h}\mathscr{W}.$$

Recall from Sections 3.3, 3.4 and 3.6 that the A_∞-categories

$$\mathcal{F}uk(\mathscr{C}; p), \quad \mathcal{F}uk_{\text{cob}}\big(\widetilde{\mathscr{C}}_{1/2}; \iota_\gamma(p, h)\big)$$

as well as the A_∞-functor $\mathcal{I}_{\gamma;p,h}$ are all weakly filtered. Moreover, by Section 3.7 the module $\mathscr{M}_{W;\gamma,p,h}$ if weakly filtered too. By Propositions 3.4, and points (3.6), (3.6) on page 69 the discrepancy of this module is bounded from above by $\boldsymbol{\epsilon}(p, h) = (\epsilon_1(p, h), \epsilon_2(p, h), \dots, \epsilon_d(p, h), \dots)$ which satisfies $\lim \epsilon_d(p, h) \to 0$ for every d, as $p \to p_0 \in \mathcal{N}$ and $h \to 0$ (the latter in the C^1-topology).

Throughout the proof we will repeatedly deal with quantities having the same asymptotics as $\epsilon_d(p, h)$. In order to simplify the text we introduce the following notation. Let $\mathcal{N}_0 \subset \mathcal{N}$ and let $(p, h) \longmapsto C(p, h)$ be a real valued function defined for p in a subset of E'_{reg} whose closure contains $\mathcal{N}_0$, and $h \in \mathscr{H}'_{\text{prof}}$.

We will write $C(p,h) \in O(\mathcal{N}_0)$ to indicate that for every $p_0 \in \mathcal{N}_0$ we have $\lim C(p,h) = 0$ as $p \to p_0$ and $h \to 0$ (the latter in the C^1-topology).

By Proposition 3.5 (and (3.32)) we have

$$S^{s_h}\mathcal{M}_{W;\gamma,p,h} = \mathcal{C}one(\mathcal{L}_k \xrightarrow{\overline{\phi}_k} \mathcal{C}one(\mathcal{L}_{k-1} \xrightarrow{\overline{\phi}_{k-1}} \mathcal{C}one(\cdots \cdots \mathcal{C}one(\mathcal{L}_2 \xrightarrow{\overline{\phi}_2} \mathcal{C}one(\mathcal{L}_1 \xrightarrow{\overline{\phi}_1} \mathcal{L}_0))\cdots))), \tag{5.9}$$

where $s_h \to 0$ as $h \to 0$. (Recall from §2.3.4 that $S^{s_h}\mathcal{M}_{V;\gamma,p,h}$ stands for the module $\mathcal{M}_{V;\gamma,p,h}$ with action-shift by s_h.) The modules $\mathcal{L}_i$ in (5.9) are the Yoneda modules of the L_i's. The notation $\overline{\phi_i}$ stands for $\overline{\phi}_i = (\phi_i, 0, \boldsymbol{\delta}^{(i)})$, with ϕ_i being a homomorphism of modules that shifts action by ≤ 0 and has discrepancy $\leq \boldsymbol{\delta}^{(i)}(p,h)$ where for every d we have $\delta_d^{(j)}(p,h) \in O(\mathcal{N})$. For simplicity of notation set

$$\boldsymbol{\delta}(p,h) := \max\left\{\boldsymbol{\delta}^{(1)}(p,h), \ldots, \boldsymbol{\delta}^{(k)}(p,h)\right\},$$

so that the discrepancy of all the ϕ_i's is $\leq \boldsymbol{\delta}(p,h)$ and we still have $\delta_d(p,h) \in O(\mathcal{N})$ for all d.

Consider the filtered chain complex

$$\mathcal{C}_{p,h} := S^{s_h}\mathcal{M}_{W;\gamma,p,h}(L_0)$$

endowed with the differential coming from the μ_1-operation of $\mathcal{M}_{W;\gamma,p,h}$. By definition $\mathcal{C}_{p,h} = S^{s_h}\,\mathrm{CF}(\gamma \times L_0, W; \mathcal{D}_{\gamma\times L_0,W})$, where $\mathcal{D}_{\gamma\times L_0,W}$ is the Floer datum prescribed by $\iota_\gamma(p,h)$. By (5.9) the Floer complex of (L_0, L_0) is a subcomplex of $\mathcal{C}_{p,h}$, or more precisely, we have an *action preserving* inclusion of chain complexes:

$$\mathrm{CF}(L_0, L_0; p) \subset \mathcal{C}_{p,h}, \tag{5.10}$$

where $\mathcal{D}_{L_0,L_0}$ is specified by p and is subject to the additional restrictions imposed earlier in the proof. To simplify the notation, we will denote from now on for a pair of Lagrangians (L', L'') by $\mathrm{CF}(L', L''; p)$ the Floer complex $\mathrm{CF}(L', L''; \mathcal{D}_{L',L''})$, where $\mathcal{D}_{L',L''}$ is the Floer datum specified by p.

Recall that we also have the curve $\gamma' \subset \mathbb{R}^2$ with $\gamma' \cap \pi(W) = \emptyset$. Choose a Floer datum $\mathcal{D}'$ for $(\gamma' \times L_0, W)$ with a sufficiently C^2-small Hamiltonian function so that $\mathrm{CF}(\gamma' \times L_0, W; \mathcal{D}') = 0$. Now $\gamma \times L_0$ can be Hamiltonian isotoped to $\gamma' \times L_0$ via an isotopy horizontal at infinity with Hofer length $\leq \mathcal{S}(W) + \frac{1}{2}\epsilon$. By standard Floer theory (see *e.g.* [**FOOO09a**, Section 5.3.2]) the identity map on $\mathcal{C}_{p,h}$ is null homotopic via a chain homotopy which shifts action by $\leq \mathcal{S}(W) + \frac{1}{2}\epsilon$. Translated to the formalism of (2.44) in Section 2.7 this means that $B_h(\mathrm{id}_{\mathcal{C}_{p,h}}) \leq \mathcal{S}(W) + \frac{1}{2}\epsilon$, hence by (2.46) we have

$$\beta(\mathcal{C}_{p,h}) \leq \mathcal{S}(W) + \tfrac{1}{2}\epsilon, \tag{5.11}$$

where $\beta(\mathcal{C}_{p,h})$ is the boundary depth of the (acyclic) chain complex $\mathcal{C}_{p,h}$ as defined in Section 2.7.

We now appeal to Theorem 2.14 applied to the weakly filtered iterated cone (5.9). We apply this theorem with $X = L_0$ and $\rho_i = 0$. We obtain a new weakly filtered module $\mathcal{M}$ such that $\mathcal{M}(L_0)$ has a differential $\mu_1^{\mathcal{M}}$ as described in that theorem together with a filtered *chain isomorphism* $\sigma_1 : \mathcal{C}_{p,h} \to \mathcal{M}(L_0)$. An inspection of the sizes of shifts and discrepancies of the various maps involved in Theorem 2.14 shows that there exists a constant $s^\sigma(p,h) \in O(\mathcal{N})$ such that σ_1 shifts filtration by $\leq s^\sigma(p,h)$. Additionally, Theorem 2.14 implies that $\mathrm{CF}(L_0, L_0; p)$ is also a filtered subcomplex of $\mathcal{M}(L_0)$ and that $\mathrm{pr}_0 \circ \sigma_1$ maps $\mathrm{CF}(L_0, L_0; p) \subset \mathscr{C}_{p,h}$ to $\mathrm{CF}(L_0, L_0; p) \subset \mathcal{M}(L_0)$ via the identity map: $(\mathrm{pr}_0 \circ \sigma_1)|_{\mathrm{CF}(L_0,L_0;p)} = \mathrm{id}$. Here $\mathrm{pr}_0 : \mathcal{M}(L_0) \to \mathrm{CF}(L_0, L_0; p)$ is the projection onto the 0-th factor of $\mathcal{M}(L_0)$.

Consider now the homology unit $e_{L_0} \in \mathrm{CF}(L_0, L_0; p)$ as constructed in (3.10). By standard Floer theory $e_{L_0} = q$ (recall that q is the unique maximum of $f : L_0 \to \mathbb{R}$).

Let $c \in \mathrm{CF}(L_0, L_0; p)$ and $\gamma \in \mathcal{O}(H^{L_0,L_0})$ a generator, where H^{L_0,L_0} is the Hamiltonian function of the Floer datum specified by p for (L_0, L_0). We denote by $\langle c, \gamma\rangle \in \Lambda$ the coefficient of γ when writing c as a linear combination of elements of $\mathcal{O}(H^{L_0,L_0})$ with coefficients in Λ.

We will need the following.

Lemma 5.3. — *For every chain $c \in \mathrm{CF}(L_0, L_0; p)$ we have $\langle \mu_1(c), q\rangle = 0$.*

We postpone the proof of the lemma and continue with the proof of Theorem 5.1.

Put $C_f := \max_{x \in L_0} |f(x)|$, $C^{(1)}(p,h) := C_f + s^\sigma(p,h)$. By (5.11) there exists

$$b' \in \mathscr{C}_{p,h} \text{ with } A(b'; \mathscr{C}_{p,h}) \leq A(e_{L_0}; \mathscr{C}_{p,h}) + \mathcal{S}(W) + \tfrac{1}{2}\epsilon \leq C_f + \mathcal{S}(W) + \tfrac{1}{2}\epsilon\ , \tag{5.12}$$

such that $e_{L_0} = \mu_1^{\mathscr{C}_{p,h}}(b')$.

Recall from point (2.14) of Theorem 2.14 that $\mathrm{pr}_0 \circ \sigma_1|_{\mathrm{CF}(L_0,L_0;p)} = \mathrm{id}$. Set $b := \sigma_1(b')$ and apply $\mathrm{pr}_0 \circ \sigma_1$ to the equality $e_{L_0} = \mu_1^{\mathscr{C}_{p,h}}(b')$. We obtain

$$e_{L_0} = \mathrm{pr}_0 \circ \mu_1^{\mathcal{M}}(b), \quad A\big(b; \mathcal{M}(L_0)\big) \leq C^{(1)}(p,h) + \mathcal{S}(W) + \tfrac{1}{2}\epsilon, \tag{5.13}$$

where $C^{(1)}(p,h) := C_f + s^\sigma(p,h)$. Obviously $C^{(1)}(p,h) \in O(\mathcal{N})$. (Note that $f \to 0$ as $p \to p_0 \in \mathcal{N}$.)

Using the splitting (2.30) write $b = b_0 + \cdots + b_k$, with $b_i \in \mathrm{CF}(L_0, L_i; p)$ and

$$A\big(b_i; \mathrm{CF}(L_0, L_i; p)\big) \leq C^{(2)}(p,h) + \mathcal{S}(W) + \tfrac{1}{2}\epsilon,$$

where $C^{(2)}(p,h)$ is a new constant such that $\lim C^{(2)}(p,h) \in O(\mathcal{N})$.

Continuing to apply Theorem 2.14 we have

$$q = \mathrm{pr}_0 \circ \mu_1^{\mathcal{M}}(b) = \sum_{j=0}^{k} a_{0,j}(b_j) = \mu_1^{\mathrm{CF}(L_0,L_0;p)}(b_0) + \sum_{j=1}^{k} a_{0,j}(b_j), \tag{5.14}$$

where the operators $a_{i,j}$ are the entries of the matrix representation of $\mu_1^{\mathcal{M}}$ with respect to the splitting (2.30). By Lemma 5.3, $\langle \mu_1(b_0), q\rangle = 0$, hence there exists $1 \leq j_0 \leq k$ such that

$$\big\langle a_{0,j_0}(b_{j_0}), q\big\rangle \neq 0, \quad \nu\big(\langle a_{0,j_0}(b_{j_0}), q\rangle\big) \leq \nu(1) = 0. \tag{5.15}$$

Here ν is the standard valuation of Λ (see (3.11)).

By Theorem 2.14 there exist chains $c_{i',i''} \in \mathrm{CF}(L_{i'}, L_{i''}; p)$, for all $i' < i''$, with $A(c_{i',i''}) \leq C^{(3)}(p,h)$, where $C^{(3)}(p,h) \in O(\mathcal{N})$ and such that

$$a_{0,j_0}(b_{j_0}) = \sum_{2 \leq d,\, \underline{i}} \mu_d^{\mathcal{F}uk(\mathcal{C};p)}(b_{j_0}, c_{i_d,i_{d-1}}, \ldots, c_{i_2,i_1}),$$

where $\underline{i} = (i_1, \ldots, i_d)$ runs over all partitions $0 = i_1 < i_2 \cdots < i_{d-1} < i_d = j_0$.

It follows that there exists a partition $\underline{i}^0 = (i_1^0, \ldots, i_d^0)$ with $d \geq 2$, for which

$$\left\langle \mu_d^{\mathcal{F}uk(\mathcal{C};p)}(b_{j_0}, c_{i_d^0,i_{d-1}^0}, \ldots, c_{i_2^0,i_1^0}), q \right\rangle \neq 0, \quad \nu\Big(\left\langle \mu_d^{\mathcal{F}uk(\mathcal{C};p)}(b_{j_0}, c_{i_d^0,i_{d-1}^0}, \ldots, c_{i_2^0,i_1^0}), q \right\rangle\Big) \leq 0.$$

Writing b_{j_0} as a linear combination (over Λ) of elements from $L_0 \cap L_{j_0}$ and similarly for the $c_{i_r^0,i_{r-1}^0}$'s, we deduce that there exist $x \in L_0 \cap L_{j_0}$, $P(T) \in \Lambda$, and $z_r \in L_{i_r^0} \cap L_{i_{r-1}^0}$, $Q_r \in \Lambda$ for $r = 2, \ldots, d$, such that

$$\begin{gathered} A(P(T)x) \leq C^{(2)}(p,h) + \mathcal{S}(W) + \tfrac{1}{2}\epsilon, \\ A(Q_r(T)z_r) \leq C^{(3)}(p,h), \quad \text{for all } 2 \leq r \leq d, \\ \left\langle \mu_d^{\mathcal{F}uk(\mathcal{C};p)}(P(T)x, Q_d(T)z_d, \ldots, Q_2(T)z_2), q \right\rangle \neq 0, \\ \nu\Big(\left\langle \mu_d^{\mathcal{F}uk(\mathcal{C};p)}(P(T)x, Q_d(T)z_d, \ldots, Q_2(T)z_2), q \right\rangle\Big) \leq 0. \end{gathered}$$

Note that d, as well as the points x, $z_d, \ldots, z_2$, q, all depend on (p,h), but for the moment we suppress this from the notation.

Since $A(P(T)x) = -\nu(P(T))$ and $A(Q_r(T)z_r) = -\nu(Q_r(T))$ we obtain

$$\nu\Big(\left\langle \mu_d^{\mathcal{F}uk(\mathcal{C};p)}(x, z_d, \ldots, z_2), q \right\rangle\Big) \leq \mathcal{S}(W) + \tfrac{1}{2}\epsilon + C^{(4)}(p,h), \tag{5.16}$$

where $C^{(4)}(p,h) \in O(\mathcal{N})$.

Denote by $\mathcal{D}(p) = (K(p), J(p))$ the perturbation datum prescribed by $p \in E''_{\mathrm{reg}}$ for the tuple of Lagrangians $(L_0, L_{j_0}, L_{i_{d-1}}, \ldots, L_{i_2}, L_0)$. It follows from (5.16) that there exists a non-constant Floer polygon $u \in \mathcal{M}(x, z_d, \ldots, z_2, q; \mathcal{D}(p))$ with

$$\omega(u) \leq \mathcal{S}(W) + \tfrac{1}{2}\epsilon + C^{(4)}(p,h).$$

Let $p_0 \in \mathcal{N}$ be any choice of perturbation data which assigns to the tuple of Lagrangians $(L_0, L_{j_0}, L_{i_{d-1}}, \ldots, L_{i_2}, L_0)$ the perturbation data $\mathcal{D}(p_0) = (K = 0, J(p_0))$, where $J(p_0)$ is a family of almost complex structures that coincide with J^B on B (see (5.8)).

Fix a generic C^1-small Morse function f as on page 88. We now choose a sequence $\{(p_n, h_n)\}$ in E''_{reg} with $(p_n, h_n) \to (p_0, 0)$ as $n \to \infty$, and with the following additional property. The Hamiltonian function $H^{L_0,L_0}(n)$ prescribed by p_n for the Floer datum $\mathcal{D}_{L_0,L_0}(p_n)$ of (L_0, L_0) is $H^{L_0,L_0}_{\frac{1}{n}f}$, *i.e.* constructed as on page 88 but with the function $\frac{1}{n}f$ instead of f. Consequently, the point q (the maximum of $\frac{1}{n}f$) does not depend on n.

Passing to a subsequence of $\{(p_n, h_n)\}$ if necessary we may assume that both d as well as the points $x, z_d, \ldots, z_2$ above do not depend on n either. (Note that by Theorem 2.14, $d \leq k$, so there are only finitely many possible values for d.)

In summary, we obtain a sequence $u_n \in \mathcal{M}(x, z_d, \dots, z_2, q; \mathcal{D}(p_n))$ with

$$\omega(u_n) \leq \mathcal{S}(W) + \tfrac{1}{2}\epsilon + C^{(4)}(p_n, h_n).$$

By a compactness result [**OZ**], [**OZ11**] (see also [**FO97**], [**Oh96b**], [**Oh96a**]) there exists a subsequence of $\{u_n\}$ which converges to a union of Floer polygons $v_0, v_1, \dots, v_l$, $l \geq 0$, together with a (possibly broken) negative gradient trajectory η of f. [9]

The Floer polygons v_i map the boundary components of their domains of definition to some of the Lagrangians in the collection $L_0, L_{j_0}, L_{i_{d-1}}, \dots, L_{i_2}, L_0$. Moreover, v_0 maps one of its boundary components to L_0. The maps v_i satisfy the Floer equation corresponding to the perturbation data prescribed by p_0. Consequently they are all genuine pseudo-holomorphic (*i.e.* without Hamiltonian perturbations) with respect to the (domain-dependent) almost complex structures prescribed by p_0. In particular, one has $\omega(v_i) \geq 0$ for every i.

As $\omega(u_n) \leq \mathcal{S}(W) + \frac{1}{2}\epsilon + C^{(4)}(p_n, h_n)$ for every n, we have $\sum_{i=0}^{l} \omega(v_i) \leq \mathcal{S}(W) + \frac{1}{2}\epsilon$, hence

$$\omega(v_0) \leq \mathcal{S}(W) + \tfrac{1}{2}\epsilon. \tag{5.17}$$

The other part of the limit of $\{u_n\}$, namely the negative gradient trajectory η of f, emanates from an L_0-boundary point of one of the polygons, say v_0, and ends at the point q.

Consider now v_0 and η. Note that η must be the constant trajectory at the point q since it goes into q which is a maximum of f. It follows that the polygon v_0 passes (along its boundary) through the point q.

We now appeal to the special form of $J(p_0)$ over the ball B. Recall that v_0 is $J(p_0)$-holomorphic. Thus restricting v_0 to the subdomain (of its definition) which is mapped to $\mathrm{Int}\,(B)$ we obtain a *proper* J^B-holomorphic curve v_0' parametrized by a non-compact Riemann surface with one boundary component. Moreover that boundary component is mapped to $B \cap L_0$, and $q \in B$ which is the center of the ball is in the image of that boundary component. Passing to the standard ball $B(r)$ via the symplectic embedding e mentioned in (5.8) we obtain from v_0' a proper J_{std}-holomorphic curve v_0'' inside $B(r)$ which passes through 0 and its boundary is mapped to $B_{\mathbb{R}}(r) \subset B^{2n}(r)$. Applying a reflection along $\mathbb{R}^n \times 0$ to v_0'', and gluing the result to v_0'' we obtain a proper J_{std}-holomorphic curve (without boundary) $\widetilde{v}_0''$ in $\mathrm{Int}\, B^{2n}(r)$ which passes through 0. By the Lelong inequality we have $\pi r^2 \leq \omega_{\mathrm{std}}(\widetilde{v}_0'')$. Putting everything together we obtain

$$\delta(L_0, S) - \epsilon \leq \pi r^2 \leq \omega(\widetilde{v}_0'') = 2\omega(v_0'') \leq 2\omega(v_0) \leq 2\mathcal{S}(W) + \epsilon.$$

Since this inequality holds for all $\epsilon > 0$ the desired inequality (5.3) follows.

Proof of Lemma 5.3. — Recall that the Hamiltonian function in the Floer data $\mathcal{D}_{L_0,L_0}$ of (L_0, L_0) is $H_f^{L_0,L_0}$ and we have $\mathcal{O}(H_f^{L_0,L_0}) = \mathrm{Crit}(f)$.

Let $u : \mathbb{R} \times [0,1] \to M$ be a Floer strip connecting x_- to x_+, where $x_\pm \in \mathrm{Crit}(f)$. Identifying $(D \setminus \{-1,+1\}, \partial D \setminus \{-1,+1\})$ with $(\mathbb{R} \times [0,1], \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\})$ we obtain

9. "Broken" means that the trajectory might pass through several critical points of f.

from u a map $u' : (D \setminus \{-1,+1\}, \partial D \setminus \{-1,+1\}) \to (M, L_0)$ that extends continuously to a map $\bar{u}' : (D, \partial D) \to (M, L_0)$. Since L_0 is weakly exact we have $\omega(\bar{u}') = 0$, hence $\omega(u) = 0$.

By (3.13) it follows that $f(x_-) = f(x_+) + E(u)$, where $E(u)$ is the energy of u (see (3.2)). As $E(u) \geq 0$ we have $f(x_-) \geq f(x_+)$ with equality iff $E(u) = 0$.

Suppose by contradiction that $\langle \mu_1(x), q \rangle \neq 0$ for some $x \in \mathrm{Crit}(f)$. Let u be a Floer strip that contributes to $\mu_1(x)$ and connects x to q. By the above, we have $f(x) \geq f(q)$. Since q is the unique maximum of f it follows that $x = q$. Moreover, $E(u) = 0$. The latter implies that $\partial_s u \equiv 0$. But this can happen only if u is the constant strip at q which contradicts the fact that u contributes to $\mu_1(x)$. □

To complete the proof of inequality (5.3) it remains only to treat the case when the Lagrangians $L_0, L_1, \ldots, L_k$ do not intersect pairwise transversely. Fix $\epsilon > 0$. We apply k Hamiltonian isotopies, one to each Lagrangian L_i, $1 \leq i \leq k$, such that the following holds:

1) The images $L'_1, \ldots, L'_k$ of $L_1, \ldots, L_k$ after these isotopies are such that $L_0, L'_1, \ldots, L'_k$ intersect pairwise transversely.
2) The Hofer length of each of these isotopies is $\leq \epsilon/k$.
3) $\delta(L_0; S) - \epsilon \leq \delta(L_0; S')$, where $S' = L'_1 \cup \cdots \cup L'_k$.

Let $V : L_0 \rightsquigarrow (L_1, \ldots, L_k)$ be a weakly exact cobordism. We now glue to each of the negative ends L_i of V the Lagrangian suspension associated to the preceding Hamiltonian isotopy used to move L_i to L'_i. The result is a new cobordism $V' : L_0 \rightsquigarrow (L'_1, \ldots, L'_k)$ whose shadow satisfies $\mathcal{S}(V') \leq \mathcal{S}(V) + \epsilon$.

Since the ends of V' intersect pairwise transversely it follows from what we have already proved that $\mathcal{S}(V') \geq \frac{1}{2}\delta(L_0; S')$. Therefore:

$$\tfrac{1}{2}\delta(L_0; S) - \tfrac{1}{2}\epsilon \leq \tfrac{1}{2}\delta(L_0; S') \leq \mathcal{S}(V') \leq \mathcal{S}(V) + \epsilon.$$

As this holds for all $\epsilon > 0$ the result readily follows.

This completes the proof of inequality (5.3).

We now turn to the proofs of the other two statements of Theorem 5.1.

5.1.1. ***Proof of statement* (i). —** As in the previous part of the proof, we first assume that the Lagrangians $L_1, \ldots, L_k$ intersect pairwise transversely.

Fix $\epsilon > 0$ small enough such that

$$\mathcal{S}(V) + \epsilon < \tfrac{1}{2}\delta^{\Sigma'}(N \cup S) - \tfrac{1}{2}\epsilon. \tag{5.18}$$

Fix also $r > 0$ with

$$\delta^{\Sigma'}(N \cup S) - \epsilon \leq \pi r^2 < \delta^{\Sigma'}(N \cup S). \tag{5.19}$$

Write $\Sigma' = N \cap S = \{x_1, \ldots, x_m\}$ for the double points of $N \cup S$, and let $e_{x_i} : B(r) \to M$, $i = 1, \ldots, m$, be a collection of symplectic embeddings with the properties as in the definition of $\delta^{\Sigma'}$ on page 85 (we take $\mathbb{L} = N \cup S$ and $Q = \varnothing$ in that definition). Denote $B := \bigcup_{i=1}^m \mathrm{image}\,(e_{x_i})$ and let J^B be the complex structure on B whose value

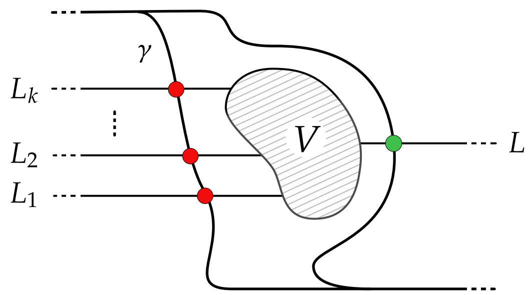

FIGURE 6. The curves γ and γ' and the cobordism V.

on image (e_{x_i}) is the push forward $(e_{x_i})_*(J_{\mathrm{std}})$ of the standard complex structure J_{std} of $B(r)$ via the map e_{x_i}.

We consider now two curves γ, γ' of the the same shape as in the earlier part of the proof (see Figure 6) and such that (similarly to (5.7)) there exists a Hamiltonian isotopy, horizontal at infinity, $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$, $t \in [0,1]$, with $\varphi_0 = \mathrm{id}$, $\varphi_1(\gamma) = \gamma'$ and with

$$\text{length}\{\varphi_t\} \leq \mathcal{S}(V) + \tfrac{1}{2}\epsilon. \tag{5.20}$$

Next we set up the Fukaya categories involved in the proof. Let $\mathscr{C}$ be the collection of Lagrangians $L_1, \ldots, L_k, L$. We will work with the Fukaya $\mathscr{F}uk(\mathscr{C}; p)$ defined with choices of perturbation data p with the following restrictions. The Floer data of (N, L_i), prescribed by p, are of the type $\mathscr{D}_{N,L_i} = (H^{N,L_i} = 0, J(p))$, where $J(p) = \{J_t(p)\}$ is a family of almost complex structures such that $J_t(p)_{|B} = J^B$ for all t. The Floer data $\mathscr{D}_{L_i,L_j}$ $i \neq j$ have the 0 Hamiltonian function. Finally, the perturbation data $\mathscr{D}_{N,L_{i_d},\ldots,L_{i_1}}$, $d \geq 2$, all have vanishing Hamiltonian form, *i.e.* they are of the type $(K = 0, J)$. Due to the assumption that $N, L_1, \ldots, L_k$ intersect pairwise transversely, regular choices of perturbation data with the above properties do exist. We denote the space of such regular choices by E''_{reg}. (It is important to note that the restriction that $J_t(p)_{|B} = J^B$ for every t does not pose any regularity problem since every Floer strip or polygon relevant for the definition of $\mathscr{F}uk(\mathscr{C}; p)$ cannot have its image lying entirely inside B, and outside of B we have not posed any restrictions on the choice of almost complex structures.)

We set up the Fukaya categories $\mathscr{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}}_{1/2}, \iota_\gamma(p,h))$ and $\mathscr{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}}_{1/2}, \iota_{\gamma'}(p,h))$ and the associated inclusion functors in the same way as in the previous part of the proof.

Let $\mathscr{V}$, $\mathscr{V}'$ be the Yoneda modules corresponding to V, one time viewed as an object $V \in \mathrm{Ob}(\mathscr{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}}_{1/2}; \iota_\gamma(p,h)))$ and one time as $V \in \mathrm{Ob}(\mathscr{F}uk_{\mathrm{cob}}(\widetilde{\mathscr{C}}_{1/2}; \iota_{\gamma'}(p,h)))$. Consider the pull-back modules

$$\mathscr{M}_{V;\gamma,p,h} := \mathscr{I}^*_{\gamma;p,h}\mathscr{V}, \quad \mathscr{M}_{V;\gamma',p,h} := \mathscr{I}^*_{\gamma;p,h}\mathscr{V}'.$$

By Proposition 3.5 (and (3.32)) we have

$$S^{s_h}\mathcal{M}_{V;\gamma,p,h} = \mathcal{C}one\big(\mathcal{L}_k \xrightarrow{\bar{\phi}_k} \mathcal{C}one\big(\mathcal{L}_{k-1} \xrightarrow{\bar{\phi}_{k-1}} \mathcal{C}one\big(\cdots \mathcal{C}one\big(\mathcal{L}_2 \xrightarrow{\bar{\phi}_2} \mathcal{L}_1\big)\big)\cdots\big)\big), \tag{5.21}$$

where $s_h \to 0$ as $h \to 0$, and similarly to what we have had on page 90, $\bar{\phi}_i = (\phi_i, 0, \boldsymbol{\delta}^{(i)})$, with ϕ_i a homomorphism of modules that shifts action by ≤ 0 and has discrepancy $\leq \boldsymbol{\delta}(p,h)$, where for every d we have $\delta_d(p,h) \in O(\mathcal{N})$. By similar arguments, $S^{s'_h}\mathcal{M}_{V;\gamma',p,h} = \mathcal{L}$, where $\mathcal{L}$ is the Yoneda module of L, and $s'_h \to 0$ as $h \to 0$.

Consider now the chain complexes

$$\mathcal{C}_{p,h} := \mathcal{M}_{V;\gamma,p,h}(N), \quad \mathcal{C}'_{p,h} := \mathcal{M}_{V;\gamma',p,h}(N)$$

endowed with the differential coming from the A_∞-modules $\mathcal{M}_{V;\gamma,p,h}$, $\mathcal{M}_{V;\gamma',p,h}$.[10] By definition

$$\mathcal{C}_{p,h} = \mathrm{CF}(\gamma \times N, V; \mathcal{D}_{\gamma\times N,V}),$$

where $\mathcal{D}_{\gamma\times N,V}$ is the Floer datum prescribed by $\iota_\gamma(p,h)$. Similarly

$$\mathcal{C}'_{p,h} = \mathrm{CF}(\gamma' \times N, V; \mathcal{D}_{\gamma'\times N,V}),$$

where $\mathcal{D}_{\gamma'\times N,V}$ is the Floer datum prescribed by $\iota_{\gamma'}(p,h)$. Consider now the Hamiltonian isotopy $\widetilde{\varphi}_t := \varphi_t \times \mathrm{id} : \mathbb{R}^2 \times M \to \mathbb{R}^2 \times M$, $t \in [0,1]$, where φ_t is the Hamiltonian isotopy from page 95. Note that $\{\widetilde{\varphi}_t\}$ is horizontal at infinity and by (5.20) has Hofer length $\leq \mathcal{S}(V) + \frac{1}{2}\epsilon$. Since $\widetilde{\varphi}_1(\gamma \times N) = \gamma' \times N$, by standard Floer theory (see *e.g.* [**FOOO09a**, Chapter 5]) this isotopy induces two chain maps

$$\phi : \mathcal{C}_{p,h} \longrightarrow \mathcal{C}'_{p,h}, \quad \psi : \mathcal{C}'_{p,h} \longrightarrow \mathcal{C}_{p,h},$$

which are both filtered and such that $\psi \circ \phi$ is chain homotopic to id by a chain homotopy that shifts action by $\leq \mathcal{S}(V) + \epsilon$. More specifically

$$\psi \circ \phi = \mathrm{id} + K d^{\mathcal{C}_{p,h}} + d^{\mathcal{C}_{p,h}} K,$$

where $d^{\mathcal{C}_{p,h}}$ is the differential of $\mathcal{C}_{p,h}$ and K is a Λ-linear map that shifts action by $\leq \mathcal{S}(V) + \epsilon$. Using the formalism of (2.44) this means that

$$B_h(\psi \circ \phi - \mathrm{id}) \leq \mathcal{S}(V) + \epsilon. \tag{5.22}$$

We now appeal to Theorem 2.14, by which we obtain the following:

- ▹ A chain complex $\mathcal{M}(N)$ whose underlying Λ-module coincides with $\mathcal{C}_{p,h}$ and whose differential $\mu_1^{\mathcal{M}(N)}$ is described by (2.31) from Theorem 2.14.
- ▹ An isomorphism of chain complexes $\sigma_1 : \mathcal{M}(N) \to \mathcal{C}_{p,h}$ such that both σ_1 and its inverse $\sigma_1^{-1} : \mathcal{C}_{p,h} \to \mathcal{M}(N)$ shift action by $\leq C^{(1)}(p,h)$, where $C^{(1)}(p,h) \in O(\mathcal{N})$.

We now estimate the action drop $\delta_{\mu_1^{\mathcal{M}(N)}}$ (as defined in (2.42)) of the differential $\mu_1^{\mathcal{M}(N)}$ of the chain complex $\mathcal{M}(N)$. By Theorem 2.14 the differential $\mu_1^{\mathcal{M}(N)}$ comprises various μ_d-operations, $1 \leq d \leq k$, associated to tuples of Lagrangians of the type $(N, L_{i_d}, L_{i_{d-1}}, \ldots, L_{i_2}, L_{i_1})$, where $i = i_1 < \cdots < i_d \leq j$, $1 \leq i \leq j \leq k$. Recall also that the perturbation data $p \in E''_{\mathrm{reg}}$ were chosen with vanishing Hamiltonian perturbation

10. Note that $\mathcal{C}_{p,h}$ defined here is different than the $\mathcal{C}_{p,h}$ from page 90.

for tuple of Lagrangians as above. Therefore, the above mentioned μ_d-operations are defined by counting (unperturbed) pseudo-holomorphic polygons u with corners mapped to intersection points between consecutive pairs of Lagrangians in tuples as above.

Each polygon u contributing to these μ_d-operations has an intersection point in $N \cap L_j$ as one of its inputs and an intersection point in $N \cap L_i$ as its output. Moreover, these polygons are J^B-holomorphic over B. We thus obtain

$$\omega(u) \geq \omega(\text{image}\,(u) \cap B) \geq \tfrac{1}{4}\pi r^2 + \tfrac{1}{4}\pi r^2 = \tfrac{1}{2}\pi r^2,$$

where the first inequality hold because u is unperturbed-pseudo-holomorphic over its entire domain, while the second inequality follows from a Lelong-inequality type of argument (see *e.g.* [BC07], [BC06]). Combining the preceding inequalities with (5.19) we deduce that every Floer polygon u that participate in the calculation of the differential $\mu_1^{\mathcal{M}(N)}$ must satisfy $\omega(u) \geq \frac{1}{2}\delta^{\Sigma'}(N \cup S) - \frac{1}{2}\epsilon$. It follows that

$$\delta_{\mu_1^{\mathcal{M}(N)}} \geq \tfrac{1}{2}\delta^{\Sigma'}(N \cup S) - \tfrac{1}{2}\epsilon. \tag{5.23}$$

In view of the map σ_1 and its inverse σ_1^{-1}, mentioned earlier in the proof, we deduce the following estimate for the action drop of the differential of $\mathscr{C}_{p,h}$:

$$\delta_{d^{\mathscr{C}_{p,h}}} \geq \tfrac{1}{2}\delta^{\Sigma'}(N \cup S) - \tfrac{1}{2}\epsilon - 2C^{(1)}(p,h).$$

As $C^{(1)}(p,h) \in O(\mathscr{N})$, by choosing $p \in E''_{\text{reg}}$ close enough to $\mathscr{N}$ and the profile function h small enough, we may assume in view of (5.18) that

$$\tfrac{1}{2}\delta^{\Sigma'}(N \cup S) - \tfrac{1}{2}\epsilon - 2C^{(1)}(p,h) > \mathcal{S}(V) + \epsilon.$$

Combining the above together with (5.22) we obtain

$$\delta_{d^{\mathscr{C}_{p,h}}} > \mathcal{S}(V) + \epsilon \geq B_h(\psi \circ \phi - \text{id}).$$

By Lemma 2.15 (applied with $C = \mathscr{C}_{p,h}$, $f = \psi \circ \phi$, $g = \text{id}$) we deduce that $\psi \circ \phi$ is injective. It follows that $\phi : \mathscr{C}_{p,h} \to \mathscr{C}'_{p,h}$ is injective too, hence $\dim_\Lambda \mathscr{C}_{p,h} \leq \dim_\Lambda \mathscr{C}'_{p,h}$. But

$$\mathscr{C}_{p,h} = \bigoplus_{i=1}^{k} \bigoplus_{x \in N \cap L_i} \Lambda \cdot x, \quad \mathscr{C}'_{p,h} = \bigoplus_{x \in N \cap L} \Lambda \cdot x,$$

which implies the desired inequality (5.4). This completes the proof of statement (i) under the additional assumption that $L_1, \ldots, L_k$ intersect pairwise transversely.

It remains to treat the case when the Lagrangians $L_1, \ldots, L_k$ do not necessarily intersect pairwise transversely.

Let V be a cobordism as in the statement of the theorem. Fix $r > 0$ and $\epsilon > 0$ with

$$\mathcal{S}(V) + \epsilon < \pi r^2 < \delta^{\Sigma'}(N \cup S). \tag{5.24}$$

Let $e_x : B(r) \to M$, $x \in \Sigma' = N \cap S$, be a collection of symplectic embeddings as in the definition of $\delta^{\Sigma'}(N \cup S)$ on page 85. Since $\Sigma' = \bigcup_{i=1}^{k}(N \cap L_i)$ and the latter union is disjoint every $x \in \Sigma'$ belongs to *precisely* one of the Lagrangians $L_1, \ldots, L_k$. Now let $y \in \Sigma'$ and assume that $y \in N \cap L_i$. Let $j \neq i$. It is easy to see from the assumptions imposed on the embeddings e_x in the definition of $\delta^{\Sigma'}(N \cup S)$ that

$$L_j \cap e_y\big(B(r)\big) = \emptyset.$$

In particular $L_j \cap L_i$ lies outside of $e_y(B(r))$. It follows that $\bigcup_{i'<i''}(L_{i'} \cap L_{i''})$ lies outside of $B := \bigcup_{x \in \Sigma'} \text{image}\, e_x(B(r))$.

Next, apply a small Hamiltonian perturbation to each of the Lagrangians $L_1, \ldots, L_k$, keeping them fixed inside B, so as to obtain new Lagrangians $L'_1, \ldots, L'_k$ that intersect pairwise transversely. By taking these perturbations small enough we may also assume that no new intersection points between S and N have been created, *i.e.* $L'_i \cap N = L_i \cap N$ for every i. Moreover, we take these Hamiltonian perturbations to be small in the Hofer metric so that the Hofer length of each of the isotopies generating the above perturbations is $\leq \frac{1}{2k}\epsilon$.

We now glue to each of the negative ends L_i of V the Lagrangian suspension associated to the preceding Hamiltonian isotopy used to move L_i to L'_i. The result is a new cobordism $V' : L \rightsquigarrow (L'_1, \ldots, L'_k)$ with $\mathcal{S}(V') \leq \mathcal{S}(V) + \frac{1}{2}\epsilon$. Combining with (5.24) we get:

$$\mathcal{S}(V') \leq \delta^{\Sigma'}(N \cup S').$$

As the ends L'_i of V' intersect pairwise transversely, by what we have proved earlier we have

$$\#(N \cap L) \geq \sum_{i=1}^{k} \#\,(N \cap L'_i).$$

Since $N \cap L'_i = N \cap L_i$ for all i, the results follows and completes the proof of statement (i).

5.1.2. ***Proof of statement*** **(ii).** — The proof is similar to the proof of statement (i) above, only that now we use Proposition 2.18 to estimate $\#(N \cap L)$ in (5.5) instead of Lemma 2.15. Below we will mainly go over the points in the proof that differ from the proof of statement (i).

Fix $\epsilon > 0$ small enough and $r > 0$ so that

$$\mathcal{S}(V) + \epsilon < \tfrac{1}{4}\delta^{\Sigma''}(S; N) - \tfrac{1}{4}\epsilon, \quad \delta^{\Sigma''}(S; N) - \epsilon \leq \pi r^2 < \delta^{\Sigma''}(S; N). \tag{5.25}$$

Next, fix symplectic embeddings $e_x : B(r) \to M$, $x \in \Sigma''$, as in the definition of $\delta^{\Sigma''}(S; N)$ on page 85. Fix also curves γ, γ' as in the proof of statement (i). We set up the Fukaya categories $\mathcal{F}uk(\mathcal{C}; p)$, $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}}_{1/2}, \iota_{\gamma}(p, h))$, $\mathcal{F}uk_{\text{cob}}(\widetilde{\mathcal{C}}_{1/2}, \iota_{\gamma'}(p, h))$ and the inclusion functors $\mathcal{I}_{\gamma;p,h}$, $\mathcal{I}_{\gamma';p,h}$, in the same way as in the proof of statement (i). We then define the chain complexes $\mathcal{C}_{p,h}$, $\mathcal{C}'_{p,h}$, and the two chain maps $\phi : \mathcal{C}_{p,h} \to \mathcal{C}'_{p,h}$, $\psi : \mathcal{C}'_{p,h} \to \mathcal{C}_{p,h}$, with

$$\psi \circ \phi = \text{id} + K \circ d^{\mathcal{C}_{p,h}} + d^{\mathcal{C}_{p,h}} \circ K, \tag{5.26}$$

where K shifts action by $\leq \mathcal{S}(V) + \epsilon$.

As before, we now use Theorem 2.14 and obtain a chain complex $\mathscr{M}(N)$ whose underlying Λ-module coincides with $\mathscr{C}_{p,h}$ and equals

$$\mathscr{M}(N) = \bigoplus_{i=1}^{k} \mathrm{CF}(N, L_i; \mathscr{D}_{N,L_i}). \tag{5.27}$$

By Theorem 2.14 the differential $\mu_1^{\mathscr{M}(n)}$ can be written with respect to the splitting (5.27) as an upper triangular matrix of operators $(a_{i,j})$ with diagonal elements

$$a_{i,i} = \mu_1^{\mathrm{CF}(N,L_i;\mathscr{D}_{N,L_i})}.$$

Write $\mu_1^{\mathscr{M}(N)} = d_0 + d_1$, where:

- $d_0 = \bigoplus_{i=1}^{k} \mu_1^{\mathrm{CF}(N,L_i;\mathscr{D}_{N,L_i})}$ with respect to (5.27) and
- $d_1 : \mathscr{M}(N) \to \mathscr{M}(N)$ is the operator represented by the part of the matrix $(a_{i,j})$ that lies strictly above the diagonal.

The operator d_1 consists of sums of μ_d-operations, $d \geq 2$, where among the inputs of each such operation there is at least one point from $L_i \cap L_j$, $i < j$. A similar argument to the one used on page 96 in estimating $\delta_{\mu_1^{\mathscr{M}(N)}}$ in the proof of statement (i) shows that $\delta_{d_1} \geq \frac{1}{4}\pi r^2$. Here, δ_{d_1} is the action drop of d_1 (see Section 2.7, page 44).

Combining with (5.25) we get

$$\delta_{d_1} \geq \tfrac{1}{4}\delta^{\Sigma''}(S; N) - \tfrac{1}{4}\epsilon. \tag{5.28}$$

Put $f' := \psi \circ \phi : \mathscr{C}_{p,h} \to \mathscr{C}_{p,h}$. By (5.26) we have $B_h(f' - \mathrm{id}) \leq \mathcal{S}(V) + \epsilon$. Recall from Theorem 2.14 the isomorphism of chain complexes $\sigma_1 : \mathscr{C}_{p,h} \to \mathscr{M}(N)$ such that both σ_1 and its inverse σ_1^{-1} shift action by $\leq C^{(1)}(p, h)$, where $C^{(1)}(p, h) \in O(\mathcal{N})$. Consider

$$f := \sigma_1 \circ f' \circ \sigma_1^{-1} : \mathscr{M}(N) \longrightarrow \mathscr{M}(N).$$

We would like to apply Proposition 2.18 to $C = \mathscr{M}(N)$, d_0, d_1 as defined above and the map f. We have

$$f - \mathrm{id} = (\sigma_1 \circ K \circ \sigma_1^{-1}) \circ d^{\mathscr{C}_{p,h}} + d^{\mathscr{C}_{p,h}} \circ (\sigma_1 \circ K \circ \sigma_1^{-1}),$$

hence $B_h(f - \mathrm{id}) \leq \mathcal{S}(V) + \epsilon + 2C^{(1)}(p, h)$. As $C^{(1)}(p, h) \in O(\mathcal{N})$, by taking p close enough to $\mathcal{N}$ and the profile function h small enough, we may assume in view of (5.25) that

$$\mathcal{S}(V) + \epsilon + 2C^{(1)}(p, h) < \tfrac{1}{4}\delta^{\Sigma''}(S; N) - \tfrac{1}{4}\epsilon.$$

Together with (5.28) we now obtain

$$B_h(f - \mathrm{id}) < \delta_{d_1}. \tag{5.29}$$

In order to apply Proposition 2.18 it remains to check that

$$\dim_\Lambda H_*\big(\mathscr{M}(N), d_0\big) \geq \dim_\Lambda H_*\big(\mathscr{M}(N), \mu_1^{\mathscr{M}(N)}\big). \tag{5.30}$$

This follows from standard results in homological algebra since

$$H_*\big(\mathscr{M}(N), d_0\big) = \bigoplus_{i=1}^{k} \mathrm{HF}(N, L_i), \quad H_*\big(\mathscr{M}(N), \mu_1^{\mathscr{M}(N)}\big) \cong H_*(\mathscr{C}_{p,h}, d^{\mathscr{C}_{p,h}})$$

and $\mathscr{C}_{p,h}$ is an iterated cone of the type

$$\mathscr{C}_{p,h} = \mathscr{C}one\big(\mathrm{CF}(N, L_k) \to \mathscr{C}one\big(\mathrm{CF}(N, L_{k-1}) \\ \to \mathscr{C}one\big(\cdots \mathscr{C}one\big(\mathrm{CF}(N, L_2) \to \mathrm{CF}(N, L_1)\big)\cdots\big)\big).$$

We are now in position to apply Proposition 2.18, by which we obtain

$$\dim_\Lambda \big(\mathrm{image}\,(f)\big) \geq \dim_\Lambda H_*(\mathscr{M}(N), d_0) = \sum_{i=1}^{k} \dim_\Lambda \mathrm{HF}(N, L_i).$$

On the other hand $\dim_\Lambda(\mathrm{image}\,(f)) \leq \dim_\Lambda \mathscr{M}(N) = \sum_{i=1}^{k} \#(N \cap L_i)$. Putting the last two inequalities together yields (5.5) and concludes the proof of statement (ii). □

5.1.3. ***Remark.*** — The following argument, due to Misha Khanevsky, leads to a more direct proof of an inequality as in the first part of Theorem 5.1 but gives a weaker estimate. We reproduce the argument here with Khanevsky's permission.

A result of Usher (Theorem 4.9 in [**Ush14**]) claims that, given two Lagrangians V and V' that intersect transversely and non-trivially, there exists $\delta > 0$ depending on V and V', such that the energy (in the sense of Hofer geometry) required to disjoin V from V' is greater than δ. This result was proven for compact or (tame at infinity) symplectic manifolds but can be adjusted without any difficulty to the case of Lagrangians with cylindrical ends in $\mathbb{C} \times M$.

Assume that $L \not\subset \bigcup_i L_i$ and let T be a small Lagrangian torus, disjoint from all L_i's, and such that T intersects L transversely and non-trivially. Let γ' be a curve as in Figure 6 and let $V' = \gamma' \times T$. Thus V' and V intersect non-trivially and transversely (see also Figure 4). The isotopy Ψ taking the curve γ to the curve γ' in Figure 6 disjoins V' from V and thus its energy $E(\Psi)$ has to exceed δ. At the same time, Ψ can be picked in such a way that $E(\Psi)$ is as close as needed to $\mathcal{S}(V)$ and thus we deduce the inequality $\mathcal{S}(V) \geq \delta$ which finishes Khanevsky's argument.

However, notice that the dependence on T of the constant δ here means that it is generally smaller than $\delta(L; S)$ from the statement of Theorem 5.1. Note also that this argument does not imply the points (i) and (ii) of the statement and it also can not be adjusted to estimate the algebraic measurements that we will see later in Corollary 6.13.

5.2. Proof of Theorem 5.2

The proof of inequality (5.4) given in Section 5.1 carries over to the monotone case without any modifications.

We now explain how to adjust the proof of (5.3) given in Section 5.1 in order to prove (5.6).

We may assume throughout the proof that $\mathcal{S}(V) < A_L$, for otherwise inequality (5.6) is trivially satisfied. We need to prove that $\mathcal{S}(V) \geq \frac{1}{2}\delta(L; S)$.

We fix $\epsilon > 0$ as in the proof of (5.3) but we require additionally that

$$\mathcal{S}(W) + \epsilon < A_L. \tag{5.31}$$

The proof now goes along the same lines as the proof of (5.3), detailed in Section 5.1, up to the point where we had to use Lemma 5.3 (see page 91). That lemma does not hold in the monotone case, and we will now use the following lemma instead:

LEMMA 5.4. — *Let $c \in \mathrm{Crit}(f)$, viewed as an element of $\mathcal{O}(H_f^{L_0,L_0})$.*

$$\textit{If } \big\langle \mu_1^{\mathrm{CF}(L_0,L_0;p)}(c), q\big\rangle \neq 0 \textit{ then } \nu\big(\big\langle \mu_1^{\mathrm{CF}(L_0,L_0;p)}(c), q\big\rangle\big) \geq A_{L_0}.$$

We postpone the proof for a while and continue with the proof of Theorem 5.2. As in the proof of Theorem 5.2 we decompose the element b from (5.13) as $b = b_0 + \cdots + b_k$ with $b_i \in \mathrm{CF}(L_0, L_i; p)$. We cannot deduce that $\langle \mu_1(b_0), q\rangle = 0$, as earlier. However by Lemma 5.4 and (5.31) we still obtain

$$\nu\big(\langle \mu_1(b_0), q\rangle\big) \geq A_{L_0} - \mathcal{S}(W) - C^{(2)}(p,h) - \tfrac{1}{2}\epsilon > \tfrac{1}{2}\epsilon - C^{(2)}(p,h),$$

where $C^{(2)}(p,h) \in O(\mathcal{N})$. By taking p close enough to $p_0 \in \mathcal{N}$ and h small enough we may assume that $\nu(\langle \mu_1(b_0), q\rangle) > 0$. In view of (5.14) we can now deduce, as before, that there exists $1 \leq j_0 \leq k$ such that (5.15) holds. From this point on, the proof continues exactly as carried out in the weakly exact case in Section 5.1.

It remains to prove the preceding lemma.

Proof of Lemma 5.4. — Let $u \in \mathcal{M}(c, q; \mathcal{D}_{L_0,L_0})$ be a Floer strip that goes from c to q and contributes to $\mu_1^{\mathrm{CF}(L_0,L_0;p)}(x)$. We need to show that $\omega(u) \geq A_{L_0}$.

Indeed, as in the proof of Lemma 5.3 on page 91, after identifying

$$\big(\mathbb{R} \times [0,1], \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\}\big) \quad \text{with} \quad \big(D \setminus \{-1,+1\}, \partial D \setminus \{-1,+1\}\big)$$

the map u extends continuously to a map $\bar{u} : (D, \partial D) \to (M, L_0)$. The dimension of the component of u in the space $\mathcal{M}^*(c, q; \mathcal{D}_{L_0,L_0})$ of *non-parametrized* Floer trajectories connecting c to q is given by

$$\dim \mathcal{M}_u^*(c, q; \mathcal{D}_{L_0,L_0}) = |c| - |q| - 1 + \mu\big([\bar{u}]\big) = |c| - n - 1 + \mu\big([\bar{u}]\big),$$

where μ is the Maslov index and $[\bar{u}] \in H_2^D(M, L_0)$ is the homology class induced by $\bar{u}$. Since $\dim \mathcal{M}_u^*(c, q; \mathcal{D}_{L_0,L_0}) \geq 0$ we must have

$$\mu\big([\bar{u}]\big) \geq n + 1 - |c| > 0.$$

By monotonicity of L_0 we have $\omega([\bar{u}]) \geq A_{L_0}$, hence $\omega(u) \geq A_{L_0}$. This concludes the proof of Lemma 5.4. □

The proof of Theorem 5.2 is now complete.

5.3. The quasi-exact and quasi-monotone cases

For the applications in Chapter 6, versions of Theorems 5.1 and 5.2 will be important for the quasi-exact and quasi-monotone cases. The definitions of these classes of Lagrangian cobordisms appear in Chapter 4 (see Definitions 4.2 and 4.4).

We have the following generalization of Theorems 5.1 and 5.2 to the quasi-exact and quasi-monotone cases.

Theorem 5.5. — *Let $L, L_1, \dots, L_k$ be weakly exact Lagrangians (resp. monotone Lagrangians in $\mathcal{L}ag^{\mathrm{mon},\mathbf{d}}(M)$) and $V : L \rightsquigarrow (L_1, \dots, L_k)$ a quasi-exact (resp. quasi-monotone) Lagrangian cobordism. Let $K_V \subset \mathbb{R}^2$ be a compact subset, homeomorphic to a closed 2-disk, which is quasi-exact (resp. quasi-monotone) admissible for V. Then all the statements of Theorem 5.1 (resp. Theorem 5.2) continue to hold with $\mathcal{S}(V)$ replaced by* $\mathrm{Area}(K_V)$.

5.3.1. ***Remarks***

1) For K_V as in the theorem we have $\mathrm{Area}(K_V) \geq \mathcal{S}(V)$.

2) If V is a connected monotone cobordism then all its ends have the same Maslov-2 disk count: $\mathbf{d}_L = \mathbf{d}_{L_i} = \mathbf{d}_V$. However, the latter is not clear if we only assume that V is quasi-monotone. The issue is that we do not know whether there exists an almost complex structure J_V for which both (J_V, K_V) is quasi-monotone admissible and *in addition* J_V is regular for all J_V-holomorphic disks of Maslov-2. A typical argument would be to perturb J_V inside K_V to achieve regularity and then try to argue by Gromov compactness that for a small enough perturbation J_V^{ϵ} all pseudo-holomorphic disks have Maslov index ≥ 2. The problem with this approach is that as $\epsilon \to 0$ there might be J_V^{ϵ}-holomorphic disks u_ϵ with $\mu(u_\epsilon) \leq 0$ and with $\omega(u_\epsilon) \to \infty$, hence we cannot apply Gromov compactness to u_ϵ as $\epsilon \to 0$.

For this reason, in Theorem 5.5 for the quasi-monotone case, we have assumed explicitly that all the Lagrangians L, L_i have the same Maslov-2 disk count $\mathbf{d}$. Of course, in the monotone case, Theorem 5.2, this is not needed as it follows from the assumption that V is monotone and connected.

5.3.2. ***Proof of Theorem 5.5.*** — In view of the theory developed in Chapter 4 (especially Propositions 4.3 and 4.6), the proof is essentially the same as the proofs of Theorems 5.1 and 5.2 as presented above. The main change is that the projection of the "non-cylindrical" part of V should now be replaced by K_V. Other than that, instead of working with the modules $\mathcal{V}, \mathcal{W}, \mathcal{M}_{V;\gamma,p,h}, \mathcal{M}_{W;\gamma,p,h}$ one uses their quasi-exact or quasi-monotone versions $\mathcal{V}_q$, $\mathcal{W}_q$, $\mathcal{M}^q_{V;\gamma,p,h}$, $\mathcal{M}^q_{W;\gamma,p,h}$, where q stands for either $q = \mathrm{qe}$ or $q = \mathrm{qm}$. □

CHAPTER 6

METRICS ON SPACES OF LAGRANGIANS AND EXAMPLES

This chapter gives some context to the phenomena reflected in Theorem 5.1 and discusses a number of applications and ramifications.

The first goal is to introduce metrics on the space of Lagrangian submanifolds that come from shadow measurements. Roughly speaking, our metrics will be defined by infimizing the shadow over all (multiply ended) Lagrangian cobordisms with two of their ends coinciding with two given Lagrangian submanifolds. As usual, the difficult part is in showing that this procedure leads to a non-degenerate measurement, and the main ingredient in establishing the non-degeneracy of our metrics will be Theorem 5.1 and its various versions.

Of course, in order to obtain non-degeneracy we need to restrict the class of Lagrangians in M and the class of cobordisms in $\mathbb{R}^2 \times M$ in our considerations. The two classes of Lagrangians (in M) that we will focus on, are weakly-exact Lagrangians and monotone ones. Naturally, we would like to use cobordisms of the same class (weakly-exact or monotone) in defining the metrics. However, here *a new problem arises*. In order to retain the triangle inequality for our metrics we need to infimize shadows over a class of Lagrangian cobordisms that is *closed* under composition (or gluing) of two cobordisms along a pair of matching ends. As it turns out, neither the class of weakly-exact cobordisms nor the class of monotone ones seems to enjoy this property (unless one imposes additional topological restrictions). It is at this point that we need to appeal to the more general class of quasi-exact and quasi-monotone cobordisms. The next section elaborates on this issue and how to solve it.

6.1. Setting up the right class of cobordisms

Let:

- $* =$ we (*weakly exact*) or
- $* = (\text{mon}, \mathbf{d})$ (*monotone* with Maslov-2 disk count equal to $\mathbf{d}$).

Denote by $\mathscr{L}ag^*(M)$ the collection of Lagrangian submanifolds $L \subset M$ of class $*$. Let Q be a class of Lagrangian cobordisms with ends in $\mathscr{L}ag^*(M)$.

We will denote by $\mathcal{L}ag^{Q,*}(\mathbb{R}^2 \times M)$ the collection of Lagrangian cobordisms of class Q with ends in $\mathcal{L}ag^{*}(M)$.

We say that Q (or $\mathcal{L}ag^{Q,*}(\mathbb{R}^2 \times M)$) is *closed under composition* if for every two cobordisms $V : L \rightsquigarrow (L_1, \ldots, L_r)$ and $W : L_j \rightsquigarrow (K_1, \ldots, K_s)$ in $\mathcal{L}ag^{Q,*}(\mathbb{R}^2 \times M)$ their composition along L_j,

$$W \circ V : L \rightsquigarrow (L_1, \ldots, L_{j-1}, K_1, \ldots, K_s, L_{j+1}, \ldots, L_r)\ ,$$

is also in $\mathcal{L}ag^{Q,*}(\mathbb{R}^2 \times M)$. Here and in what follows we always assume that the matching end L_j is *connected*.

It is easy to see by an application of the Van Kampen theorem that if we consider W, V monotone and both inclusions $W \to M$ and $V \to M$ are trivial in π_1, then $W \circ V$ is again monotone with $\pi_1(W \circ V) \to \pi_1(M)$ trivial (these are the assumptions in [**BC14**]). Moreover, if W, V are weakly exact, connected cobordisms with a single positive end and a single negative end both connected, then $W \circ V$ is weakly exact. Indeed, by results in [**BS19**] a weakly exact simple cobordism V has the property that the map induced on π_1 by the inclusion of an end of V into V is epimorphic. Using this fact, the Van Kampen theorem again implies the claim. However, these conditions are quite restrictive and, without them, the class of weakly exact cobordisms generally seems not to be closed under composition, and the same for monotone cobordisms.

At the same time we will see soon that the classes of quasi-exact cobordisms (with weakly exact ends) and the class of quasi-monotone cobordisms (with ends in $\mathcal{L}ag^{\text{mon},\mathbf{d}}(M)$) are closed under composition. This is the reason we will appeal to these classes of cobordisms.

However, for our applications we will actually need to somewhat restrict the classes of quasi-exact and quasi-monotone cobordisms as follows.

Definition 6.1 *(Tightly quasi-exact and quasi-monotone cobordisms)*. — Let $V \subset \mathbb{R}^2 \times M$ be a quasi-exact (resp. quasi-monotone) cobordism with ends in $\mathcal{L}ag^{\text{we}}(M)$ (resp. $\mathcal{L}ag^{\text{mon},\mathbf{d}}(M)$). In the "(mon, $\mathbf{d}$)" case assume in addition that not all the ends of V are void. We say that V is *tightly* quasi-exact (resp. tightly quasi-monotone) if *for every* compact subset $K_V \subset \mathbb{R}^2$, homeomorphic to a closed 2-disk, for which V is cylindrical over $\mathbb{R}^2 \setminus \text{Int}(K_V)$ (see Definition 4.1), there exists J_V such that (J_V, K_V) is quasi-exact (resp. quasi-monotone) admissible (see Definitions 4.2 and 4.4).

We will elaborate more on the reasons for introducing the classes of tightly quasi-exact/quasi-monotone cobordisms in Remark 6.2.1 below.

6.1.1. *Remarks*

1) If V is tightly quasi-exact, then for every $\epsilon > 0$ there exists a quasi-exact admissible (J_V, K_V) with $\mathcal{S}(V) \leq \text{Area}(K_V) \leq \mathcal{S}(V) + \epsilon$. The same holds also in the tightly quasi-monotone case.

2) Every weakly exact cobordism is tightly quasi-exact. As we will see below compositions of weakly exact cobordism along one pair of matching ends is tightly quasi-exact. A similar remark applies to quasi-monotone cobordisms.

3) In principle it seems that the class of tightly quasi-exact (resp. quasi-monotone) cobordisms is smaller than the quasi-exact (resp. quasi-monotone) ones. However, we are not aware of any concrete examples of quasi-exact (resp. monotone) cobordisms that are not tightly quasi-exact (resp. quasi-monotone).

PROPOSITION 6.2. — *Each of the following classes Q of cobordisms is closed under composition:*

(i) *Exact cobordisms with exact ends.*
(ii) *Quasi-exact cobordisms with weakly exact ends.*
(iii) *Quasi-monotone cobordisms with ends in $\mathscr{L}ag^{\mathrm{mon},\mathbf{d}}(M)$.*
(iv) *Tightly quasi-exact cobordisms with weakly exact ends.*
(v) *Tightly quasi-monotone cobordisms with ends in $\mathscr{L}ag^{\mathrm{mon},\mathbf{d}}(M)$.*

For the proof of this proposition we need the following variant of Lemma 4.5, which can be proved by similar way.

LEMMA 6.3. — *Let (V, J_V, K_V) be quasi-exact with weakly exact ends (resp. quasi-monotone with ends in $\mathscr{L}ag^{\mathrm{mon},\mathbf{d}}(M)$). Let $\sigma : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a diffeomorphism which coincides with the identity in a neighborhood of K_V. Assume that σ sends the horizontal rays of $\pi(V)$ to other horizontal half-lines so that $V' = (\sigma \times \mathrm{id})(V)$ is also a cobordism. Under these assumptions, (V', J_V, K_V) is also quasi-exact (resp. quasi-monotone).*

The proof is similar to the proof of Lemma 4.5.

We now prove the previous proposition.

Proof of Proposition 6.2. — Let $V : L \rightsquigarrow (L_1, \dots, L_r)$ and $W : L_j \rightsquigarrow (K_1, \dots, K_s)$ be two cobordisms of class Q.

▹ Case "*Q = exact cobordisms with exact ends*"

Denote by $\widetilde{\lambda} = \lambda \oplus \lambda_{\mathbb{R}^2}$ the primitive of $\widetilde{\omega} = \omega \oplus \omega_{\mathbb{R}^2}$ with respect to which we define exactness (here λ is the given primitive of ω and $d\lambda_{\mathbb{R}^2} = \omega_{\mathbb{R}^2}$). Let $F_V : V \longrightarrow \mathbb{R}$ and $F_W : W \longrightarrow \mathbb{R}$ be primitives of $\widetilde{\lambda}_{|V}$ and $\widetilde{\lambda}_{|W}$ respectively. Let $\alpha \approx (-1, 1)$ be the projection to $\mathbb{R}^2$ of the neck of $W \circ V$ resulting from the gluing of W and V along their L_j-ends. By adding a suitable constant we can arrange that F_V and F_W agree along $\alpha \times L_j$, hence $\widetilde{\lambda}_{|W \circ V}$ is exact. (Note that L_j is connected, as by assumption composition of cobordisms is performed only along a pair of *connected* matching ends. See the beginning of Section 6.1.)

▹ Case "*Q = quasi-exact*"

We first use Lemma 6.3 to rearrange conveniently the ends of V to get a quasi-exact cobordism V' whose ends are all positive except for a single negative end that coincides with L_j. We then glue V to V' along the end L_j. More explicitly, we translate V' (together with the almost complex structure J_V) along the plane so that $K_V \subset \pi^{-1}((1, \infty) \times \mathbb{R})$ and $V' \cap \pi^{-1}([0, 1] \times \mathbb{R}) = [0, 1] \times \{0\} \times L_j$. Similarly, we translate W (together with J_W) along the plane so that $K_W \subset \pi^{-1}((-\infty, -1) \times \mathbb{R})$ and $W \cap \pi^{-1}([-1, 0] \times \mathbb{R}) = [-1, 0] \times \{0\} \times L_j$. We then define $W \circ V'$ as the union

$$W \circ V' = \big(V' \cap \big([0, \infty) \times \mathbb{R} \times M\big)\big) \cup \big(W \cap \big((-\infty, 0] \times \mathbb{R} \times M\big)\big).$$

In the region $(-1,1)\times\mathbb{R}\times M$ both almost complex structures J_V and J_W are fiberwise split so we can interpolate between corresponding fiber structures thus getting a new almost complex structure $\widetilde{J}$ that is split in the exterior of $K_V\cup K_W$ and coincides with J_V on $[1,\infty)\times\mathbb{R}$ and with J_W on $(-\infty,-1]\times\mathbb{R}$. The cobordism $(W\circ V',\widetilde{J})$ is quasi-exact by an immediate application of the open mapping theorem combined with the fact that L_j is weakly exact and that (V',J_V) and (W,J_W) are quasi-exact. Finally, we use Lemma 6.3 again to move the remaining L_i ends of $W\circ V'$ to the left, thus getting that the cobordism $W\circ V$ in the statement is quasi-exact.

The case of tight quasi-exact cobordisms follows immediately from the previous argument. Indeed, if $K_{W\circ V}\subset\mathbb{R}^2$ is a compact subset homeomorphic to a closed 2-disk with $W\circ V$ being cylindrical over $\mathbb{R}^2\setminus\mathrm{Int}\,(K_{W\circ V})$, then $\mathrm{Int}\,(K_{W\circ V})$ must contain the (bounded!) plane curve forming the neck of the gluing of W with V along L_j. We now take two disjoint subsets $K_V, K_W\subset K_{W\circ V}$ each homeomorphic to a closed 2-disk and such that V and W are cylindrical over $\mathbb{R}^2\setminus\mathrm{Int}\,(K_V)$ and over $\mathbb{R}^2\setminus\mathrm{Int}\,(K_W)$ respectively. By the tightness assumption there exists J_V and J_W for which (V,J_V,K_V) and (W,J_W,K_W) are both quasi-exact. Let $S\subset\mathbb{R}^2$ be a thin strip (homeomorphic to $[-1,1]\times[-\epsilon,\epsilon]$) that contains in its interior the neck of the gluing of W and V along L_j and such that $S\subset\mathrm{Int}\,(K_{W\circ V})$. By positioning the strip S appropriately we may assume that $K_V\cup K_W\cup S$ is homeomorphic to a closed 2-disk. By the previous argument (for "Q= quasi-exact"), $J_{V\,|K_V\times M}$ and $J_{W\,|K_W\times M}$ extend to an almost complex structure $J_{W\circ V}$ on $\mathbb{R}^2\times M$ which makes $(J_{W\circ V}, K_V\cup K_W\cup S)$ quasi-exact admissible for $W\circ V$. Since $K_V\cup K_W\cup S\subset K_{W\circ V}$, the pair $(J_{W\circ V}, K_{W\circ V})$ is also quasi-exact admissible for $W\circ V$.

Finally, the proof of the statements in the quasi-monotone and tightly quasi-monotone cases is the same as for the classes of quasi-exact and tightly quasi-exact cobordisms. □

6.1.2. ***Remark.*** — Let $V: L\rightsquigarrow(L_1,\dots,L_r)$ and $W: L_j\rightsquigarrow(K_1,\dots,K_s)$ be two quasi-exact cobordisms with weakly exact ends, and let

$$W\circ V: L\rightsquigarrow(L_1,\dots,L_{j-1},K_1,\dots,K_s,L_{j+1},\dots,L_r)$$

be their composition along L_j, which by Proposition 6.2 is again quasi-exact. Let $K_V, K_W\subset\mathbb{R}^2$ be compact subsets which are quasi-exact admissible for V and W respectively. The proof of Proposition 6.2 shows that for every $\epsilon>0$ there exists a compact subsets $K^{\epsilon}_{W\circ V}\subset\mathbb{R}^2$ which is quasi-exact admissible for $W\circ V$ and such that

$$\mathrm{Area}(K^{\epsilon}_{W\circ V})\leq\mathrm{Area}(K_V)+\mathrm{Area}(K_W)+\epsilon.$$

6.2. Shadow metrics on spaces of Lagrangian submanifolds

Let (M,ω) be a symplectic manifold. Fix a class $\mathcal{L}ag^*(M)$ of Lagrangian submanifolds of M, where $*$ can be either "we" (*i.e.* weakly exact) or "(mon, $\mathbf{d}$)" (*i.e.* monotone with a fixed Maslov-2 disk count $\mathbf{d}$, see Section 3.5). In case $(M,\omega=d\lambda)$ is an exact symplectic manifold we allow also $*=$ ex, *i.e.* exact Lagrangians. In case $*=$ we, let Q be the class of Lagrangian cobordisms which are *tightly quasi-exact*, and in case $* = (\mathrm{mon}, \mathbf{d})$ let Q be the class of tightly quasi-monotone Lagrangian cobordisms with ends in $\mathcal{L}ag^{\mathrm{mon},\mathbf{d}}(M)$. Finally, if $* = \mathrm{ex}$ we can take Q to be either the class of exact Lagrangian cobordisms with exact ends or the class of quasi-exact cobordisms with exact ends. For the definition of exact cobordisms we fix a primitive $\lambda_{\mathbb{R}^2}$ of $\omega_{\mathbb{R}^2}$ and take $\widetilde{\lambda} = \lambda \oplus \lambda_{\mathbb{R}^2}$ as the primitive of $\widetilde{\omega}$ for the purpose of defining exact cobordisms.

Fix a family $\mathcal{F} \subset \mathcal{L}ag^*(M)$ of Lagrangian submanifolds of M. Let L and L' be two other Lagrangians in $\mathcal{L}ag^*(M)$. Theorem 5.1 and its various generalizations (Theorems 5.2 and 5.5) suggest the definition of the following two sequences of numbers. The definition of these numbers has a geometric underpinning in that it is based on the existence of certain cobordisms.

First, for each $a > 0$, define the *a-cone-length* of L' relative to L (with respect to $\mathcal{F}$) as

$$l_a^{\mathcal{F}}(L', L) := \min \big\{ k \in \mathbb{N} \ ; \ \exists V : L' \rightsquigarrow (L_1, \ldots, L_{s-1}, L, L_s, \ldots, L_k), \\ L_i \in \mathcal{F}, \ \mathcal{S}(V) \leq a \big\}. \tag{6.1}$$

Here, the minimum is taken only over cobordisms $V \in \mathcal{L}ag^Q(\mathbb{R}^2 \times M)$, *i.e.* in the class Q. We stress that we allow V to be disconnected, and that $V \in \mathcal{L}ag^Q(\mathbb{R}^2 \times M)$ means that every path connected component of V is of class Q. We use the convention that the number $l_a^{\mathcal{F}}(L', L)$ equals 0 if L and L' are related by a simple cobordism $V : L' \rightsquigarrow L$ of shadow $\leq a$ (a cobordism with just two possibly non-void ends, one positive and one negative, is called simple). We set $l_a^{\mathcal{F}}(L', L) = \infty$ if no cobordism V as above exists. We will omit $\mathcal{F}$ from the notation when there is no risk of confusion. It is clear that $l_a^{\mathcal{F}}(L', L)$ is non-increasing in a and symmetric with respect to L, L'. Next, define $l^{\mathcal{F}}(L', L) := \lim_{a\to\infty} l_a^{\mathcal{F}}(L', L)$ to be the *absolute* cone length of L' relative to L and $l_0^{\mathcal{F}}(L', L) := \lim_{a\to 0} l_a^{\mathcal{F}}(L', L)$.

In view of Theorem 5.1 it is natural to also estimate the minimal shadow required for splittings as in the definition of $l_a^{\mathcal{F}}$ and thus define a second family of natural measurements as follows. For every $k \in \mathbb{N}$ define:

$$d_k^{\mathcal{F}}(L', L) := \inf \big\{ \mathcal{S}(V) \ ; \ V : L' \rightsquigarrow (L_1, \ldots, L_{s-1}, L, L_s, \ldots, L_r), \ L_i \in \mathcal{F}, \ r \leq k \big\}. \tag{6.2}$$

Again, the infimum is taken only over cobordisms V of class Q and we allow V to be disconnected. This is significant as, for instance, if $\mathcal{F}$ contains a representative in each Hamiltonian isotopy class of the Lagrangians in $\mathcal{L}ag^*(M)$, then $d_2^{\mathcal{F}}(L', L)$ is finite for all $L, L' \in \mathcal{L}ag^*(M)$ (one can take V as an appropriate union $V_0 \cup V_1$ of two disjoint Lagrangian suspensions $V_0 : L' \rightsquigarrow L'_1$, $V_1 : \varnothing \rightsquigarrow (L, L_1)$ with L_1 and L'_1 respectively Hamiltonian isotopic to L and to L'). We take $d_k^{\mathcal{F}}(L', L) = \infty$ if no cobordisms V as in (6.2) exist. Again, $d_k^{\mathcal{F}}$ is symmetric in (L, L'), and $d_k^{\mathcal{F}}(L', L)$ is non-increasing in k. Note that $d_0^{\mathcal{F}}$ is the "shadow" metric on elementary cobordism equivalence classes as defined in [**CS19**]. Thus for a Hamiltonian diffeomorphism ϕ, we have

$$d_0^{\mathcal{F}}(\phi(L), L) \leq \|\phi\|_H,$$

where $\|\bullet\|_H$ denotes the Hofer norm of ϕ.

The following inequality is immediate in view of Proposition 6.2:

$$d^{\mathcal{F}}_{k+k'}(L, L'') \leq d^{\mathcal{F}}_{k}(L, L') + d^{\mathcal{F}}_{k'}(L', L''). \tag{6.3}$$

Obviously, we also have $d^{\mathcal{F}}_{l^{\mathcal{F}}_a(L',L)}(L', L) \leq a$ and $l^{\mathcal{F}}_{d^{\mathcal{F}}_k(L',L)}(L', L) \leq k$.

Finally, we define also the following measurement:

$$d^{\mathcal{F}}(L, L') = \lim_{k\to\infty} d^{\mathcal{F}}_{k}(L, L') = \inf_{k\geq 0} d^{\mathcal{F}}_{k}(L, L'). \tag{6.4}$$

Or more explicitly

$$d^{\mathcal{F}}(L, L') = \inf\big\{\mathcal{S}(V)\ ;\ V : L' \rightsquigarrow (L_1, \dots, L_{s-1}, L, L_s, \dots, L_r), \quad L_i \in \mathcal{F},\ V \in \mathcal{L}ag^{Q}(\mathbb{R}^2 \times M)\big\}. \tag{6.5}$$

From the above it follows that $d^{\mathcal{F}}(\bullet, \bullet)$ is a pseudo-metric called the shadow pseudo-metric associated to $\mathcal{F}$. By definition, $d^{\mathcal{F}}(L, L')$ is infinite only if there are no cobordisms relating L to L' and with all the other ends in $\mathcal{F}$.

Theorem 5.5 implies:

Corollary 6.4. — *If $d^{\mathcal{F}}(L', L) = 0$, then $L \subset L' \cup \overline{\bigcup_{K\in\mathcal{F}} K}$ and $L' \subset L \cup \overline{\bigcup_{K\in\mathcal{F}} K}$.*

Proof. — If the first inclusion in this statement does not hold, then

$$\delta(L; L' \cup \overline{\bigcup_{K\in\mathcal{F}} K}) > 0. \tag{6.6}$$

If Q is the class of exact cobordisms with exact ends, then the first point of Theorem 5.1 implies that $d^{\mathcal{F}}(L', L)$ cannot vanish.

If Q is either "tightly quasi-exact Lagrangians with weakly exact ends" or "tightly quasi-monotone Lagrangians with ends in $\mathcal{L}ag^{\mathrm{mon},\mathbf{d}}(M)$", then again, by Theorem 5.5 together with point 1) of Remark 6.1.1 it follows that $d^{\mathcal{F}}(L', L)$ can not vanish.

The argument for the second inclusion is the same. □

It is easy to see (Remark 6.3.2 below) that the pseudo-metric $d^{\mathcal{F}}$ given by (6.4) is in general degenerate. However, we have

Corollary 6.5. — *Let $\mathcal{F}$ and $\mathcal{F}'$ be two families of Lagrangians in $\mathcal{L}ag^*(M)$ such that the intersection $(\overline{\bigcup_{K\in\mathcal{F}} K}) \cap (\overline{\bigcup_{K'\in\mathcal{F}'} K'})$ is totally disconnected (e.g. discrete). Then the pseudo-metric on $\mathcal{L}ag^*(M)$ defined by*

$$\widehat{d}^{\,\mathcal{F},\mathcal{F}'} := \max\{d^{\mathcal{F}}, d^{\mathcal{F}'}\}$$

is non-degenerate.

Proof. — If $\widehat{d}^{\,\mathcal{F},\mathcal{F}'}(L, L') = 0$ we deduce from Corollary 6.4 that $L \subset L' \cup \overline{\bigcup_{K\in\mathcal{F}} K}$ and $L \subset L' \cup \overline{\bigcup_{K'\in\mathcal{F}'} K'}$. Assume that there is a point $x \in L$ such that $x \notin L'$. Then there is an open disk $D \subset L$ with $D \cap L' = \emptyset$. It follows that one has $D \subset \overline{\bigcup_{K\in\mathcal{F}} K}$ as well as $D \subset \overline{\bigcup_{K'\in\mathcal{F}'} K'}$ which is not possible because the set $(\overline{\bigcup_{K\in\mathcal{F}} K}) \cap (\overline{\bigcup_{K'\in\mathcal{F}'} K'})$ is totally disconnected. We conclude that $L \subset L'$. The roles of L and L' being symmetric, we deduce that $L = L'$. □

Notice that if $L' = \phi(L)$ with ϕ a Hamiltonian diffeomorphism, then

$$\widehat{d}^{\,\mathcal{F},\mathcal{F}'}(L, L') \leq \|\phi\|_H.$$

Given a family $\mathcal{F}$ that is finite (but this can also work in more general instances) it is easy to produce an additional family $\mathcal{F}'$ that satisfies the assumption of Corollary 6.5. This can be achieved, for instance, by transporting each element of $\mathcal{F}$ by an appropriate Hamiltonian isotopy.

We will not analyze here in detail the properties of the metrics $\widehat{d}^{\,\mathcal{F},\mathcal{F}'}$ but there are two simple observations that we include.

Corollary 6.6. — *For every Hamiltonian diffeomorphism ϕ of M we have*

$$\begin{aligned} &\left|\widehat{d}^{\,\mathcal{F},\mathcal{F}'}(L, L') - \widehat{d}^{\,\mathcal{F},\mathcal{F}'}(\phi(L), \phi(L'))\right| \leq 2\|\phi\|_H, \\ &\left|\widehat{d}^{\,\phi(\mathcal{F}),\phi(\mathcal{F}')}(L, L') - \widehat{d}^{\,\mathcal{F},\mathcal{F}'}(L, L')\right| \leq 2\|\phi\|_H. \end{aligned} \tag{6.7}$$

Therefore, $\mathrm{Ham}(M, \omega)$ *acts by quasi-isometries on the metric space* $(\mathcal{L}ag^*(M), \widehat{d}^{\,\mathcal{F},\mathcal{F}'})$*. Moreover, the identity is a quasi-isometry between the two metric spaces*

$$\big(\mathcal{L}ag^*(M), \widehat{d}^{\,\phi(\mathcal{F}),\phi(\mathcal{F}')}\big), \ \big(\mathcal{L}ag^*(M), \widehat{d}^{\,\mathcal{F},\mathcal{F}'}\big).$$

Proof. — A cobordism $V : L \rightsquigarrow (F_1, \ldots, L', \ldots, F_k)$ can be extended, by gluing appropriate Lagrangian suspensions to the ends L and L', to a cobordism

$$V' : \phi(L) \rightsquigarrow (F_1, \ldots, \phi(L'), \ldots, F_k)$$

of shadow $\mathcal{S}(V') \leq \mathcal{S}(V) + 2\|\phi\|_H$. The first inequality in the statement then follows rapidly, by applying the same argument to ϕ^{-1}. Similarly, to deduce the second inequality, consider $V : L \rightsquigarrow (F_1, \ldots, L', \ldots, F_k)$. By applying ϕ to V we get

$$\phi(V) : \phi(L) \rightsquigarrow \big(\phi(F_1), \ldots, \phi(L'), \ldots, \phi(F_k)\big).$$

Extend both ends $\phi(L)$ and $\phi(L')$ by Lagrangian suspensions thus getting $V'' \rightsquigarrow (\phi(F_1), \ldots, L', \ldots, \phi(F_k))$ of shadow bounded by $\mathcal{S}(V) + 2\|\phi\|_H$ and the desired inequality follows easily. □

6.2.1. ***Remark.*** — Consider the case $* = \mathrm{we}$. As indicated at the beginnig of Chapter 6 we could not have defined the pseudo-metric $d^{\mathcal{F}}$ by infimizing in (6.5) only over weakly exact cobordisms. The reason is that compositions of weakly exact cobordisms might not be weakly exact, hence the triangle inequality (6.3) might not hold. It is for this reason that we needed to enlarge the class of cobordisms to quasi-exact.

Next we explain why infimizing shadows over quasi-exact cobordisms still does not give the correct definition and why we need to appeal to tightly quasi-exact ones. The reason is that Theorem 5.5 (as opposed to Theorem 5.1) gives us only a lower bound for $\mathrm{Area}(K_V)$ rather than for $\mathcal{S}(V)$. (Here $K_V \subset \mathbb{R}^2$ is a compact subset which is quasi-exact admissible for V.) However, if V is tightly quasi-exact then by point 1) of Remark 6.1.1 we have

$$\inf\big\{\,\mathrm{Area}(K_V)\,;\ K_V \subset \mathbb{R}^2 \text{ is quasi-exact admissible for } V\big\} = \mathcal{S}(V).$$

Therefore the definition in (6.5) has the desired properites.

Of course, one could attempt to define another pseudo-metric similar to $d^{\mathcal{F}}$, by

$$d^{\mathcal{F},\mathrm{qe}}(L,L') = \inf\big\{\mathrm{Area}(K_V)\ ;\ V : L' \rightsquigarrow (L_1,\dots,L_{s-1},L,L_s,\dots,L_r),\\ L_i \in \mathcal{F}, V \in \mathcal{L}ag^{\mathrm{qe}}(\mathbb{R}^2\times M),\\ K_V \text{ is quasi-exact admissible for } V\big\}, \tag{6.8}$$

where the infimum is taken over all quasi-exact cobordisms as in (6.8). In view of Theorem 5.5 and Remark 6.1.2, this yields a pseudo-metric with similar properties to $d^{\mathcal{F}}$.

Similar remarks apply to the case $* = (\mathrm{mon}, \mathbf{d})$ and to quasi-monotone versus tightly quasi-monotone cobordisms.

Note that in the case $*$ = ex (*exact Lagrangians*) one can safely take Q to be the class of exact cobordisms, since compositions of exact cobordisms is exact and moreover Theorem 5.1 applies to exact cobordisms. □

The construction of the metrics $\widehat{d}^{\mathcal{F},\mathcal{F}'}$ admits several variations. For instance, let $\mathcal{U} = \{U_i\}_{i\in I}$ be a family of open sets $U_i \subset M$ and let $\mathcal{F}_i = \{L \in \mathcal{L}ag^*(M)\ ;\ L\cap U_i = \varnothing\}$. For each index $i \in I$ we then have a shadow pseudo-metric $d^{\mathcal{F}_i}$. Define a new pseudo-metric:

$$D^{\mathcal{U}} = \sup\big\{d^{\mathcal{F}_i}\ ;\ i \in I\big\}.$$

For the next corollary we will make use of the following. For $L \in \mathcal{L}ag^*(M)$ let:

$$\Delta(L;\mathcal{U}) = \inf\big\{s\ ;\ \forall i \in I, \exists\phi \text{ Hamiltonian diffeomorphism}\\ \text{with } \phi(L)\cap U_i = \varnothing, \|\phi\|_H \le s\big\}.$$

Corollary 6.7. — *With the notation above we have*

(i) *If $\mathcal{U}$ is a covering of M in the sense that $\bigcup_i U_i = M$, then $D^{\mathcal{U}}$ is non-degenerate.*

(ii) *For all $L, L' \in \mathcal{L}ag^*(M)$ such that $\Delta(L;\mathcal{U})$ and $\Delta(L';\mathcal{U})$ are finite, we have*

$$D^{\mathcal{U}}(L,L') \le \Delta(L;\mathcal{U}) + \Delta(L';\mathcal{U}).$$

Proof. — The first point follows immediately from Corollary 6.4. For the second point fix some $s > \Delta(L;\mathcal{U})$, $s' > \Delta(L';\mathcal{U})$ and pick one family $\mathcal{F}_i$. There is a cobordism $V : L \rightsquigarrow (L'_1, L', L_1)$ such that V is a disjoint union of two Lagrangian suspensions $V_0 : L \rightsquigarrow L_1$ and $V_1 : \varnothing \rightsquigarrow (L'_1, L')$ such that $L_1, L'_1 \in \mathcal{F}_i$ and $\mathcal{S}(V_0) \le s$, $\mathcal{S}(V_1) \le s'$. This means that $d^{\mathcal{F}_i}(L,L') \le s + s'$ which implies the claim. □

There are other variants of the definition of the metric $\widehat{d}^{\mathcal{F},\mathcal{F}'}$ that have interesting features. For instance, by considering in (6.2) only cobordisms $V : L' \rightsquigarrow (L_1,\dots,L_k,L)$, in other words cobordisms for which L' is the positive end and L is the *top* negative end, one gets a measurement $t^{\mathcal{F}}_k(L',L)$. It has similar properties to $d^{\mathcal{F}}_k$, except that it is not symmetric. We define $t^{\mathcal{F}}(L',L)$ as in (6.4) and we symmetrize by putting

$$r^{\mathcal{F}}(L',L) = \tfrac{1}{2}\big(t^{\mathcal{F}}(L',L) + t^{\mathcal{F}}(L,L')\big),$$

thus obtaining a new pseudo-metric. This pseudo-metric satifies the conclusion of Corollary 6.4 and can be used in the rest of the preceding constructions, leading to metrics $\widehat{r}^{\mathcal{F},\mathcal{F}'}$ that satisfy the conclusions of Corollaries 6.5, 6.6 and 6.7, where in 6.7 the pseudo-metric $D^{\mathcal{U}}$ is replaced with $R^{\mathcal{U}} = \sup\{r^{\mathcal{F}_i} \ ; \ i \in I\}$.

An additional interesting feature of the pseudo-metrics $R^{\mathcal{U}}$ is the following:

Corollary 6.8. — *With the notation above fix $L \in \mathcal{L}ag^*(M)$ and assume that $\mathcal{U}$ is a covering of M. There exists a constant $\delta > 0$ depending on L and $\mathcal{U}$ such that, if $L' \in \mathcal{L}ag^*(M)$ is disjoint from L, then $R^{\mathcal{U}}(L, L') \geq \delta$.*

Proof. — The crucial remark is that, by inspecting the proof of the first part of Theorem 5.1, we see that given $V : L \rightsquigarrow (L_1, \ldots, L_k, L')$ in $\mathcal{L}ag^{Q}(\mathbb{R}^2 \times M)$ and such that $L \cap L' = \emptyset$, then $\mathcal{S}(V) \geq \frac{1}{2}\delta(L; S)$ where $S = \bigcup_i L_i$ but S – and thus $\delta(L, S)$ – does not depend on L'. As $\mathcal{U}$ is a covering of M there exists some index $i \in I$ and an open set $U \subset U_i$ such that U is the image of an embedding $e : B(r) \to M$ with $e^{-1}(L) = B_{\mathbb{R}}(r)$. Obviously, U is disjoint from all the elements of $\mathcal{F}_i$ and thus $R^{\mathcal{U}}(L, L') \geq \frac{1}{4}\pi r^2$. □

To ilustrate Corollary 6.8, consider $M = \mathbb{T}^2 = S^1 \times S^1$ with $L = \{x_*\} \times S^1$ and $L_k = \{x_k\} \times S^1$, where x_k is a sequence in S^1 with $x_k \to x_*$ as $k \to \infty$ and $x_k \neq x_*$ for all k. Clearly L_k converges to L in the Hausdorff distance. By Corollary 6.8 all the Lagrangians L_k remain at a bounded distance from L in the $R^{\mathcal{U}}$-metric.

6.2.2. *Remarks*

1) It is well-known that there are other natural metrics defined on $\mathcal{L}ag^*(M)$. The most famous is Hofer's Lagrangian metric, used since the work of Chekanov [**Che00**], which infimizes the Hofer energy needed to carry one Lagrangian to the other. Another interesting more algebraic metric, smaller than the Hofer metric, is the spectral metric due to Viterbo. Both these metrics are infinite as soon as the two Lagrangians compared are not Hamiltonian isotopic. A metric smaller than the Hofer norm, and based on simple Lagrangian cobordism has been introduced in [**CS19**]: it measures the distance between L and L' by infimizing the shadow of cobordisms having only L and L' as ends. This metric is finite on each simple cobordism class and, with the notation above, it coincides with $d^{\emptyset} = \widehat{d}^{\emptyset,\emptyset}$. This metric is again often infinite. For instance, in the exact case, as soon as L and L' have non-isomorphic homologies, the simple shadow distance between L and L' is infinite. Indeed, if L and L' are related by an exact simple cobordism, then L and L' have isomorphic singular homologies [**BC13**] (more rigidity is actually true, see [**Sua17**]). For other results on the simple shadow metric see [**Bis**], [**Bis19a**], [**Bis19b**]. It is already known that without appropriate constraints on the class of admissible Lagrangians and cobordisms, such as those imposed here, even the simple cobordism metric $d^{\emptyset}$ is degenerate [**CS19**].

2) A notion of cone-length is familiar in homotopy theory as a measure of complexity for topological spaces [**Cor94**].

6.3. Some examples and calculations

6.3.1. ***Curves on tori and related examples.*** — We fix a family of Lagrangians $\mathcal{F}$, to be specified later, and omit it from the notation of the measurements $l_a^{\mathcal{F}}$, $d_k^{\mathcal{F}}$, $l^{\mathcal{F}}$, $d^{\mathcal{F}}$.

If ϕ is a Hamiltonian diffeomorphism, then clearly $l_a(\phi(L), L) = 0$ as soon as $a \geq \|\phi\|_H$ and so $l(\phi(L), L) = 0$. However, we will see below classes of examples with $0 < l_a(\phi(L), L) < \infty$. Intuitively, an inequality of the type $1 \leq l_a(\phi(L), L) < \infty$ seems to indicate that ϕ distorts L (at least for our choices of classes $\mathcal{F}$).

The examples below that satisfy $1 \leq l_a(\phi(L), L) < \infty$ also satisfy

$$d_{l_a(\phi(L),L)}(\phi(L), L) < d_0(\phi(L), L) \leq \|\phi\|_H.$$

In other words, in these examples the "optimal" (in the sense of minimizing the shadow) approximation of $\phi(L)$ through elements of the set $\{L\} \cup \mathcal{F}$ requires more elements than just L. Moreover, the relevant d_k's are small enough so that inequality (5.4) of Theorem 5.1 applies and indeed, as predicted by the theorem, in these examples the number of intersection points $\phi(L) \cap N$, where N is an appropriate other Lagrangian $N \in \mathcal{L}ag^*(M)$, is much higher than the usual lower bound, given by the rank of the Floer homology group $\mathrm{HF}(N, L)$.

Consider the 2-dimensional torus $M = T^2$ endowed with an area form. We identify T^2 with the square $[-1, 1] \times [-1, 1]$ with the usual identifications of the edges. We consider five Lagrangians on T^2, described on the square $[-1, 1] \times [-1, 1]$ by

$$\begin{aligned} L = [-1, 1] \times \{0\}, \qquad & S_1 = \{-\tfrac{1}{2} - \epsilon\} \times [-1, 1], S_2 = \{-\tfrac{1}{2} + \epsilon\} \times [-1, 1], \\ & S_3 = \{\tfrac{1}{2} - \epsilon\} \times [-1, 1], S_4 = \{\tfrac{1}{2} + \epsilon\} \times [-1, 1]. \end{aligned}$$

Here $0 < \epsilon \leq \frac{1}{8}$. We will construct a new Lagrangian obtained through surgery between L and the S_i's. We use the surgery conventions from [**BC13**] and define – see Figure 7:

$$L' = S_3 \# \left[(S_2 \# (L \# S_1)) \# S_4 \right]. \tag{6.9}$$

In the surgeries above we use handles of equal size in the sense that the area enclosed by each handle is equal to a fixed $\delta > 0$ with δ very small. We will also make use of the two rectangles

$$K_1 = [-\tfrac{1}{2} - 2\epsilon, -\tfrac{1}{2} + 2\epsilon] \times [-\epsilon, \epsilon], \quad K_2 = [\tfrac{1}{2} - 2\epsilon, \tfrac{1}{2} + 2\epsilon] \times [-\epsilon, \epsilon]$$

and we put $K = K_1 \cup K_2$ (see again Figure 7).

Lemma 6.9. — *Let $\mathcal{F} = \{S_1, S_2, S_3, S_4\}$ and assume that $\delta < \frac{1}{2}\epsilon^2$. We have*

(i) $d_0(L', L) = 4\epsilon$, $d_4(L', L) \leq 2\delta$, $l(L', L) = 0$, $l_{2\delta}(L', L) = 4$.

(ii) *For any weakly-exact Lagrangian $N \subset T^2$ with $N \cap K = \varnothing$, we have*

$$\#(N \cap L') \geq rk\big(\mathrm{HF}(N, L)\big) + \sum_{i=1}^{4} rk\big(\mathrm{HF}(N, S_i)\big). \tag{6.10}$$

(iii) *If N' is a weakly exact Lagrangian $N' \subset T^2$, then either $N' \cap L \neq \varnothing$ or, for any Hamiltonian diffeomorphism ϕ with $\phi(L) = L'$ we have $\phi(N') \cap K \neq \varnothing$.*

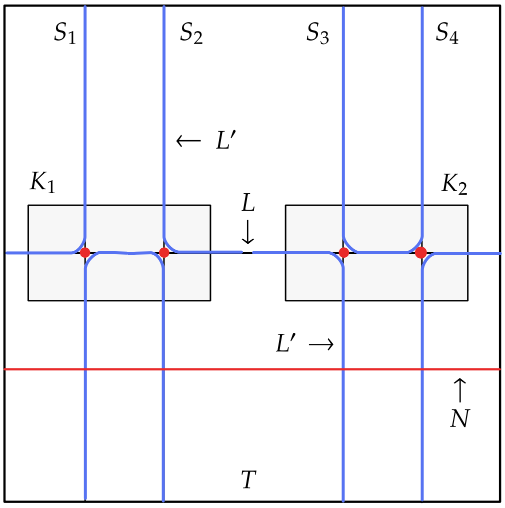

FIGURE 7. The Lagrangians L, $L' = S_3 \# [(S_2 \# (L \# S_1)) \# S_4]$, and N in T^2.

Floer homology is considered here with coefficients in $\mathbb{Z}/2$. Notice that $\mathrm{HF}(N, L') \cong \mathrm{HF}(N, L)$ so the inequality (6.10) indicates an "excess" of intersection points. An example of a Lagrangian N as at point ii is simply $N = [-1, 1] \times \{-2\epsilon\}$.

Proof. — By inspecting again Figure 7 and possibly extending the representation of the torus by adding vertically two additional fundamental domains to the square $[-1, 1] \times [-1, 1]$ one can see that there is a Hamiltonian isotopy $\phi : T^2 \longrightarrow T^2$ so that $L' = \phi(L)$ (this is because the upper and lower "bends" in the picture encompass equal areas). The expression in (6.9) show that there is a cobordism

$$V : L' \longrightarrow (S_3, S_2, L, S_1, S_4)$$

given as the trace of the respective surgeries (as given in [**BC13**]) and because the S_i's are disjoint and all the handles are of area δ we have $\mathcal{S}(V) \leq 2\delta$. The reason for the factor 2 is that the handles associated to the surgeries on the "left" and those on the "right" can not be assumed to have a superposing projection; the constant is 2 and not 4 because the two handles on the left (and similarly for the two handles on the right) can be assumed to have overlapping projections. It is a simple exercise to show that $\delta(L'; L) = 8\epsilon$.

From the first part of Theorem 5.1 we deduce $d_0(L', L) \geq 4\epsilon$. It is also easy to see that one can find a Hamiltonian $H : T^2 \longrightarrow \mathbb{R}$ with variation equal to 4ϵ and so that $\phi_1^H(L) = L'$. Therefore, $d_0(L', L) = 4\epsilon$.

On the other hand, recall the assumption $\delta < \epsilon^2/2$. Therefore we have

$$d_4(L', L) \leq 2\delta.$$

We now estimate cone-length. Clearly, the absolute number is $l(L', L) = 0$. From the existence of the cobordism V we deduce $l_{2\delta}(L', L) \leq 4$. We want to show $l_{2\delta}(L', L) = 4$. Assume that $l_{2\delta}(L', L) \leq 3$. Therefore there exists a cobordism $V' : L' \longrightarrow (L_1, L_2, L_3, L_4)$ where one of the L_i's equals L and the other three are picked among the S_i's (or are void) and the shadow of V' is at most 2δ. Without loss of generality, assume that S_1 is not among the L_i's. We now consider the number $\delta(L'; L \cup S_2 \cup S_3 \cup S_4)$. By using a disk centered along the part of S_1 contained in L' we see that $\delta(L'; L \cup S_1 \cup S_2) \geq 8\epsilon$. By the first part of Theorem 5.1 it follows $\mathcal{S}(V') \geq 4\epsilon$ which contradicts $\delta \leq 2\epsilon^2$.

The two other points of the Lemma also follow from Theorem 5.1 (they possibly admit also more elementary, direct proofs). Point (b) of the Theorem implies that for any weakly-exact Lagrangian $N \subset T^2$ so that $N \cap K = \emptyset$, we have (6.10). Indeed, we may find disks around the (unique) intersection point of each of the S_i's with L that are of area $4\epsilon^2$, have the real part along L and the imaginary part along S_i, are contained in K, and any two of these disks have disjoint interiors. As N avoids K, this means $\delta^{\Sigma}(\mathbb{L}; N) \geq 4\epsilon^2$ for $\mathbb{L} = L \bigcup \bigcup_i S_i$ and Σ the intersection points of the S_i's with L. The last point of the Lemma follows in a similar way. Assuming also that $N' \cap L = \emptyset$ we also have $\phi(N') \cap L' = \emptyset$. If we also have $\phi(N') \cap K = \emptyset$, then $\phi(N')$ satisfies inequality (6.10) (with $\phi(N')$ in the place of N). From the fact that N' is weakly exact we deduce that the singular homology class of N' is the same as that of L and thus $\mathrm{HF}(N', S_i)$ does not vanish. But this leads to a contradiction with $\phi(N') \cap L' = \emptyset$. □

It is easy to construct examples similar to the one above in higher dimensions. For instance, one can consider $M = (T^2 \times T^2, \omega \oplus \omega)$ and take $\bar{L} = L \times L$, $\bar{S}_i = S_i \times S_i$ etc. We will see some less trivial extensions in the next subsection.

6.3.2. ***Remarks.*** — The examples above also point out two deficiencies of the pseudo-metric $d^{\mathcal{F}}$.

1) $d^{\mathcal{F}}$ is generally degenerate. For example, $d_3^{\mathcal{F}}(S_1, S_2) = 0$, hence $d^{\mathcal{F}}(S_1, S_2) = 0$. Indeed, let $V : S_1 \rightsquigarrow (S_1, S_2, S_2)$ be the cobordism $V = \gamma_0 \times S_1 \coprod \gamma_1 \times S_2$ where $\gamma_0 = \mathbb{R} + i \subset \mathbb{C}$ and γ_1 is a curve in $\mathbb{C}$ that has two horizontal negative ends, one at height 2 and the other at height 3 and is disjoint from γ_0. The same construction shows that for any family $\mathcal{F}$ with more than one element the resulting pseudo-metric is degenerate.

In the above examples the cobordisms V are disconnected and they also have vanishing shadow. However, there are also examples of connected cobordisms W_ϵ with constant ends and *positive* shadow such that $\lim_{\epsilon\to 0} \mathcal{S}(W_\epsilon) = 0$. For instance, with the notation above, consider a curve $\gamma \subset \mathbb{R}^2$ which has a "$\supset$" shape with its lower end going to $-\infty$ along the horizontal line $y = -1$ and its upper end going to $-\infty$ along the horizontal line $y = 1$. Let γ' be the x-axis, $y = 0$. Consider now the surgery $W_\epsilon := (\gamma \times S_1) \#_\epsilon (\gamma' \times L) \subset \mathbb{R}^2 \times T^2$. (Note that, in contrast to the construction of *e.g.* L' above, the surgery here is performed in the space $\mathbb{R}^2 \times T^2$.) Clearly W_ϵ is a (connected) weakly exact Lagrangian cobordism $W_\epsilon : L \rightsquigarrow (S_1, L, S_1)$ and $\lim_{\epsilon\to 0} \mathcal{S}(W_\epsilon) = 0$.

2) In general, even if both L and L' belong to the triangulated completion of the family $\mathcal{F}$, it can be difficult to know whether $d^{\mathcal{F}}(L, L')$ is finite because there might not be any practical way to construct cobordisms with ends L, L' and elements of $\mathcal{F}$.

6.3.3. ***Matching cycles in simple Lefschetz fibrations.*** — We revisit here the phenomena described above in a different context and we also present examples of symplectic diffeomorphisms $\phi : M \to M$ with $l(\phi^k(L), L) = k$ (in these examples ϕ is a Dehn twist).

The manifold M is now taken to be the total space of a Lefschetz fibration

$$\pi : M \longrightarrow \mathbb{C}$$

over $\mathbb{C}$ with general fiber the cotangent bundle of a sphere K (in particular M is not compact). We will assume that the Lefschetz fibration has exactly three singularities x_1, x_2, x_3, whose projection on $\mathbb{C}$ is arranged as in Figure 8 below. We also assume that there are two matching cycles relating the three singularities that we denote by S, from x_1 to x_2, and L, from x_2 to x_3 – as in the same figure.

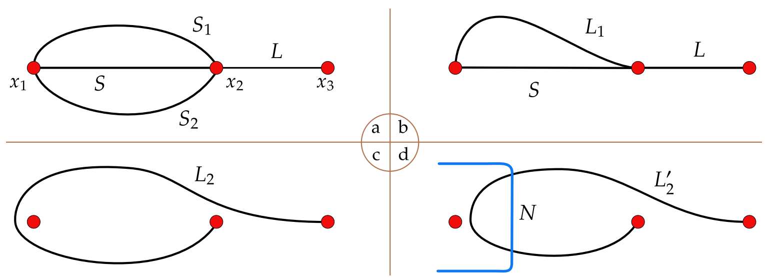

FIGURE 8. The matching cycles S, S_1, S_2 and L and the Lagrangians L_1, L_2, L_2' constructed by surgery (and small perturbation) from them.

Notice that L and S intersect (transversely) in a single point. Moreover, recall that with the notation in [BC13], [BC17] we have that $S \# L$ is Hamiltonian isotopic to the Dehn twist $\tau_S(L)$, and, similarly, $L \# S$ is Hamiltonian isotopic to $\tau_S^{-1}(L)$. An important point to emphasize here is that the Dehn twist $\tau_S(L)$ is only well defined up to Hamiltonian isotopy. On the other hand, the models for $\tau_S(L)$ (and $\tau_S^{-1}(L)$) given by surgery, as before, are precisely determined as soon as the local data of the surgery is fixed (the surgery handle and the precise Darboux chart around the intersection point). We will also need two other matching cycles S_1 and S_2 with a projection as in Figure 8a.

They are both Hamiltonian isotopic to S. The two spheres S_1 and S_2 intersect transversely at the points x_1 and x_2 and each of them intersects transversely L at the point x_2. We now consider the following three Lagrangians: L_1 which is obtained from $S_1 \# L$ after a small Hamiltonian isotopy such that its projection is as in Figure 8b, L_2 given as a small deformation of $S_2 \# L_1$ and L_2', a small deformation of $L_1 \# S_2$ such that their projections are as in the same figure, part c and d, respectively. Notice that L_1 is a model for $\tau_S(L)$ and that L_2 and L'_2 are models for $\tau_S^2(L)$ and $L = \tau_S^{-1}\tau_S(L)$, respectively. In particular, there is a Hamiltonian isotopy ϕ such that $L'_2 = \phi(L)$.

Fix the family $\mathscr{F} = \{S_1, S_2\}$. The first remark is that by taking the surgery handles sufficiently small we have $d_2(L'_2, L) < d_0(L'_2, L) < \infty$. Further, let K' be a Hamiltonian perturbation of the vanishing sphere K in the general fiber. Let N be the trail of K' along a curve as in Figure 8d. We now claim that $l(L_2, L) = 2$. Indeed, by construction we have a cobordism $L_2 \rightsquigarrow (S_2, S_1, L)$, hence $l(L_2, L) \leq 2$. Now, it is not hard to see that

$$\mathrm{HF}(N, L_2) = \mathrm{HF}(N, S_1) \oplus \mathrm{HF}(N, S_2) = \mathrm{HF}(K, K) \oplus \mathrm{HF}(K, K)$$

(one can use Seidel's exact triangle associated to a Dehn twist for this computation, or alternatively Theorem A from [**BC17**] with $V = L_2$). Since $\mathrm{HF}(N, L) = 0$ and $\mathrm{HF}(N, L_2) \neq 0$ it follows that L_2 is not Hamiltonian isotopic to L, hence $l(L_2, L) \geq 1$. Moreover, $l(L_2, L) \neq 1$ for otherwise we would have either a cobordism $L_2 \rightsquigarrow (L, F)$ or $L_2 \rightsquigarrow (F, L)$ with $F \in \{S_1, S_2\}$. As $\mathrm{HF}(N, L) = 0$, the latter would imply that

$$\mathrm{HF}(N, L_2) \cong \mathrm{HF}(N, F) \cong \mathrm{HF}(N, S) \cong \mathrm{HF}(K, K),$$

a contradiction. This proves that $l(L_2, L) = 2$.

On the other hand, $\mathrm{HF}(N, L'_2) = 0$. However, by taking the surgery handles in the constructions above sufficiently small we see that $\#(N \cap L'_2) \geq 2\,\mathrm{rk}(\mathrm{HF}(K, K))$, as predicted by Theorem 5.1. Notice also that if the surgery handle is not small enough, or, alternatively, N avoids L'_2 by passing closer to x_1, then N is disjoint from L'_2.

The last remark in this setting is the following. By taking more copies of the sphere S, (for instance four, as on the left of Figure 9), we can construct, in a way similar to the above, models L_k for $\tau_S^k(L)$. In Figure 9, on the right, we represent $\tau_S^4(L)$ in this way. As before, it is easy to compute

$$\mathrm{HF}(N, L_k) = \bigoplus_{i=1}^{k} \mathrm{HF}(K, K).$$

This shows that $l(L_k, L) = k$ (this is a reflection of the well-known fact that the class of τ_S is not a torsion element in $\pi_0\,\mathrm{Symp}(M)$, see [**KS02**], [**Sei00**]).

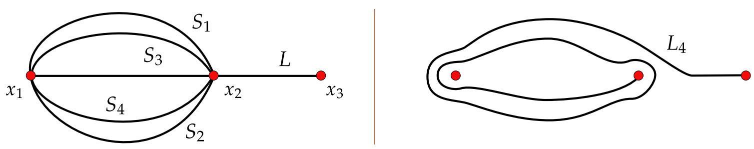

FIGURE 9. A model for $\tau_S^4(L)$.

6.3.4. ***Trace of surgery.*** — The numbers d_k are hard to compute as it is difficult in general to identify cobordisms with fixed ends and with minimal shadow. However, we will see here how to use inequality (5.5) to show the "optimality" of decompositions given by the trace of certain surgeries at one point.

We focus on just one example. As in §6.3.1 we take $M = T = S^1 \times S^1$ and we fix S_1 an L as in that subsection. We now consider $L'' = L \# S_1$ and, again as in §6.3.1, we assume that the area of the handle used in the surgery giving L'' is equal to δ. We fix $\mathcal{F} = \{S_1, S_2, S_3, S_4\}$ as in Lemma 6.9. Notice that the shadow of the trace of the surgery $V : L'' = L \# S^1 \longrightarrow (L, S_1)$ is equal to δ.

Corollary 6.10. — *For δ small enough we have $d_1(L'', L) = \delta$.*

In other words, there is no decomposition of L'' in terms of the family $L \cup \mathcal{F}$ through a cobordism with two negative ends and of shadow smaller than δ.

Proof. — Suppose that there is a cobordism $V' : L'' \longrightarrow (L_1, L_2)$ such that one of the L_i's equals L, the other equals one of the S_i's and

$$\mathcal{S}(V') = \delta' < \delta.$$

We first notice that S_1 needs to appear among the L_i's. Indeed, suppose, for instance that $(L_1, L_2) = (L, S_2)$. In this case, consider a disk based on the part of L'' that coincides with S_1 and is disjoint from S_2 as well as from L and whose real part is along L''. The area of such a disk can be assumed to be as close as needed to $2(4\epsilon - \delta)$, where ϵ is defined in §6.3.1. By now applying the first part of Theorem 5.1 we deduce that $\delta > \mathcal{S}(V') \geq 4\epsilon - \delta$ which is a contradiction if δ is small enough. In conclusion, we deduce that the two negative ends of V' coincide with L and S_1. Consider now the Lagrangian N as in Figure 10 and denote by o the intersection of L and S_1.

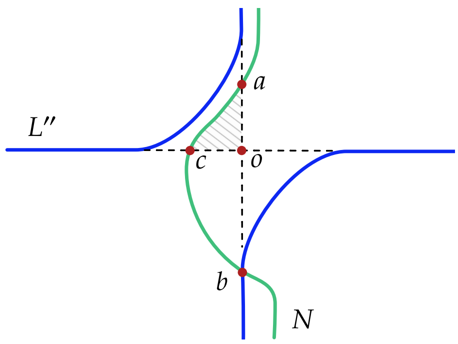

Figure 10. The triangle coa is of area A with $\delta > A > \delta'$.

The properties of N are the following: N is Hamiltonian isotopic to S_1; it intersects S_1 transversely at precisely two points a and b and it intersects L transversely at one point c; N intersects L'' transversely at the point b; the small triangle of vertices c, o, a is of area A with $\delta' < A < \delta$. We use the Lagrangian N as follows. First, notice that by assuming δ small enough, and writing $\mathbb{L} = L \cup S_1$, we can find the relevant disks centered at o so as to estimate $\delta^{\Sigma \mathbb{L}}(\mathbb{L}; N) \geq 4A$. By applying (5.5) in Theorem 5.1 we deduce

$$1 = \#(N \cap L'') \geq \dim \mathrm{HF}(N, L) + \dim \mathrm{HF}(N, S_1) = 3$$

which is a contradiction and thus proves that V' does not exist. □

6.4. Algebraic metrics on $\mathscr{L}ag^*(M)$

The main purpose of this subsection is to notice that it is possible to define pseudo-metrics similar to those in Section 6.2 but that only exploit the algebraic structures involved and that do not appeal to cobordism. We emphasize that, as before, our pseudo-metrics may take infinite values. The proof of the first part of Theorem 5.1 implies not only the non-degeneracy of $\widehat{d}^{\mathscr{F},\mathscr{F}'}$ but also that of its algebraic counterpart. The main advantage of these pseudo-metrics is that when $\mathscr{F}$ generates $D\mathscr{F}uk^*(M)$ some of these algebraic pseudo-metrics are finite by definition, independently of the existence of cobordisms – see Remark 6.4.6. Additionally, the construction of both metrics $\widehat{d}^{\mathscr{F},\mathscr{F}'}$ as well as that of their algebraic counterparts fits a more general, abstract pattern, potentially useful in other contexts, that we outline here.

6.4.1. ***Weighted triangulated categories.*** **—** Let $\mathscr{X}$ be a triangulated category and let $\mathscr{X}_0$ be a family of objects of $\mathscr{X}$ that generate $\mathscr{X}$ through triangular completion. The purpose of this subsection is to describe a procedure leading to a (pseudo) metric on $\mathscr{X}_0$. The pseudo-metrics $d^{\mathscr{F}}$ in Section 6.2 are of this type but, as we shall see further below, other choices are possible.

There is a category denoted by $T^S\mathscr{X}$ that was introduced in [BC13], [BC14]. This category is monoidal and its objects are finite ordered families $(K_1, \ldots, K_r)$ with K_i in $\mathscr{O}b(\mathscr{X})$ with the operation given by concatenation. Up to a certain natural equivalence relation, the morphisms in $T^S\mathscr{X}$ are direct sums of basic morphisms $\bar{\phi}$ from a family formed of a single object of $\mathscr{X}$ to a general family, $\bar{\phi} : K \to (K_1, \ldots, K_s)$. Such a morphism $\bar{\phi}$ is a triple (ϕ, a, η), where $a \in \mathrm{Ob}(\mathscr{X})$, η is a decomposition of a through iterated distinguished triangles, namely:

$$a = \mathscr{C}one\big(K_s \to \mathscr{C}one\big(K_{s-1} \to \cdots \to \mathscr{C}one\big(K_2 \to K_1\big)\cdots\big)\big) \tag{6.11}$$

and $\phi : K \to a$ is an isomorphism. The tuple $(K_1, \ldots, K_s)$ is called the linearization of the cone decomposition (6.11). In essence, the morphisms in $T^S\mathscr{X}$ parametrize all the cone-decompositions of the objects in $\mathscr{X}$. Composition in $T^S\mathscr{X}$ comes down to refinement of cone-decompositions. Denote by $T^S\mathscr{X}_0$ the full subcategory of $T^S\mathscr{X}$ that has objects $(K_1, \ldots, K_r)$ with $K_i \in \mathscr{X}_0$, $1 \le i \le r$. Assume that we are given a weight $w : \mathrm{Mor}_{T^S\mathscr{X}_0} \to [0, \infty]$ such that

$$w(\bar{\phi} \circ \bar{\psi}) \le w(\bar{\phi}) + w(\bar{\psi}), \quad w(\mathrm{id}_X) = 0, \quad \text{for all } X, \tag{6.12}$$

where id_X is the identity morphism viewed as defined on the family formed by the single object X and with values in the same family. We will refer to this w as a weight on $\mathscr{X}$. Fix also a family $\mathscr{F} \subset \mathscr{X}_0$. In this setting, we define (compare to (6.2)):

$$s^{\mathscr{F}}(K', K) = \inf\big\{w(\bar{\phi})\ ;\ \bar{\phi} : K' \to (F_1, \ldots, K, \ldots, F_r),\ F_i \in \mathscr{F}, \forall i\big\}. \tag{6.13}$$

We set $s^{\mathscr{F}}$ to be ∞ if there are no morphisms as in (6.13). If w is finite and if $\mathscr{F}$ generates $\mathscr{X}$, then $s^{\mathscr{F}}$ is finite. Clearly $s^{\mathscr{F}}$ satisfies the triangle inequality but it is not symmetric in general. Defining

$$\bar{s}^{\mathscr{F}}(K', K) := \tfrac{1}{2}\big(s^{\mathscr{F}}(K', K) + s^{\mathscr{F}}(K, K')\big),$$

we obtain a pseudo-metric on the set of objects of $\mathcal{X}$. We will refer to the pseudo-metrics obtained by this procedure as *weighted fragmentation* pseudo-metrics.

The case of interest in this paper is $\mathcal{X} = D\mathcal{F}uk^*(M)$ with $\mathcal{X}_0$ consisting of all the Yoneda modules associated to the Lagrangians in $\mathcal{L}ag^*(M)$. In our notation, the category $\mathcal{F}uk^*(M)$ is defined as described at the beginning of Chapter 3, without reference to filtrations.

The shadow pseudo-metric $d^{\mathcal{F}}$ from Section 6.2 is a first example of a (class) of weighted fragmentation pseudo-metrics associated to a weight w defined as follows. Recall from [**BC14**], [**CC16**] that there is a monoidal cobordism category $\mathcal{C}ob^{Q}(M)$ whose objects are families $(L_1, \ldots, L_s)$ with $L_i \in \mathcal{L}ag^*(M)$ and with morphisms (formal sums) of cobordisms in the class Q that are the type $V : L \rightsquigarrow (L_1, \ldots, L_s)$ (modulo an appropriate equivalence relation; the monoidal operation is concatenation). There is a monoidal functor, denoted in [**BC14**] by $\widetilde{\mathcal{F}}$ but that, to avoid confusion in notation, we will denote here by $\widetilde{\Phi}$:

$$\widetilde{\Phi} : \mathcal{C}ob^{Q}(M) \longrightarrow T^{S}\big(D\mathcal{F}uk^*(M)\big). \tag{6.14}$$

On objects, this functor associates to a Lagrangian L its Yoneda module $\mathcal{L}$ and its properties have been used extensively earlier in the paper, starting from Section 3.7. In the setting, $\mathcal{X} = D\mathcal{F}uk^*(M)$, for a morphism $\bar{\phi} \in \mathrm{Mor}_{T^S\mathcal{X}_0}$ we define the *shadow* weight of $\bar{\phi}$ by

$$w_{\mathcal{S}}(\bar{\phi}) = \inf\big\{\mathcal{S}(V)\ ;\ \widetilde{\Phi}(V) = \bar{\phi}\big\} \tag{6.15}$$

and it is easy to see from the various definitions involved that $d^{\mathcal{F}}$ coincides with the weighted fragmentation pseudo-metric $\bar{s}^{\mathcal{F}}$ associated to $w_{\mathcal{S}}$. Additionally, recall from Corollary 6.5 that, by using an appropriate perturbation $\mathcal{F}'$, we obtain an actual metric $\widehat{d}^{\,\mathcal{F},\mathcal{F}'} = \max\{d^{\mathcal{F}}, d^{\mathcal{F}'}\}$.

6.4.2. ***Remark.*** — The category $T^S\mathcal{X}$ was inspired by the work on Lagrangian cobordism and might seem artificial in itself. However, we will remark here that (in a slightly modified form) it is the natural categorification of the Grothendieck group $K(\mathcal{X})$. This group is defined as the quotient of the free abelian group generated by the objects in $\mathcal{X}$ modulo the relations $B = A + C$ whenever $A \to B \to C$ is a distinguished triangle in $\mathcal{X}$. We will work here in a simplified setting and take the identity for the shift functor. As a consequence $K(\mathcal{X})$ is a $\mathbb{Z}/2$ vector space. Alternatively, $K(\mathcal{X})$ can also be defined as the free monoid of finite ordered families $(K_1, \ldots, K_r)$ where $K_i \in \mathcal{O}b(\mathcal{X})$, with the operation being given by concatenation of families, modulo the relations $K_1 + K_2 + \cdots + K_s = 0$ whenever there exists a cone decomposition of 0 with linearization $(K_1, \ldots, K_s)$. When $\mathcal{X}$ is small, there is a category, $\widehat{T}^S\mathcal{X}$, closely associated to $T^S\mathcal{X}$, that categorifies $K(\mathcal{X})$ in the usual sense (meaning that it is a monoidal category with the property that the monoid formed by the isomorphism classes of its objects is $K(\mathcal{X})$). The basic idea is that, in $\widehat{T}^S\mathcal{X}$, families that are linearizations of acyclic cones are declared isomorphic to 0. More formally, $\widehat{T}^S\mathcal{X}$ is defined if the category $\mathcal{X}$ is small and is constructed in three steps: first we add to the morphisms in $T^S\mathcal{X}$ the morphism $0 \to \emptyset$ (and the relevant compositions) thus getting $T^S\mathcal{X}^+$; secondly, we localize $T^S\mathcal{X}^+$ at the family of morphisms

$$\mathcal{A} = \left\{\phi \in \mathrm{Mor}_{T^S\mathcal{X}^+} \; ; \; \phi : 0 \to (K_1, \ldots, K_s)\right\}$$

(here 0 is viewed as a family formed by the single element 0; this is equivalent to adding inverses to all the morphisms having 0 as domain and adding relations so that associativity of composition is still satisfied); finally, we complete in the monoidal sense by allowing formal sums for all the new and old morphisms.

6.4.3. ***Energy of retracts of weakly filtered modules.*** **—** Assume that $\mathcal{M}$ is a weakly filtered module over the weakly filtered A_∞-category $\mathcal{A}$ with discrepancy $\leq \epsilon^m$ as in §2.3.1 and that $\psi : \mathcal{M} \to \mathcal{M}$ is a weakly filtered module homomorphism with discrepancy $\leq \epsilon^h$ which is null-homotopic. Following the terminology in Section 2.7, we consider the homotopical boundary level of ψ,

$$B_h(\psi; \epsilon^h) := \beta_h(\psi; \epsilon^h) + A(\psi).$$

Let $f : \mathcal{M}_0 \to \mathcal{M}_1$ be a morphism of weakly filtered modules and define:

$$\rho(f) = \inf_g \left(\max \left\{B_h(g \circ f - \mathrm{id}; \epsilon^h), A(g) + A(f), 0\right\}\right) \tag{6.16}$$

where the infimum is taken over all weakly filtered module morphisms

$$g : \mathcal{M}_1 \longrightarrow \mathcal{M}_0, \quad \text{with } g \circ f \in \hom^{\epsilon^h}(\mathcal{M}_0, \mathcal{M}_0)\ ,\ g \circ f \simeq \mathrm{id}_{\mathcal{M}_0}\,.$$

In case no such g exists we put $\rho(f) = \infty$. The measurement ρ estimates the minimal energy required to find a left homotopy inverse for f.

6.4.4. ***Remark.*** **—** Similar notions are familiar in Floer theory, generally to compare two quasi-isomorphic chain complexes, and in that case the infimum above is taken also over all morphisms f and one also takes into account a homotopy $f \circ g \simeq \mathrm{id}_{\mathcal{M}_1}$. For instance, see [**UZ16**].

The next two lemmas give simple properties of ρ that will be useful below.

Lemma 6.11. — *Given* $\mathcal{M}_0 \xrightarrow{f} \mathcal{M}_1$, $\mathcal{M}_1 \xrightarrow{f'} \mathcal{M}_2$, *one has*

$$\rho(f' \circ f) \leq \rho(f) + \rho(f'). \tag{6.17}$$

Proof. — Indeed, assume $\mathcal{M}_1 \xrightarrow{g} \mathcal{M}_0$, $\mathcal{M}_2 \xrightarrow{g'} \mathcal{M}_1$ are weakly filtered module maps and $\eta : g \circ f \simeq \mathrm{id}_{\mathcal{M}_0}$, $\eta : g' \circ f' \simeq \mathrm{id}_{\mathcal{M}_1}$ are the respective homotopies. Assume that $f, g, \eta, f', g', \eta'$ shift filtrations by $\leq s, r, k, s', r', k'$, respectively. These numbers can be taken larger but as close as desired to the respective action levels. Notice that $f' \circ f$ shifts filtrations by $\leq s + s'$, $g \circ g'$ shifts filtrations by $\leq r + r'$ and, moreover, the homotopy

$$\bar{\eta} = g \circ \eta' \circ f + \eta : g \circ g' \circ f' \circ f \simeq \mathrm{id}_{\mathcal{M}_0}$$

shifts filtrations by $\leq \max\{r + s + k', k\}$. This implies the claim. □

To state the second property, assume that the weakly filtered module $\mathcal{M}_1$ can be written as a weakly filtered iterated cone

$$\mathcal{M}_1 = \mathcal{C}one(K_s \to \mathcal{C}one\big(K_{s-1} \to \cdots \\ \cdots \to \mathcal{C}one\big(\mathcal{N} \to \mathcal{C}one\big(K_{i-1} \to \cdots \mathcal{C}one(K_2 \to K_1\big) \cdots)\big)$$

and that there is another weakly filtered module $\mathcal{N}'$ together with weakly filtered maps $u : \mathcal{N} \to \mathcal{N}'$ and $v : \mathcal{N}' \to \mathcal{N}$ and a weakly filtered homotopy $\xi : v \circ u \simeq \mathrm{id}_{\mathcal{N}}$.

Lemma 6.12. — *There is another weakly filtered module $\mathcal{M}_1'$ that can be written as a filtered iterated cone of the same form as the decomposition for $\mathcal{M}_1$ except with $\mathcal{N}'$ replacing $\mathcal{N}$ and there is an associated map $u' : \mathcal{M}_1 \to \mathcal{M}_1'$ so that $\rho(u') \leq \max\{A(u) + A(v), A(\xi), 0\}$.*

As a corollary we deduce that given $\mathcal{M}_1$, $\mathcal{N}$ as well as $\mathcal{N}'$ and a weakly filtered map $u : \mathcal{N} \to \mathcal{N}'$ with $\rho(u) < \infty$, then, for any $\epsilon > 0$, there exists a weakly filtered module $\mathcal{M}_1'$ and a map $u' : \mathcal{M}_1 \to \mathcal{M}_1'$ as in the lemma such that

$$\rho(u') \leq \rho(u) + \epsilon. \tag{6.18}$$

Proof of Lemma 6.12. — By recurrence, the proof is easily reduced to showing the statement for two particular types of decompositions:

$$\mathcal{M}_1 = \mathcal{C}one\big(\mathcal{N} \xrightarrow{\phi} K_1\big) \quad \text{and} \quad \mathcal{M}_1 = \mathcal{C}one\big(K_2 \xrightarrow{\phi} \mathcal{N}\big).$$

We will only treat here the first case the second being entirely similar. Without loss of generality, we may assume that ϕ does not shift action filtrations. Assume that the map $v : \mathcal{N}' \to \mathcal{N}$ shifts filtrations by $\leq r$, the map u shifts filtrations by $\leq s$ and ξ shifts filtration by $\leq k$. Following the definitions of weakly filtered cones in Section 2.4 we construct $\mathcal{M}_1'$ as follows.

Let $\bar{v} : S^{-r}\mathcal{N}' \to \mathcal{N}$ be given by the map v after shifting the filtration of its domain up by r. Define

$$\phi' := \phi \circ \bar{v} : S^{-r}\mathcal{N}' \longrightarrow K_1$$

and put $\mathcal{M}_1' = \mathcal{C}one(\phi')$. With the notation in (2.7), this cone is defined by taking the action shift of ϕ' to be 0. There are module morphisms $v' : \mathcal{M}_1' \to \mathcal{M}_1$ defined as $v' = (\bar{v}, \mathrm{id}_{K_1})$ and $u' : \mathcal{M}_1 \to \mathcal{M}_1'$, $u' = (\bar{u}, \phi \circ \xi + \mathrm{id}_{K_1})$ where $\bar{u} : \mathcal{N} \to S^{-r}\mathcal{N}'$ is the map u with its target with a shifted filtration (these equations have to be interpreted component by component, as in the definition of the structure maps of cones of A_∞-modules). There is also a homotopy $\xi' : \mathcal{M}_1 \to \mathcal{M}_1$, $\xi' : v' \circ u' \simeq \mathrm{id}$ given by the formula $\xi' = (\xi, 0)$. Notice that v' does not shift filtrations; u' shifts action filtrations by $\leq \max\{r + s, k\}$; ξ' shifts filtration by $\leq k$. As we can take r, s, k larger but as close as desired to, respectively, $A(v), A(u), A(\xi)$ this implies the claim. □

6.4.5. *Algebraic weights on $T^S D\mathcal{F}uk^*(M)$.* — We now use the measurement ρ introduced in §6.4.3 to define an algebraic weight w, in the sense of §6.4.1. We will assume here that

$$\mathcal{X} = D\mathcal{F}uk^*(M)$$

and that $\mathcal{X}_0$ consists of the Yoneda modules associated to the Lagrangians in $\mathcal{L}ag^*(M)$. We will appeal to the constructions from Section 3.3. Recall from Proposition 3.1

that to a system of coherent perturbation data $p \in E'_{\text{reg}}$ we associate a weakly filtered A_∞-category $\mathcal{F}uk(\mathcal{C};p)$, where $\mathcal{C} = \mathcal{L}ag^*(M)$. We also recall that we denote by $\mathcal{N}$ the family of coherent perturbation data $\mathcal{D} = (K, J)$ with $K \equiv 0$. Proposition 3.1 also shows that for $p_0 \in \mathcal{N}$ the discrepancies of the categories $\mathcal{F}uk(\mathcal{C};p)$ tend to zero when $p \to p_0$.

We will denote by $\mathcal{F}uk(\mathcal{C};p)^\Delta$ the category of all (finite) iterated weakly filtered cones that one can construct – as in Section 2.4 – out of the objects of $\mathcal{F}uk(\mathcal{C};p)$. There is clearly a functor

$$\mathcal{F}uk(\mathcal{C};p)^\Delta \longrightarrow D\mathcal{F}uk^*(M)$$

that forgets filtrations on objects and associates to each morphism its homology class (again, at the same time forgetting the filtration).

We denote by $[X]$ the image of an object X through this functor and similarly for morphisms.

The distinguished triangles in $D\mathcal{F}uk^*(M)$ are associated through this functor to the cone attachments in $\mathcal{F}uk(\mathcal{C};p)^\Delta$ and there is a similar correspondence for the iterated cones.

Let $\bar{\phi} : \mathcal{L} \to (\mathcal{L}_1, \ldots, \mathcal{L}_k)$, $\bar{\phi} = (\phi, a, \eta)$ be a morphism in $T^S\mathfrak{X}_0$ (see §6.4.1) and

$$\begin{aligned}(6.19)\qquad w_p(\bar{\phi}) := \inf\big\{\rho(\alpha)\,;\ &\alpha \in \mathrm{Mor}_{\mathcal{F}uk(\mathcal{C};p)^\Delta}, \alpha : \mathcal{L} \to \mathcal{M},\\ &\text{such that } \mathcal{M} \text{ admits an iterated cone}\\ &\quad\text{decomposition } \bar{\eta} \text{ with } [\alpha] = \phi, [\mathcal{M}] = a, [\bar{\eta}] = \eta\big\}.\end{aligned}$$

In summary, $w_p(\bar{\phi})$ infimizes ρ among all the filtered models of the morphism $\bar{\phi}$ inside $\mathcal{F}uk(\mathcal{C};p)$. The weight w_p satisfies (6.12), hence can be used as in §6.4.1 to define a pseudo-metric $\bar{s}^{\mathcal{F}}_p$. It is useful to define also similar notions for points $p_0 \in \mathcal{N}$. For this purpose, we set

$$w_{p_0}(\bar{\phi}) = \limsup_{p \to p_0}(w_p(\bar{\phi})).$$

It is easy to see that w_{p_0} continues to satisfy (6.12) and therefore there is a corresponding weighted fragmentation pseudo-metric $\bar{s}^{\mathcal{F}}_{p_0}$. It follows from the proof of the first part of Theorem 5.1 that

Corollary 6.13. — *Let $\bar{\phi} : \mathcal{L} \to (\mathcal{L}_1, \ldots, \mathcal{L}_k)$ be a morphism in $T^S D\mathcal{F}uk^*(M)$.*

(i) *There exists $p_0 \in \mathcal{N}$ such that, with the notation in Theorem 5.1, we have*

$$w_{p_0}(\bar{\phi}) \geq \tfrac{1}{2}\delta(L;S).$$

(ii) *If there exists a Lagrangian cobordism $V : L \rightsquigarrow (L_1, \ldots, L_k)$ with $\widetilde{\Phi}(V) = \bar{\phi}$ (where $\widetilde{\Phi}$ is the functor from (6.14)), then for any $p \in E'_{\text{reg}}$ we have*

$$\mathcal{S}(V) \geq w_p(\bar{\phi}).$$

For $\bar{\phi}$ a morphism as in Corollary 6.13, define

$$w_{\text{alg}}(\bar{\phi}) := \sup_{p_0 \in \mathcal{N}} w_{p_0}(\bar{\phi}).$$

The weight w_{alg} still satisfies (6.12) and, as in §6.4.1, there is an associated pseudo-metric, $\bar{s}^{\mathcal{F}}_{\mathrm{alg}}$ on $\mathcal{L}ag^*(M)$. Point (i) of 6.13 implies that Corollary 6.4 remains valid with $\bar{s}^{\mathcal{F}}_{\mathrm{alg}}$ taking the place of $d^{\mathcal{F}}$. Moreover, if $\mathcal{F}$, $\mathcal{F}'$ satisfy the assumption in Corollary 6.5, then the formula

$$\widehat{s}^{\mathcal{F},\mathcal{F}'}_{\mathrm{alg}} = \max\left\{\bar{s}^{\mathcal{F}}_{\mathrm{alg}}, \bar{s}^{\mathcal{F}'}_{\mathrm{alg}}\right\} \tag{6.20}$$

defines a metric on $\mathcal{L}ag^*(M)$. Point (ii) of Corollary 6.13 shows that $\widehat{s}^{\mathcal{F},\mathcal{F}'}_{\mathrm{alg}}$ is bounded from above by the shadow metric $\widehat{d}^{\,\mathcal{F},\mathcal{F}'}$ from 6.5.

6.4.6. *Remark.* — Assume that $\mathcal{F}$ and $\mathcal{F}'$ satisfy the hypothesis in Corollary 6.5 and that they both generate $D\mathcal{F}uk^*(M)$. In this case, the weights w_p are finite and thus the pseudo-metrics $\bar{s}^{\mathcal{F}}_p$ as well as $\widehat{s}^{\mathcal{F},\mathcal{F}'}_p$ (which is defined by the obvious analogue of (6.20)) are also finite. On the other hand, for a fixed p it is not clear that the pseudo-metric $\widehat{s}^{\mathcal{F},\mathcal{F}'}_p$ is non-degenerate. By contrast, $\widehat{s}^{\mathcal{F},\mathcal{F}'}_{\mathrm{alg}}$ is non-degenerate but might be infinite.

Proof of Corollary 6.13. — Let $\bar{\phi} = (\phi, a, \eta)$ and consider a category $\mathcal{F}uk(\mathcal{C}, p)$ and a map $\alpha : \mathcal{L} \to \mathcal{M}$ so that $[\alpha] = \phi$, $[\mathcal{M}] = a$, and so that the cone-decomposition η corresponds to the writing of $\mathcal{M}$ as a weakly filtered iterated cone:

$$\mathcal{M} = \mathcal{C}one\big(\mathcal{L}_k \to \mathcal{C}one\big(\mathcal{L}_{k-1} \cdots \to \mathcal{C}one\big(\mathcal{L}_2 \to \mathcal{L}_1\big)\big) \cdots\big)\big).$$

Let $\beta : \mathcal{M} \to \mathcal{L}$ be another map and assume that $\zeta : \mathcal{L} \to \mathcal{L}$ is a homotopy so that $\zeta : \beta \circ \alpha \simeq \mathrm{id}_{\mathcal{L}}$. Assume that α shifts filtrations by $\leq s$, β shifts filtrations by $\leq r$ and ζ shifts filtrations by $\leq k$. Consider

$$\mathcal{M}_1 = \mathcal{C}one(\mathcal{M} \xrightarrow{\beta} \mathcal{L}) \text{ and the inclusion } i : \mathcal{L} \to \mathcal{M}_1,\ i = (0, \mathrm{id}_{\mathcal{L}}).$$

As described in Section 2.4, when defining the cone $\mathcal{M}_1$ we use the value r to write

$$\mathcal{M}_1 = S^{-r}\mathcal{M} \oplus \mathcal{L}.$$

We now notice that the map $\bar{\zeta} = (\alpha, \zeta) : \mathcal{L} \to \mathcal{M}_1$ is a homotopy $\bar{\zeta} : i \simeq 0$ and we see that $\bar{\zeta}$ shifts filtrations by $\leq \max\{r + s, k\}$. We deduce:

$$B_h(i) \leq \rho(\alpha). \tag{6.21}$$

Using this remark we now return to the setting of the proof of Theorem 5.1. In particular, we pick the choice of perturbation data p as in (5.10) and, for coherence of notation, we put $L_0 = L$. Instead of the complex $\mathcal{C}_{p,h}$ which has a geometric construction we use the complex $\mathcal{M}_1(L_0)$ constructed above. Inequality (5.12) is a consequence of (5.11). If we replace inequality (5.11) with (6.21) , we can still deduce an inequality similar to (5.12) but with $\rho(\alpha)$ instead of $\mathcal{S}(W)$. In other words, there is

$$b' \in \mathcal{M}_1(L_0) \quad \text{with} \quad A\big(b'; \mathcal{M}_1(L_0)\big) \leq A\big(e_{L_0}; \mathcal{M}_1(L_0)\big) + \rho(\alpha) + \tfrac{1}{2}\epsilon. \tag{6.22}$$

The reason is that we do not need to use in this argument the boundary depth of the chain complex $\mathcal{M}_1(L_0)$ (which in our algebraic setting might not even be acyclic) but only the boundary depth of the element e_{L_0} which is controled by the boundary depth of the map $i : \mathscr{L} = \mathscr{L}_0 \to \mathscr{M}_1$ which in turn is controled by $\rho(\alpha)$. Given that $w_p(\bar{\phi}) = \inf_{[\alpha]=\phi} \rho(\alpha)$ we may assume that

$$\rho(\alpha) \le w_p(\bar{\phi}) + \epsilon'''$$

and by continuing as in the proof of Theorem 5.1 we obtain, after making $p \to p_0$ that there is a Floer polygon v_0 (compare to (5.17)) such that

$$\omega(v_0) \le w_{p_0}(\bar{\phi}) + \tfrac{1}{2}\epsilon + \epsilon'''.$$

The argument ends by the same type of application of the Lelong inequality as in the proof of the Theorem 5.1.

The proof of the second point of the corollary is again basically contained in the proof of Theorem 5.1. It uses the isotopy pictured in Figure 5 but applies the construction there directly to the cobordism V in Figure 4 (and not to W). As in (5.9) we deduce the existence of a weakly filtered module

$$\mathscr{M}_{V;\gamma,p,h} = \mathscr{C}one\big(\mathscr{L}_k \xrightarrow{\phi_k} \mathscr{C}one\big(\mathscr{L}_{k-1} \xrightarrow{\phi_{k-1}} \mathscr{C}one\big(\cdots \mathscr{C}one\big(\mathscr{L}_2 \xrightarrow{\phi_2} \mathscr{L}_1)\big)\cdots\big)\big)\big), \tag{6.23}$$

(where we neglect a small shift that can be made to $\to 0$). There is also a similar module $\mathscr{M}_{V;\gamma',p,h}$ which is identified with the Yoneda module of L. Isotopy $\gamma' \to \gamma$ of Hofer length $\le \mathcal{S}(V) + \frac{1}{2}\epsilon$ (see above (5.11)) induces module homomorphisms (see for instance [**FOOO09a**, Chapter 5], at least for modules over an A_∞-algebra, the case of A_∞-categories is similar; alternatively, a direct argument based on moving boundary conditions is also possible)

$$\alpha : \mathscr{M}_{V;\gamma',p,h} \longrightarrow \mathscr{M}_{V;\gamma,p,h}, \quad \beta : \mathscr{M}_{V;\gamma,p,h} \longrightarrow \mathscr{M}_{V;\gamma',p,h}$$

as well as homotopies $\eta : \beta \circ \alpha \simeq \mathrm{id}$, $\eta' : \alpha \circ \beta \simeq \mathrm{id}$ that are all shifting actions by not more than $\mathcal{S}(V) + \frac{1}{2}\epsilon$. By the definition of the functor $\widetilde{\Phi}$ we have $\widetilde{\Phi}(V) = (\phi, a, \eta)$ and $[\alpha] = \phi$, $a = [\mathscr{M}_{V;\gamma,p,h}]$ and, as we just indicated, we also have $\rho(\alpha) \le \mathcal{S}(V) + \frac{1}{2}\epsilon$. This means that by definition (6.19), $w_p(\bar{\phi}) \le \mathcal{S}(V) + \frac{1}{2}\epsilon$. □

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We introduce new metrics on spaces of Lagrangian submanifolds, not necessarily in a fixed Hamiltonian isotopy class. Our metrics arise from measurements involving Lagrangian cobordisms. We also show that splitting Lagrangians through cobordism has an energy cost and, from this cost being smaller than certain explicit bounds, we deduce some forms of rigidity of Lagrangian intersections. We also fit these constructions in the more general algebraic setting of triangulated categories, independent of Lagrangian cobordism. As a main technical tool, we develop aspects of the theory of (weakly) filtered A_∞-categories.