QUANTUM STRUCTURES FOR LAGRANGIAN SUBMANIFOLDS

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Contents

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1. Introduction.

It is well-known that Gromov-Witten invariants are not defined, in general, in the Lagrangian (or relative case): bubbling of disks is a co-dimension one phenomenon and thus counting J-holomorphic disks, possibly with various incidence conditions, produces numbers that strongly depend on the particular choices of the almost complex structure J and of the geometry of the incidences. However, this lack of invariance of the direct counts combined with the rich combinatorial properties of the moduli spaces of disks indicates that invariance can still be achieved by defining an appropriate homology theory.

The purpose of this paper is to discuss systematically such a homology theory and the related algebraic structures in the case of monotone Lagrangians with minimal Maslov class at least 2 (which we will shortly call the *monotone case* below). We will also discuss its relations with Floer homology as well as various computations, examples and applications. It is already important to underline the fact that the point of view of this paper is not that of intersection theory and, thus, not that of Floer theory. In particular, this homology theory (and most of the other structures involved) is associated to a single Lagrangian submanifold, never vanishes and is invariant with respect to ambient symplectic isotopy. Therefore, this is a very rich structure and most of our various applications reflect its rigidity.

It should be mentioned at the outset that there are two systematic models for dealing with the general, non-monotone context: the A_∞ approach of Fukaya-Oh-Ohta-Ono [34] and the cluster homology approach of Cornea-Lalonde [23] (the last one being closer to the point of view we take here). The monotone case, while remaining reasonably rich, has the property that many of the technical complications which are present in these two models disappear. Indeed, as we shall see in this case, transversality issues can be dealt with by elementary means and the various invariants defined are all based on counting J-curves (for generic almost complex structures J) and not perturbed objects, a fact which is of invaluable help in computations. Hence, this is a case which is worth exploring in detail not only because many of the most relevant examples of Lagrangians fit in this context but also because it shows clearly what type of results and applications can be expected in general and computations are efficient.

There are many relations between this work and the extensive literature of the subject and they will be discussed explicitly later in the paper. We feel we should mention at this point that at the center of the construction is a chain complex called here the "pearl complex" initially described by Oh in [50]. Thus, this complex was known before and it was sometimes used in the literature. Our intention in preparing this paper has been to focus on the various structures related to this complex - a good number of which are first introduced here - and on their applications. However, we soon realized that no complete proofs are available in what concerns even the most basic parts of the construction, for example, for $d^2 = 0$ in the "pearl complex". Therefore we have decided to provide essentially complete arguments here. The structure of the paper is as follows. Section 2 contains the statement of the main algebraic properties of the structures that we are interested in here. The next three sections §3, §4, §5 are focused, respectively, on transversality, gluing and, finally, the proof of the algebraic properties announced in §2. Then, in §6 we describe our applications of this structure together with their proofs. A cautionary word to the reader: while the paper is written in the logical order needed to reduce redundancies, to more rapidly grasp the power and the motivation behind the algebraic structures described in the paper it might be useful to read the statements of the applications in §6 immediately after §2 and before the technical chapters in between.

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While working on this project our two children, Zohar and Robert, have been born. We would like to dedicate this work to them and to their lovely mothers, Michal and Alina.

2. Algebraic structures.

2.1. **The main algebraic statement.** We shall only work in this paper with, connected, closed, monotone Lagrangians $L \subset (M^{2n}, \omega)$ where (M, ω) is a tame monotone symplectic manifold. This means that the two homomorphisms:

$$\omega : \pi_2(M, L) \to \mathbb{Z}, \quad \mu : \pi_2(M, L) \to \mathbb{R}$$

given respectively by integration of ω and by the Maslov index satisfy

$$\omega(A) > 0 \quad \text{iff} \quad \mu(A) > 0, \quad \forall\, A \in \pi_2(M, L).$$

It is easy to see that this is equivalent to the existence of a constant $\tau > 0$ such that

$$\omega(A) = \tau\mu(A), \ \forall\, A \in \pi_2(M, L)\ . \tag{1}$$

We shall refer to τ as the *monotonicity constant* of $L \subset (M, \omega)$. Define the *minimal Maslov number* of L to be the integer

$$N_L = \min\{\mu(A) > 0 \mid A \in \pi_2(M, L)\}.$$

Throughout this paper we shall assume that L is monotone with $N_L \geq 2$. Since the Maslov numbers come in multiples of N_L we shall use sometimes the following notation:

$$\bar{\mu} = \tfrac{1}{N_L}\mu : \pi_2(M, L) \to \mathbb{Z}. \tag{2}$$

Let us now introduce the type of coefficient rings we shall work with. Let $\Lambda = \mathbb{Z}_2[t, t^{-1}]$ the ring of Laurent polynomials in the varible t. The grading on Λ is given by $\deg t = -N_L$. We shall also work with the positive version of Λ, namely $\Lambda^+ = \mathbb{Z}_2[t]$ with the same grading. The ring Λ should be viewed as a simplified version of the Novikov ring over $\pi_2(M, L)$, commonly used in Floer theory. Since we work in the monotone case the simplified Novikov rings Λ, Λ^+ are enough for our purposes.

There is a natural decreasing filtration of both Λ^+ and Λ by the degrees of t, i.e.

$$\mathcal{F}^k\Lambda = \{P \in \mathbb{Z}_2[t, t^{-1}] \mid P(t) = a_k t^k + a_{k+1}t^{k+1} + \dots\} \ . \tag{3}$$

We shall call this filtration the *degree filtration*. It induces an obvious filtration on any free module over this ring.

Let $f : L \to \mathbb{R}$ be a Morse function on L and let ρ be a Riemannian metric on L so that the pair (f, ρ) is Morse-Smale. Fix also a generic almost complex structure J compatible with ω. It is well known that, under the above assumption of monotonicity, the Floer homology of the pair (L, L) is well defined (see [48]) and we denote it by $HF(L)$ (the construction will be rapidly reviewed later in the paper).

In what follows we shall also use the following version of the quantum homology of M. Put $Q^+H(M) = H_*(M; \mathbb{Z}_2)\otimes\Lambda^+$, $QH(M) = H_*(M; \mathbb{Z}_2)\otimes\Lambda$ with the grading induced from $H_*(M; \mathbb{Z}_2)$ and Λ^+ (respectively Λ). We endow $Q^+H(M)$ and $QH(M)$ with the quantum cap product (see [44] for the definition). There are a few slight differences in our convention in comparison to the ones common in the literature. The first is that the degree of the variable t in the quantum homology of M is usually minus the minimal Chern number $-N_M$ of (M, ω). In our setting we have $\deg t = -N_L$. Since we are in the monotone case we have $N_L | N_M$, thus our $QH(M)$ is actually a kind of extension of the usual quantum homology of M. The second difference is that we work here with coefficients over $\mathbb{Z}_2$ rather than $\mathbb{Q}$ or $\mathbb{Z}$ which are more common in quantum homology theory. This is not essential and has to do with technical issues concerning the definition of the Floer homology of L (see Remark 2.1.2 below). Finally note that we work here with quantum *homology* (not cohomology), hence the quantum product $QH_k(M) \otimes QH_l(M) \to QH_{k+l-2n}(M)$ has degree $-2n$. The unit is $[M] \in QH_{2n}(M)$, thus of degree $2n$.

Theorem 2.1.1. *For a generic choice of the triple (f, ρ, J) there exists a chain complex*

$$\mathcal{C}^+(L; f, \rho, J) = (\mathbb{Z}_2\langle \mathrm{Crit(f)}\rangle \otimes \Lambda^+, \mathrm{d})$$

with the following properties:

i. *The homology of this chain complex is independent of the choices of J, f, ρ. It will be denoted by $Q^+H_*(L)$. There exists a canonical (degree preserving) augmentation $\epsilon_L : Q^+H_*(L) \to \Lambda^+$ which is a Λ^+-module map.*

ii. *The homology $Q^+H(L)$ has the structure of a two-sided algebra with unit over the quantum homology of M, $Q^+H(M)$. More specifically, for every i, j, k there exist Λ^+-bilinear maps:*

$$Q^+H_i(L) \otimes Q^+H_j(L) \to Q^+H_{i+j-n}(L), \quad \alpha \otimes \beta \mapsto \alpha * \beta,$$
$$Q^+H_k(M) \otimes Q^+H_i(L) \to Q^+H_{k+i-2n}(L), \quad a \otimes \alpha \mapsto a * \alpha.$$

The first map endows $Q^+H(L)$ with the structure of a ring with unit. This ring is in general not commutative. The second map endows $Q^+H(L)$ with the structure of a module over the quantum homology ring $Q^+H(M)$. Moreover, when viewing these two structures together, the ring $Q^+H(L)$ becomes a two-sided algebra over the ring $Q^+H(M)$. The unit $[M]$ of $Q^+H(M)$ has degree $2n = \dim M$ and the unit of $Q^+H(L)$ has degree $n = \dim L$.

iii. *There exists a map*

$$i_L : Q^+H_*(L) \to Q^+H_*(M)$$

which is a $Q^+H_*(M)$-module morphism and which extends the inclusion in singular homology. This map is determined by the relation:

$$\langle h^*, i_L(x)\rangle = \epsilon_L(h * x) \tag{4}$$

for $x \in Q^+H(L)$, $h \in H_*(M)$, with $(-)^*$ Poincaré duality and $\langle -, - \rangle$ the Kronecker pairing.

iv. *The differential d respects the degree filtration and all the structures above are compatible with the resulting spectral sequences.*

v. *The homology of the complex:*

$$\mathcal{C}(L; f, \rho, J) = \mathcal{C}^+(L; f, \rho, J) \otimes_{\Lambda^+} \Lambda$$

is denoted by $QH_*(L)$ and all the points above remain true if using $QH(-)$ instead of $Q^+H(-)$. The map $Q^+H(L) \to QH(L)$ induced in homology by the change of coefficients above is canonical. Moreover, there is an isomorphism

$$QH_*(L) \to HF_*(L)$$

which is also canonical up to a shift in grading.

By a two-sided algebra A over a ring R we mean that A is a module over R which has an internal product $A \times A \to A$ so that for any $r \in R$ and $a, b \in A$ we have $r(ab) = (ra)b = a(rb)$. The last equality is non-trivial, of course, only when the product in A is not commutative. A more natural description is the following. If A is a (left)-module over R, define a right-action of R on A by $ar = ra$. Then the "two-sidedness" of A over R means that *both* actions give A the structure of a module over R.

Before going on any further we would like to point out that, the existence of a module structure asserted by Theorem 2.1.1 has already some non-trivial consequences. For example, the fact that $QH_*(L) \cong HF_*(L)$ is a module over $QH_*(M)$ implies that if $a \in QH_k(M)$ is an invertible element, then the map $a * (-)$ gives rise to *isomorphisms*

$HF_i(L) \to HF_{i+k-2n}(L)$ for every $i \in \mathbb{Z}$. This clearly follows from the general algebraic definition of a "module over a ring with unit".

We shall call the complex $\mathcal{C}(L; f, \rho, J)$ (respectively, $\mathcal{C}^+(L; f, \rho, J)$) the *(positive) pearl complex* associated to f, ρ, J and we shall call the resulting homology the (positive) *quantum homology* of L. In the perspective of [23, 24] the complex $\mathcal{C}(L; f, \rho, J)$ corresponds to the *linear cluster complex*.

Parts of Theorem 2.1.1 appear already in the literature and have been verified up to various degrees of rigor. The complex $\mathcal{C}(L; f, \rho, J)$ has been first introduced by Oh [50] (see also Fukaya [33]) and is a particular case of the cluster complex as described in Cornea-Lalonde [23]. The module structure over $Q^+H(M)$ discussed at point ii. is probably known by experts - at least in the Floer homology setting - but has not been explicitly described yet in the literature. The product at ii. is a variant of the usual pair of pants product - it might not be widely known in this form. The map i_L at point iii. is the analogue of a map first studied by Albers in [5] in the absence of bubbling. The spectral sequence appearing at iv. is a variant of the spectral sequence introduced by Oh [49]. The compatibility of this spectral sequence with the product at point ii. has been first mentioned and used by Buhovsky [16] and independently by Fukaya-Oh-Ohta-Ono [34]. The positive Novikov ring Λ^+ is commonly used in algebraic geometry as well as in the closed case and has appeared in the Lagrangian setting in Fukaya-Oh-Ohta-Ono [34]. The comparison map at v. is an extension of the Piunikin-Salamon-Schwarz construction [54], it extends also the partial map constructed by Albers in [4] and a more general such map was described in [23] in the "cluster" context. We also remark that this comparison map identifies all the algebraic structures described above with the corresponding ones defined in terms of the Floer complex.

Remark 2.1.2. a. It is quite clear that, with rather obvious modifications, all the structure described in this statement should carry over to the case when L is non-monotone but orientable and relative spin. The coefficients in that case have to be rational - obviously this requires that the various moduli spaces involved be oriented coherently. One option to pursue the construction in this case is to further replace the Novikov ring Λ (or the positive Novikov ring Λ^+) with a cluster complex $\mathcal{C}l(L; J)$ of L [23]. Using these "cluster" coefficients means that the complex $\mathcal{C}(L; f, \rho, J)$ is replaced with the fine Floer complex of [23] and, with the exception that $QH_*(L)$ is replaced in all places by the fine Floer homology of L, $I\!\!F H_*(L)$, the statement of Theorem 2.1.1 should remain true and even have a "positive" version.

b. Another interesting point that we want to emphasize here - and will be exemplified later in the paper - is that the structures discussed in the statement of the theorem lead to the definition of certain Gromov-Witten type invariants. The procedure is as follows. Suppose first that $k \in \mathbb{Z}_2$ (or $\in \mathbb{Z}$, if we assume orientations) is some numerical invariant defined out of the algebra structure of $Q^+H_*(L)$ (this means that this number is left invariant by isomorphisms of the structure). Assume also that, under certain circumstances, for special choices of the function f and the almost complex structure J, the chain complex $\mathcal{C}^+(L; f, \rho, J)$ has a trivial differential. This could happen, for example, if f is a perfect Morse function and if J is a special "symmetric" structure or, for example, as we shall see further in this paper, if L is a torus with non trivial Floer homology. In that case, the "counting" leading to k, which is invariant, in general, only after passage to homology will be invariant already at the chain level simply because, for these special choices of f, J, the chain level is isomorphic with the homology one. But this means that, with these special choices, the count giving k is invariant and this is exactly what is needed to define Gromov-Witten type invariants. It is then another matter to interpret these numbers geometrically in a meaningful way.

2.2. **Other algebraic structures.**

2.2.1. *Duality.* The first point that we want to discuss here is a form of *duality* which extends Poincaré duality. We first fix some notation. Suppose that $(\mathcal{C}, \partial)$ is a chain complex over Λ^+. In particular, it is a free module over Λ^+, $\mathcal{C} = G \otimes \Lambda^+$ with G some $\mathbb{Z}_2$ vector space. We let

$$\mathcal{C}^{\odot} = \hom_{\Lambda^+}(\mathcal{C}, \Lambda^+)$$

graded so that the degree of a morphism $g : \mathcal{C} \to \Lambda^+$ is k if g takes $\mathcal{C}_l$ to Λ^+_{l+k} for all l.

Let $\mathcal{C}' = \hom_{\mathbb{Z}_2}(G, \mathbb{Z}_2) \otimes \Lambda^+$ be graded such that if x is a basis element of G, then its dual $x^* \in \mathcal{C}'$ has degree $|x^*| = -|x|$. There is an obvious degree preserving isomorphism $\psi : \mathcal{C}^{\odot} \to \mathcal{C}'$ defined by $\psi(f) = \sum_i f(g_i) g_i^*$ where (g_i) is a basis of G and (g_i^*) is the dual basis. We define the differential of $\mathcal{C}^{\odot}$, ∂^*, as the adjoint of ∂:

$$\langle \partial^* y^*, x \rangle = \langle y^*, \partial x \rangle \ , \ \forall x, y \in G \ .$$

Clearly, $\mathcal{C}^{\odot}$ continues to be a chain complex (and not a co-chain complex).

An additional algebraic notion will be useful: the co-chain complex $\mathcal{C}^*$ associated to $\mathcal{C}$. To define it we first let $(\Lambda^+)^*$ be the ring Λ^+ with the reverse grading: the degree of each element in $(\Lambda^+)^*$ is the opposite of the degree of the same element in Λ^+. For the free chain complex $\mathcal{C} = G \otimes \Lambda^+$ as before, we define $\mathcal{C}^* = \hom_{\mathbb{Z}_2}(G, \mathbb{Z}_2) \otimes (\Lambda^+)^*$ where the grading of the dual x^* of a basis element $x \in G$ is $|x^*| = |x|$. The differential in $\mathcal{C}^*$ is given as usual as the adjoint of the differential in $\mathcal{C}$. The complex $\mathcal{C}^*$ is obviously a co-chain complex. The difference between this complex and $\mathcal{C}^{\odot}$ is just that the grading is reversed in the sense that if an element $x \otimes \lambda$ has degree k in one complex, then it has degree $-k$ in the other. The co-homology of $\mathcal{C}$ is then defined as $H^k(\mathcal{C}) = H^k(\mathcal{C}^*)$. Obviously, there is a canonical isomorphism: $H_{-k}(\mathcal{C}^{\odot}) \cong H^k(\mathcal{C}^*)$.

A particular case of interest here is when $\mathcal{C} = \mathcal{C}(L; f, \rho, J)$. In this case we denote:

$$Q^+H^{n-k}(L) = H^k(\mathcal{C}^+(L; f, \rho, J)^*) \ .$$

Notice that the chain morphisms $\eta : \mathcal{C} \to \mathcal{C}^{\odot}$ of degree $-n$ are in correspondence with the chain morphisms of degree $-n$:

$$\tilde{\eta} : \mathcal{C} \otimes_{\Lambda^+} \mathcal{C} \to \Lambda^+ \ .$$

via the formula $\tilde{\eta}(x \otimes y) = \eta(x)(y)$. Here the ring Λ^+ on the right handside is considered as a chain complex with trivial differential. Moreover, if η induces an isomorphism in homology, then the pairing induced in homology by $\tilde{\eta}$ is non-degenerate.

Fix now $n \in \mathbb{N}^*$. For any chain complex $\mathcal{C}$ as before we let $s^n\mathcal{C}$ be its n-fold suspension. This is a chain complex which coincides with $\mathcal{C}$ but its graded so that the degree of x in $s^n\mathcal{C}$ is $n+$ the degree of x in $\mathcal{C}$.

A particular useful case where these notions appear is in the following sequence of obvious isomorphisms: $H_k(s^n\mathcal{C}^{\odot}) \cong H_{k-n}(\mathcal{C}^{\odot}) \cong H^{n-k}(\mathcal{C}^*)$.

Corollary 2.2.1. *Set $n = \dim L$. There exists a degree preserving morphism of chain complexes:*

$$\eta : \mathcal{C}^+(L; f, \rho, J) \to s^n(\mathcal{C}^+(L; f, \rho, J))^{\odot}$$

which induces an isomorphism in homology. In particular, we have an isomorphism: $\eta : Q^+H_k(L) \to Q^+H^{n-k}(L)$. The corresponding (degree $-n$) bilinear map

$$H(\tilde{\eta}) : Q^+H(L) \otimes Q^+H(L) \to \Lambda^+$$

coincides with the product described in Theorem 2.1.1-ii composed with the augmentation ϵ_L. The same result continues to hold with Λ^+, $\mathcal{C}^+$, QH^+ replaced by Λ, $\mathcal{C}$, QH respectively.

Remark 2.2.2. a. The relation of the Corollary above with Poincaré duality is as follows: in case $\mathcal{C}^+(-)$ in the statement is replaced with the Morse complex $C(f)$ of some Morse function $f : L \to \mathbb{R}$ we may define the morphism $\eta : C(f) \to s^n(C(f)^{\odot})$ as a composition of two morphisms with the first being the usual comparison morphism $C(f) \to C(-f)$ and the second $C(-f) \to s^n(C(f)^{\odot})$ given by $x \in \mathrm{Crit(f)} \to \mathrm{x}^* \in \mathrm{hom}_{\mathbb{Z}_2}(\mathrm{C(f)}, \mathbb{Z}_2)$. We have the identifications $H_k(s^n(C(f)^{\odot})) = H_{k-n}(C(f)^{\odot}) = H^{n-k}(C(f))$ and the morphism η described above induces in homology the Poincaré duality map: $H_k(L) \to H^{n-k}(L)$.

b. The last Corollary also obviously shows that $Q^+H(L)$ together with the bilinear map $\epsilon_L \circ (- * -)$ is a Frobenius algebra, though not necessarily commutative.

2.2.2. *Action of the symplectomorphism group.* This property is very useful in computations when symmetry is present.

Corollary 2.2.3. *Let $\phi : L \to L$ be a diffeomorphism which is the restriction to L of an ambient symplectic diffeomorphism $\bar{\phi}$ of M. Let f, ρ, J be so that the pearl complex $\mathcal{C}^+(L; f, \rho, J)$ is defined. There exists a chain map:*

$$\tilde{\phi} : \mathcal{C}^+(L; f, \rho, J) \to \mathcal{C}^+(L; f, \rho, J)$$

which respects the degree filtration, induces an isomorphism in homology, and so that the morphism $E^2(\tilde{\phi})$ induced by $\tilde{\phi}$ at the E^2 level of the degree spectral sequence coincides with $H_*(\phi) \otimes id_{\Lambda^+}$. The map $\bar{\phi} \to \tilde{\phi}$ induces a representation:

$$\hbar : Symp(M, L) \to Aut(Q^+H_*(L))$$

where $Aut(Q^+H_*(L))$ are the augmented ring automorphisms of $Q^+H_*(L)$ and $Symp(M, L)$ are the symplectomorphisms of M which restrict to diffeomorphisms of L. The restriction of $\hbar$ to $Symp_0(M) \cap Symp(M, L)$ takes values in the automorphisms of $Q^+H(L)$ as an algebra over $Q^+H(M)$.

The same result continues to hold with Λ^+, $\mathcal{C}^+$, QH^+ replaced by Λ, $\mathcal{C}$, QH respectively.

2.2.3. *Minimal pearl complexes.* It is easy to see that all the calculations with the structures described above are much more efficient if the Lagrangian L admits a perfect Morse function - that is a Morse function $f : L \to \mathbb{R}$ so that the Morse differential vanishes. We now want to notice that there exists an algebraic procedure which allows one to treat any general L in the same way. Moreover, we will see that this produces another a chain complex which is a quantum invariant of L and contains all the quantum specific properties that we generally want to study (a similar construction in the cluster set-up has been sketched in [23]).

Let G be a finite dimensional graded $\mathbb{Z}_2$-vector space and let $\mathcal{C} = (G \otimes \Lambda^+, d)$ be a chain complex. For an element $x \in G$ let $d(x) = d_0(x) + d_1(x)t$ with $d_0(x) \in G$. In other words d_0 is obtained from $d(x)$ by treating t as a polynomial variable and putting $t = 0$. Clearly $d_0 : G \to G$, $d_0^2 = 0$. Let $\mathcal{H}$ be the homology of the complex (G, d_0). Similarly, for a chain morphism ξ we denote by ξ_0 the d_0-chain morphism obtained by making $t = 0$.

Proposition 2.2.4. *With the notation above there exists a chain complex*

$$\mathcal{C}_{min} = (\mathcal{H} \otimes \Lambda^+, \delta), \text{ with } \delta_0 = 0$$

and chain maps $\phi : \mathcal{C} \to \mathcal{C}_{min}$, $\psi : \mathcal{C}_{min} \to \mathcal{C}$ *so that* $\phi \circ \psi = id$ *and* ϕ *and* ψ *induce isomorphisms in d-homology and* ϕ_0 *and* ψ_0 *induce an isomorphism in* d_0*-homology. Moreover, the properties above characterize* $\mathcal{C}_{min}$ *up to isomorphism.*

Here is an important consequence of this result:

Corollary 2.2.5. *There exists a complex* $\mathcal{C}_{min}(L) = (H_*(L;\mathbb{Z}_2) \otimes \Lambda^+, \delta)$, *with* $\delta_0 = 0$ *and so that for any* (L, f, ρ, J) *such that* $\mathcal{C}(L; f, \rho, J)$ *is defined there are chain morphisms* $\phi : \mathcal{C}(L; f, \rho, J) \to \mathcal{C}_{min}(L)$ *and* $\psi : \mathcal{C}_{min}(L) \to \mathcal{C}(L; f, \rho, J)$ *which both induce isomorphisms in quantum homology as well as in Morse homology and so that* $\phi \circ \psi = id$. *The complex* $\mathcal{C}_{min}(L)$ *with these properties is unique up to isomorphism.*

We call the complex provided by this corollary the *minimal pearl complex*. This terminology is justified by the use of minimal models in rational homotopy where a somewhat similar notion is central. There is a slight abuse in this notation as, while any two complexes as provided by the corollary are isomorphic this isomorphism is not canonical. Obviously, in case a perfect Morse function exists on L any pearl complex associated to such a function is already minimal.

Remark 2.2.6. a. An important consequence of the existence of the chain morphisms ϕ and ψ is that all the algebraic structures described before (product, module structure etc) can be transported and computed on the minimal complex. For example, the product is the composition:

$$\mathcal{C}_{min}(L) \otimes \mathcal{C}_{min}(L) \xrightarrow{\psi \otimes \psi} \mathcal{C}(L; f, \rho, J) \otimes \mathcal{C}(L; f, \rho, J) \xrightarrow{*} \mathcal{C}(L; f, \rho, J) \xrightarrow{\phi} \mathcal{C}_{min}(L) \ .$$

It is easy to see that - in homology - the resulting product has as unit the fundamental class $[L] \in H_n(L)$.

b. A consequence of point a. is that $HF(L) \cong QH(L) = 0$ iff there is some $x \in H_*(L;\mathbb{Z}_2)$ so that $\delta x = [L]t^k$ in $\mathcal{C}_{min}(L)$. Indeed, suppose that $QH(L) = 0$. Then, as for degree reasons $[L]$ is a cycle in $\mathcal{C}_{min}(L)$, we obtain that it has to be also a boundary. Conversely, if $[L]$ is a boundary (which means $\delta x = [L]t^k$ for some x and k) we have for any other cycle $c \in \mathcal{C}_{min}(L)$: $[c] = [c]*[L] = [c*[L]] = [\delta(c*x)] = 0$ where we have denoted by $- * -$ the product on $\mathcal{C}_{min}(L)$ as defined above (see also §6.1.1 for other criteria of similar nature).

c. It is also useful to note that there is an isomorphism $Q^+H(L) \cong H(L;\mathbb{Z}_2) \otimes \Lambda^+$ iff the differential δ in $\mathcal{C}_{min}(L)$ is identically zero.

2.2.4. *Large and small coefficient rings.* We have seen before that both the ring Λ^+ and the ring Λ can be used in our constructions. Indeed, the interest of Λ^+ is mainly that the resulting homology never vanishes while the ring Λ is needed for the comparison with Floer homology.

Example 2.2.7. Consider $S^1 \subset \mathbb{C}$ the standard circle in the complex plane. Obviously, as S^1 is displaceable we have that $QH(S^1) = HF(S^1) = 0$. However, the positive quantum homology $Q^+H(S^1)$ verifies: $Q^+H_*(S^1) = 0$ for $* \neq 1$ and $Q^+H_1(S^1) = \mathbb{Z}_2$. Indeed, we may take on S^1 a Morse function with a single minimum P and a single maximum Q. The standard almost complex structure is regular and the standard disk which fills the circle is of Maslov class two. This disks obviously goes through the minimum and the maximum and this shows that $dP = Qt$ in the pearl complex. It is easy to see that $dQ = 0$ for degree reasons. Therefore, Q is a cycle but not a boundary and this implies the claim.

This example generalizes easily to show that for any monotone Lagrangian L we have $Q^+H(L) \neq 0$. Indeed, as L is assumed connected we may work with a Morse function $f : L \to \mathbb{R}$ with a single maximum which we will again denote by Q. In this case we again have $dQ = 0$. Indeed, the Morse differential of Q is null because Q is the unique maximum of f. Moreover, if $dQ = Rt^k + ...$ we need $|R| - kN_L = Q - 1$ which is not possible for $k \neq 0$ because $N_L \geq 2$. Thus, the unique maximum of such a function represents a cycle in the pearl complex and, given that t is not invertible in Λ^+, it follows that the homology class represented by the maximum is non-trivial.

In a rather obvious way these are the minimal rings that one can use for these purposes. Indeed, let $\pi_2(M, L)^+$ be the semi-group of all the elements u so that $\omega(u) \geq 0$. Then $\Lambda^+ = \mathbb{Z}_2[\pi_2(M, L)^+/ \sim]$ with $\sim$ the equivalence relation $u \sim v$ iff $\mu(u) = \mu(v)$ and similarly $\Lambda = \mathbb{Z}_2[\pi_2(M, L)/ \sim]$. For certain other applications it can be useful to also use large rings which distinguish explicitely elements in $\pi_2(M, L)$. For this purpose we remark now that all the arguments in the paper carry over when replacing Λ^+ with $\tilde{\Lambda}^+ = \mathbb{Z}_2[\pi_2(M, L)^+]$ and Λ with $\tilde{\Lambda} = \mathbb{Z}_2[\pi_2(M, L)]$.

Indeed, with a single exception to be discussed below, for all the constructions in the paper to hold the coefficient ring needs to satisfy just two conditions: it needs to behave additively with respect to gluing and bubbling and it needs to distinguish disks with different symplectic areas (due to our monotonicity assumption this is, of course, the same as distinguishing disks with different Maslov classes). In particular, any ring $\mathcal{R}$ such that there is a ring morphism $r : \mathbb{Z}_2[\pi_2(M, L)^+] \to \mathcal{R}$ with $r(u) \neq r(v)$ whenever $\omega(u) \neq \omega(v)$ will do. The single exception is the comparison with Floer homology - and, in particular, to show that $QH(-)$ vanishes for a displaceable Lagrangian. For these additional properties to hold, the ring $\mathcal{R}$ has to be stable with respect to the invertion of the elements in $\pi_2(M, L)$. For example, in the case described above, r needs also to extend to a ring morphism $\mathbb{Z}_2[\pi_2(M, L)] \to \mathcal{R}$.

2.3. **Action estimates.** All the elements of the moduli spaces involved in our various algebraic structures admit meaningful energy notions. It is therefore easy (and essentially standard) to deduce various action estimates from the non-triviality of these structures.

We shall give here a single example of such an application which is an extension of the action estimates that appeared in the work of Albers [5] and apply to a version of the quantum inclusion map i_L.

Let $\widetilde{Ham}(M)$ be the universal cover of the group of Hamiltonian diffeomorphisms and fix $\phi \in \widetilde{Ham}(M)$. Recall that for any $\alpha \in H_*(M;\mathbb{Z}_2)$ there are spectral invariants $\sigma(\alpha;\phi) \in \mathbb{R}$ and $\sigma(\alpha^*;\phi)$ where $\alpha^* \in H^*(M;\mathbb{Z}_2)$ is the Poincaré dual of α. We refer the reader to [59, 53, 51, 52, 46, 45, 44] for the foundations of the theory of spectral invariants. We shall also recall the basic definitions in §5.10. We now define the *depth* of ϕ on L by

$$\mathrm{depth}_L(\phi) = \sup_{[H]=\phi} \int_0^1 (\inf_{x\in L} H(x,t))dt \ .$$

Similarly, we let the *height* of ϕ on L be defined by:

$$\mathrm{height}_L(\phi) = \inf_{[H]=\phi} \int_0^1 (\sup_{x\in L} H(x,t))dt \ .$$

Corollary 2.3.1. *Assume that $\alpha \in H_*(M;\mathbb{Z}_2)$, $x, y \in Q^+H(L)$ are so that $y \neq 0$ and $\alpha * x = yt^k + \text{higher order terms}$. Then we have the following inequalities:*

$$\sigma(\alpha;\phi) - \mathrm{depth}_L(\phi) \geq -k\tau \leq \mathrm{height}_L(\phi) - \sigma(\alpha^*;\phi)$$

where τ is the monotonicity constant.

As before, the same result continues to hold with $QH^+(L)$ replaced by $QH(L) = HF(L)$.

3. Transversality

The purpose of this section is to deal with the main transversality issues that appear in the definition of our algebraic structures. The pearl moduli spaces - they are at the heart of the definition of the pearl complex - are introduced here and we shall see that transversality is not difficult to achieve for them using the structural results of Lazzarini [42, 41] combined with some combinatorial arguments. The main ideas and technical lemmas of this section will then be used for these and various other moduli spaces of similar type in §5.

3.1. **Transversality for strings of pearls.** Let (M^{2n},ω) be a tame symplectic manifold and $L^n \subset (M^{2n},\omega)$ a closed Lagrangian submanifold. Assume that L is monotone with minimal Maslov number $N_L \geq 2$. Denote by $\mathcal{J}(M,\omega)$ the space of almost complex structures on M which are compatible with ω. Given a homology class $F \in H_2(M,L;\mathbb{Z})$, denote by $\mathcal{M}(F, J)$ the space of J-holomorphic disks $u : (D, \partial D) \to (M, L)$ in the class F. (Here and in what follows $D \subset \mathbb{C}$ stands for the closed unit disk.)

Definition 3.1.1. (1) A J-holomorphic disk $u : (D, \partial D) \to (M, L)$ is called simple if there exists an open dense subset $S \subset D$ such that for every $z \in S$ we have $u^{-1}(u(z)) = \{z\}$ and $du_z \neq 0$. We denote by $\mathcal{M}^*(F, J) \subset \mathcal{M}(F, J)$ the space of simple J-holomorphic disks $u : (D, \partial D) \to (M, L)$ in the class F.

(2) Let $v_i : (D, \partial D) \to (M, L)$, $i = 1, \ldots, k$ be a sequence of J-holomorphic disks. We say that $(v_1, \ldots, v_k)$ are *absolutely distinct* if for every $1 \leq i \leq k$ we have $v_i(D) \not\subset \bigcup_{j \neq i} v_j(D)$.

Let $f : L \to \mathbb{R}$ be a Morse function and ρ a Riemannian metric on L. We denote by $\Phi_t : L \to L$, $t \in \mathbb{R}$, the *negative* gradient flow of (f, ρ) (i.e. the flow of the vector field $-\mathrm{grad}_\rho f$). Given critical points $x, y \in \mathrm{Crit}(f)$ denote by W^u_x, W^s_y the unstable and stable submanifolds of x and y with respect to the negative gradient flow of f.

Consider the (non-proper) embedding

$$(L \setminus \mathrm{Crit}(f)) \times \mathbb{R}_{>0} \hookrightarrow L \times L, \quad (x, t) \longmapsto (x, \Phi_t(x)). \tag{5}$$

Denote the image of this embedding by $Q_{f,\rho} \subset L \times L$.

Denote by $G = \mathrm{Aut}(D) \cong PSL(2, \mathbb{R})$ the group of biholomorphisms of D. Given points $p_1, \ldots, p_m \in D$ we denote by $G_{p_1,\ldots,p_m} \subset G$ the subgroup of all the automorphisms $\sigma \in G$ that fix all the p_i's.

Let $\mathbf{A} = (A_1, \ldots, A_l)$ be a sequence of *non-zero* homology classes $A_1, \ldots, A_l \in H_2(M, L; \mathbb{Z})$, $l \geq 1$. Set $\mu(\mathbf{A}) = \sum_{i=1}^{l} \mu(A_i)$. Put:

$$\mathcal{M}(\mathbf{A}, J) = \mathcal{M}(A_1, J)/G_{-1,1} \times \cdots \times \mathcal{M}(A_l, J)/G_{-1,1}. \tag{6}$$

Denote by $\mathcal{M}^{*,\mathrm{d}}(\mathbf{A}, J)$ the subspace of all $(u_1, \ldots, u_l) \in \mathcal{M}(\mathbf{A}, J)$ which are simple and absolutely distinct. Consider the following evaluation map:

$$ev_{\mathbf{A}} : \mathcal{M}(\mathbf{A}, J) \longrightarrow L^{\times 2l}, \quad ev_{\mathbf{A}}(u_1, \ldots, u_l) = \big(u_1(-1), u_1(1), \ldots, u_l(-1), u_l(1)\big). \tag{7}$$

For every $x, y \in \mathrm{Crit}(f)$, put

$$\mathcal{P}(x, y, \mathbf{A}; J, f, \rho) = ev_{\mathbf{A}}^{-1}\Big(W^u_x \times \big(Q_{f,\rho}\big)^{\times(l-1)} \times W^s_y\Big).$$

We call $\mathcal{P}(- - -)$ the "moduli space of pearls". They consist of objects as in figure 1.

Finally, write $\mathcal{P}^{*,\mathrm{d}}(x, y, \mathbf{A}; J, f, \rho) = \mathcal{P}(x, y, \mathbf{A}; J, f, \rho) \cap \mathcal{M}^{*,\mathrm{d}}(\mathbf{A}, J)$, namely the subspace of all $(u_1, \ldots, u_l) \in \mathcal{P}(x, y, \mathbf{A}; J, f, \rho)$ which are simple and absolutely distinct. Standard arguments [44] show that:

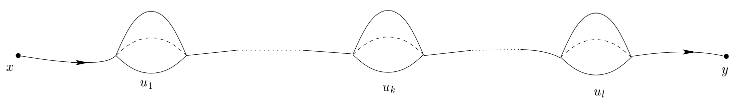

Figure 1. An element of $\mathcal{P}(x, y; \mathbf{A}, J)$

3.1.2. *For every pair (f, ρ) there exists a subset $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}(M, \omega)$ of second category such that for every $J \in \mathcal{J}_{\mathrm{reg}}$, for every sequence of non-zero homology classes $\mathbf{A}$ and every $x, y \in \mathrm{Crit}(f)$, the restriction of $ev_{\mathbf{A}}$ to $\mathcal{M}^{*,\mathrm{d}}(\mathbf{A}, J)$ is transverse to $W_x^u \times \big(Q_{f,\rho}\big)^{\times(l-1)} \times W_y^s$ at every $\mathbf{u} \in \mathcal{M}^{*,\mathrm{d}}(\mathbf{A}, J)$. In particular the space $\mathcal{P}^{*,\mathrm{d}}(x, y, \mathbf{A}; J, f, \rho)$ is either empty or a smooth manifold of dimension $\mu(\mathbf{A}) + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1$.*

In this section we prove the following.

Proposition 3.1.3. *Let $f : L \to \mathbb{R}$ be a Morse function and ρ a Riemannian metric on L such that the pair (f, ρ) is Morse-Smale. Then there exists a subset of second category $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}(M, \omega)$ with the following property. For every sequence of non-zero homology classes $\mathbf{A} = (A_1, \ldots, A_l)$ and every $x, y \in \mathrm{Crit}(f)$ with $\mu(\mathbf{A}) + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 \leq 1$:*

(1) $\mathcal{P}(x, y, \mathbf{A}; J, f, \rho) = \mathcal{P}^{*,\mathrm{d}}(x, y, \mathbf{A}; J, f, \rho)$. In other words all elements $(u_1, \ldots, u_l) \in \mathcal{P}(x, y, \mathbf{A}; J, f, \rho)$ are simple and absolutely distinct. Thus $\mathcal{P}(x, y, \mathbf{A}; J, f, \rho)$ is either empty or a smooth manifold of dimension $\mu(\mathbf{A}) + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1$. In particular, if $\mu(\mathbf{A}) + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 < 0$ we have $\mathcal{P}(x, y, \mathbf{A}; J, f, \rho) = \emptyset$.

(2) If $\mu(\mathbf{A}) + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 = 0$ then $\mathcal{P}(x, y, \mathbf{A}; J, f, \rho)$ is a compact *0-dimensional manifold hence consists of finite number of points.*

The proof of Proposition 3.1.3 is given in Section 3.3 below.

Remark 3.1.4. Since a countable intersection of second category subset of $\mathcal{J}(M, \omega)$ is of second category too we shall denote various second category subsets of almost complex structure in this section by the same notation $\mathcal{J}_{\mathrm{reg}}$.

Let $\mathbf{B}' = (B'_1, \ldots, B'_{l'})$, $\mathbf{B}'' = (B''_1, \ldots, B''_{l''})$ be two vectors of non-zero homology classes in $H_2(M, L; \mathbb{Z})$. Put

$$\mathcal{M}(\mathbf{B}', \mathbf{B}'', J) = \prod_{i=1}^{l'} \frac{\mathcal{M}(B'_i, J)}{G_{-1,1}} \times \prod_{j=1}^{l''} \frac{\mathcal{M}(B''_j, J)}{G_{-1,1}}.$$

Denote by $\mathcal{M}^{*,\mathrm{d}}(\mathbf{B}', \mathbf{B}'', J)$ the subspace of all $(\mathbf{u}', \mathbf{u}'') \in \mathcal{M}(\mathbf{B}', \mathbf{B}'', J)$ for which the J-holomorphic disks $(u'_1, \ldots, u'_{l'}, u''_1, \ldots, u''_{l''})$ are simple and absolutely distinct. Define an evaluation map $ev_{\mathbf{B}',\mathbf{B}''} : \mathcal{M}(\mathbf{B}', \mathbf{B}'', J) \to L^{\times(2l'+2l'')}$ by:

$$ev_{\mathbf{B}',\mathbf{B}''}(u'_1, \ldots, u'_{l'}, u''_1, \ldots, u''_{l''}) = \big(ev_{\mathbf{B}'}(u'_1, \ldots, u'_{l'}),\ ev_{\mathbf{B}''}(u''_1, \ldots, u''_{l''})\big),$$

where $ev_{\mathbf{B}'}$, $ev_{\mathbf{B}''}$ are defined as in (7). Put

$$\mathcal{P}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho) = ev^{-1}_{\mathbf{B}', \mathbf{B}''}\Big(W^u_x \times \big(Q_{f,\rho}\big)^{\times(l'-1)} \times \mathrm{diag}(L) \times \big(Q_{f,\rho}\big)^{\times(l''-1)} \times W^s_y\Big).$$

Finally, write $\mathcal{P}^{*,\mathrm{d}}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho) = \mathcal{P}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho) \cap \mathcal{M}^{*,\mathrm{d}}(\mathbf{B}', \mathbf{B}'', J)$. Standrad arguments [44] show that:

3.1.5. *For every pair (f, ρ) there exists a subset $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}(M, \omega)$ of second category such that for every $J \in \mathcal{J}_{\mathrm{reg}}$, for every two sequences of non-zero homology classes $\mathbf{B}'$, $\mathbf{B}''$ and every $x, y \in \mathrm{Crit}(f)$ the restriction of the map $ev_{\mathbf{B}',\mathbf{B}''}$ to $\mathcal{M}^{*,\mathrm{d}}(\mathbf{B}', \mathbf{B}'', J)$ is transverse to*

$$W^u_x \times \big(Q_{f,\rho}\big)^{\times(l'-1)} \times \mathrm{diag}(L) \times \big(Q_{f,\rho}\big)^{\times(l''-1)} \times W^s_y.$$

In particular, the space $\mathcal{P}^{*,\mathrm{d}}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho)$ is either empty or a smooth manifold of dimension $\mu(\mathbf{B}') + \mu(\mathbf{B}'') + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 2$.

With the above notation we have

Proposition 3.1.6. *Let $f : L \to \mathbb{R}$ be a Morse function and ρ a Riemannian metric on L such that the pair (f, ρ) is Morse-Smale. Then there exists a subset of second category $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}(M, \omega)$ with the following property. For every two sequences of non-zero homology classes $\mathbf{B}' = (B'_1, \ldots, B'_{l'})$, $\mathbf{B}'' = (B''_1, \ldots, B''_{l''})$ and every $x, y \in \mathrm{Crit}(f)$ with $\mu(\mathbf{B}') + \mu(\mathbf{B}'') + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 \leq 1$:*

(1) $\mathcal{P}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho) = \mathcal{P}^{*,\mathrm{d}}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho)$. In other words for every

$$(u'_1, \ldots, u'_{l'}, u''_1, \ldots, u''_{l''}) \in \mathcal{M}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho)$$

the disks $(u'_1, \ldots, u'_{l'}, u''_1, \ldots, u''_{l''})$ are simple and absolutely distinct.

(2) If $\mu(\mathbf{B}') + \mu(\mathbf{B}'') + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 \leq 0$ then $\mathcal{P}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho) = \emptyset$.

(3) If $\mu(\mathbf{B}') + \mu(\mathbf{B}'') + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 = 1$ then $\mathcal{P}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho)$ is either empty or a compact 0-dimensional smooth manifold, hence consists of finite number of points.

We shall not give a proof for Proposition 3.1.6 since it can be proved in a very similar way to Proposition 3.1.3 which will be proved below.

3.2. **Reduction to simple disks.** Let $F \in H_2(M, L; \mathbb{Z})$, $J \in \mathcal{J}(M, \omega)$. Denote by $\mathcal{M}^*(F, J)$ the space of simple J-holomorphic disks $u : (D, \partial D) \to (M, L)$ in the class F. According to the general theory of pseudo-holomorphic curves [44] for a generic choice of $J \in \mathcal{J}(M, \omega)$ the space $\mathcal{M}^*(F, J)$ is either empty or a smooth manifold of dimension $\mu(F) + n$. This fails to be true for the space $\mathcal{M}(F, J)$ of all disks in the class F. Therefore a crucial ingredient in the proof of Proposition 3.1.3 is a procedure which enables to decompose a (general) J-holomorphic disk to simple "pieces". This will make it possible to obtain transversality and to control dimensions of moduli spaces of pseudo-holomorphic disks. There are two (essentially equivalent) approaches to this decomposition, one due to Kwon and Oh [40] and the other to Lazzarini [42, 41]. Below we shall follow Lazzarini's approach.

Let $u : (D, \partial D) \to (M, L)$ be a non-constant J-holomorphic disk. Put $\mathcal{C}(u) = u^{-1}(\{du = 0\})$. Define a relation $\mathcal{R}_u$ on pairs of points $z_1, z_2 \in \operatorname{Int} D \setminus \mathcal{C}(u)$ in the following way:

$$z_1 \mathcal{R}_u z_2 \iff \begin{cases} \forall \text{ neighbourhoods } V_1, V_2 \text{ of } z_1, z_2, \\ \exists \text{ neighbourhoods } U_1, U_2 \text{ such that:} \\ \text{(i) } z_1 \in U_1 \subset V_1, z_2 \in U_2 \subset V_2. \\ \text{(ii) } u(U_1) = u(U_2). \end{cases}$$

Denote by $\overline{\mathcal{R}}_u$ the closure of $\mathcal{R}_u$ in $D \times D$. Note that $\overline{\mathcal{R}}_u$ is reflexive and symmetric but it may fail to be transitive (see [41] for more details on this). Define the *non-injectivity graph* of u to be:

$$\mathcal{G}(u) = \{z \in D \,|\, \exists z' \in \partial D \text{ such that } z \overline{\mathcal{R}}_u z'\}.$$

It is proved in [41, 42] that $\mathcal{G}(u)$ is indeed a graph and its complement $D \setminus \mathcal{G}(u)$ has finitely many connected components. In what follows we shall use the following theorem due to Lazzarini (See Proposition 4.1 in [41]. See also [42]).

Theorem 3.2.1 (Decomposition of disks)**.** *Let $u : (D, \partial D) \to (M, L)$ be a non-constant J-holomorphic disk. Then for every connected component $\mathfrak{D} \subset D \setminus \mathcal{G}(u)$ there exists a surjective map $\pi_{\overline{\mathfrak{D}}} : \overline{\mathfrak{D}} \to D$, holomorphic on $\mathfrak{D}$ and continuous on $\overline{\mathfrak{D}}$, and a simple J-holomorphic disk $v_{\mathfrak{D}} : (D, \partial D) \to (M, L)$ such that $u|_{\overline{\mathfrak{D}}} = \pi_{\overline{\mathfrak{D}}} \circ v_{\mathfrak{D}}$. The map $\pi_{\overline{\mathfrak{D}}} : \overline{\mathfrak{D}} \to D$ has a well defined degree $m_{\mathfrak{D}} \in \mathbb{N}$ and we have in $H_2(M, L; \mathbb{Z})$:*

$$[u] = \sum_{\mathfrak{D}} m_{\mathfrak{D}} [v_{\mathfrak{D}}],$$

where the sum is taken over all connected components $\mathfrak{D} \subset D \setminus \mathcal{G}(u)$.

Remark. Some of the connected components $\mathfrak{D} \subset D \setminus \mathcal{G}(u)$ may not be disks. This happens if and only if $\mathcal{G}(u)$ is not connected. Nevertheless, $\overline{\mathfrak{D}}/\overline{\mathcal{R}}_{\mathfrak{D}}$ is a disk, where $\overline{\mathcal{R}}_{\mathfrak{D}}$ is the relation defined similarly to $\overline{\mathcal{R}}_u$ but for pairs of points in $\overline{\mathfrak{D}}$.

For the proof of Proposition 3.1.3 we shall need the following two lemmas. We state them here but defer their proofs to Section 3.4.

Lemma 3.2.2. *Suppose $n = \dim L \geq 3$. Then there exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$ the following holds. If $u, v : (D, \partial D) \to (M, L)$ are simple J-holomorphic disks such that $u(D) \cap v(D)$ is an infinite set then:*

- *either $u(D) \subset v(D)$ and $u(\partial D) \subset v(\partial D)$; or*
- *$v(D) \subset u(D)$ and $v(\partial D) \subset u(\partial D)$.*

Lemma 3.2.3. *Suppose $n = \dim L \geq 3$. Then there exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$ the following holds. For every non-simple J-holomorphic disk $u : (D, \partial D) \to (M, L)$ with $u(-1) \neq u(1)$ there exists a J-holomorphic disk $u' : (D, \partial D) \to (M, L)$ with the following properties:*

(1) $u'(-1) = u(-1)$, $u'(1) = u(1)$.
(2) $u'(D) = u(D)$ and $u'(\partial D) = u(\partial D)$.
(3) u' is simple.
(4) $\omega([u']) < \omega([u])$. In particular, if L is monotone we also have $\mu([u']) < \mu([u])$.

Remark 3.2.4. None of the Lemmas 3.2.2 and 3.2.3 require L to be monotone.

3.3. **Proof of Proposition 3.1.3.** We separate the proof of Proposition 3.1.3 into two cases: $n = \dim L \geq 3$ and $n = \dim L \leq 2$. We start with $n \geq 3$.

Proof of Proposition 3.1.3 for $n \geq 3$. We start with statement (1) of the Proposition. Let $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ be the intersection of the sets given by Lemma 3.2.3 and by Statement 3.1.2. The proof is carried out by induction over the integer $\mu(\mathbf{A})/N_L$.

Suppose $\mu(\mathbf{A}) = N_L$. Let $\mathbf{u} = (u_1, \ldots, u_l) \in \mathcal{P}(x, y, \mathbf{A}; J, f, \rho)$. As L is monotone we have $l = 1$ (i.e. $\mathbf{u}$ consists of just one disk u_1). Since $\mu([u_1]) = N_L$ it follows from Theorem 3.2.1 that u_1 is simple, hence $\mathbf{u} \in \mathcal{P}^{*,\text{d}}(x, y, \mathbf{A}; J, f, \rho)$.

Suppose that statement (1) of our Proposition holds for every $\mathbf{A}$ with $\mu(\mathbf{A}) \leq kN_L$. Let $\mathbf{A} = (A_1, \ldots, A_l)$ be a sequence of non-zero homology classes with $\mu(\mathbf{A}) = (k+1)N_L$, and such that $\mu(\mathbf{A}) + \text{ind}_f(x) - \text{ind}_f(y) - 1 \leq 1$. Let $\mathbf{u} \in \mathcal{P}(x, y, \mathbf{A}; J, f, \rho)$ and suppose by contradiction that $\mathbf{u} \notin \mathcal{P}^{*,\text{d}}(x, y, \mathbf{A}; J, f, \rho)$.

First note that for every $1 \leq i \leq l$ we have $u_i(-1) \neq u_i(1)$. Indeed if $u_j(-1) = u_j(1)$ for some j then let $\mathbf{u}'$ be the sequence of disks obtained from $\mathbf{u}$ by omitting u_j. Let $\mathbf{A}'$ be obtained from $\mathbf{A}$ by omitting A_j. As $u_j(-1) = u_j(1)$ we have $\mathbf{u}' \in \mathcal{P}(x, y, \mathbf{A}'; J, f, \rho)$. But $\mu(\mathbf{A}') \leq \mu(\mathbf{A}) - N_L$ hence by the induction hypothesis we have $\mathbf{u}' \in \mathcal{P}^{*,\text{d}}(x, y, \mathbf{A}'; J, f, \rho)$. This leads to contradiction since

$$\dim \mathcal{P}^{*,\text{d}}(x, y, \mathbf{A}'; J, f, \rho) = \mu(\mathbf{A}) - \mu(A_j) + \text{ind}_f(x) - \text{ind}_f(y) - 1 \leq 1 - N_L \leq -1.$$

We assume from now on that $u_i(-1) \neq u_i(1)$ for every i.

Case 1. There exists $1 \leq i_0 \leq l$ such that u_{i_0} is not simple.

Apply Lemma 3.2.3 with $u = u_{i_0}$ to obtain a (simple) disk u'_{i_0} with $u'_{i_0}(-1) = u_{i_0}(-1)$, $u'_{i_0}(1) = u_{i_0}(1)$ and such that $\mu([u'_{i_0}]) < \mu([u_{i_0}])$. Let $\mathbf{u}'$ be the sequence of disks obtained from $\mathbf{u}$ by replacing u_{i_0} by u'_{i_0}. Let $\mathbf{A}'$ be obtained from $\mathbf{A}$ by replacing A_{i_0} by $[u'_{i_0}]$. Clearly $\mathbf{u}' \in \mathcal{P}(x, y, \mathbf{A}'; J, f, \rho)$. By the induction hypothesis we have $\mathbf{u}' \in \mathcal{P}^{*,\mathrm{d}}(x, y, \mathbf{A}'; J, f, \rho)$. But this leads to contradiction since

(8)
$$\dim \mathcal{P}^{*,\mathrm{d}}(x, y, \mathbf{A}'; J, f, \rho) = \mu(\mathbf{A}') + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 \leq \mu(\mathbf{A}) - N_L + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 \leq -1.$$

Case 2. The disks $(u_1, \ldots, u_l)$ are simple but not absolutely distinct.

In this case there exists i_0 such that $u_{i_0}(D) \subset \cup_{i \neq i_0} u_i(D)$. It follows that there exists j_0 such that $u_{i_0}(D) \cap u_{j_0}(D)$ is an infinite set. By Lemma 3.2.2 we have:

- either $u_{i_0}(D) \subset u_{j_0}(D)$ and $u_{i_0}(\partial D) \subset u_{j_0}(\partial D)$; or
- $u_{j_0}(D) \subset u_{i_0}(D)$ and $u_{j_0}(\partial D) \subset u_{i_0}(\partial D)$.

Without loss of generality assume that the first possibility occurs.

Subcase i. $i_0 < j_0$**.**

Denote by $\mathbf{u}'$ the sequence of disks obtained from $\mathbf{u}$ by omitting all the disks $u_{i_0}, \ldots, u_{j_0 - 1}$. Denote by $\mathbf{A}'$ the corresponding vector of homology classes. There exists a point $p \in \partial D$ such that $u_{j_0}(p) = u_{i_0}(-1)$.

In case $p \neq 1$ we can replace u_{j_0} by $u_{j_0} \circ \sigma$, where $\sigma \in \mathrm{Aut}(D)$ is such that $\sigma(1) = 1$ and $\sigma(-1) = p$. Note that now $\mathbf{u}' \in \mathcal{P}(x, y, \mathbf{A}'; J, f, \rho)$.

In case $p = 1$ omit from $\mathbf{u}'$ also the disk u_{j_0}. If the resulting sequence of disks $\mathbf{u}'$ is empty we obtain a trajectory (of $-\mathrm{grad}_\rho f$) connecting x to y. But this is impossible since $\mathrm{ind}_f(x) - \mathrm{ind}_f(y) \leq 1 - \mu(\mathbf{A}) \leq -1$ and (f, ρ) is Morse-Smale. Thus we may assume that $\mathbf{u}'$ is not empty and we have $\mathbf{u}' \in \mathcal{P}(x, y, \mathbf{A}'; J, f, \rho)$.

Summing up, in both cases, $p = 1$ and $p \neq 1$, we have $\mathbf{u}' \in \mathcal{P}(x, y, \mathbf{A}'; J, f, \rho)$ and $\mu(A') < \mu(A)$. The induction hypothesis implies that $\mathbf{u}' \in \mathcal{P}^{*,\mathrm{d}}(x, y, \mathbf{A}'; J, f, \rho)$. We now obtain contradiction in the same way as in inequality (8) above.

Subcase ii. $i_0 > j_0$**.**

We argue similarly to Subcase i only that now we omit from $\mathbf{u}$ the disks $u_{j_0+1}, \ldots, u_{i_0}$.

This completes the proof of statement (1) of Proposition 3.1.3 in the case $n \geq 3$.

The proof of statement (2) of Proposition 3.1.3 is based on similar arguments to the above. Note however that we shall need to reduce further the space $\mathcal{J}_{\mathrm{reg}}$ (e.g by intersecting it with the subset coming from statement 3.1.5). □

Proof of Proposition 3.1.3 for $n \leq 2$. Again, we prove only statement (1).

Denote by $\mathcal{J}'$ be the set of all $J \in \mathcal{J}(M, \omega)$ for which the following holds: for every class $A \in H_2(M, L; \mathbb{Z})$ with $\mu(A) = 2$ and every $x, y \in \mathrm{Crit}(f)$ the evaluation maps

$$
\begin{aligned}
&ev'_A : \Big(\mathcal{M}^*(A, J) \times \mathrm{Int}\, D\Big)/G_1 \longrightarrow M \times L, && ev'_A(u, p) = (u(p), u(1)),\\
&ev''_A : \Big(\mathcal{M}^*(A, J) \times \mathrm{Int}\, D\Big)/G_{-1} \longrightarrow M \times L, && ev''_A(u, p) = (u(-1), u(p)),
\end{aligned} \tag{9}
$$

are transverse to $W^u_x \times W^s_y$. Standard arguments [44] show that $\mathcal{J}' \subset \mathcal{J}(M, \omega)$ is of second category. We define the set $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}(M, \omega)$ to be the intersection of $\mathcal{J}'$ with the sets given by Statements 3.1.2, 3.1.5.

Let $\mathbf{A} = (A_1, \ldots, A_l)$ be a sequence of non-zero classes, and $x, y \in \mathrm{Crit}(f)$ with $\mu(\mathbf{A}) + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 \leq 1$. Since $n \leq 2$ we have $\mu(\mathbf{A}) \leq 4$.

Suppose first that $\mu(\mathbf{A}) \leq 3$. Since $N_L \geq 2$, for every $(u_1, \ldots, u_l) \in \mathcal{P}(x, y, \mathbf{A}; J, f, \rho)$ we must have $l = 1$. By Theorem 3.2.1 u_1 is simple. This completes the proof in the case $\mu(\mathbf{A}) \leq 3$.

Suppose now that $\mu(\mathbf{A}) = 4$. Note that in this case we must have $n = 2$. Let $\mathbf{u} \in \mathcal{P}(x, y, \mathbf{A}; J, f, \rho)$ and assume by contradiction that $\mathbf{u} \notin \mathcal{P}^{*,\mathrm{d}}(x, y, \mathbf{A}; J, f, \rho)$. By monotonicity of L we either have $l = 2$, $\mu(A_1) = \mu(A_2) = 2$ or $l = 1$, $\mu(A_1) = 4$. Also, by a similar argument to the ones at the beginning of the proof for the case $n \geq 3$ we may assume that $u_i(-1) \neq u_i(1)$ for every i.

The case $l = 2$, $\mu(A_1) = \mu(A_2) = 2$.

Since $\mu(u_1) = \mu(u_2) = 2$, both u_1 and u_2 are simple. Thus u_1, u_2 are not absolutely distinct. Without loss of generality assume that $u_1(D) \subset u_2(D)$.

Suppose first that $u_1(-1) \in u_2(\mathrm{Int}\, D)$. Let $p \in \mathrm{Int}\, D$ such that $u_2(p) = u_1(-1)$. Then $(u_2, p) \in (ev'_{A_2})^{-1}(W^u_x \times W^s_y)$, where ev'_{A_2} is the evaluation map defined in (9). Since ev'_{A_2} is transverse to $W^u_x \times W^s_y$ a simple computation shows that

$$\dim (ev'_{A_2})^{-1}(W^u_x \times W^s_y) = \mu(A_2) - n + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) = \mathrm{ind}_f(x) - \mathrm{ind}_f(y) \leq 2 - \mu(\mathbf{A}) = -2,$$

a contradiction.

Suppose now that $u_1(-1) \in u_2(\partial D)$. If $u_1(-1) \neq u_2(1)$ then after a suitable reparametrization of u_2 we may assume that $u_2 \in \mathcal{P}^{*,\mathrm{d}}(x, y, A_2; J, f, \rho)$ which is impossible since

$$\dim \mathcal{P}^{*,\mathrm{d}}(x, y, A_2; J, f, \rho) = \mu(A_2) + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 = \mu(\mathbf{A}) + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 - \mu(A_1) \leq -1.$$

The remaining case to consider is $u_1(-1) = u_2(1)$. In this case we can omit both u_1 and u_2 and obtain a trajectory of $-\mathrm{grad}_\rho f$ going form x to y. But this is impossible since $\mathrm{ind}_f(x) < \mathrm{ind}_f(y)$ and (f, ρ) is Morse-Smale.

The case $l = 1$, $\mu(A_1) = 4$.

In this case u_1 is not simple. Let $\mathcal{G} = \mathcal{G}(u_1)$ be the non-injectivity graph of u_1. Since $\mu([u_1]) = 4$, $D \setminus \mathcal{G}$ may have at most two connected components.

Subcase i. $D \setminus \mathcal{G}$ ***is connected.***

By Theorem 3.2.1, u_1 factors through a simple J-holomorphic disk $v : (D, \partial D) \to (M, L)$ via a holomorphic map $\pi : D \to D$ of degree ≥ 2. (In fact the degree is exactly 2 here). It follows that $\mu([v]) = 2$. Since $u_1(-1) \neq u_2(1)$ there exists two *distinct* points $p', p'' \in \partial D$ such that $v(p') = u_1(-1)$, $v(p'') = u_1(1)$. After a suitable reparametrization of v we may assume that $p' = -1$, $p'' = 1$ and we have $v \in \mathcal{P}^{*,\mathrm{d}}(x, y, [v]; J, f, \rho)$. But this leads to contradiction since

(10)
$$\dim \mathcal{P}^{*,\mathrm{d}}(x, y, [v]; J, f, \rho) = \mu([v]) + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 \leq \mu(A_1) - 2 + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1 \leq -1.$$

It remains to deal with the case that $D \setminus \mathcal{G}$ has two connected components $\mathfrak{D}_1, \mathfrak{D}_2$. Let $\pi_i = \pi_{\overline{\mathfrak{D}}_i}$, $v_{\mathfrak{D}_i}$, $m_{\mathfrak{D}_j}$, $i = 1, 2$ be the maps and multiplicities given by Theorem 3.2.1. (In fact, since $\mu(A_1) = 4$ we must have $m_1 = m_2 = 1$ and $\mu([v_1]) = \mu([v_2]) = 2$.)

Subcase ii. $-1, 1 \in \overline{\mathfrak{D}}_1$ ***(see the left part of figure 2).***

There exists two *distinct* points $p', p'' \in \partial D$ such that $v_1(p') = u_1(-1)$, $v_1(p'') = u_1(1)$. After a suitable reparametrization of v we may assume that $p' = -1$, $p'' = 1$ and we have $v_1 \in \mathcal{P}^{*,\mathrm{d}}(x, y, [v_1]; J, f, \rho)$. We now obtain contradiction by a dimension count similar to (10).

Subcase iii. $-1 \in \overline{\mathfrak{D}}_1$***,*** $1 \in \overline{\mathfrak{D}}_2$ ***(see the right part of figure 2).***

Put $B_1 = [v_1]$, $B_2 = [v_2]$. In case $v_1(D) \subset v_2(D)$ or $v_2(D) \subset v_1(D)$ we can argue in a similar way to *"The case $l = 2$, $\mu(A_1) = \mu(A_2) = 2$"* above and arrive to contradiction.

It remains to deal with the case that v_1, v_2 are absolutely distinct. Put $B_1 = [v_1]$, $B_2 = [v_2]$. After suitable reparametrizations of v_1, v_2 we may assume that $v_1(-1) = u_1(-1)$ and $v_2(1) = u_1(1)$. Since $\overline{\mathfrak{D}}_1 \cap \overline{\mathfrak{D}}_2$ must contain at a 1-dimensional component there exists two arcs $\gamma_1, \gamma_2 \subset \partial D$ such that for every $p_1 \in \gamma_1$, $p_2 \in \gamma_2$ we have $v_1(p_1) = v_2(p_2)$. It follows that $\{(v_1, p_1, p_2, v_2)\}_{p_1 \in \gamma_1, p_2 \in \gamma_2}$ lies in $\dim \mathcal{P}^{*,\mathrm{d}}(x, y, B_1, B_2; J, f, \rho)$ hence the latter space is at least 1-dimensional. But this is impossible since

$$\dim \mathcal{P}^{*,\mathrm{d}}(x, y, B_1, B_2; J, f, \rho) = \mu(B_1) + \mu(B_2) + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 2 \leq 0.$$

This completes the proof of Statement (1) of Proposition 3.1.3 for the case $n \leq 2$.

□

3.4. **Proof of Lemmas 3.2.3, 3.2.2.**

Proof of Lemma 3.2.2. The Lemma is an immediate consequence of the following.

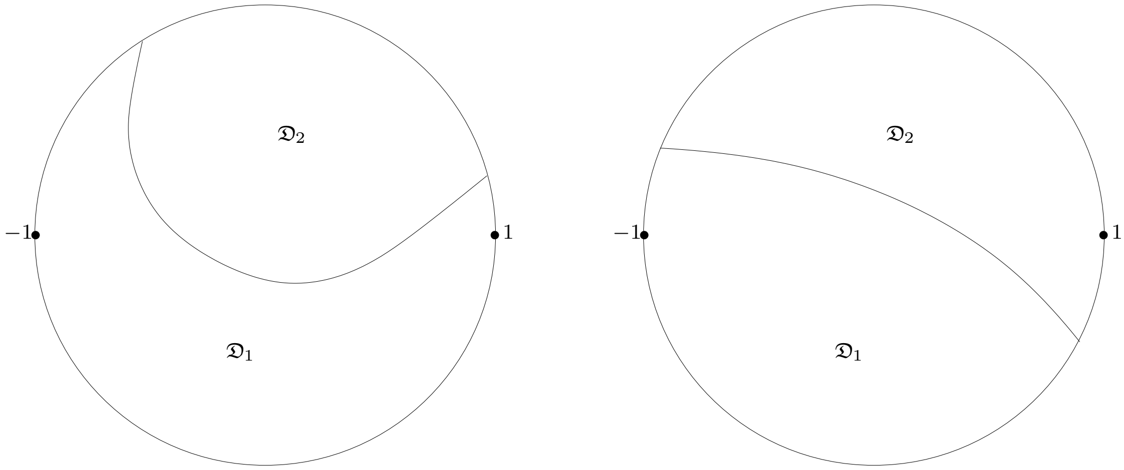

FIGURE 2. Subcases ii and iii.

Proposition 3.4.1. *Suppose $n = \dim L \geq 3$. Then there exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$ the following holds:*

(1) For every simple J-holomorphic disk $u : (D, \partial D) \to (M, L)$ the set $u^{-1}(L) \cap \operatorname{Int} D$ is finite.

(2) For every two J-holomorphic disks $u, v : (D, \partial D) \to (M, L)$ which are simple and absolutely distinct the set $u(D) \cap v(D)$ is finite.

Indeed, suppose that u, v satisfy the assumptions of Lemma 3.2.2, i.e. u, v are simple and $u(D) \cap v(D)$ is an infinite set. Statement (2) of Proposition 3.4.1 implies that u, v are not absolutely distinct, namely $u(D) \subset v(D)$ or $v(D) \subset u(D)$. Without loss of generality assume that $v(D) \subset u(D)$. It remains to show that $v(\partial D) \subset u(\partial D)$. To prove this, note that by statement (1) of Proposition 3.4.1 only finite number of points in ∂D can be mapped by v to $u(\operatorname{Int} D)$ for otherwise $u^{-1}(L) \cap \operatorname{Int} D$ would be an infinite set. Thus $v(\partial D \setminus \text{finite set}) \subset u(\partial D)$. Since v is continuous it easily follows that $v(\partial D) \subset u(\partial D)$. $\square$

Proof of Proposition 3.4.1. Given two non-zero classes $F_1, F_2 \in H_2(M, L; \mathbb{Z})$, $J \in \mathcal{J}(M, \omega)$ and $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ denote by $\mathcal{M}^{*,\text{d}}(F_1, F_2; J) \subset \mathcal{M}(F_1, J) \times \mathcal{M}(F_2, J)$ the subspace consisting of all pairs (u, v) which are *simple and absolutely distinct.*

Define $\mathcal{J}_{\text{reg}}$ to be the subset of all $J \in \mathcal{J}(M, \omega)$ for which the following holds:

- For every non-zero homology class $F \in H_2(M, L; \mathbb{Z})$:
  - The space $\mathcal{M}^*(F, J)$ is (either empty or) a smooth manifold of dimension $\mu(F) + n$.

  - For every $\alpha \geq 1$ the evaluation map

$$ev_\alpha : \mathcal{M}^*(F, J) \times (\mathrm{Int}\, D)^{\times \alpha} \longrightarrow M^{\times \alpha},$$
$$ev_\alpha(u, p_1, \ldots, p_\alpha) = \big(u(p_1), \ldots, u(p_\alpha)\big)$$

    is transverse to $L^{\times \alpha}$.

- For every pair of non-zero homology classes $F_1, F_2 \in H_2(M, L; \mathbb{Z})$:
  - The space $\mathcal{M}^{*,\mathrm{d}}(F_1, F_2; J)$ is either empty or a smooth manifold of dimension $\leq 2n + \mu(F_1) + \mu(F_2)$.
  - For every $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ the evaluation map

$$ev_{\alpha,\beta} : \mathcal{M}^{*,\mathrm{d}}(F_1, F_2, \alpha, \beta; J) \times (\mathrm{Int}\, D)^{\times 2\alpha} \times (\partial D)^{\times 2\beta} \longrightarrow M^{\times 2\alpha} \times L^{\times 2\beta},$$

$$\begin{aligned} &ev_{\alpha,\beta}(u, v, p_1, q_1, \ldots, p_\alpha, q_\alpha, p'_1, q'_1, \ldots, p'_\beta, q'_\beta) \\ &= \big(u(p_1), v(q_1), \ldots, u(p_\alpha), v(q_\alpha), u(p'_1), v(q'_1), \ldots, u(p'_\beta), v(q'_\beta)\big) \end{aligned}$$

    is transverse to $\mathrm{diag}(M)^{\times \alpha} \times \mathrm{diag}(L)^{\times \beta}$.

Standard arguments [44] show that the above subset $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}(M, \omega)$ is indeed of second category.

We now prove statement (1) of Proposition 3.4.1. Let $J \in \mathcal{J}_{\mathrm{reg}}$ and let $u : (D, \partial D) \to (M, L)$ be a simple J-holomorphic curve. Put $F = [u]$. By the transversality of the map ev_α we have $\dim ev_\alpha^{-1}(L^{\times \alpha}) = \mu(F) + n - \alpha(n-2)$. As $n \geq 3$ it follows that for $\alpha \gg 1$, $\dim ev_\alpha^{-1}(L^{\times \alpha}) < 0$ hence $\dim ev_\alpha^{-1}(L^{\times \alpha}) = \emptyset$. This proves statement (1).

We turn to the proof of statement (2) of Proposition 3.4.1. Let $J \in \mathcal{J}_{\mathrm{reg}}$ and let $u, v : (D, \partial D) \to (M, L)$ be two simple J-holomorphic disks which are absolutely distinct. Put $F_1 = [u]$, $F_2 = [v]$. In view of statement (1) of our proposition it is enough to prove that each of the following sets

$$\big\{(z_1, z_2) \in \mathrm{Int}\, D \times \mathrm{Int}\, D \,|\, u(z_1) = v(z_2)\big\}, \quad \big\{(z_1, z_2) \in \partial D \times \partial D \,|\, u(z_1) = v(z_2)\big\}$$

is finite. By the transversality of the map $ev_{\alpha,\beta}$ we have

$$\dim ev_{\alpha,\beta}^{-1}\big(\mathrm{diag}(M)^{\times \alpha} \times \mathrm{diag}(L)^{\times \beta}\big) = \mu(F_1) + \mu(F_2) - \beta(n-2) - 2\alpha(n-2) + 2n.$$

As $n \geq 3$ it following that if $\alpha \gg 1$ or $\beta \gg 1$ then

$$\dim ev_{\alpha,\beta}^{-1}\big(\mathrm{diag}(M)^{\times \alpha} \times \mathrm{diag}(L)^{\times \beta}\big) = \emptyset.$$

This proves statement (2).

The proof of Proposition 3.4.1 (hence of Lemma 3.2.2 too) is complete. □

Proof of Lemma 3.2.3. Take $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M,\omega)$ to be the subset defined by Proposition 3.4.1 and Lemma 3.2.2. Let $u : (D, \partial D) \to (M, L)$ be a non-simple J-holomorphic disk.

Put $\mathcal{G} = \mathcal{G}(u)$. Let $\mathfrak{D}_1, \ldots, \mathfrak{D}_r \subset D \setminus \mathcal{G}$ be the connected components of the complement of $\mathcal{G}$. Let $\pi_{\overline{\mathfrak{D}}_j} : \overline{\mathfrak{D}}_j \to D$, $v_{\mathfrak{D}_j} : (D, \partial D) \to (M, L)$, $m_{\mathfrak{D}_j} \in \mathbb{N}$, $1 \leq j \leq r$, be the maps and multiplicities given by Theorem 3.2.1. For simplicity we shall denote them by π_j, v_j, m_j, respectively.

Case 1: $D \setminus \mathcal{G}$ has only one connected component (i.e. $r = 1$).

Since u is not simple, Theorem 3.2.1 implies that $m_1 \geq 2$. Put $u' = v_1$. Clearly $\omega([u']) < \omega([u])$. We also have $u'(\pi_1(-1)) = u(-1)$ and $u'(\pi_1(1)) = u(1)$. Finally, note that $\pi_1(-1) \neq \pi_1(1)$ (since $u(-1) \neq u(1)$) hence after a reparametrization u' by an element of $\mathrm{Aut}(D)$ we may assume that $u'(-1) = u(-1)$, $u'(1) = u(1)$. The other properties claimed by the Lemma are obvious.

Case 2: $D \setminus \mathcal{G}$ has more than one connected component (i.e. $r \geq 2$).

Define an abstract graph Γ as follows (see figure 3). For each domain $\mathfrak{D}_i$ we assign a vertex $i \in \{1, \ldots, r\}$. We assign an edge between vertex i' and vertex i'' if $\overline{\mathfrak{D}}_{i'} \cap \overline{\mathfrak{D}}_{i''}$ contains a 1-dimensional component (in other words if the two domains have a 1-dimensional common border). Note that Γ is connected.

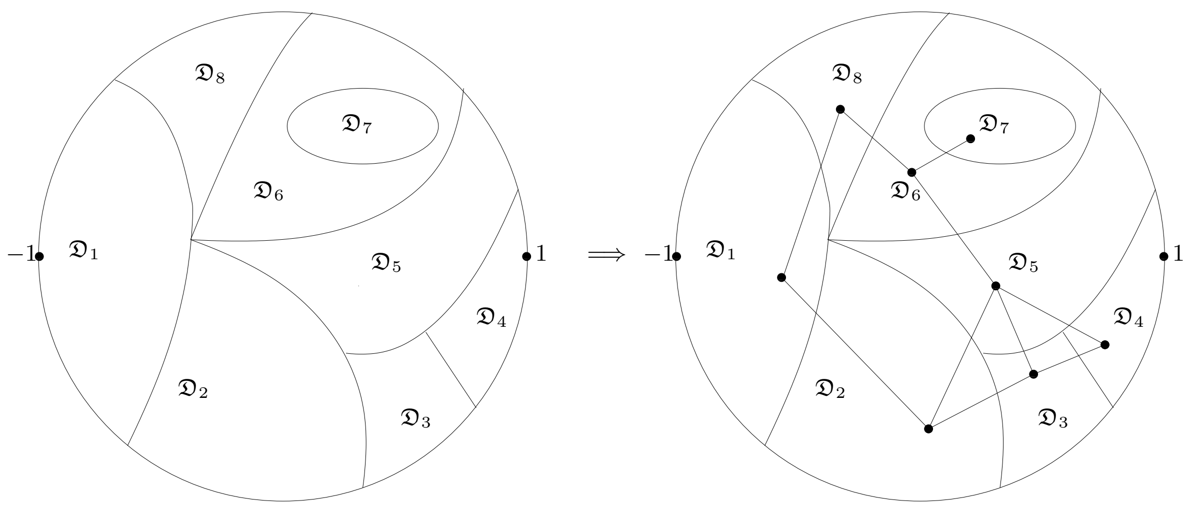

FIGURE 3. The graph Γ.

Choose a path in Γ that passes through each vertex of Γ at least once. Denote the vertices of this path (in the order they appear in the path) by $t_1, \ldots, t_\nu$, $\nu \geq 2$. We shall now construct a sequence of indices $1 \leq k_1 \leq \ldots \leq k_\nu \leq \nu$ such that for every $1 \leq q \leq \nu$:

(I) $v_{t_j}(D) \subset v_{t_{k_q}}(D)$ for every $1 \leq j \leq q$.

(II) $v_{t_j}(\partial D) \subset v_{t_{k_q}}(\partial D)$ for every $1 \leq j \leq q$.

We construct the sequence $1 \leq k_1 \leq \ldots \leq k_\nu \leq \nu$ by induction as follows. Put $k_1 = 1$. By construction $v_{t_1}(D) \cap v_{t_2}(D)$ is an infinite set, hence Lemma 3.2.2 implies that:

- either $v_{t_2}(D) \subset v_{t_1}(D)$ and $v_{t_2}(\partial D) \subset v_{t_1}(\partial D)$; or
- $v_{t_1}(D) \subset v_{t_2}(D)$ and $v_{t_1}(\partial D) \subset v_{t_2}(\partial D)$.

In the first case define $k_2 = k_1 = 1$ and in the second case $k_2 = 2$. Suppose that we have already constructed $k_1 \leq \ldots \leq k_q$ with properties I, II. We define k_{q+1} as follows. Since $v_{t_q}(D) \cap v_{t_{q+1}}(D)$ is infinite then $v_{t_{k_q}}(D) \cap v_{t_{q+1}}(D)$ is also an infinite set. By Lemma 3.2.2 we have:

- either $v_{t_{q+1}}(D) \subset v_{t_{k_q}}(D)$ and $v_{t_{q+1}}(\partial D) \subset v_{t_{k_q}}(\partial D)$; or
- $v_{t_{k_q}}(D) \subset v_{t_{q+1}}(D)$ and $v_{t_{k_q}}(\partial D) \subset v_{t_{q+1}}(\partial D)$.

In the first case put $k_{q+1} = k_q$ and in the second case $k_{q+1} = q+1$. Clearly I, II hold now with q replaced by $q+1$. By induction we get the desired sequence $1 \leq k_1 \leq \ldots \leq k_\nu \leq \nu$.

Put $u' = v_{t_{k_\nu}}$. Properties (2),(3) claimed by the lemma are obvious. Property (4) follows from the fact that $r \geq 2$. Finally, since $u(-1) \neq u(1)$ we must have two *distinct* points $z_1, z_2 \in \partial D$ with $u'(z_1) = u(-1)$, $u'(z_2) = u(1)$. Thus after a suitable reparametrization of u' we may assume that $z_1 = -1$ and $z_2 = 1$. This proves property (1) claimed by the lemma. The proof of Lemma 3.2.3 is complete. □

4. Gluing

In essence, the gluing of J-holomorphic disks appears already in the literature in the work of Fukaya-Oh-Ohta-Ono [34] (see also [2]). However for the purposes of this paper we need a small variation of the gluing theorem of [34], and moreover we also need the surjectivity of the gluing map which is not explicitly discussed in [34]. Therefore, for the sake of completeness we felt it useful to include a detailed argument for gluing in which we closely follow the original proof of Fukaya-Oh-Ohta-Ono [34] as well as a proof for the surjectivity of the gluing map. We also discuss here the gluing for the pearls introduced in §3 and for some other of the elements of the moduli spaces which will be used in §5. Other variants of these gluing statements will be used sometimes in the paper - in particular, we focus here on the case of a fixed almost complex structure but there is sometimes a need to allow this structure to vary inside a family. All of them are obtained by rather direct modifications of the gluing arguments presented here.

4.1. **Main statements.** Let (M^{2n}, ω) be a tame symplectic manifold endowed with an ω-compatible almost complex structure J. Let $L \subset M$ be a closed Lagrangian submanifold.

Let $u_1, u_0 : (D, \partial D) \to (M, L)$ be two J-holomorphic disks. Put $A_i = [u_i] \in H_2(M, L)$. Denote by $\mathcal{M}(A, J)$ the space of J-holomorphic disks with boundary on L, in the class A.

Let W be a manifold and $\mathbf{h} : W \to L \times M \times M \times L$ a smooth map. We shall denote the components of $\mathbf{h}$ by h_-, h_1, h_0, h_+ so that $\mathbf{h}(q) = (h_-(q), h_1(q), h_0(q), h_+(q)) \in L \times M \times M \times L$ for every $q \in W$. Fix two points lying on the *real* part of the disk $z_1, z_0 \in (\operatorname{Int} D) \cap \mathbb{R}$ and a point $q_* \in W$. In what follows we shall put the following assumption on u_1, u_0 and J:

Assumption 4.1.1. (1) $u_1(1) = u_0(-1)$.
(2) $\mathbf{h}(q_*) = \big(u_1(-1), u_1(z_1), u_0(z_0), u_0(1)\big)$.
(3) J is regular for both u_1 and u_0 in the sense that the linearizations D_{u_1}, D_{u_0} of the $\overline{\partial}$ operator at u_1, u_0 are surjective.
(4) Let $ev : \mathcal{M}(A_1, J) \times \mathcal{M}(A_0, J) \to L \times M \times L \times L \times M \times L$ be the evaluation map

$$ev(v_1, v_0) = \big(v_1(-1), v_1(z_1), v_1(1), v_0(-1), v_0(z_0), v_0(1)\big).$$

Define a map $\mathbf{h}_{\Delta_L} : W \times L \to L \times M \times L \times L \times M \times L$ by

$$\mathbf{h}_{\Delta_L}(q, p) = \big(h_-(q), h_1(q), p, p, h_0(q), h_+(q)\big).$$

Put $p_* = u_1(1) = u_0(-1)$. Then we assume that ev and $\mathbf{h}_{\Delta_L}$ are mutually transverse at the points $(u_1, u_0) \in \mathcal{M}(A_1, J) \times \mathcal{M}(A_0, J)$ and $(q_*, p_*) \in W \times L$.

Put $A = A_1 + A_0$. Consider the space of all $(u, r, q) \in \mathcal{M}(A, J) \times (0, 1) \times W$ such that

$$\big(u(-1), u(-r), u(r), u(1)\big) = \mathbf{h}(q). \tag{11}$$

We denote the space of (u, r, q)'s described in (11) by $\mathcal{M}(A, J; \mathcal{C}(\mathbf{h}))$ (Here $\mathcal{C}(\mathbf{h})$ stands for the configuration described by conditions (11).)

Theorem 4.1.2 (Gluing). *Under Assumption 4.1.1 there exists a path $\{(v_s, a(s), q_s)\}_{0<s} \subset \mathcal{M}(A, J; \mathcal{C}(\mathbf{h}))$ with the following properties:*
(1) $q_s \xrightarrow[s\to\infty]{} q_*$ and $a(s) \xrightarrow[s\to\infty]{} 1$.
(2) v_s converges with the marked points $(-1, -a(s), a(s), 1)$, as $s \to \infty$, to (u_1, u_0) with the marked points $(-1, z_1)$, $(z_0, 1)$ in the Gromov topology.

In particular, the point $\big((u_1, z_1), (u_0, z_0), q_*\big)$ lies in the boundary of the closure $\overline{\mathcal{M}}(A, J; \mathcal{C}(\mathbf{h}))$ of the space $\mathcal{M}(A, J; \mathcal{C}(\mathbf{h}))$ in the sense of (1), (2) above. Furthermore, if $\mu(A) + \dim W - 5n = 0$ then the above path is unique in the following sense. There exists a neighbourhood $\mathcal{U}$ of the point $\big((u_1, z_1), (u_0, z_0), q_*\big)$ in $\overline{\mathcal{M}}(A, J; \mathcal{C}(\mathbf{h}))$ such that $\mathcal{U} \setminus \big\{\big((u_1, z_1), (u_0, z_0), q_*\big)\big\}$ coincides with the path $\{(v_s, a(s), q_s)\}$ for $s \gg 0$. In other words, every path $\{(w_s, a'(s), q'_s)\} \subset \mathcal{M}(A, J; \mathcal{C}(\mathbf{h}))$ with $q'_s \xrightarrow[s\to\infty]{} q_*$ and such that $w_s \xrightarrow[s\to\infty]{}$

(u_1, u_0) *with marked points as in (2) is obtained from* $\{(v_s, a(s), q_s)\}$ *by reparametrization in* s*, for* $s \gg 0$.

The proof of Theorem 4.1.2 will occupy Section 4.3- 4.9 below.

Remarks. (1) The uniqueness statement seems to hold without the assumption $\mu(A) + \dim W - 5n = 0$, however the proof is much more complicated in that case. Anyway, we shall not need this more general statement.

(2) The requirement that the points z_1, z_0 lie on the real axis of D is not crucial. Indeed a similar theorem holds for any choice of $z_1, z_0 \in \operatorname{Int} D$ but the marked points $-r, r$ used to define $\mathcal{M}(A, J; \mathcal{C}(\mathbf{h}))$ must be changed accordingly.

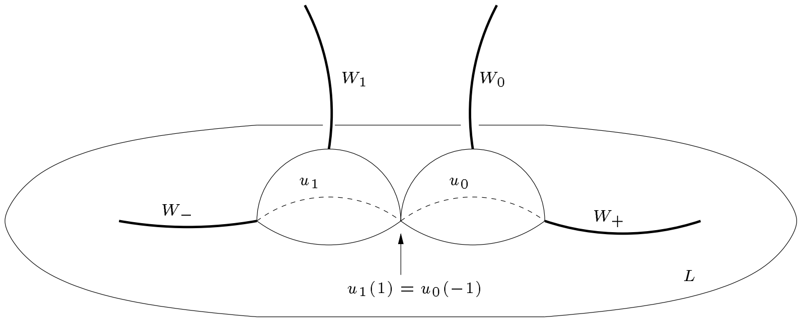

FIGURE 4. Before gluing

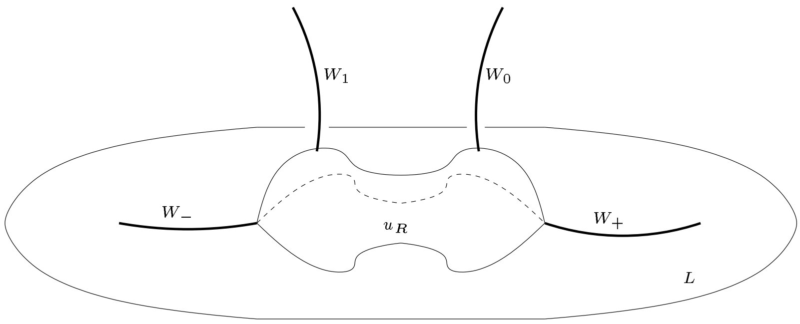

FIGURE 5. After gluing

4.2. **Examples.** Before going to the proof of Theorem 4.1.2 we present several typical applications of Theorem 4.1.2. These will occupy Section 4.2.1- 4.2.4 below.

In most of the situations the manifold W will be a taken to be a product $W = W_- \times W_1 \times W_0 \times W_+$ and $\mathbf{h} = h_- \times h_1 \times h_0 \times h_+$ with $h_\pm : W_\pm \to L$, $h_i : W_i \to M$, $i = 1, 0$.

A realistic illustration of the gluing process is given in figures 4, 5. In these figures $W_\pm$ are taken to be submanifold of L, W_i submanifolds of M and the maps $h_\pm$, h_i are the inclusions.

4.2.1. *Gluing two trajectories of pearls.* Let $f : L \to \mathbb{R}$ be a Morse function and ρ a Riemannian metric on L. Assume that (f, ρ) is Morse-Smale. Given two vectors of non-zero classes in $H_2(M, L; \mathbb{Z})$, $\mathbf{B}' = (B'_1, \ldots, B'_{l'})$, $\mathbf{B}'' = (B''_1, \ldots, B''_{l''})$ we denote $\mathbf{B}'\#\mathbf{B}'' = (B'_1, \ldots, B'_{l'-1}, B'_{l'} + B''_1, B''_2, \ldots, B''_{l''})$. For $x, y \in \mathrm{Crit}(f)$ and a vector $\mathbf{C}$ of non-zero classes we use the notation $\mathcal{P}(x, y, \mathbf{C}; J, f, \rho)$, $\mathcal{P}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho)$ introduced in Section 3.1.

The following is a corollary of Theorem 4.1.2.

Corollary 4.2.1. *Let (f, ρ) be as above. There exists a second category subset $\mathcal{J}_{\mathrm{reg}} \subset \mathcal{J}(M, \omega)$ such that for every $J \in \mathcal{J}_{\mathrm{reg}}$, every pair of vectors of non-zero classes $\mathbf{B}', \mathbf{B}''$ and every $x, y \in \mathrm{Crit}(f)$ with $\mathrm{ind}_f(x) - \mathrm{ind}_f(y) + \mu(\mathbf{B}') + \mu(\mathbf{B}'') - 1 = 1$ the following holds. For every $(\mathbf{v}', \mathbf{v}'') \in \mathcal{P}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho)$ there exists a path $\{\mathbf{u}_s\} \subset \mathcal{P}(x, y, \mathbf{B}'\#\mathbf{B}''; J, f, \rho)$ which converges in the Gromov topology as $s \to \infty$ to $(\mathbf{v}', \mathbf{v}'')$. Moreover, the end of the 1-dimensional manifold $\mathcal{P}(x, y, \mathbf{B}'\#\mathbf{B}''; J, f, \rho)$ parametrized by $\{\mathbf{u}_s\}$ is unique in the sense that every other path $\{\mathbf{w}_s\}$ in $\mathcal{P}(x, y, \mathbf{B}'\#\mathbf{B}''; J, f, \rho)$ that converges to $(\mathbf{v}', \mathbf{v}'')$ as $s \to \infty$, lies in the same end.*

Proof. Fix an element $(\mathbf{v}', \mathbf{v}'') = (v'_1, \ldots, v'_{l'}, v''_1, \ldots, v''_{l''}) \in \mathcal{P}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho)$. Recall from Proposition 3.1.6 that by taking J to be generic we may assume that $\mathcal{P}(x, y, \mathbf{B}', \mathbf{B}''; J, f, \rho)$ is a finite set and that the disks $(v'_1, \ldots, v'_{l'}, v''_1, \ldots, v''_{l''})$ are simple and absolutely distinct.

We now define the manifold W and the map $\mathbf{h}$ used for applying Theorem 4.1.2. The manifold W will be a product $W = W_- \times W_1 \times W_0 \times W_+$ defined as follows. If $l' = 1$ put $W_- = W^u_x$. If $l' \geq 2$ define first $\widehat{W}_-$ to be the space of all $(w'_1, \ldots, w'_{l'-1}, p')$ such that:

- $w'_i \in \mathcal{M}(B'_i, J)$ for every $1 \leq i \leq l' - 1$.
- $w'_1(-1) \in W^u_x$.
- $(w'_i(1), w'_{i+1}(-1)) \in Q_{(f,\rho)}$ for every $1 \leq i \leq l' - 2$. (See formula (5) in Section 3.1 for the definition of $Q_{(f,\rho)}$.)
- $(w'_{l'-1}(1), p') \in Q_{(f,\rho)}$.
- $(w'_1, \ldots, w'_{l'-1})$ are simple and absolutely distinct.

Finally put $W_- = \widehat{W}_- / G_{-1,1}^{\times(l'-1)}$. Define $h_- : W_- \to L$ as follows. If $l' = 1$, let h_- be the inclusion. If $l'' \geq 2$ define $h_-(w'_1, \ldots, w'_{l'-1}, p') = p'$.

We define W_+ and $h_+ : W_+ \to L$ in an analogous way. We write elements of W_+ as $(p'', w''_2, \ldots, w''_{l''})$. Standard transversality results imply that for generic J, the spaces W_-, W_+ are smooth manifold of dimensions $\mathrm{ind}_f(x) + \sum_{i=1}^{l'-1} \mu(B'_i)$ and $n - \mathrm{ind}_f(y) + \sum_{i=2}^{l''} \mu(B''_i)$ respectively.

Put $u_1 = v'_{l'}$, $u_0 = v''_1$, $p_* = u_1(1) = u_0(-1) \in L$ and set:

- $q^*_- = (v'_1, \dots, v'_{l'-1}, p')$, where $p' = u_1(-1)$,
- $q^*_+ = (p'', v''_2, \dots, v''_{l''})$, where $p'' = u_0(1)$.

Note that $q^*_- \in W_-$, $q^*_+ \in W_+$. Set $A_1 = B'_{l'}$ and $A_0 = B''_1$. Consider the following maps:

$$\begin{aligned} &h' : W_- \times W_+ \times L \to L \times L \times L \times L, \quad h'(q_-, q_+, p) = (h_-(q_-), p, p, h_+(q_+)), \\ &ev_{1,0} : \mathcal{M}^*(A_1, J) \times \mathcal{M}^*(A_0, J) \to L \times L \times L \times L, \\ &ev_{1,0}(v_1, v_0) = (v_1(-1), v_1(1), v_0(-1), v_0(1)). \end{aligned}$$

Here $\mathcal{M}^*(A_i, J)$, $i = 1, 0$, stands for the space of *simple* J-holomorphic disks in the class A_i. Again, by taking J to be generic we may assume that $\mathcal{M}^*(A_i, J)$ are smooth manifolds and that the maps h' and $ev_{1,0}$ are mutually transverse at the points $(q^*_-, q^*_+, p_*) \in W_- \times W_+ \times L$ and $(u_1, u_0) \in \mathcal{M}^*(A_1, J) \times \mathcal{M}^*(A_0, J)$. (For this to hold it is crucial to know that the disks $(v'_1, \dots, v'_{l'}, v''_1, \dots, v''_{l''})$ are simple and absolutely distinct.)

We turn to the manifolds W_1, W_0. Let $z_1, z_0 \in (\operatorname{Int} D) \cap \mathbb{R}$ be two points for which $du_{1(z_1)}, du_{0(z_0)} \neq 0$. (Note that since u_1, u_0 are J-holomorphic and not constant, such two points z_1, z_0 do exist.) Fix two $(2n-1)$-dimensional manifolds $W_1, W_0 \subset M$ with $u_1(z_1) \in W_1$, $u_0(z_0) \in W_0$ and such that $u_1, u_0 : D \to M$ are transverse to W_1, W_0 at the points z_1, z_0 respectively. Define $h_i : W_i \to M$ to be the inclusions.

Define $W = W_- \times W_1 \times W_0 \times W_+$ and $\mathbf{h} : W \to L \times M \times M \times L$ to be $\mathbf{h} = (h_-, h_1, h_0, h_+)$. Put $p_* = u_1(1) = u_0(-1) \in M$ and $q_* = (q^*_-, q^*_1, q^*_0, q^*_+) \in W$ where q^*_-, q^*_+ are defined above and $q^*_1 = u_1(z_1)$, $q^*_0 = u_0(z_0)$. A simple computation shows that the maps $\mathbf{h}_{\Delta_L}$ and ev (defined in Assumption 4.1.1) are mutually transverse at the points $(q_*, p_*) \in W \times L$ and $(u_1, u_0) \in \mathcal{M}(A_1, J) \times \mathcal{M}(A_0, J)$. Clearly the rest of the assumptions in 4.1.1 are also satisfied. Note that in the above construction once J is fixed the manifolds W_-, W_+ are determined in a canonical way. The manifolds W_1, W_0 on the other hand are chosen a posteriori and depend on the element $(\mathbf{v}', \mathbf{v}'')$.

We now apply Theorem 4.1.2. We obtain from this Theorem a family of J-holomorphic disks $v_s \in \mathcal{M}(A_1 + A_0, J)$ together with marked points $-a(s), a(s) \in (\operatorname{Int} D) \cap \mathbb{R}$ and points $q_s = (q_{-,s}, q_{1,s}, q_{0,s}, q_{+,s}) \in W_- \times W_1 \times W_0 \times W_+$, such that:

- $q_{-,s} = (u'_{1,s}, \dots, u'_{l'-1,s}, p'_s) \xrightarrow[s\to\infty]{} q^*_- = (v'_1, \dots, v'_{l'-1}, p_*)$.
- $q_{+,s} = (p''_s, u''_{2,s}, \dots, u'_{l'',s}) \xrightarrow[s\to\infty]{} q^*_+ = (p_*, v''_2, \dots, v''_{l''})$.
- $v_s(-1) = p'_s$, $v_s(1) = p''_s$.
- $v_s(-a(s)) = q_{1,s} \xrightarrow[s\to\infty]{} u_1(z_1)$, $v_s(a(s)) = q_{0,s} \xrightarrow[s\to 1\infty]{} u_0(z_0)$.
- v_s converges with the marked points $(-1, -a(s), a(s), 1)$ to (u_1, u_0) with the marked points $(-1, z_1)$, $(z_0, 1)$.

Put $\mathbf{u}_s = (u'_{1,s}, \ldots, u'_{l'-1,s}, v_s, u''_{2,s}, \ldots, u'_{l'',s})$. Clearly $\mathbf{u}_s \in \mathcal{P}(x, y, \mathbf{B}'\#\mathbf{B}''; J, f, \rho)$ and $\mathbf{u}_s$ converges as $s \to \infty$ to $(\mathbf{v}', \mathbf{v}'')$.

We turn to the uniqueness statement. Let $\{\mathbf{w}_s\}$ be a path in $\mathcal{P}(x, y, \mathbf{B}'\#\mathbf{B}''; J, f, \rho)$ such that $\mathbf{w}_s \xrightarrow[s\to\infty]{} (\mathbf{v}', \mathbf{v}'')$. Write $\mathbf{w}_s = (w_{1,s}, \ldots, w_{l'-1,s}, w_{l',s}, \ldots, w_{l'+l''-1,s})$. Then we have $w_{l',s} \xrightarrow[s\to\infty]{} (u_1, u_0)$ in the Gromov topology. By applying a suitable family of holomorphic reparametrization to $w_{l',s}$ we may assume that the maps $w_{l',s}$ uniformly converge in the C^∞-topology, as $s \to \infty$, to u_1 on compact subsets of $D \setminus \{1\}$. Similarly, after (other) reparametrizations, $w_{l',s}$ uniformly converges in the C^∞-topology, as $s \to \infty$, to u_0 on compact subsets of $D \setminus \{-1\}$. Due to the transversality between u_i and W_i at $z_i \in D$, $i = 1, 0$, it follows that there exists points $b_1(s), b_0(s) \in (\operatorname{Int} D) \cap \mathbb{R}$ with $b_1(s) \xrightarrow[s\to\infty]{} -1$, $b_0(s) \xrightarrow[s\to\infty]{} 1$ such that $w_{l',s}(b_i(s)) \in W_i$ and $w_{l',s}(b_i(s)) \xrightarrow[s\to\infty]{} u_i(z_i)$, $i = 1, 0$.

After further reparametrizations we may assume that $b_1(s) = -b_0(s)$. As before we construct elements $q'_s \in W$ using $(w_{1,s}, \ldots, w_{l'-1,s})$, $(w_{l'+1,s}, \ldots, w_{l'+l''-1,s})$ and $w_{l',s}(\pm b_0(s))$ such that $(w_{l',s}, b_0(s), q'_s) \in \mathcal{M}(A, J; \mathcal{C}(\mathbf{h}))$, $q'_s \xrightarrow[s\to\infty]{} q_*$ and such that $w_{l',s}$ converges with the marked points $(-1, -b_0(s), b_1(s), 1)$ as $s \to \infty$ to (u_1, u_0) with the marked points $(-1, z_1)$, $(z_0, 1)$. Noting that

$$\begin{aligned}
&\mu(A_1) + \mu(A_0) + \dim W - 5n = \\
&\mu(B'_{l'}) + \mu(B''_1) + \Big(\operatorname{ind}_f(x) + \sum_{i=1}^{l'-1} \mu(B'_i)\Big) + 2(2n-1) + \Big(n - \operatorname{ind}_f(y) + \sum_{i=2}^{l''} \mu(B''_i)\Big) - 5n = \\
&\operatorname{ind}_f(x) - \operatorname{ind}_f(y) + \mu(\mathbf{B}') + \mu(\mathbf{B}'') - 2 = 0,
\end{aligned}$$

it follows from the uniqueness statement of Theorem 4.1.2 that for $s \gg 0$, $(w_{l',s}, b_0(s), q'_s)$ coincides with $(v_s, a(s), q_s)$ up to reparametrizations in s. It follows that the path $\mathbf{w}_s$ and $\mathbf{u}_s$ are the same up to reparametrization for $s \gg 0$. This completes the proof of Corollary 4.2.1.

Remark. The manifolds W_1, W_0 above were important only for the uniqueness statement. We had to choose them to be $(2n-1)$-dimensional in order to reduce the dimension of the space $\mathcal{M}(A, J; \mathcal{C}(\mathbf{h}))$ to be 1, i.e. to assure that $\mu(A_1) + \mu(A_0) + \dim W - 5n = 0$, which is the assumption we need for the uniqueness statement in Theorem 4.1.2.

□

4.2.2. *Gluing a trajectory of pearls to a trajectory with external constrains I.* Here we show how to apply Theorem 4.1.2 in order to glue a trajectory of pearls to a trajectory from the space $\mathcal{P}_{III_i}(a, x, y; \mathbf{B}', \mathbf{B}'', J)$, $i = 1, 2$, defined in Section 5.3.2 in the context of the quantum module structure.

Let $h : M \to \mathbb{R}$, $f : L \to \mathbb{R}$ be Morse functions and ρ_M, ρ_L Riemannian metrics on M, L. Assume that (f, ρ_L) and (h, ρ_M) satisfy Assumption 5.3.1.

Corollary 4.2.2. *Let (f, ρ_L), (h, ρ_M) be as above. There exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$, every pair of vectors of non-zero classes $\mathbf{B}', \mathbf{B}''$ and every $x, y \in \text{Crit}(f)$, $a \in \text{Crit}(h)$ with $\text{ind}_h(a) + \text{ind}_f(x) - \text{ind}_f(y) + \mu(\mathbf{B}') + \mu(\mathbf{B}'') - 2n = 1$ the following holds. For every $(\mathbf{v}', \mathbf{v}'') \in \mathcal{P}_{III_i}(a, x, y, \mathbf{B}', \mathbf{B}''; J)$ there exists a path $\{\mathbf{u}_s\} \subset \mathcal{P}_I(a, x, y, \mathbf{B}'\#\mathbf{B}''; J)$ which converges in the Gromov topology as $s \to \infty$ to $(\mathbf{v}', \mathbf{v}'')$. Moreover, the end of the 1-dimensional manifold $\mathcal{P}_I(a, x, y, \mathbf{B}'\#\mathbf{B}''; J)$ parametrized by $\{\mathbf{u}_s\}$ is unique in the sense that every other path $\{\mathbf{w}_s\}$ in $\mathcal{P}_I(a, x, y, \mathbf{B}'\#\mathbf{B}''; J)$ that converges to $(\mathbf{v}', \mathbf{v}'')$ as $s \to \infty$, lies in the same end.*

Proof. We outline the proof for the space $\mathcal{P}_{III_1}(a, x, y; \mathbf{B}', \mathbf{B}'', J)$. Suppose that $\mathbf{B}' = (B'_1, \ldots, B'_{l'})$, $\mathbf{B}'' = (B''_1, \ldots, B''_{l''})$. Recall that $\mathcal{P}_{III_1}(a, x, y; \mathbf{B}', \mathbf{B}'', J)$ is disjoint union of the spaces $\mathcal{P}_{III_1}(a, x, y; (\mathbf{B}', k'), \mathbf{B}'', J)$ where k' goes from 1 to l'.

In case $(\mathbf{v}', \mathbf{v}'') \in \mathcal{P}_{III_1}(a, x, y; (\mathbf{B}', k'), \mathbf{B}'', J)$ with $1 \leq k' \leq l' - 1$ the proof is very similar to the proof of Corollary 4.2.1. The only difference is that the manifold W_- is now defined by elements $(w'_1, \ldots, w'_{l'-1}, p')$ with the additional condition that $w'_{k'}(0) \in W^u_a$. The rest of the proof goes in the same way as for Corollary 4.2.1.

In case $k' = l'$, i.e. $(\mathbf{v}', \mathbf{v}'') \in \mathcal{P}_{III_1}(a, x, y; (\mathbf{B}', l'), \mathbf{B}'', J)$ the proof is even easier. We define the manifold W_1 to be W^u_a and take $z_1 = 0 \in \text{Int}\, D$. The manifolds $W_\pm$, W_0 are defined in the same way as in the proof of Corollary 4.2.1. □

4.2.3. *Gluing trajectories with external constrains and Hamiltonian perturbations.* Here we show how to perform gluing on elements from the space $\mathcal{P}_{III}(a, x, y; \mathbf{B}', \mathbf{B}'', J, H)$ introduced in Section 5.3.9, in order to obtain elements from the space $\mathcal{P}_I(a, x, y; \mathbf{B}'\#\mathbf{B}'', J, H)$.

Let $h : M \to \mathbb{R}$, $f : L \to \mathbb{R}$ be Morse functions and ρ_M, ρ_L Riemannian metrics on M, L. Assume that (f, ρ_L) and (h, ρ_M) satisfy Assumption 5.3.1.

Corollary 4.2.3. *Let (f, ρ_L), (h, ρ_M) be as above. There exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$ there exists a second category subset $\mathcal{H}_{\text{reg}}(J) \subset \mathcal{H}$ with the following properties. Let $H \in \mathcal{H}_{\text{reg}}(J)$, $a \in \text{Crit}(h)$, $x, y \in \text{Crit}(f)$, $\mathbf{B}' = (B'_1, \ldots, B'_{l'})$, $\mathbf{B}'' = (B''_1, \ldots, B''_{l''})$ be two vectors of non-zero classes with $\text{ind}_h(a) + \text{ind}_f(x) - \text{ind}_f(y) + \mu(\mathbf{B}') + \mu(\mathbf{B}'') - 2n = 1$. Let $(\mathbf{v}', \mathbf{v}'') \in \mathcal{P}_{III_i}(a, x, y; \mathbf{B}', \mathbf{B}'', J, H)$, $i = 1, 2$. Then there exists a path $\{\mathbf{u}_s\} \subset \mathcal{P}_I(a, x, y; \mathbf{B}'\#\mathbf{B}'', J, H)$ which converges in the Gromov topology, as $s \to \infty$, to $(\mathbf{v}', \mathbf{v}'')$. Moreover, the end of the 1-dimensional manifold $\mathcal{P}_I(a, x, y; \mathbf{B}'\#\mathbf{B}'', J, H)$ parametrized by this path is unique in the sense that every other path with the same property for $s \to \infty$, lies in the same end.*

Remark 4.2.4. A similar statement holds for the spaces $\mathcal{P}_{II_i}(a,x,y;\mathbf{A},J,H)$ from Section 5.3.9, whenever $\mathrm{ind}_h(a)+\mathrm{ind}_f(x)-\mathrm{ind}_f(y)+\mu(\mathbf{A})-2n=1$. The proof is almost the same as the one given below.

Proof of Corollary 4.2.3. We prove the Corollary for the space $\mathcal{P}_{III_1}(a,x,y;\mathbf{B}',\mathbf{B}'',J,H)$. Recall that this space is a disjoint union of $\mathcal{P}_{III_1}(a,x,y;(\mathbf{B}',k'),\mathbf{B}'',J,H)$, where k' goes from 1 to l'. Assume that $(\mathbf{v}',\mathbf{v}'')$ lies in the k''th component of this space for some $1\leq k'\leq l'$.

If $k'<l'$ the proof is essentially the same as for Corollary 4.2.2.

Assume $k'=l'$. Write $\mathbf{v}'=(v'_1,\ldots,v'_{l'})$, $\mathbf{v}''=(v''_1,\ldots,v''_{l''})$. We have to glue the (J,H)-holomorphic disk $v'_{l'}$ to the (genuine) J-holomorphic disk v''_1, preserving the constrains imposed by $\mathcal{P}_I(a,x,y;(\mathbf{B}'\#\mathbf{B}'',k'),J,H)$. (Recall from Section 5.3.9 that the space $\mathcal{P}_I(a,x,y;\mathbf{B}'\#\mathbf{B}'',J,H)$ is a disjoint union of $\mathcal{P}_I(a,x,y;(\mathbf{B}'\#\mathbf{B}'',j),J,H)$ for $1\leq j\leq l'+l''-1$.)

We shall perform the gluing in $\widetilde{M}=D\times M$. Let $\widetilde{J}_H$ be the almost complex structure in $D\times M$ associated to J and H (see Section 5.3.7). Put $\widetilde{L}=\partial D\times L$. Let $\widetilde{u}_1(z)=(z,v'_{l'}(z))$ be the graph of $v'_{l'}$ and let $u_0(z)=(1,v''_1(z))$ be a copy of v''_1 lying in the fibre over $1\in D$. Clearly both u_1,u_0 are $\widetilde{J}_H$-holomorphic disks with boundary on $\widetilde{L}$ and $u_1(1)=u_0(-1)$. Moreover $A_1=[u_1]=B'_{l'}+[D]\in H_2(\widetilde{M},\widetilde{L})$, $A_0=[u_0]=B''_1\in H_2(\widetilde{M},\widetilde{L})$.

Again we take W to be a product $W=W_-\times W_1\times W_0\times W_+$. The manifolds $W_\pm$ and the maps $h_\pm$ are defined in a similar way as in the proof of Corollaries 4.2.1, 4.2.2 (only that now h_- maps W_- to $\{-1\}\times L$ and h_+ maps W_+ to $\{1\}\times L$). As before, we have: $\dim W_-=\mathrm{ind}_f(x)+\sum_{i=1}^{l'-1}\mu(B'_i)$, $\dim W_+=n-\mathrm{ind}_f(y)+\sum_{i=2}^{l''}\mu(B''_i)$.

Next, take W_1 to be $\{0\}\times W^u_a\subset\widetilde{M}$ and $W_0=\widetilde{M}$. We take $h_i:W_i\to\widetilde{M}$ to be the inclusions. Note that

$$\dim W=\mathrm{ind}_h(a)+\mathrm{ind}_f(x)-\mathrm{ind}_f(y)+\sum_{i=1}^{l'-1}\mu(B'_i)+\sum_{i=2}^{l''}\mu(B''_i)+3n+2. \tag{12}$$

As before, by taking J and H to be generic we may assume that the assumptions of Theorem 4.1.2 are satisfied for u_1,u_0. We obtain from this theorem a path $(\widehat{v}_s,a(s),q_s)$, where $\widehat{v}_s:(D,\partial D)\to(\widetilde{M},\widetilde{L})$ is in the class A_1+A_0 and satisfies:

- $(\widehat{v}_s(-1),\widehat{v}_s(-a(s)),\widehat{v}_s(a(s)),\widehat{v}_s(1))=q_s\in W$.
- $\widehat{v}_s$ with the marked points $(-1,-a(s),a(s),1)$ converges, as $s\to\infty$, to (u_1,u_0) with the marked points $(-1,0)$, $(0,1)$ in the Gromov topology.

In order to extract from $\widehat{v}_s$ a (J,H)-holomorphic disk write $\widehat{v}_s(z)=(\varphi_s(z),v_s(z))$, where $\varphi_s:(D,\partial D)\to(D,\partial D)$ and $v_s:(D,\partial D)\to(M,L)$. Note that

$$\varphi_s(-1)=-1,\quad \varphi_s(1)=1,\quad \varphi_s(-a(s))=0.$$

As $\widehat{v}_s$ is $\widetilde{J}_H$-holomorphic, $pr_D : (\widetilde{M}, \widetilde{J}_H) \to (D, i)$ is holomorphic, and $pr_{D_*}[\widehat{v}_s] = [D] \in H_2(D, \partial D)$ it follows that $\varphi_s \in \mathrm{Aut}(D)$. Put $w_s = v_s \circ \varphi_s^{-1}$. Then $\widetilde{w}_s(z) = (z, w_s(z))$ is a $\widetilde{J}_H$-holomorphic section of $D \times M \to D$ hence by Proposition 5.3.10 w_s is (J, H)-holomorphic. Also note that

$$w_s(\pm 1) = v_s(\pm 1), \quad w_s(0) = v_s(-a(s)) \in W_a^u.$$

As in the proof of Corollary 4.2.1 we extract from q_s a path of chains of J-holomorphic disks $(u_{1,s}, \ldots, u_{k'-1,s}, u_{k'+1,s}, \ldots, u_{l'+l''-1,s})$ such that when we insert the disk w_s to the k''th entry we obtain

$$\mathbf{u}_s = (u_{1,s}, \ldots, u_{k'-1,s}, w_s, u_{k'+1,s}, \ldots, u_{l'+l''-1,s}) \in \mathcal{P}_I(a, x, y; (\mathbf{B}' \# \mathbf{B}'', k'), J, H),$$

and $\mathbf{u}_s$ converges in the Gromov topology as $s \to \infty$ to $(\mathbf{v}', \mathbf{v}'')$. This concludes the proof of the existence statement.

The uniqueness statement follows from Proposition 4.1.2 since by (12) we have:

$$\mu(A_1) + \mu(A_0) + \dim W - 5 \dim \widetilde{M} = \mathrm{ind}_h(a) + \mathrm{ind}_f(x) - \mathrm{ind}_f(y) + \mu(\mathbf{B}') + \mu(\mathbf{B}'') - 2n - 1 = 0.$$

□

4.2.4. *Gluing a trajectory of pearls to a trajectory with external constrains II.* Consider the following situation encountered in the proof of Proposition 5.3.18 in Section 5.3. Let $f : L \to \mathbb{R}$, $h', h'' : M \to \mathbb{R}$ be Morse functions and $\rho_L, \rho'_M, \rho''_M$ be Riemannian metrics on L, M. Let $\mathbf{B}' = (B'_1, \ldots, B'_{l'})$, $\mathbf{B}'' = (B''_1, \ldots, B''_{l''})$ be two vectors of non-zero classes and $1 \le k' \le l'$, $1 \le k'' \le l''$. Given $x, y \in \mathrm{Crit}(f)$, $a' \in \mathrm{Crit}(h')$, $a'' \in \mathrm{Crit}(h'')$ define $\widehat{\mathcal{P}}_V(a', a'', x, y, (\mathbf{B}', k'), (\mathbf{B}'', k''); J)$ to be the space of all $(u'_1, \ldots, u'_{l'}, u''_1, \ldots, u''_{l''})$ such that:

- $u'_i \in \mathcal{M}(B'_i, J)$ for every $1 \le i \le l'$, $u''_j \in \mathcal{M}(B''_j, J)$ for every $1 \le j \le l''$.
- $u'_1(-1) \in W_x^u$, $u''_{l''}(1) \in W_y^s$.
- $(u'_i(1), u'_{i+1}(-1)) \in Q_{(f,\rho_L)}$ for every $1 \le i \le l' - 1$, $(u''_j(1), u''_{j+1}(-1)) \in Q_{(f,\rho_L)}$ for every $1 \le j \le l'' - 1$.
- $u'_{l'}(1) = u''_1(-1)$.
- $u'_{k'}(0) \in W_{a'}^u$, $u''_{k''}(0) \in W_{a''}^u$.

Define now

$$\mathcal{P}_V(a', a'', x, y, (\mathbf{B}', k'), (\mathbf{B}'', k''); J) = \widehat{\mathcal{P}}_V(a', a'', x, y, (\mathbf{B}', k'), (\mathbf{B}'', k''); J)/\mathbf{G}' \times \mathbf{G}'',$$

where $\mathbf{G}'$ is $G_{-1,1}^{\times l'}$ with the k''th factor replaced by the trivial group, and $\mathbf{G}''$ is defined in a similar way.

Similarly, for a vector of non-zero classes $\mathbf{C} = (C_1, \ldots, C_l)$ and $1 \le p' < p'' \le l$ consider the space of all $(u_1, \ldots, u_l)$ such that:

(i) $u_i \in \mathcal{M}(C_i, J)$ for every $1 \le i \le l$.

(ii) $u_1(-1) \in W^u_x$, $u_l(1) \in W^s_y$.
(iii) $(u_i(1), u_{i+1}(-1)) \in Q_{(f,\rho_L)}$ for every $1 \leq i \leq l-1$.
(iv) $u_{p'}(0) \in W^u_{a'}$, $u_{p''}(0) \in W^u_{a''}$.

The group $\mathbf{G}^{\times(l-2)}_{-1,1}$ acts on this space by reparametrizations. Denote the quotient space by $\mathcal{P}_{IV}(a', a'', x, y; (\mathbf{C}, p', p''); J)$.

Finally, let $1 \leq p \leq l$. Consider the space of all $(u_1, \ldots, u_l, r)$ such that:

- $(u_1, \ldots, u_l)$ satisfy conditions (i)-(iii) above.
- $r \in (0, 1)$.
- $u_p(-r) \in W^u_{a'}$, $u_p(r) \in W^u_{a''}$.

The group $\mathbf{G}^{\times(l-1)}_{-1,1}$ acts on this space by reparametrizations. Denote the quotient space by $\mathcal{P}_{IV}(a', a'', x, y; (\mathbf{C}, p); J)$.

Corollary 4.2.5. *Let (f, ρ_L), (h', ρ'_M), (h'', ρ''_M) be as above where the triple of metrics $\rho_L, \rho'_M, \rho''_M$ is assumed to be generic. There exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$, every pair of vectors of non-zero classes $\mathbf{B}', \mathbf{B}''$ and every $x, y \in \text{Crit}(f)$, $a' \in \text{Crit}(h')$ $a'' \in \text{Crit}(h'')$ with $\text{ind}_{h'}(a') + \text{ind}_{h''}(a'') + \text{ind}_f(x) - \text{ind}_f(y) + \mu(\mathbf{B}') + \mu(\mathbf{B}'') - 4n = 0$ the following holds. Let $1 \leq k' \leq l'$, $1 \leq k'' \leq l''$ and $(\mathbf{v}', \mathbf{v}'') \in \mathcal{P}_V(a', a'', x, y, (\mathbf{B}', k'), (\mathbf{B}'', k''); J)$.*

(1) If $(k', k'') \neq (l', 1)$ then there exists a path $\{\mathbf{u}_s\} \subset \mathcal{P}_{IV}(a', a'', x, y, (\mathbf{B}' \# \mathbf{B}'', k', l' + k'' - 1); J)$ which converges in the Gromov topology as $s \to \infty$ to $(\mathbf{v}', \mathbf{v}'')$.

(2) If $k' = l'$, $k'' = 1$ then there exists a path $\{(\mathbf{u}_s, a(s))\} \subset \mathcal{P}_{IV}(a', a'', x, y, (\mathbf{B}' \# \mathbf{B}'', k'); J)$ such that:

(a) $\mathbf{u}_s$ converges in the Gromov topology as $s \to \infty$ to $(\mathbf{v}', \mathbf{v}'')$.

(b) $a(s) \xrightarrow[s \to \infty]{} 1$.

(c) The k''th disk, $u_{k',s}$, in $\mathbf{u}_s$, converges with the marked points $(-1, -a(s), a(s), 1)$, as $s \to \infty$ to $(v'_{k'}, v''_1)$ with the marked points $(-1, 0)$, $(0, 1)$, in the Gromov topology.

Moreover, in both cases (1) and (2) above the end of the 1-dimensional manifold $\mathcal{P}_{IV}$ parametrized by the path $\{\mathbf{u}_s\}$ (resp. $\{(\mathbf{u}_s, a(s))\}$) is unique in the sense that every other path satisfying the same properties for $s \to \infty$, lies in the same end.

Proof. The case $(k', k'') \neq (l', 1)$ is proved in a similar way to Corollaries 4.2.1 and 4.2.2.

As for the case $k' = l'$, $k'' = 1$, the manifolds $W_{\pm}$ are defined in a similar way as in the proofs of Corollaries 4.2.1, 4.2.2 and we take $W_1 = W^u_{a'}$, $W_0 = W^u_{a''}$. $\square$

4.3. **Overview of the proof of Theorem 4.1.2.** The proof of Theorem 4.1.2 is built from the following steps. We first transform the domains of u_1, u_0 to the strip $S =$

$\mathbb{R}\times[0,1]$. The main reason for this is convenience, especially when performing translations along the $\mathbb{R}$-axis. In Section 4.4 we introduce the analytic setup for performing the gluing. In particular we shall have to work with weighted Sobolev spaces in order to make the linearization D of the non-linear $\overline{\partial}$ operator Fredholm.

The second step is usually called pregluing. Here we build an approximate solutions u_R of the $\overline{\partial}$ equation depending on a parameter $R \gg 0$. The u_R's are glued from u_1, u_0 using partition of unity and coincide with suitable (larger and larger) translates of u_1, u_0 near the ends of S. The u_R's are approximate solutions in the sense that $\overline{\partial} u_R$ becomes smaller and smaller as $R \to \infty$ in a suitable norm. The pregluing and the needed estimates of the u_R's are done in Section 4.5. In order for certain operators related to u_R to become uniformly bounded we shall have to deform our weighted norms with R. These norms are introduced in Section 4.5. For the reader convenience we included in Section 4.9 a Sobolev-type inequality that will be used frequently in the proof.

The third step is to construct a right inverse to the linearization D_{u_R} of the $\overline{\partial}$ operator at u_R. More precisely we construct a family of operators $\{Q_R\}_{R\gg 0}$ such that $D_{u_R}\circ Q_R = \mathbb{1}$ for every $R \gg 1$ and such that the Q_R's are *uniformly bounded*. This is done in Section 4.7.

The next step is to use an implicit function theorem which will correct the approximate solutions u_R to genuine J-holomorphic solutions v_R. This is carried out in Section 4.8. The operators Q_R's are needed for the implicit function theorem to work.

In Section 4.8 we also prove that the v_R's verify the marked points conditions (11) and that v_R converges as $R \to \infty$ to (u_1, u_0) with marked points in the Gromov topology.

The steps above prove the existence statement of Theorem 4.1.2. The final step is devoted to proving the uniqueness statement of Theorem 4.1.2. This is usually called in the "gluing literature" *surjectivity of the gluing map*. It occupies the rest of Section 4.8.

Throughout the proof of the existence part we essentially follow the work of Fukaya-Oh-Ohta-Ono [34], however we occasionally use slightly different notation and normalizations.

4.4. **Analytic setting.** From now on we shall identify the disk D with the compactified strip $\widehat{S} = S \cup \{-\infty, \infty\}$ where $S = \mathbb{R} \times [0,1] \approx \{z \in \mathbb{C} | \, 0 \leq \mathrm{Im} z \leq 1\}$. This is done via the biholomorphism

$$\lambda : \widehat{S} \to D, \quad \lambda(z) = \frac{e^{\pi z} - i}{e^{\pi z} + i}.$$

Note that $\lambda(\infty) = 1, \lambda(-\infty) = -1$. Denote $S[a,b] = [a,b] \times [0,1] \subset S$. Similarly we have $S[a,\infty)$ etc. The reason for this change of coordinates is that it is more handy for using translations. The price we pay for this is that the Banach space norms the are well suited to the strip S are not conformally invariant. Moreover, our elliptic boundary problem becomes an "asymptotic boundary" problem at the ends of S. In order to control this asymptotic we shall need to endow our Banach spaces with some weighted norms.

Denote by $g_{\omega,J}(\cdot,\cdot) = \omega(\cdot, J\cdot)$ be the Riemannian metric associated to (ω, J). Choose a new metric $g_{\omega,J,L}$ on M for which L is totally geodesic and which coincides with $g_{\omega,J}$ outside a small neighbourhood of L. *Henceforth, we shall use the metric $g_{\omega,J,L}$ as our main Riemannian metric rather than $g_{\omega,J}$.* Thus all pointwise norms, connections, and distances are to be understood with respect to $g_{\omega,J,L}$, unless explicitly stated otherwise.

Fix $\delta > 0$ small enough (see remark below) and $p > 2$.

Definition 4.4.1. Denote by $W^{1,p;\delta}(M, L)$ the space of all pairs $(u, \underline{p})$ where:

(1) $u : (S, \partial S) \to (M, L)$ is of class $W^{1,p}_{\text{loc}}$.
(2) $\underline{p} = (p_{-\infty}, p_\infty), p_{\pm\infty} \in L$.
(3) $\int_{S[0,\infty)} e^{\delta|\tau|}\big(\text{dist}(u(\tau, t), p_\infty)^p + |du_{(\tau,t)}|^p\big)d\tau dt < \infty$,
$\int_{S(-\infty,0]} e^{\delta|\tau|}\big(\text{dist}(u(\tau, t), p_{-\infty})^p + |du_{(\tau,t)}|^p\big)d\tau dt < \infty$.

Remark 4.4.2. (1) As $p > 2 = \dim S$ all $u \in W^{1,p;\delta}(M, L)$ are actually continuous.
(2) Standard arguments show that for every $(u, \underline{p}) \in W^{1,p;\delta}(M, L)$ we have

$$\lim_{\tau\to\pm\infty} u(\tau, t) = p_{\pm\infty} \quad \text{uniformly in } t.$$

Therefore the $\underline{p}$ in $(u, \underline{p})$ is superfluous as $\underline{p}$ can be recovered from u.
(3) A simple computation shows that there exists $\delta_0 > 0$ such that for every smooth map $u_D : (D, \partial D) \to (M, L)$, the map $u = u_D \circ \lambda : (S, \partial S) \to (M, L)$ belongs to $W^{1,p;\delta}(M, L)$ for every $0 < \delta < \delta_0$. This follows from the fact that there exists a constant C such that $|\lambda'(R + it)| \leq Ce^{-\pi|R|}$ for $|R| \gg 0$.
(4) Hölder's inequality implies that each $u \in W^{1,p;\delta}(M, L)$ has finite energy $E(u) = \int_S u^*\omega = \frac{1}{2}\int_S |du|^2_{g_{\omega,J}} < \infty$.

We shall now endow $W^{1,p;\delta}(M, L)$ with a structure of a Banach manifold. For this end fix $r_M > 0$ small enough so that around every $y \in L$ there exists a geodesic ball $B_y(r_M) \subset M$ of radius r_M. Define a map $P : T(L) \to \text{Vector fields}(M)$ in the following way: given $y \in L$, $v \in T_y(L)$, define the vector field $P(v)$ by:

$$P(v)(x) = \begin{cases} \chi(\text{dist}(y, x))\text{Pal}_x(v) & x \in B_y(r_M) \\ 0 & x \notin B_y(r_M) \end{cases} \tag{13}$$

Here $\chi : [0, \infty) \to [0, 1]$ is a cutoff function which equals 1 on $[0, r_M/3]$ and 0 on $[r_M/2, \infty)$. $\text{Pal}_x(v)$ stands for parallel transport of v along the minimal geodesic that connects y to x. Given a map $u : S \to M$ and $v \in T_y(L)$, define $P_u(v) : S \to T(M)$ by $P_u(v)(\tau, t) = P(v)(u(\tau, t))$.

Put $s_\delta(\tau) = e^{\delta|\tau|}$. Let $\xi \in \Gamma(u^*T(M)$ be a section for which the limits $\xi_{\pm\infty} = \lim_{\tau\to\pm\infty} \xi(\tau, t)$ exist independently of t. Define:

$$\|\xi\|^p_{1,p;s_\delta} = \int_{S(-\infty,0]} s_\delta(\tau)\Big(|\xi - P_u(\xi_{-\infty})|^p + |\nabla(\xi - P_u(\xi_{-\infty}))|^p\Big) d\tau dt$$
$$+ \int_{S[0,\infty)} s_\delta(\tau)\Big(|\xi - P_u(\xi_{\infty})|^p + |\nabla(\xi - P_u(\xi_{\infty}))|^p\Big) d\tau dt + |\xi_{-\infty}|^p + |\xi_\infty|^p.$$

Here and in what follows ∇ stands for the Levi-Civita connection of our metric $g_{\omega,J,L}$.

Proposition 4.4.3 (See [34]). *$W^{1,p;\delta}(M, L)$ is a Banach manifold with respect to the norm $\|\cdot\|_{1,p;s_\delta}$ on its tangent spaces. Its tangent space $T^{1,p;\delta}_u = T_u(W^{1,p;\delta}(M, L))$ at u consists of all sections $\xi \in \Gamma(u^*T(M))$ such that:*

(1) $\xi \in W^{1,p}_{\mathrm{loc}}$.
(2) $\xi(x) \in T_{u(x)}(L)$ for every $x \in \partial S$.
(3) $\lim_{\tau\to\pm\infty} \xi(\tau, t)$ converges to a vector $\xi_{\pm\infty} \in T_{p_{\pm\infty}}(L)$ independently of t, where $p_{\pm\infty} = \lim_{\tau\to\pm\infty} u(\tau, t)$.
(4) $\|\xi\|_{1,p;s_\delta} < \infty$.

Again, note that since $p > 2 = \dim S$ all $\xi \in T^{1,p;\delta}_u$ are actually continuous.

Given $u : (S, \partial S) \to (M, L)$ we also define the Banach space $\mathcal{E}^{0,p;\delta}_u$ consisting of all L^p_{loc}-section $\eta \in \Gamma(\Lambda^{0,1}(S) \otimes u^*T(M))$ such that:

$$\|\eta\|^p_{0,p;s_\delta} = \int_S s_\delta(\tau)|\eta(\tau, t)|^p d\tau dt < \infty.$$

Remark. If $0 < \delta_1 < \delta_2$ then $W^{1,p;\delta_1}(M, L) \supset W^{1,p;\delta_2}(M, L)$, and $\|\cdot\|_{1,p;s_{\delta_1}} \leq \|\cdot\|_{1,p;s_{\delta_2}}$, $\|\cdot\|_{0,p;s_{\delta_1}} \leq \|\cdot\|_{0,p;s_{\delta_2}}$. In particular for every $u \in W^{1,p;\delta_2}(M, L)$ we have $T^{1,p;\delta_1}_u \supset T^{1,p;\delta_2}_u$, $\mathcal{E}^{0,p;\delta_1}_u \supset \mathcal{E}^{0,p;\delta_2}_u$.

In what follows we shall need to decrease the size of δ several times for various estimates to hold. Nevertheless we shall do so only a finite number of times and the final range of admissible δ's will depend only on $(M, L, \omega, J, g_{\omega,J,L})$ and on our initial J-holomorphic disks u_1, u_0.

4.4.1. *The linearization of the $\overline{\partial}$ operator.* In order to linearize the $\overline{\partial}$ operator at an arbitrary $u \in W^{1,p;\delta}(M, L)$ we follow [44]. For this purpose we need to introduce a different connection $\widetilde{\nabla}$ on $T(M)$ as follows. Let ∇' be the Levi-Civita connection of the metric $g_{\omega,J}$ (this is the only instance where we use the metric $g_{\omega,J}$ instead of $g_{\omega,J,L}$). Define $\widetilde{\nabla}_v X = \nabla'_v X - \frac{1}{2} J(\nabla'_v J)X$. Note that $\widetilde{\nabla}$ preserves J. Define

$$\mathcal{F}_u : T^{1,p;\delta}_u \to \mathcal{E}^{0,p;\delta}_u, \quad \mathcal{F}_u(\xi) = \Phi_u(\xi)^{-1}\overline{\partial}_J(\exp_u \xi),$$

where $\Phi_u(\xi) : u^*T(M) \to \exp_u(\xi)^*T(M)$ is defined using parallel transport with respect to $\widetilde{\nabla}$. (See [44] for more details.) However, in contrast to [44], the exp is defined here with respect to our main metric $g_{\omega,J,L}$. With the above notation the linearization of $\overline{\partial}$ at u is the linearization of $\mathcal{F}_u$ at $\xi = 0$, i.e. the operator

$$D_u := d\mathcal{F}_u(0) : T_u^{1,p;\delta} \to \mathcal{E}_u^{0,p;\delta}.$$

We have the following expression for D_u (see [44]):

$$D_u\xi = \frac{1}{2}(\nabla'\xi + J(u)\nabla'\xi \circ j) - \frac{1}{2}J(u)(\nabla'_\xi J)\partial_J u,$$

where j is the standard complex structure on S. Note that when u is J-holomorphic D_u does not depend on the choice of the connections. Therefore assumption 4.1.1-(3) makes sense independently of any metric.

4.4.2. *Fredholm property.* Without the weights in the norms (i.e. for $\delta = 0$), the operator D_u is in general not Fredholm. The reason for this is that D_u has degenerate asymptotic at the ends of S. The weighted norms $\|\cdot\|_{1,p;s_\delta}$, $\|\cdot\|_{0,p;s_\delta}$ are introduced mainly in order to rectify this problem. Let us briefly explain how the weights are related to Fredholmness. To distinguish between $\delta > 0$ and $\delta = 0$ denote by $\mathcal{D}_u^0$, $\mathcal{D}_u^\delta$ the operator D_u on the spaces $T_u^{1,p;0}$, $T_u^{1,p;\delta}$ respectively. (Here $1, p; 0$ means that we take $\delta = 0$, i.e. there is no weight.) Let $\mathcal{H}_u^{1,p;0} \subset T_u^{1,p;0}$, $\mathcal{H}_u^{1,p;\delta} \subset T_u^{1,p;\delta}$ be the subspaces consisting of all ξ with $\xi(\pm\infty) = 0$. Note that $\mathcal{H}_u^{1,p;0}, \mathcal{H}_u^{1,p;\delta}$ have finite codimension, hence Fredholmness is not affected. Define

$$\Theta_\delta : \mathcal{H}_u^{1,p;0} \to \mathcal{H}_u^{1,p;\delta}, \quad \Theta_\delta(\xi)(\tau, t) = e^{-\delta|\tau|/p}\xi(\tau, t),$$
$$\rho_\delta : \mathcal{E}_u^{0,p;0} \to \mathcal{E}_u^{0,p;\delta}, \quad \rho_\delta(\eta)(\tau, t) = e^{-\delta|\tau|/p}\eta(\tau, t).$$

These maps are bounded isomorphisms between the corresponding Banach spaces. A simple computation shows that on $S[0, \infty)$ we have:

$$(\rho_\delta^{-1} \circ \mathcal{D}_u^\delta \circ \Theta_\delta)\xi = \mathcal{D}_u^0\xi + \tfrac{\delta}{p}(d\tau \otimes \xi - dt \otimes J\xi).$$

An analogous formula holds for $S(-\infty, 0]$. In other words, we get a perturbation of $\mathcal{D}_u^0$ by a small 0-order operator. This perturbation term gives non-degenerate asymptotic at the ends of S and so for generic δ (and under certain assumptions on u) the operator $(\rho_\delta^{-1} \circ \mathcal{D}_u^\delta \circ \Theta_\delta)$ becomes Fredholm. Thus $\mathcal{D}_u^\delta$ will be Fredholm too. Moreover for $\delta > 0$ small enough the index of this operator does not depend on δ. We shall not use the notation $\mathcal{D}_u^\delta$ anymore and continue to denote by D_u our operator defined on $T_u^{1,p;\delta}$ for $\delta > 0$.

Conditions which assure Fredholmness. Assume that $u \in W^{1,p;\delta}(M, L)$ has exponential decay at the ends of S, (i.e. $|du_{(\tau,t)}| \leq Ce^{-c|\tau|}$ when $|\tau| \gg 0$ for some constants $C, c > 0$). Then there exists $\delta_{\text{Fred}}(u) > 0$ such that for every $0 < \delta \leq \delta_{\text{Fred}}(u)$, the operator D_u is Fredholm. Moreover for such δ's, $\text{index}(D_u) = \dim L + \mu([u])$ and the kernel and cokernel of D_u are independent of δ. The proofs of these statements follow in a straightforward way from the theory developed in e.g. [30, 29, 55, 57].

An important situation in which we have exponential decay (hence also Fredholmness for $0 < \delta \ll 1$) is when u is J-holomorphic near the ends of S. To see this recall that $u \circ \lambda^{-1} : (D \setminus \{-1, 1\}, \partial D \setminus \{-1, 1\}) \to (M, L)$ has finite energy (see Remark 4.4.2-(4)). Then by the removal of boundary singularities theorem of Oh [47], $u \circ \lambda^{-1}$ extends smoothly to D. It follows that u has exponential decay at the ends of S. (See Remark 4.4.2-(3).)

4.4.3. *A simplification.* In order to simplify the notation we shall henceforth restrict ourselves to the special case when $W = W_- \times W_1 \times W_0 \times W_+$ where $W_\pm$ are *sumbanifolds* of L, W_1, W_0 are *submanifolds* of M, and furthermore the map $\mathbf{h}$ is the inclusion $W_- \times W_1 \times W_0 \times W_+ \hookrightarrow L \times M \times M \times L$. The proof of the general case is not more complicated from the analytic point of view, however the notation becomes heavier.

In view of the above simplification we denote from now on

$$k_\pm = \dim W_\pm, \qquad k_i = \dim W_i, \quad i = 1, 0.$$

Since $\mathbf{h}$ is now the inclusion we can write elements of $\mathcal{M}(A, J; \mathcal{C}(\mathbf{h}))$ as pairs (u, r) (instead of triples (u, r, q)) since $q = (u(-1), u(-r), u(r), u(1))$ is determined by (u, r).

Finally, note that the third point in Assumption 4.1.1 can be now simplified to *"ev is transverse to $W_- \times W_1 \times \text{diag}(L) \times W_0 \times W_+$ at the point $(u_1, u_0) \in \mathcal{M}(A_1, J) \times \mathcal{M}(A_0, J)$"*.

4.4.4. *Assumption 4.1.1 in the new coordinates.* We return to our J-holomorphic disks u_1, u_0. Put $\tau_1 = \text{Re}\, \lambda^{-1}(z_1)$, $\tau_0 = \text{Re}\, \lambda^{-1}(z_0)$. Note that $\text{Im}\, \lambda^{-1}(z_i) = \frac{1}{2}$ and $\lambda^{-1}(\pm 1) = \pm\infty$. Replace $u_1, u_0 : (D, \partial D) \to (M, L)$ by $u_1 \circ \lambda, u_0 \circ \lambda : (\widehat{S}, \partial\widehat{S}) \to (M, L)$, respectively. We claim that for $\delta > 0$ small enough Assumption 4.1.1 implies the following one:

Assumption 4.4.4. (1) $u_1(\infty) = u_0(-\infty)$.

(2) $u_1(-\infty) \in W_-, u_1(\tau_1, \frac{1}{2}) \in W_1, u_0(\tau_0, \frac{1}{2}) \in W_0, u_0(\infty) \in W_+$.

(3) J is regular for both u_1 and u_0 in the sense that the linearizations D_{u_1}, D_{u_0} of the $\overline{\partial}$ operator at u_1, u_0 are surjective.

(4) The evaluation map $ev : \mathcal{M}(A_1, J) \times \mathcal{M}(A_0, J) \to L \times M \times L \times L \times M \times L$,

$$ev(v_1, v_0) = \big(v_1(-\infty), v_1(\tau_1, \tfrac{1}{2}), v_1(\infty), v_0(-\infty), v_0(\tau_0, \tfrac{1}{2}), v_0(\infty)\big)$$

is transverse to $W_- \times W_1 \times \text{diag}(L) \times W_0 \times W_+$ at the point (u_1, u_0). (See the remarks in Section 4.4.3.)

That property (3) follows from the corresponding property in Assumption 4.1.1 is not completely obvious. This is because the Banach spaces which are the domains and targets of D_{u_1}, D_{u_0} change when we pass from the disk D to the strip S. The other properties follow trivially from Assumption 4.1.1.

Proof that Assumption 4.1.1 $\Longrightarrow$ Assumption 4.4.4-(3). Let $u_D : (D, \partial D) \to (M, L)$ be a J-holomorphic disk and denote $u = u_D \circ \lambda$. Let $T^{1,p}_{u_D}$ be the space of $W^{1,p}$-sections ξ_D of the bundle $u_D^* T(M) \to D$ which satisfy $\xi_D(x) \in T(L)$ for every $x \in \partial D$. Put

$$\mathcal{E}^{0,p}_{u_D} = L^p\big(\Lambda^{0,1}(D) \otimes u_D^* T(M)\big).$$

We have to prove that if $D_{u_D} : T^{1,p}_{u_D} \to \mathcal{E}^{0,p}_{u_D}$ is surjective then for $\delta > 0$ small enough the operator $D_u : T^{1,p;\delta}_u \to \mathcal{E}^{0,p;\delta}_u$ is surjective too.

Choose $0 < \delta < \pi p$ small enough so that D_u is Fredholm. As D_u has closed image and the subspace of smooth compactly supported sections $C_0^\infty\big(\Lambda^{0,1}(S) \otimes u^* T(M)\big) \subset \mathcal{E}^{0,p;\delta}_u$ is dense it is enough to show that this subspace lies in the image of D_u.

Let $\eta \in \mathcal{E}^{0,p;\delta}_u$ be a smooth compactly supported section. Put $\eta_D = (\lambda^{-1})^* \eta \in \mathcal{E}^{0,p}_{u_D}$. By assumption 4.1.1 there exists $\xi_D \in T^{1,p}_{u_D}$ such that $D_{u_D} \xi_D = \eta_D$. Put $\xi = \xi_D \circ \lambda$. Then

$$D_u \xi = D_u(\xi_D \circ \lambda) = \lambda^* D_{u_D} \xi_D = \lambda^* \eta_D = \eta.$$

It remains to prove that $\xi \in T^{1,p;\delta}_u$, i.e. $\|\xi\|_{1,p;s_\delta} < \infty$. To see this note that by elliptic regularity $\xi_D : D \to u_D^* T(M)$ is smooth. It follows that for $\tau \gg 0$ we have

$$\begin{aligned} |\xi(\tau, t) - P_u \xi(\infty)| &= |\xi_D \circ \lambda(\tau + it) - \mathrm{Pal}_{u_D \circ \lambda(\tau + it)} \xi_D(1)| \\ &\leq C_1(\xi_D) |\lambda(\tau + it) - 1| \leq C_1'(\xi_D) e^{-\pi \tau}, \end{aligned} \tag{14}$$

where the constants $C_1(\xi_D), C_1'(\xi_D)$ depend on ξ_D. Similarly, for $\tau \gg 0$ we have:

$$|\nabla(\xi(\tau, t) - P_u \xi(\infty))| \leq C_2(\xi_D) |\lambda(\tau + it) - 1| \leq C_2'(\xi_D) e^{-\pi\tau}, \tag{15}$$

for some constants $C_2(\xi_D), C_2'(\xi_D)$ that depend on ξ_D. Analogous estimates to (14) and (15) hold for $\tau \ll 0$ too. As $\delta < \pi p$ it easily follows that $\|\xi\|_{1,p;s_\delta} < \infty$. □

4.5. **Pregluing.** Let $u_1, u_0 : (\widehat{S}, \partial \widehat{S}) \to (M, L)$ be two J-holomorphic disks satisfying Assumption 4.4.4. Put $p = u_1(\infty) = u_0(-\infty)$. At this point it will be convenient to reparametrize u_1, u_0 in the following way. Pick $\tau^1_{\text{shift}}, \tau^0_{\text{shift}} > 0$ very large so that $u_1(\tau + \tau^1_{shift}, t), u_0(-\tau - \tau^0_{shift}, t) \in B_p(r_M/2)$ for every $\tau > 0$. Here r_M is the radius used to define the map P in formula (13) of Section 4.4. We replace $u_1(\tau, t), u_0(\tau, t)$ by $u_1(\tau + \tau^1_{\text{shift}}, t), u_0(\tau - \tau^0_{\text{shift}}, t)$ respectively. In order for the transversality assumption in 4.4.4 to continue to hold, replace $(\tau_1, \frac{1}{2})$ by $(\tau_1 - \tau^1_{\text{shift}}, \frac{1}{2})$ and $(\tau_0, \frac{1}{2})$ by $(\tau_0 + \tau^0_{\text{shift}}, \frac{1}{2})$. After these replacements we may assume without loss of generality that $u_1(\tau, t), u_0(-\tau, t) \in B_p(r_M/2)$ for every $\tau > 0$ and that the transversality assumption in 4.4.4 holds with $(\tau_1, \frac{1}{2})$ and

$(\tau_0, \frac{1}{2})$. Moreover, by choosing $\tau^1_{\text{shift}}, \tau^0_{\text{shift}}$ large enough and so that $\tau^0_{\text{shift}} - \tau^1_{\text{shift}} = -(\tau_0 + \tau_1)$ we may assume that may assume that now we have $\tau_1 = -\tau_0$ and that $\tau_0 > 0$. We shall write from now on $\tau_* = \tau_0 = -\tau_1$.

Let $\zeta_1(\tau, t), \zeta_0(\tau, t) \in T_p(M)$ be such that $u_1(\tau, t) = \exp_p(\zeta_1(\tau, t))$ for every $\tau \geq 0$ and $u_0(\tau, t) = \exp_p(\zeta_0(\tau, t))$ for every $\tau \leq 0$. Let $\sigma^+_R, \sigma^-_R : \mathbb{R} \to [0, 1]$, $R \in \mathbb{R}$, be a family of smooth functions with the properties:

(1) $\sigma^-_R(\tau) = 0$ for every $\tau \geq R + 1$, $\sigma^-_R(\tau) = 1$ for every $\tau \leq R - 1$.
(2) ${\sigma^-_R}'(\tau) \leq 0$ for every τ.
(3) $\sigma^+_R = 1 - \sigma^-_R$.

For every $R > 0$ define $u_R : (\widehat{S}, \partial\widehat{S}) \to (M, L)$ by:

$$u_R(\tau, t) = \begin{cases} u_1(\tau + 5R, t) & \tau \leq -4R, \\ \exp_p\Big(\sigma^+_{-R}(\tau)\zeta_0(\tau - 5R, t) + \sigma^-_R(\tau)\zeta_1(\tau + 5R, t)\Big) & |\tau| \leq 4R, \\ u_0(\tau - 5R, t) & \tau \geq 4R. \end{cases} \tag{16}$$

Note that $u_R(\tau, t)$ is well defined for $R \gg 0$, and that $u_R(\tau, t) = u_0(\tau - 5R, t)$ for every $\tau \geq R + 1$ while $u_R(\tau, t) = u_1(\tau + 5R, t)$ for every $\tau \leq -R - 1$. Also note that u_R maps ∂S to L. This is because L is totally geodesic with respect to the metric we use hence $\zeta_i(\tau, 0), \zeta_i(\tau, 1) \in T_p(L)$.

4.5.1. *The Fredholm property revisited.* There exists $\delta_{\text{Fred}}(u_1, u_0) > 0$ and $R_0 > 0$ such that for every $R_0 \leq R$, $0 < \delta \leq \delta_{\text{Fred}}(u_1, u_0)$ the operator D_{u_R} is Fredholm. Moreover its index is $\dim L + \mu(A_1 + A_0)$. This follows from the arguments in [55, 30]. The reason for this independence of δ on R is roughly speaking as follows. First note that each of the u_R's has exponential decay at the ends of S since they are holomorphic near the ends of S (see 4.4.2). For a given R, the set of δ's for which D_{u_R} is not Fredholm coincides with the spectrum of an (unbounded) self-adjoint operator A_R which can be derived from the asymptotic behavior of D_{u_R} as $|\tau| \to \infty$ (see [55, 30] for more details. See also [56, 57] for expositions of the Fredholm property in the case of cylindrical Floer trajectories). The point is that, due to our definition of u_R, the asymptotic operators A_R do not depend on R, hence their spectrum remains constant. In this case this spectrum is a discrete set (which contains 0 !) hence there exists $\delta_{\text{Fred}}(u_1, u_0) > 0$ such that $(0, \delta_{\text{Fred}}(u_1, u_0)]$ lies in its complement.

4.5.2. *Deformed weighted norms.* Given $R > 1$, $\delta > 0$ define $\alpha_{R,\delta} : S \to \mathbb{R}$ by

$$\alpha_{R,\delta}(\tau, t) = \alpha_{R,\delta}(\tau) = \begin{cases} e^{\delta(|\tau| - 5R)} & |\tau| \geq 5R, \\ e^{\delta(5R - |\tau|)} & |\tau| \leq 5R. \end{cases} \tag{17}$$

Using the weight $\alpha_{R,\delta}$ we define a new norm on $T^{1,p;\delta}_{u_R}$ by:

$$\begin{aligned}(18)\qquad \|\xi\|^p_{1,p;\alpha_{R,\delta}} = &\int_{S(-\infty,-5R]} \alpha_{R,\delta}\big(|\xi - P_{u_R}(\xi_{-\infty})|^p + |\nabla(\xi - P_{u_R}(\xi_{-\infty}))|^p\big)\\ &+ \int_{S[5R,\infty)} \alpha_{R,\delta}\big(|\xi - P_{u_R}(\xi_{\infty})|^p + |\nabla(\xi - P_{u_R}(\xi_{\infty}))|^p\big)\\ &+ \int_{S[-5R,5R]} \alpha_{R,\delta}\big(|\xi - \mathrm{Pal}_{u_R}\xi(0,\tfrac{1}{2})|^p + |\nabla(\xi - \mathrm{Pal}_{u_R}\xi(0,\tfrac{1}{2}))|^p\big)\\ &+|\xi_{-\infty}|^p + |\xi_{\infty}|^p + |\xi(0,\tfrac{1}{2})|^p.\end{aligned}$$

We endow $\mathcal{E}^{0,p;\delta}_{u_R}$ with the weighted norm:

$$\|\eta\|_{0,p;\alpha_{R,\delta}} = \Big(\int_S \alpha_{R,\delta}|\eta|^p\Big)^{1/p}.$$

Remark. Using the inequalities from Section 4.9 it is easy to see that $\|\cdot\|_{1,p;\alpha_{R,\delta}}$ is equivalent to $\|\cdot\|_{1,p;s_\delta}$ however they are not uniformly equivalent as $R \to \infty$, i.e. it is impossible to find one constant C such that $\|\cdot\|_{1,p;\alpha_{R,\delta}} \leq C\|\cdot\|_{1,p;s_\delta}$ for every R.

4.5.3. *Comparison of norms.*

Proposition 4.5.1. *Let $u \in W^{1,p;\delta}(M,L)$. There exists a constant $C_{4.5.1} > 0$ depending on (u_1,u_0) and on p, but not on R,δ, such that for every $\xi \in T^{1,p;\delta}_{u_R}$, $R,\delta > 0$, we have:*

$$\|\xi\|_{L^\infty} \leq C_{4.5.1}\|\xi\|_{1,p;s_\delta}, \quad \|\xi\|_{L^\infty} \leq C_{4.5.1}\|\xi\|_{1,p;\alpha_{R,\delta}}.$$

Proof. Let $x \in S[5R,\infty)$. Since $\alpha_{R,\delta} \geq 1$, we get from Proposition 4.9.1:

$$\begin{aligned}|\xi(x)| &\leq |\xi(x) - P_{u_R}(\xi_\infty)(x)| + |\xi_\infty|\\ &\leq C_{4.9.1}\Big(\int_{S[5R,\infty)} \alpha_{R,\delta}\big(|\xi - P_{u_R}\xi_\infty|^p + |\nabla(\xi - P_{u_R}\xi_\infty)|^p\big)\Big)^{1/p} + |\xi_\infty|\\ &\leq C\|\xi\|_{1,p;\alpha_{R,\delta}},\end{aligned}$$

where $C = 2(C_{4.9.1}+1)$.

Let $x \in S[-5R,5R]$. Then by Proposition 4.9.1:

$$\begin{aligned}|\xi(x)| &\leq |\xi(x) - \mathrm{Pal}_{u_R(x)}\xi(0,\tfrac{1}{2})| + |\xi(0,\tfrac{1}{2})|\\ &\leq C_{4.9.1}\Big(\int_{S[-5R,5R]} \big(|\xi - \mathrm{Pal}_{u_R}\xi(0,\tfrac{1}{2})|^p + |\nabla(\xi - \mathrm{Pal}_{u_R}\xi(0,\tfrac{1}{2}))|^p\big)\Big)^{1/p} + |\xi(0,\tfrac{1}{2})|,\\ &\leq C\|\xi\|_{1,p;\alpha_{R,\delta}}.\end{aligned}$$

In a similar way one shows that for every $x \in S(-\infty,-5R]$, $|\xi(x)| \leq C\|\xi\|_{1,p;\alpha_{R,\delta}}$. This proves that $\|\xi\|_{L^\infty} \leq C\|\xi\|_{1,p;\alpha_{R,\delta}}$. The proof of the inequality for $\|\xi\|_{1,p;s_\delta}$ is similar. $\square$

4.5.4. *Estimates on u_R.*

Proposition 4.5.2. *There exists constants $\delta_0 > 0$, $R_0 > 1$, $C'_{4.5.2}, c'_{4.5.2}, C''_{4.5.2} > 0$ that depend only on (u_1, u_0) such that for every $0 < \delta \leq \delta_0$ and $R_0 \leq R$ we have:*

$$\|\overline{\partial}_J u_R\|_{0,p;\alpha_{R,\delta}} \leq C'_{4.5.2} e^{-c'_{4.5.2}R}, \quad \|du_R\|_{0,p;\alpha_{R,\delta}} \leq C''_{4.5.2}, \quad \|du_R\|_{L^\infty} \leq C''_{4.5.2}.$$

Outline of the proof. The proof is a straightforward computation combined with the following points:

(1) We have an exponential decay estimate for du_i, $i = 1, 0$, namely there exists constants $C, c > 0$ such that $|du_{i(\tau,t)}| \leq Ce^{-c|\tau|}$. Similar exponential decay estimates hold also for ζ_1, ζ_0. It follows that $|du_{0(\tau-5R,t)}| \leq Ce^{-c|\tau-5R|}$ and $|du_{1(\tau+5R,t)}| \leq Ce^{-c|\tau+5R|}$.

(2) $\overline{\partial}_J u_R = 0$ outside $S[-R-1, R+1]$, thus for estimating $\|\overline{\partial}_J u_R\|_{0,p;\alpha_{R,\delta}}$ it is enough to estimate $\|du_R\|_{0,p;\alpha_{R,\delta}}$ on $S[-R-1, R+1]$.

(3) There exists $K > 0$ such that $|d(\exp_p)_v| \leq K$ for every $v \in T_p(M)$ with $|v| \leq \epsilon$. Note that for R large enough the vectors appearing inside the $\exp_p$ in expression (16) have norm $\leq \epsilon$ due to exponential decay of the ζ_i's.

□

4.6. **The main operators.** In view of the simplification in Section 4.4.3, fix four smooth maps $G_\pm : L \to \mathbb{R}^{n-k_\pm}$, $G_i : M \to \mathbb{R}^{2n-k_i}$, $i = 1, 0$ such that $G_\pm$ are submersions near $W_\pm, W_i$ and such that near $W_\pm, W_i$ we have $G_\pm^{-1}(0) = W_\pm$, $G_i^{-1}(0) = W_i$. Put

$$T^{1,p;\delta}_{u_1,u_0} = \{(\xi_1, \xi_0) \in T^{1,p;\delta}_{u_1} \oplus T^{1,p;\delta}_{u_0} \mid \xi_1(\infty) = \xi_0(-\infty)\}. \tag{19}$$

Define the following operator:

$$D'_{u_1,u_0} : T^{1,p;\delta}_{u_1,u_0} \longrightarrow \mathcal{E}^{0,p;\delta}_{u_1} \oplus \mathcal{E}^{0,p;\delta}_{u_0} \oplus \mathbb{R}^{n-k_-} \oplus \mathbb{R}^{2n-k_1} \oplus \mathbb{R}^{2n-k_0} \oplus \mathbb{R}^{n-k_+},$$
$$D'_{u_1,u_0}(\xi_1, \xi_0) = \big(D_{u_1}\xi_1, D_{u_0}\xi_0, dG_-\xi_1(-\infty), dG_1\xi_1(-\tau_*, \tfrac{1}{2}), dG_0\xi_0(\tau_*, \tfrac{1}{2}), dG_+\xi_0(\infty)\big).$$

For brevity denote by $E = \mathbb{R}^{n-k_-} \oplus \mathbb{R}^{2n-k_1} \oplus \mathbb{R}^{2n-k_0} \oplus \mathbb{R}^{n-k_+}$ the summand of the last four linear spaces in the target of D'_{u_1,u_0}. We also write $D_{u_1,u_0}(\xi_1, \xi_0)$ for the first two coordinates of $D'_{u_1,u_0}(\xi_1, \xi_0)$ and $d\mathcal{G}_{1,0}(\xi_1, \xi_0)$ for its last four coordinates.

Proposition 4.6.1. *Under Assumption 4.4.4, the operator D'_{u_1,u_0} is surjective.*

Proof. Let $\eta_i \in \mathcal{E}^{0,p;\delta}_{u_i}$, $i = 1, 0$ and $\overrightarrow{a} = (a_-, a_1, a_0, a_+) \in E$. Choose $w_- \in T_{u_1(-\infty)}(W_-)$, $w_1 \in T_{u_1(-\tau_*, \frac{1}{2})}(W_1)$, $w_0 \in T_{u_0(\tau_*, \frac{1}{2})}(W_0)$, $w_+ \in T_{u_0(\infty)}$ such that $dG_\pm(w_\pm) = a_\pm$, $dG_i(w_i) = a_i$, $i = 1, 0$.

Since D_{u_1}, D_{u_0} are surjective there exist ξ_1', ξ_0' such that $D_{u_i}\xi_i' = \eta_i$. By Assumption 4.4.4 there exist $\zeta_i \in T_{u_i}(\mathcal{M}(A_i, J))$ such that

$$\Big(\zeta_1(-\infty), \zeta_1(-\tau_*, \tfrac{1}{2}), \zeta_1(\infty), \zeta_0(-\infty), \zeta_0(\tau_*, \tfrac{1}{2}), \zeta_0(\infty)\Big) \in \big(w_-, w_1, \xi_1'(\infty), \xi_0'(-\infty), w_0, w_+\big) \\ + T_{\underline{x}}\big(W_- \times W_1 \times \mathrm{diag}(L) \times W_0 \times W_+\big),$$

where $\underline{x} = (u_1(-\infty), u_1(-\tau_*, \frac{1}{2}), u_1(\infty), u_0(-\infty), u_0(\tau_*, \frac{1}{2}), u_0(\infty))$.

Put $\xi_i = \xi_i' - \zeta_i$, $i = 1, 0$. Clearly $(\xi_1, \xi_0) \in T^{1,p;\delta}_{u_1,u_0}$ and $D'_{u_1,u_0}(\xi_1, \xi_0) = (\eta_1, \eta_0, \overrightarrow{a})$. □

Define a map

$$\mathcal{F}'_{u_R} : T^{1,p;\delta}_{u_R} \longrightarrow \mathcal{E}^{0,p;\delta}_{u_R} \times \mathbb{R}^{n-k_-} \times \mathbb{R}^{2n-k_1} \times \mathbb{R}^{2n-k_0} \times \mathbb{R}^{n-k_+},$$

$$\mathcal{F}'_{u_R}(\xi) = (\mathcal{F}_{u_R}(\xi), G_-(\exp_{u_R} \xi(-\infty)), G_1(\exp_{u_R} \xi(-5R - \tau_*)), \\ G_0(\exp_{u_R} \xi(5R + \tau_*)), G_+(\exp_{u_R} \xi(\infty))).$$

Let $D'_{u_R} = d\mathcal{F}'_{u_R}(0) : T^{1,p;\delta}_{u_R} \to \mathcal{E}^{0,p;\delta}_{u_R} \oplus E$ be its linearization at $\xi = 0$. We have $D'_{u_R}\xi = (D_{u_R}\xi, d\mathcal{G}_R\xi)$, where

$$d\mathcal{G}_R\xi = (dG_-\xi(-\infty), dG_1\xi(-5R - \tau_*, \tfrac{1}{2}), dG_0\xi(5R + \tau_*, \tfrac{1}{2}), dG_+\xi(\infty)).$$

From now on we shall often endow the space $\mathcal{E}^{0,p;\delta}_{u_R} \oplus E$ with the norm

$$\|(\eta, \overrightarrow{a})\|'_{0,p;\alpha_{R,\delta}} = \|\eta\|_{0,p;\alpha_{R,\delta}} + |\overrightarrow{a}|. \tag{20}$$

Clearly the operators D'_{u_R} have the same Fredholm properties as D_{u_R} as described in Sections 4.4.2 and 4.5.1.

4.7. A right inverse to D_{u_R}.

Proposition 4.7.1. *There exist $\delta_0 > 0$, $R_0 > 0$ such that for every $0 < \delta \leq \delta_0$, there exists a family of operators $\{Q_R : \mathcal{E}^{0,p;\delta}_{u_R} \oplus E \to T^{1,p;\delta}_{u_R}\}_{R_0 \leq R}$, with the following properties:*

(1) $D'_{u_R} \circ Q_R = \mathbb{1}$.

(2) Q_R is uniformly bounded in the $\alpha_{R,\delta}$-weighted norms, i.e. there exists a constant $C_{4.7.1}(\delta) > 0$ that does not depend on R such that for every $R_0 \leq R$, $(\eta, \overrightarrow{a}) \in \mathcal{E}^{0,p;\delta}_{u_R} \oplus E$ we have $\|Q_R(\eta, \overrightarrow{a})\|_{1,p;\alpha_{R,\delta}} \leq C_{4.7.1}(\delta)(\|\eta\|_{0,p;\alpha_{R,\delta}} + |\overrightarrow{a}|)$.

(3) Q_R depends smoothly on R. (See remark below.)

Remarks. (1) Uniform boundedness of the Q_R's does not seem to hold for the s_δ-weighted norms.

(2) In order to make sense of the smooth dependence of Q_R on R one has to identify the spaces $T^{1,p;\delta}_{u_R}$ for nearby $R's$. This can be done using parallel transport along short geodesics.

The rest of this section is devoted to the proof of Proposition 4.7.1. The reader who wishes to get a less technical account of the gluing may skip to Section 4.8 in which the proof of Theorem 4.1.2 is carried out.

4.7.1. *Some auxiliary operators.* Put

$$T^{1,p;\delta}_{u_1,u_0} = \{(\xi_1, \xi_0) \in T^{1,p;\delta}_{u_1} \oplus T^{1,p;\delta}_{u_0} \mid \xi_1(\infty) = \xi_0(-\infty)\}.$$

Define a linear map $I^R : T^{1,p;\delta}_{u_1,u_0} \to T^{1,p;\delta}_{u_R}$

$$I^R(\xi_1, \xi_0)(\tau, t) =$$

$$= \begin{cases} \xi_1(\tau + 5R, t), & \tau \leq -4R, \\ \mathrm{Pal}_{u_R(\tau,t)}\Big(v + \sigma^+_{-R}\big(\mathrm{Pal}_p(\xi_0(\tau - 5R, t)) - v\big) + \sigma^-_R\big(\mathrm{Pal}_p(\xi_1(\tau + 5R, t)) - v\big)\Big), & |\tau| \leq 4R, \\ \xi_0(\tau - 5R, t) & \tau \geq 4R, \end{cases}$$

where:

(1) $p = u_1(\infty) = u_0(-\infty)$.
(2) $v = \xi_1(\infty) = \xi_0(-\infty)$.
(3) $\mathrm{Pal}_x(\eta)$ is defined for every $x \in B_p(r_M)$ and every $\eta \in T_y(M)$, $y \in B_p(r_M)$. It is the parallel transport of η along the minimal geodesic connecting y to x.

Define also the following maps:

$$J^R_{0,*} : \mathcal{E}^{0,p;\delta}_{u_0} \to \mathcal{E}^{0,p;\delta}_{u_R}, \qquad J^R_{0,*}\eta(\tau, t) = \mathrm{Pal}_{u_R(\tau,t)}\big(\sigma^+_{-R}(\tau)\eta(\tau - 5R, t)\big)$$
$$J^R_{1,*} : \mathcal{E}^{0,p;\delta}_{u_1} \to \mathcal{E}^{0,p;\delta}_{u_R}, \qquad J^R_{1,*}\eta(\tau, t) = \mathrm{Pal}_{u_R(\tau,t)}\big(\sigma^-_R(\tau)\eta(\tau + 5R, t)\big)$$

Note that $J^R_{0,*}\eta(\tau, t) = \eta(\tau - 5R, t)$ for every $\tau \geq R + 1$ while $J^R_{0,*}\eta(\tau, t) = 0$ for every $\tau \leq -R - 1$. Similarly define

$$J^R_{*,0} : \mathcal{E}^{0,p;\delta}_{u_R} \to \mathcal{E}^{0,p;\delta}_{u_0}, \qquad J^R_{*,0}\eta(\tau, t) = \mathrm{Pal}_{u_0(\tau,t)}\big(\sigma^+_{-R}(\tau + 5R)\eta(\tau + 5R, t)\big)$$
$$J^R_{*,1} : \mathcal{E}^{0,p;\delta}_{u_R} \to \mathcal{E}^{0,p;\delta}_{u_1}, \qquad J^R_{*,1}\eta(\tau, t) = \mathrm{Pal}_{u_1(\tau,t)}\big(\sigma^-_R(\tau - 5R)\eta(\tau - 5R, t)\big)$$

Note that $J^R_{*,0}\eta(\tau, t) = \eta(\tau + 5R, t)$ for every $\tau \geq -4R + 1$. The following identities are easily verified:

$$\begin{aligned} & J^R_{0,*} \circ J^R_{*,0}(\sigma^+_0(\tau)\eta) = \sigma^+_0\eta, \\ & J^R_{1,*} \circ J^R_{*,1}(\sigma^-_0(\tau)\eta) = \sigma^-_0\eta, \\ & J^R_{0,*} \circ J^R_{*,0}(\sigma^+_0(\tau)\eta) + J^R_{1,*} \circ J^R_{*,1}(\sigma^-_0(\tau)\eta) = \eta. \end{aligned} \tag{21}$$

We now have the following estimate:

Proposition 4.7.2. *There exist constants $C_{4.7.2}(R) > 0$, with $\lim_{R\to\infty} C_{4.7.2}(R) = 0$ and such that:*

$$\|D_{u_R} \circ I^R(\xi_1, \xi_0) - J^R_{0,*} D_{u_0}\xi_0 - J^R_{1,*} D_{u_1}\xi_1\|_{0,p;\alpha_{R,\delta}} \leq C_{4.7.2}(R)(\|\xi\|_{1,p;s_\delta} + \|\xi_0\|_{1,p;s_\delta}).$$

The proof is a straightforward computation, based on the definition of the weighted norms $s_\delta, \alpha_{R,\delta}$. See [34] for more details.

4.7.2. *The approximate right inverse.* Let $Q_0' : \mathcal{E}^{0,p;\delta}_{u_1} \oplus \mathcal{E}^{0,p;\delta}_{u_0} \oplus E \to T^{1,p;\delta}_{u_1,u_0}$ be a bounded right inverse to D'_{u_1,u_0} (see Proposition 4.6.1). Write Q_0' as

$$Q_0'(\eta_1, \eta_0, \overrightarrow{a}) = Q_0(\eta_1, \eta_0) + A(\overrightarrow{a}). \tag{22}$$

Define $\widetilde{Q}_R : \mathcal{E}^{0,p;\delta}_{u_R} \oplus E \to T^{1,p;\delta}_{u_R}$ by

$$\widetilde{Q}_R(\eta, \overrightarrow{a}) = I^R\Big(Q_0\big(J^R_{*,0}(\sigma_0^+\eta), J^R_{*,1}(\sigma_0^-\eta)\big) + A(\overrightarrow{a})\Big). \tag{23}$$

Proposition 4.7.3. *There exist constants $R_0, \delta_0 > 0$ and constants $C_{4.7.3}(\delta)$ such that for every $R_0 \leq R$, $0 < \delta \leq \delta_0$, $\eta \in \mathcal{E}^{0,p;\delta}_{u_R}$, $\overrightarrow{a} \in E$ we have*

$$\|\widetilde{Q}_R(\eta, \overrightarrow{a})\|_{1,p;\alpha_{R,\delta}} \leq C_{4.7.3}(\delta)(\|\eta\|_{0,p;\alpha_{R,\delta}} + |\overrightarrow{a}|).$$

Proof of Proposition 4.7.3. A simple computation shows that:

Lemma 4.7.4. *There exists $C_{4.7.4} > 0$ such that for every $R \gg 0$:*

$$\|J^R_{*,0}(\sigma_0^+\eta)\|_{0,p;s_\delta} \leq C_{4.7.4}\|\eta\|_{0,p;\alpha_{R,\delta}},$$
$$\|J^R_{*,1}(\sigma_0^-\eta)\|_{0,p;s_\delta} \leq C_{4.7.4}\|\eta\|_{0,p;\alpha_{R,\delta}}$$

To prove Proposition 4.7.3 it suffices to prove the following:

Proposition 4.7.5. *There exist constants R_0, $\delta_0 > 0$ and $C_{4.7.5}(\delta)$ such that for every $R_0 \leq R$, $0 < \delta \leq \delta_0$, $(\xi_1, \xi_0) \in T^{1,p;\delta}_{u_1,u_0}$, we have*

$$\|I^R(\xi_1, \xi_0)\|_{1,p;\alpha_{R,\delta}} \leq C_{4.7.5}(\delta)(\|\xi_1\|_{1,p;s_\delta} + \|\xi_0\|_{1,p;s_\delta}).$$

□

Proof of Proposition 4.7.5. Put $\xi_{\mathrm{new}} = I^R(\xi_1, \xi_0)$, $v = \xi_0(-\infty) = \xi_1(\infty)$. It is easy to see that

$$\begin{aligned}
&\int_{S[5R,\infty)} \alpha_{R,\delta}\Big(|\xi_{\mathrm{new}} - P_{u_R}\xi_0(\infty)|^p + |\nabla(\xi_{\mathrm{new}} - P_{u_R}\xi_0(\infty))|^p\Big) \\
+ &\int_{S(-\infty,5R]} \alpha_{R,\delta}\Big(|\xi_{\mathrm{new}} - P_{u_R}\xi_1(-\infty)|^p + |\nabla(\xi_{\mathrm{new}} - P_{u_R}\xi_1(-\infty))|^p\Big) \\
\leq &\|\xi_0\|^p_{1,p;s_\delta} + \|\xi_1\|^p_{1,p;s_\delta} + |\xi_0(\infty)|^p + |\xi_1(-\infty)|^p.
\end{aligned}$$

It remains to prove that

$$\begin{aligned}I := \int_{S[-5R,5R]} \alpha_{R,\delta}\Big(|\xi_{\mathrm{new}} - \mathrm{Pal}_{u_R}\xi_{\mathrm{new}}(0,\tfrac{1}{2})|^p + |\nabla(\xi_{\mathrm{new}} - \mathrm{Pal}_{u_R}\xi_{\mathrm{new}}(0,\tfrac{1}{2}))|^p\Big) \\ + |\xi_{\mathrm{new}}(0,\tfrac{1}{2})|^p \leq C_1(\|\xi_0\|^p_{1,p;s_\delta} + \|\xi_1\|^p_{1,p;s_\delta}),\end{aligned} \tag{24}$$

for some constant C_1 that does not depend on R. Note that due to the reparametrization we have made in the beginning of Section 4.5 we can write Pal_{u_R} in the two terms under the integral of (24) instead of P_{u_R}. Now

$$I \leq C_2(I_1 + I_2 + |\xi_{\mathrm{new}}(0,\tfrac{1}{2})|^p), \tag{25}$$

for some constant $C_2 > 0$, where

$$\begin{aligned}I_1 &= \int_{S[-5R,5R]} \alpha_{R,\delta}\Big(|\xi_{\mathrm{new}} - \mathrm{Pal}_{u_R}v|^p + |\nabla(\xi_{\mathrm{new}} - \mathrm{Pal}_{u_R}v)|^p\Big), \\ I_2 &= \int_{S[-5R,5R]} \alpha_{R,\delta}\Big(|\mathrm{Pal}_{u_R}\xi_{\mathrm{new}}(0,\tfrac{1}{2}) - \mathrm{Pal}_{u_R}v|^p + |\nabla(\mathrm{Pal}_{u_R}\xi_{\mathrm{new}}(0,\tfrac{1}{2}) - \mathrm{Pal}_{u_R}v)|^p\Big).\end{aligned}$$

We begin by estimating I_2. Put $V(x) = \mathrm{Pal}_{u_R(x)}\xi_{\mathrm{new}}(0,\frac{1}{2}) - \mathrm{Pal}_{u_R(x)}v$. Then, by the definition of ξ_{new} we have:

$$\begin{aligned}V(x) = \mathrm{Pal}_{u_R(x)}\mathrm{Pal}_{u_R(0,\frac{1}{2})}\mathrm{Pal}_p\xi_0(-5R,\tfrac{1}{2}) + \mathrm{Pal}_{u_R(x)}\mathrm{Pal}_{u_R(0,\frac{1}{2})}\mathrm{Pal}_p\xi_1(5R,\tfrac{1}{2}) \\ - (\mathrm{Pal}_{u_R(x)}v + \mathrm{Pal}_{u_R(x)}\mathrm{Pal}_{u_R(0,\frac{1}{2})}v).\end{aligned}$$

Using the fact that for every $(\tau,t) \in S[-5R,5R]$ we have $\mathrm{dist}(u_R(\tau,t),p) \leq C_3 e^{-c_3(5R-|\tau|)}$ for some constants $C_3, c_3 > 0$ (that do not depend on R) we obtain by standard arguments from ode's that:

$$|V(\tau,t)| \leq |\xi_0(-5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_0(-5R,\frac{1}{2})}v| + |\xi_1(5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_1(5R,\frac{1}{2})}v| + C_4 e^{-c_4(5R-|\tau|)}|v|, \tag{26}$$

for some constants $C_4, c_4 > 0$. Similar arguments show that:

$$\begin{aligned}|\nabla V(\tau,t)| \leq & C_5\big(|\xi_0(-5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_0(-5R,\frac{1}{2})}v| + |\xi_1(5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_1(5R,\frac{1}{2})}v| + |v|\big) \\ & + C_5 e^{-c_5(5R-|\tau|)}(|\xi_0(-5R,\tfrac{1}{2})| + |\xi_1(5R,\tfrac{1}{2})| + |v|)\end{aligned} \tag{27}$$

for some constants $C_5, c_5 > 0$. Note that in both (26) and (27) the constants C_4, c_4, C_5, c_5 do not depend on R.

We claim that there exists a constant $\delta_0 > 0$, independent of R, such that for every $0 < \delta \leq \delta_0$ there exists $C_6 = C_6(\delta)$ (independent of R) such that:

$$I_2 = \int_{S[-5R,5R]} \alpha_{R,\delta}(|V|^p + |\nabla V|^p) \leq C_6(\|\xi_0\|^p_{1,p;s_\delta} + \|\xi_1\|^p_{1,p;s_\delta} + |v|^p). \tag{28}$$

Indeed, by (26), (27) there exist constants C_7, c_7 (independent of R and of δ) such that:

$$
\begin{aligned}
(29)\qquad & \int_{S[-5R,5R]} \alpha_{R,\delta}(|V|^p + |\nabla V|^p) \\
\leq & C_7\Big(|\xi_0(-5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_0(-5R,\frac{1}{2})} v|^p + |\xi_1(5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_1(5R,\frac{1}{2})} v|^p\Big) \int_{-5R}^{5R} \alpha_{R,\delta}(\tau)d\tau \\
+ & C_7\big(|\xi_0(-5R,\tfrac{1}{2})|^p + |\xi_1(5R,\tfrac{1}{2})|^p + |v|^p\big) \int_{-5R}^{5R} e^{-c_7(5R-|\tau|)}\alpha_{R,\delta}(\tau)d\tau \\
\leq & \frac{2C_7}{\delta} e^{\delta 5R}\big(|\xi_0(-5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_0(-5R,\frac{1}{2})} v|^p + |\xi_1(5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_1(5R,\frac{1}{2})} v|^p\big) \\
+ & \big(|\xi_0(-5R,\tfrac{1}{2})|^p + |\xi_1(5R,\tfrac{1}{2})|^p + |v|^p\big) C_7 \int_{-5R}^{5R} e^{-c7(5R-|\tau|)+\delta(5R-|\tau|)} d\tau.
\end{aligned}
$$

Now $\int_{-5R}^{5R} e^{-c7(5R-|\tau|)+\delta(5R-|\tau|)} d\tau = \frac{2}{c_7-\delta}(1-e^{(\delta-c_7)5R})$, hence if we take $\delta_0 = c_7/2$ we obtain from (29) that for every $0 < \delta < \delta_0$:

$$
\begin{aligned}
(30)\qquad & \int_{S[-5R,5R]} \alpha_{R,\delta}(|V|^p + |\nabla V|^p) \\
\leq & \frac{2C_7}{\delta} e^{\delta 5R}\big(|\xi_0(-5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_0(-5R,\frac{1}{2})} v|^p + |\xi_1(5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_1(5R,\frac{1}{2})} v|^p\big) \\
+ & \big(|\xi_0(-5R,\tfrac{1}{2})|^p + |\xi_1(5R,\tfrac{1}{2})|^p + |v|^p\big) \frac{4C_7}{c_7}.
\end{aligned}
$$

Applying Proposition 4.9.1 to (30) we obtain for every $0 < \delta < \delta_0$:

(31)

$$
\begin{aligned}
& \int_{S[-5R,5R]} \alpha_{R,\delta}(|V|^p + |\nabla V|^p) \\
\leq & \frac{2C_7}{\delta} e^{\delta 5R} C_{4.9.1} \int_{S[-5R-1,-5R]} \big(|\xi_0(\tau,t) - \mathrm{Pal}_{u_0(\tau,t)} v|^p + |\nabla(\xi_0 - \mathrm{Pal}_{u_0} v)|^p\big) e^{\delta(|\tau|-5R)} d\tau dt \\
+ & \frac{2C_7}{\delta} e^{\delta 5R} C_{4.9.1} \int_{S[5R,5R+1]} \big(|\xi_1(\tau,t) - \mathrm{Pal}_{u_1(\tau,t)} v|^p + |\nabla(\xi_1 - \mathrm{Pal}_{u_1} v)|^p\big) e^{\delta(|\tau|-5R)} d\tau dt \\
+ & \big(|\xi_0(-5R,\tfrac{1}{2})|^p + |\xi_1(5R,\tfrac{1}{2})|^p + |v|^p\big) \frac{4C_7}{c_7} \\
= & \frac{2C_7}{\delta} C_{4.9.1} \int_{S[-5R-1,-5R]} \big(|\xi_0(\tau,t) - \mathrm{Pal}_{u_0(\tau,t)} v|^p + |\nabla(\xi_0 - \mathrm{Pal}_{u_0} v)|^p\big) s_\delta(\tau) d\tau dt \\
+ & \frac{2C_7}{\delta} C_{4.9.1} \int_{S[5R,5R+1]} \big(|\xi_1(\tau,t) - \mathrm{Pal}_{u_1(\tau,t)} v|^p + |\nabla(\xi_1 - \mathrm{Pal}_{u_1} v)|^p\big) s_\delta(\tau) d\tau dt \\
+ & \big(|\xi_0(-5R,\tfrac{1}{2})|^p + |\xi_1(5R,\tfrac{1}{2})|^p + |v|^p\big) \frac{4C_7}{c_7}.
\end{aligned}
$$

By Corollary 4.9.1

$$|\xi_0(-5R,\tfrac{1}{2})| \le |\xi_0(-5R,\tfrac{1}{2}) - P_{u_0(-5R,\frac{1}{2})}v| + |v| \le C_{4.9.1}\|\xi_0\|_{1,p;s_\delta} + |v|,$$
$$|\xi_1(5R,\tfrac{1}{2})| \le |\xi_1(5R,\tfrac{1}{2}) - P_{u_1(5R,\frac{1}{2})}v| + |v| \le C_{4.9.1}\|\xi_1\|_{1,p;s_\delta} + |v|$$

Hence we obtain from (31)

$$\int_{S[-5R,5R]} \alpha_{R,\delta}(|V|^p + |\nabla V|^p) \le C_6(\|\xi_0\|^p_{1,p;s_\delta} + \|\xi_1\|^p_{1,p;s_\delta} + |v|^p),$$

for some $C_6 = C_6(\delta)$. This proves inequality (28) which estimates I_2.

We turn to estimating $|\xi_{\mathrm{new}}(0,\frac{1}{2})|$:

$$\begin{aligned}
|\xi_{\mathrm{new}}(0,\tfrac{1}{2})| &= |\mathrm{Pal}_{u_R(0,\frac{1}{2})}\big(\mathrm{Pal}_p\xi_0(-5R,\tfrac{1}{2}) + \mathrm{Pal}_p\xi_1(5R,\tfrac{1}{2})\big) - \mathrm{Pal}_{u_R(0,\frac{1}{2})}v| \\
&\le |\xi_0(-5R,\tfrac{1}{2})| + |\xi_1(5R,\tfrac{1}{2})| + |v| \\
&\le |\xi_0(-5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_0(-5R,\frac{1}{2})}v| + |\xi_1(5R,\tfrac{1}{2}) - \mathrm{Pal}_{u_1(5R,\frac{1}{2})}v| + 3v \\
&\le (C_{4.9.1}+3)(\|\xi_0\|_{1,p;s_\delta} + \|\xi_1\|_{1,p;s_\delta} + |v|).
\end{aligned} \tag{32}$$

Finally we estimate I_1.

$$\xi_{\mathrm{new}}(\tau,t) - \mathrm{Pal}_{u_R(\tau,t)}v = \mathrm{Pal}_{u_r(\tau,t)}\big(\sigma^+_{-R}(\mathrm{Pal}_p\xi_0(\tau-5R,t)-v) + \sigma^-_R(\mathrm{Pal}_p\xi_1(\tau+5R,t)-v)\big)$$

Now $\mathrm{Pal}_{u_R(\tau,t)}(\mathrm{Pal}_p\xi_0(\tau-5R,t)-v) = \mathrm{Pal}_{u_R(\tau,t)}\mathrm{Pal}_p(\xi_0(\tau-5R,t) - \mathrm{Pal}_{u_0(\tau-5R,t)}v)$ and similarly for the term involving ξ_1, hence we have:

$$|\xi_{\mathrm{new}} - \mathrm{Pal}_{u_R(\tau,t)}| \le |\xi_0(\tau-5R,t) - \mathrm{Pal}_{u_0(\tau-5R,t)}v| + |\xi_1(\tau+5R,t) - \mathrm{Pal}_{u_1(\tau+5R,t)}v|.$$

Using that fact that $\|du_R\|_{L^\infty}$ is uniformly bounded (see Proposition 4.5.2), a straightforward computation shows that on $S[-5R,5R]$ we have:

$$\begin{aligned}
&|\nabla(\xi_{\mathrm{new}} - \mathrm{Pal}_{u_R}v)(\tau,t)| \\
&\le C_8\Big(|\xi_0(\tau-5R,t) - \mathrm{Pal}_{u_0(\tau-5R,t)}v| + |\xi_1(\tau+5R,t) - \mathrm{Pal}_{u_1(\tau+5R,t)}v| \\
&+ |\nabla(\xi_0(\tau-5R,t) - \mathrm{Pal}_{u_0(\tau-5R,t)}v)| + |\nabla(\xi_1(\tau+5R,t) - \mathrm{Pal}_{u_1(\tau+5R,t)}v)|\Big),
\end{aligned}$$

where C_8 does not depend on R. Therefore

$$
\begin{aligned}
I_1 &= \int_{S[-5R,5R]} \alpha_{R,\delta}\big(|\xi_{\text{new}} - \mathrm{Pal}_{u_R} v|^p + |\nabla(\xi_{\text{new}} - \mathrm{Pal}_{u_R} v)|^p\big) \\
&\leq C_9\Big(\int_0^1 \int_{-5R}^{5R} \alpha_{R,\delta}(\tau)\big(|\xi_0(\tau - 5R, t) - \mathrm{Pal}_{u_0(\tau-5R,t)} v|^p \\
&\qquad\qquad + |\nabla(\xi_0(\tau - 5R, t) - \mathrm{Pal}_{u_0(\tau-5R,t)} v)|^p\big) d\tau dt \\
&\quad + \int_0^1 \int_{-5R}^{5R} \alpha_{R,\delta}(\tau)\big(|\xi_1(\tau + 5R, t) - \mathrm{Pal}_{u_1(\tau+5R,t)} v|^p \\
&\qquad\qquad + |\nabla(\xi_1(\tau + 5R, t) - \mathrm{Pal}_{u_1(\tau+5R,t)} v)|^p\big) d\tau dt\Big) \\
&\leq C_{10}(\|\xi_0\|^p_{1,p;s_\delta} + \|\xi_1\|^p_{1,p;s_\delta}),
\end{aligned}
\tag{33}
$$

for some constants C_9, C_{10}. The last inequality here follows by comparing the weights s_δ and $\alpha_{R,\delta}$.

Summing up, we obtain from (25), (28), (32), (33) the desired inequality (24). This completes the proof of Proposition 4.7.5, hence also of Proposition 4.7.3. □

Proof of Proposition 4.7.1. Fix $\delta > 0$ small enough (so that all the previous estimates hold for this δ). We first claim that $\|D'_{u_R} \circ \widetilde{Q}_R - \mathbb{1}\|'_{0,p;\alpha_{R,\delta}} \to 0$ when $R \to \infty$. To see this, we start with the following identities that easily follow from (22):

$$
\begin{aligned}
D_{u_1,u_0} \circ Q_0 = \mathbb{1}, &\quad D_{u_1,u_0} \circ A = 0 \\
d\mathcal{G}_{1,0} \circ Q_0 = 0, &\quad d\mathcal{G}_{1,0} \circ A = \mathbb{1}.
\end{aligned}
\tag{34}
$$

Let $\eta \in \mathcal{E}^{0,p;\delta}_{u_R}$, $\overrightarrow{a} \in E$. A simple computation gives:

$$
\begin{aligned}
&D'_{u_R}\widetilde{Q}_R(\eta, \overrightarrow{a}) - (\eta, \overrightarrow{a}) = \\
&\qquad \Big(D_{u_R} \circ I^R \circ Q_0(J^R_{*,1}\sigma_0^- \eta, J^R_{*,0}\sigma_0^+ \eta) - \eta + D_{u_R} \circ I^R \circ A(\overrightarrow{a}), \\
&\qquad d\mathcal{G}_R \circ I^R \circ Q_0(J^R_{*,1}\sigma_0^- \eta, J^R_{*,0}\sigma_0^+ \eta) + d\mathcal{G}_R \circ I^R \circ A(\overrightarrow{a}) - \overrightarrow{a}\Big).
\end{aligned}
\tag{35}
$$

Using the definition of I^R and (34) we have

$$
d\mathcal{G}_R \circ I^R \circ Q_0 = d\mathcal{G}_{1,0} \circ Q_0 = 0, \quad d\mathcal{G}_R \circ I^R \circ A(\overrightarrow{a}) = d\mathcal{G}_{1,0} \circ A(\overrightarrow{a}) = \overrightarrow{a}.
$$

Substituting this in (35) we see that we have to estimate only the first component of (35). For the last term in the first component we have:

$$
\begin{aligned}
(36)\quad & \|D_{u_R} \circ I^R \circ A(\vec{a})\|_{0,p;\alpha_{R,\delta}} \\
& \leq \|D_{u_R} \circ I^R \circ A(\vec{a}) - (J^R_{1,*} D_{u_1} \oplus J^R_{0,*} D_{u_0}) \circ A(\vec{a})\|_{0,p;\alpha_{R,\delta}} \\
& \quad + \|(J^R_{1,*} D_{u_1} \oplus J^R_{0,*} D_{u_0}) \circ A(\vec{a})\|_{0,p;\alpha_{R,\delta}}.
\end{aligned}
$$

By (34) the last term in (36) is 0. Applying Proposition 4.7.2 to the first summand of inequality (36) we get:

$$
\|D_{u_R} \circ I^R \circ A(\vec{a})\|_{0,p;\alpha_{R,\delta}} \leq C_{4.7.2}(R)\|A\||\vec{a}| \tag{37}
$$

for some constants $C_{4.7.2}(R)$ that satisfy $\lim_{R\to\infty} C_{4.7.2}(R) = 0$.

As for the first two terms in the first component of (35) we have by Proposition 4.7.2:

$$
\begin{aligned}
(38)\quad & \|D_{u_R} \circ \widetilde{Q}_R(\eta, \vec{0}) - \eta\|_{0,p;\alpha_{R,\delta}} = \|D_{u_R} \circ I^R \circ Q_0(J^R_{*,1}\sigma_0^- \eta, J^R_{*,0}\sigma_0^+ \eta) - \eta\|_{0,p;\alpha_{R,\delta}} \\
& \leq \|(J^R_{1,*} \oplus J^R_{0,*}) \circ D_{u_1,u_0} \circ Q_0(J^R_{*,1}\sigma_0^- \eta, J^R_{*,0}\sigma_0^+ \eta) - \eta\|_{0,p;\alpha_{R,\delta}} \\
& \quad + C_{4.7.2}(R)\|Q_0(J^R_{*,1}\sigma_0^- \eta, J^R_{*,0}\sigma_0^+ \eta)\|_{1,p;s_\delta} \\
& \leq \|J^R_{1,*}J^R_{*,1}\sigma_0^- \eta + J^R_{0,*}J^R_{*,0}\sigma_0^+ \eta - \eta\|_{0,p;\alpha_{R,\delta}} \\
& \quad + C_{4.7.2}(R)\|Q_0\|_{s_\delta}\big(\|J^R_{*,1}\sigma_0^- \eta\|_{0,p;s_\delta} + \|J^R_{*,0}\sigma_0^+ \eta\|_{0,p;s_\delta}\big).
\end{aligned}
$$

By the 3'rd identity in (21) the first term in the second to last line of (38) is 0, hence by Lemma 4.7.4 we now get

$$
\|D_{u_R} \circ \widetilde{Q}_R \eta - \eta\|_{0,p;\alpha_{R,\delta}} \leq 2C_{4.7.2}(R)\|Q_0\|_{s_\delta} C_{4.7.4}\|\eta\|_{0,p;\alpha_{R,\delta}}.
$$

This together with (37) proves that $\lim_{R\to\infty} \|D'_{u_R} \circ \widetilde{Q}_R - \mathbb{1}\|'_{0,p;\alpha_{R,\delta}} = 0$.

Let $R_0 > 0$ be large enough so that $\|D'_{u_R} \circ \widetilde{Q}_R - \mathbb{1}\|'_{0,p;\alpha_{R,\delta}} \leq \frac{1}{2}$ for every $R \geq R_0$. Define

$$
Q_R : \mathcal{E}^{0,p;\delta}_{u_R} \oplus E \to T^{0,p;\delta}_{u_R}, \qquad Q_R = \widetilde{Q}_R \circ \sum_{k=0}^{\infty} (\mathbb{1} - D'_{u_R} \circ \widetilde{Q}_R)^k.
$$

Clearly Q_R is a right inverse to D'_{u_R} and $\|Q_R\|_{\alpha_{R,\delta}} \leq 2\|\widetilde{Q}_R\|_{\alpha_{R,\delta}} \leq 2C_{4.7.3}(\delta)$ for $R \gg 0$, i.e. Q_R is uniformly bounded. □

4.8. **The implicit function theorem.**

Theorem 4.8.1 (See Proposition A.3.4 from [44]). *Let X, Y be Banach spaces, $\mathcal{U} \subset X$ an open subset and $f : \mathcal{U} \to Y$ a C^1-map. Let $x_0 \in \mathcal{U}$ be such that $df(x_0) : X \to Y$ is surjective and has a bounded right inverse $Q : Y \to X$. Let $r, K > 0$ be such that $B_{x_0}(r) \subset \mathcal{U}$, $\|Q\| \leq K$ and such that:*

(1) $\|x - x_0\| < r \Longrightarrow \|df(x) - df(x_0)\| < \frac{1}{2K}$.

(2) $\|f(x_0)\| < \frac{r}{4K}$.

Then there exists $x \in X$ *with the following properties:*

(1) $f(x) = 0$.
(2) $\|x - x_0\| \leq r$.

In fact we have $\|x - x_0\| \leq 2K\|f(x_0)\|$.

4.8.1. *Quadratic estimates.* In the following proposition we endow the space $\mathcal{E}^{0,p;\delta}_{u_R} \oplus E$ with the norm $\|(\eta, \overrightarrow{a})\|'_{0,p;\alpha_{R,\delta}} = \|\eta\|_{0,p;\alpha_{R,\delta}} + |\overrightarrow{a}|$.

Proposition 4.8.2 (See Proposition 3.5.3 and Remark 3.5.5 in [44])**.** *For every* $c_0 > 0$ *there exist constants* $C_{4.8.2} = C_{4.8.2}(c_0) > 0$, $R_0 = R_0(c_0) > 1$, $\delta_0 = \delta_0(c_0) > 0$ *such that for every* $0 < \delta \leq \delta_0$, $R \geq R_0$, $\xi \in T^{1,p;\delta}_{u_R}$ *with* $\|\xi\|_{L^\infty} \leq c_0$ *we have:*

$$\|d\mathcal{F}'_{u_R}(\xi)\xi' - D'_{u_R}\xi'\|'_{0,p;\alpha_{R,\delta}} \leq C_{4.8.2}\|\xi\|_{1,p;\alpha_{R,\delta}}\|\xi'\|_{1,p;\alpha_{R,\delta}}, \quad \forall \xi' \in T^{1,p;\delta}_{u_R}. \tag{39}$$

In other words, the norm of the operator $d\mathcal{F}'_{u_R}(\xi) - D'_{u_R}$ *(with respect to* $\|\cdot\|_{1,p;\alpha_{R,\delta}}$ *and* $\|\cdot\|'_{0,p;\alpha_{R,\delta}}$*) satisfies* $\|d\mathcal{F}'_{u_R}(\xi) - D'_{u_R}\| \leq C_{4.8.2}\|\xi\|_{1,p;\alpha_{R,\delta}}$ *whenever* $\|\xi\|_{L^\infty} \leq c_0$.

Moreover, whenever $\xi_0, \xi \in T^{1,p;\delta}_{u_R}$ *satisfy* $\|\xi\|_{L^\infty}, \|\xi_0\|_{L^\infty} \leq c_0$ *we have for every* $R \geq R_0$, $0 < \delta \leq \delta_0$*:*

$$\|\mathcal{F}'_{u_R}(\xi_0 + \xi) - \mathcal{F}'_{u_R}(\xi_0) - d\mathcal{F}'_{u_R}(\xi_0)\xi\|'_{0,p;\alpha_{R,\delta}} \leq C_{4.8.2}\|\xi\|_{L^\infty}\|\xi\|_{1,p;\alpha_{R,\delta}}. \tag{40}$$

Outline of the proof. The proof of the estimate (39) is essentially given in [44] (Proposition 3.5.3). To adapt that proof to our case we use the following additional ingredients:

(1) In [44] the pointwise norms $|\cdot|$ are taken with respect to the metric $g_{\omega,J}$ but we work with the metric $g_{\omega,J,L}$. However L is compact and $g_{\omega,J,L} = g_{\omega,J}$ outside a small neighbourhood of L hence the two pointwise norms are comparable.
(2) $\|du_R\|_{0,p;\alpha_{R,\delta}}$, $\|du_R\|_{L^\infty}$ are uniformly bounded in R by Proposition 4.5.2.
(3) Due to Proposition 4.5.1 we have $\|\cdot\|_{L^\infty} \leq C_{4.5.1}\|\cdot\|_{1,p;\alpha_{R,\delta}}$.
(4) Standard arguments involving ode's show that for every $\xi_\infty \in T_{u_1(\infty)}(L)$ we have $|\nabla P_{u_R}\xi_\infty(\tau,t)| \leq C_1 e^{-c_1(\tau-5R)}|\xi_\infty|$ for every $\tau \geq 5R$ for some constants C_1, c_1 that do not depend on R. An analogous estimate holds for $|\nabla P_{u_R}\xi_{-\infty}(\tau,t)|$.

It follows that $|\nabla\xi(\tau,t)| \leq |\nabla(\xi - P_{u_R}\xi_\infty(\tau,t))| + C_1 e^{-c_1(\tau-5R)}|\xi_\infty|$ for every $\tau \geq 5R$. By a simple computation it now follows that $\int_{S[5R,\infty)} \alpha_{R,\delta}|\nabla\xi|^p \leq C_2\|\xi\|^p_{1,p;\alpha_{R,\delta}}$ for some constant C_2 that does not depend on R. A similar estimate holds for the integral over $S(-\infty,-5R]$. The rest of the proof is as in [44] using the additional estimates 1-4. The proof of estimate (40) is similar (see Remark 3.5.5 in [44]). $\square$

4.8.2. *Proof of Theorem 4.1.2.*

Proof of the existence statement of Theorem 4.1.2. Fix $\delta > 0$ small enough so that all the previous estimates hold for this δ. By Proposition 4.7.1 there exists $K > 0$ such that $\|Q_R\|_{\alpha_{R,\delta}} \leq K$ for $R \gg 0$. Applying Proposition 4.8.2 with $c_0 = K$ we obtain a constant $C(K)$ such that for every $R \gg 0$ we have

$$\|\xi\|_{L^\infty} \leq C(K) \Longrightarrow \|d\mathcal{F}'_{u_R}(\xi) - D'_{u_R}\|_{\alpha_{R,\delta}} \leq C(K)\|\xi\|_{1,p;\alpha_{R,\delta}}. \tag{41}$$

By Proposition 4.5.1, there exists $C_{4.5.1} > 0$ such that $\|\xi\|_{L^\infty} \leq C_{4.5.1}\|\xi\|_{1,p;\alpha_{R,\delta}}$ for every $R > 0$, $\xi \in T^{1,p;\delta}_{u_R}$. Fix $0 < r_0 < \min\{\frac{1}{2KC(K)}, \frac{K}{C_{4.5.1}}\}$. If $\|\xi\|_{1,p;\alpha_{R,\delta}} < r_0$ then $\|\xi\|_{L^\infty} \leq C_{4.5.1}r_0 < K$ hence

$$\|d\mathcal{F}'_{u_R}(\xi) - D'_{u_R}\|_{\alpha_{R,\delta}} \leq C(K)\|\xi\|_{1,p;\alpha_{R,\delta}} < \frac{1}{2K}.$$

Note that for $R \gg 0$, $\mathcal{F}'_{u_R}(0) = \big(\overline{\partial}_J u_R, 0\big)$, hence by Proposition 4.5.2

$$\|\mathcal{F}'_{u_R}(0)\|'_{0,p;\alpha_{R,\delta}} = \|\overline{\partial}_J u_R\|_{0,p;\alpha_{R,\delta}} \leq C'_{4.5.2}e^{-c'_{4.5.2}R} \tag{42}$$

for some constants $C'_{4.5.2}, c'_{4.5.2} > 0$. Thus by taking $R \gg 0$ we may assume that $\|\mathcal{F}'_{u_R}(0)\|'_{0,p;\alpha_{R,\delta}} \leq \frac{r_0}{4K}$. We now apply Theorem 4.8.1 with:

$$\mathcal{U} = X = \big(T^{1,p;\delta}_{u_R}, \|\cdot\|_{1,p;\alpha_{R,\delta}}\big), \quad Y = \big(\mathcal{E}^{0,p;\delta}_{u_R} \oplus E, \|\cdot\|'_{0,p;\alpha_{R,\delta}}\big), \quad x_0 = 0, \quad f = \mathcal{F}'_{u_R}.$$

By Theorem 4.8.1 we obtain $\xi_R \in T^{1,p;\delta}_{u_R}$ with $\|\xi_R\|_{1,p;\alpha_{R,\delta}} \leq r_0$ such that $\mathcal{F}'_{u_R}(\xi_R) = (0,0)$, i.e. $v_R = \exp_{u_R}(\xi_R)$ satisfies:

(1) $\overline{\partial}_J v_R = 0$.
(2) $G_\pm(v_R(\pm\infty)) = 0$.
(3) $G_1(v_R(-5R - \tau_*, \frac{1}{2})) = 0$, $G_0(v_R(5R + \tau_*, \frac{1}{2})) = 0$.

By elliptic regularity v_R is actually smooth. Note that v_R has finite energy (see Remark 4.4.2-(4)) hence by the removal of boundary singularities theorem of Oh [47], $v_R \circ \lambda^{-1}$ extends smoothly to D.

Next, note that by Theorem 4.8.1 we actually have $\|\xi_R\|_{1,p;\alpha_{R,\delta}} \leq 2K\|\mathcal{F}'_{u_R}(0)\|'_{0,p;\alpha_{R,\delta}}$. From (42) we obtain $\|\xi_R\|_{1,p;\alpha_{R,\delta}} \xrightarrow[R\to\infty]{} 0$. It follows from Proposition 4.5.1 that we also have $\|\xi_R\|_{L^\infty} \xrightarrow[R\to\infty]{} 0$. Clearly for $R \gg 0$, $[v_R] = [u_R] = A_1 + A_0 \in H_2(M, L)$.

We claim that for $R \gg 0$ we have $v_R(\pm\infty) \in W_\pm$, $v_R(-5R-\tau_*, \frac{1}{2}) \in W_1$, $v_R(5R+\tau_*, \frac{1}{2}) \in W_0$. Indeed, since $\|\xi_R\|_{L^\infty} \xrightarrow[R\to\infty]{} 0$ then for $R \gg 0$ the points $v_R(-\infty)$, $v_R(-5R - \tau_*, \frac{1}{2})$, $v_R(5R + \tau_*, \frac{1}{2})$, $v_R(\infty)$ lie in arbitrarily small neighbourhoods of $u_1(-\infty)$, $u_1(-\tau_*, \frac{1}{2})$, $u_0(\tau_*, \frac{1}{2})$, $u_0(\infty)$ respectively. By the definition of $G_\pm, G_1, G_0$ (see Section 4.6) it follows that $v_R(\pm\infty) \in W_\pm$, $v_R(-5R - \tau_*, \frac{1}{2}) \in W_1, v_R(5R + \tau_*, \frac{1}{2}) \in W_0$.

Put $s_R = \lambda^{-1}(5R + \tau_*, \frac{1}{2}) \in D$. Then $s_R \xrightarrow[R\to\infty]{} 1$ and $(v_R \circ \lambda^{-1}, s_R) \in \mathcal{M}(A, J; \mathcal{C}(\mathbf{h}))$ for $R \gg 0$.

To complete the proof of the existence part of the theorem it remains to show that v_R together with the marked points $(-\infty, (-5R-\tau_*, \frac{1}{2}), (5R+\tau_*, \frac{1}{2}), \infty)$ converges to (u_1, u_0) with the marked points $(-\infty, (-\tau_*, \frac{1}{2})), ((\tau_*, \frac{1}{2}), \infty)$ in the Gromov topology as $R \to \infty$. (See [32] for more details on Gromov compactness for disks.)

For this aim we have to find biholomorphisms $\rho^1_R, \rho^0_R : \widehat{S} \to \widehat{S}$ such that:

(v-1) $v_R \circ \rho^1_R \xrightarrow[R\to\infty]{} u_1$, $v_R \circ \rho^0_R \xrightarrow[R\to\infty]{} u_0$ uniformly on compact subsets of $\widehat{S} \setminus \{\infty\}$, of $\widehat{S} \setminus \{-\infty\}$ respectively.

(v-2) $(\rho^1_R)^{-1} \circ \rho^0_R \xrightarrow[R\to\infty]{} \infty$, $(\rho^0_R)^{-1} \circ \rho^1_R \xrightarrow[R\to\infty]{} -\infty$ uniformly on compact subsets of $\widehat{S} \setminus \{-\infty\}$, of $\widehat{S} \setminus \{\infty\}$ respectively.

(v-3) $(\rho^1_R)^{-1}(-\infty) \xrightarrow[R\to\infty]{} -\infty$, $(\rho^1_R)^{-1}(-5R - \tau_*, \frac{1}{2}) \xrightarrow[R\to\infty]{} (-\tau_*, \frac{1}{2})$.

(v-4) $(\rho^0_R)^{-1}(\infty) \xrightarrow[R\to\infty]{} \infty$, $(\rho^0_R)^{-1}(5R + \tau_*, \frac{1}{2}) \xrightarrow[R\to\infty]{} (\tau_*, \frac{1}{2})$.

To prove this define $\rho^1_R(\tau, t) = (\tau - 5R, t)$, $\rho^0_R(\tau, t) = (\tau + 5R, t)$. Clearly ρ^1_R, ρ^0_R satisfy properties (v-2)–(v-4) above. Property (v-1) follows easily from the definition of u_R (see (16) in Section 4.5) and the fact that $v_R = \exp_{u_R}(\xi_R)$ with $\|\xi_R\|_{L^\infty} \xrightarrow[R\to\infty]{} 0$. This concludes the proof of the existence statement in Theorem 4.1.2.

□

Proof of the uniqueness statement of Theorem 4.1.2. We continue to use the notation introduced above in the proof of the existence part.

Let $(w_n, \tau_n) \in \mathcal{M}(A, J; \mathcal{C}(\mathbf{h}))$ be a sequence that converges with the marked points $(-\infty, (-\tau_n, \frac{1}{2}), (\tau_n, \frac{1}{2}), \infty)$ to (u_1, u_0) with the marked points $(-\infty, (-\tau_*, \frac{1}{2})), ((\tau_*, \frac{1}{2}), \infty)$ in the Gromov topology. We shall prove that for every large enough n we have $(w_n, \tau_n) = (v_{R_n}, 5R_n + \tau_*)$ for some R_n's with $R_n \xrightarrow[n\to\infty]{} \infty$.

The implicit function Theorem 4.8.1 is not enough to prove this statement although there is an extension of Theorem 4.8.1 which states that the solution x is unique among solutions that satisfy $x \in \operatorname{image} Q$ and $x \in B_{x_0}(r)$. In our case this would require to prove that $w_n = \exp_{u_{R_n}}(\xi^{(w)}_n)$ for some $\xi^{(w)}_n \in T^{1,p;\delta}_{u_{R_n}}$ with $\|\xi^{(w)}_n\|_{1,p;\alpha_{R_n},\delta} \xrightarrow[n\to\infty]{} 0$. Instead, we shall prove a similar statement in the C^0-topology and use the quadratic estimates of Proposition 4.8.2 to deduce the uniqueness.

We first fix $\delta > 0$ small enough so that all the previous estimates hold for this δ and such that all the operators D'_{u_R}, $R \gg 0$, are Fredholm for this δ (see Section 4.5.1.) Note that u_R, v_R, w_n, all extend smoothly to $\widehat{S} \approx D$ hence they belong to $W^{1,p;\delta}$ for *every* small enough $\delta > 0$. (See Remark 4.4.2-(3).)

By the definition of Gromov convergence with marked points (see e.g. [32]) we have two sequences of biholomorphisms $\varphi_n^1, \varphi_n^0 : \widehat{S} \to \widehat{S}$ such that:

(w-1) $w_n \circ \varphi_n^1 \xrightarrow[n\to\infty]{} u_1$, $w_n \circ \varphi_n^0 \xrightarrow[n\to\infty]{} u_0$ uniformly on compact subsets of $\widehat{S} \setminus \{\infty\}$, of $\widehat{S} \setminus \{-\infty\}$ respectively.

(w-2) $(\varphi_n^1)^{-1} \circ \varphi_n^0 \xrightarrow[n\to\infty]{} \infty$, $(\varphi_n^0)^{-1} \circ \varphi_n^1 \xrightarrow[n\to\infty]{} -\infty$ uniformly on compact subsets of $\widehat{S} \setminus \{-\infty\}$, of $\widehat{S} \setminus \{\infty\}$ respectively.

(w-3) $(\varphi_n^1)^{-1}(-\infty) \xrightarrow[n\to\infty]{} -\infty$, $(\varphi_n^0)^{-1}(\infty) \xrightarrow[n\to\infty]{} \infty$.

(w-4) $(\varphi_n^1)^{-1}(-\tau_n, \frac{1}{2}) \xrightarrow[n\to\infty]{} (-\tau_*, \frac{1}{2})$, $(\varphi_n^0)^{-1}(\tau_n, \frac{1}{2}) \xrightarrow[n\to\infty]{} (\tau_*, \frac{1}{2})$.

Define
$$R_n = \frac{\tau_n - \tau_*}{5}, \quad \text{so that } 5R_n + \tau_* = \tau_n.$$
We have:

- $(\varphi_n^1)^{-1} \circ \rho_{R_n}^1(-\infty) \xrightarrow[n\to\infty]{} -\infty$, $(\varphi_n^1)^{-1} \circ \rho_{R_n}^1(-\tau_*, \frac{1}{2}) \xrightarrow[n\to\infty]{} (-\tau_*, \frac{1}{2})$.
- $(\varphi_n^0)^{-1} \circ \rho_{R_n}^0(\infty) \xrightarrow[n\to\infty]{} \infty$, $(\varphi_n^1)^{-1} \circ \rho_{R_n}^0(\tau_*, \frac{1}{2}) \xrightarrow[n\to\infty]{} (\tau_*, \frac{1}{2})$.

A simple argument involving Möbius transformations implies that

$$(\varphi_n^1)^{-1} \circ \rho_{R_n}^1 \xrightarrow[n\to\infty]{} \mathbb{1}, \quad (\varphi_n^0)^{-1} \circ \rho_{R_n}^0 \xrightarrow[n\to\infty]{} \mathbb{1} \quad \text{uniformly.} \tag{43}$$

Moreover, it follows from (w-2) that $R_n, \tau_n \xrightarrow[n\to\infty]{} \infty$.

We now have the following

Lemma 4.8.3. $\sup_{z\in\widehat{S}} \mathrm{dist}\big(w_n(z), u_{R_n}(z)\big) \xrightarrow[n\to\infty]{} 0$.

We defer the proof of this lemma to Section 4.8.3 and continue with the proof of our theorem. By Lemma 4.8.3, for $n \gg 1$ there exist $\xi_n^{(w)} \in T^{1,p;\delta}_{u_{R_n}}$ with $\|\xi_n^{(w)}\|_{L^\infty} \xrightarrow[n\to\infty]{} 0$ such that $w_n = \exp_{u_{R_n}}(\xi_n^{(w)})$. We shall prove that $\xi_n^{(w)} = \xi_{R_n}$ for $n \gg 1$, hence $w_n = v_{R_n}$.

We start with the observation that for $n \gg 1$ the operator $Q_{R_n} : \mathcal{E}^{0,p;\delta}_{u_{R_n}} \oplus E \to T^{1,p;\delta}_{u_{R_n}}$ is bijective. To see this, recall that $D'_{u_{R_n}} : T^{1,p;\delta}_{u_{R_n}} \to \mathcal{E}^{0,p;\delta}_{u_{R_n}} \oplus E$ is Fredholm and its index is $\mu(A_1 + A_0) + k_- + k_+ + k_1 + k_0 - 5 \dim L$ which by our assumptions is 0. As $D'_{u_{R_n}}$ is surjective for $n \gg 1$ it must be bijective, hence its inverse $Q_{u_{R_n}}$ is bijective too.

To finish the proof, we use an argument due to McDuff and Salamon (see the proof of Corollary 3.5.6 in [44]). Put $\xi'_n = \xi_n^{(w)} - \xi_{R_n}$. Since $Q_{u_{R_n}}$ is bijective we have $\xi'_n = Q_{u_{R_n}} \circ D'_{u_{R_n}} \xi'_n$ and $\mathcal{F}'_{u_{R_n}}(\xi_{R_n}) = 0$, $\mathcal{F}'_{u_{R_n}}(\xi_{R_n} + \xi'_n) = 0$. By Proposition 4.7.1 there exists a constant $C_{4.7.1}(\delta)$ such that:

$$\begin{aligned} \|\xi'_n\|_{1,p;\alpha_{R_n},\delta} \leq & C_{4.7.1}(\delta)\|D'_{u_{R_n}}\xi'_n\|'_{0,p;\alpha_{R_n},\delta} \\ \leq & C_{4.7.1}(\delta)\|\mathcal{F}'_{u_{R_n}}(\xi_{R_n} + \xi'_n) - \mathcal{F}'_{u_{R_n}}(\xi_{R_n}) - d\mathcal{F}'_{u_{R_n}}(\xi_{R_n})\xi'_n\|'_{0,p;\alpha_{R_n},\delta} \\ & + C_{4.7.1}(\delta)\|\big(d\mathcal{F}'_{u_{R_n}}(\xi_{R_n}) - D'_{u_{R_n}}\big)\xi'_n\|'_{0,p;\alpha_{R_n},\delta}. \end{aligned} \tag{44}$$

Applying Proposition 4.8.2 to both terms of the very right-hand side of (44) we get

$$\begin{aligned}\|\xi'_n\|_{1,p;\alpha_{R_n},\delta} &\leq C_{4.7.1}(\delta)C_{4.8.2}\big(\|\xi'_n\|_{L^\infty} + \|\xi_{R_n}\|_{1,p;\alpha_{R_n},\delta}\big)\|\xi'_n\|_{1,p;\alpha_{R_n},\delta} \\ &\leq C_{4.7.1}(\delta)C_{4.8.2}\big(\|\xi_n^{(w)}\|_{L^\infty} + \|\xi_{R_n}\|_{L^\infty} + \|\xi_{R_n}\|_{1,p;\alpha_{R_n},\delta}\big)\|\xi'_n\|_{1,p;\alpha_{R_n},\delta}.\end{aligned} \tag{45}$$

Recall that $\|\xi_n^{(w)}\|_{L^\infty}, \|\xi_{R_n}\|_{L^\infty}, \|\xi_{R_n}\|_{1,p;\alpha_{R_n},\delta} \xrightarrow[n\to\infty]{} 0$. Therefore for $n \gg 1$,

$$C_{4.7.1}(\delta)C_{4.8.2}\big(\|\xi_n^{(w)}\|_{L^\infty} + \|\xi_{R_n}\|_{L^\infty} + \|\xi_{R_n}\|_{1,p;\alpha_{R_n},\delta}\big) < 1.$$

It immediately follows from (45) that $\|\xi'_n\|_{1,p;\alpha_{R_n},\delta} = 0$ for $n \gg 1$. This completes the proof of the uniqueness statement of Theorem 4.1.2 (modulo the proof of Lemma 4.8.3). $\square$

4.8.3. *Proof of Lemma 4.8.3.* We first claim that for every $a \in \mathbb{R}$:

$$\sup_{z\in\widehat{S}[-\infty,-5R_n+a]} \operatorname{dist}\big(w_n(z), u_{R_n}(z)\big) \xrightarrow[n\to\infty]{} 0, \tag{46}$$

$$\sup_{z\in\widehat{S}[5R_n-a,\infty]} \operatorname{dist}\big(w_n(z), u_{R_n}(z)\big) \xrightarrow[n\to\infty]{} 0. \tag{47}$$

Proof of (46), (47)*.* We start with the inequality:

$$\begin{aligned}&\sup_{z\in\rho^1_{R_n}(\widehat{S}[-\infty,a])} \operatorname{dist}\big(w_n(z), v_{R_n}(z)\big) \\ &\leq \sup_{z\in\rho^1_{R_n}(\widehat{S}[-\infty,a])} \operatorname{dist}\big(w_n(z), u_1\circ(\varphi^1_n)^{-1}(z)\big) + \sup_{z\in\rho^1_{R_n}(\widehat{S}[-\infty,a])} \operatorname{dist}\big(v_{R_n}(z), u_1\circ(\rho^1_{R_n})^{-1}(z)\big) \\ &+ \sup_{z\in\rho^1_{R_n}(\widehat{S}[-\infty,a])} \operatorname{dist}\big(u_1\circ(\varphi^1_n)^{-1}(z), u_1\circ(\rho^1_{R_n})^{-1}(z)\big).\end{aligned} \tag{48}$$

The last term in the right-hand side of this inequality equals

$$\sup_{z\in\widehat{S}[-\infty,a]} \operatorname{dist}\big(u_1\circ(\varphi^1_n)^{-1}\circ\rho^1_{R_n}(z), u_1(z)\big),$$

hence by (43) it tends to 0 as $n \to \infty$. Similarly by (v-1) the second term in the right-hand side of (48) tends to 0 too as $n \to \infty$. As for the first term, due to (43) we have $\rho^1_{R_n}(\widehat{S}[-\infty,a]) \subset \varphi^1_n(\widehat{S}[-\infty,a+1])$ for $n \gg 1$, therefore

$$\begin{aligned}&\sup_{z\in\rho^1_{R_n}(\widehat{S}[-\infty,a])} \operatorname{dist}\big(w_n(z), u_1\circ(\varphi^1_n)^{-1}(z)\big) \\ &\leq \sup_{z\in\varphi^1_n(\widehat{S}[-\infty,a+1])} \operatorname{dist}\big(w_n(z), u_1\circ(\varphi^1_n)^{-1}(z)\big) = \sup_{z\in\widehat{S}[-\infty,a+1]} \operatorname{dist}\big(w_n\circ\varphi^1_n(z), u_1(z)\big) \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

It follows that

$$\sup_{z\in\widehat{S}[-\infty,-5R_n+a]} \operatorname{dist}\big(w_n(z), v_{R_n}(z)\big) = \sup_{z\in\rho^1_{R_n}(\widehat{S}[-\infty,a])} \operatorname{dist}\big(w_n(z), v_{R_n}(z)\big) \xrightarrow[n\to\infty]{} 0.$$

But $\sup_{z\in\widehat{S}} \operatorname{dist}\big(v_{R_n}(z), u_{R_n}(z)\big) \xrightarrow[n\to\infty]{} 0$, hence (46) follows. The proof of (47) is analogous. This concludes the proof of (46), (47). □

In the rest of the proof we shall need to use some estimates concerning energy of pseudo-holomorphic strips. Given a subset $T \subset \widehat{S}$ and a pseudo-holomorphic map u defined in a neighbourhood of T we denote by $E(u;T) = \int_T u^*\omega$ the energy of u along T. We first claim that:

$$\lim_{d\to\infty} \lim_{n\to\infty} E\big(w_n; \widehat{S}[-5R_n + d, 5R_n - d]\big) = 0, \tag{49}$$

$$\lim_{d\to\infty} \lim_{n\to\infty} E\big(v_{R_n}; \widehat{S}[-5R_n + d, 5R_n - d]\big) = 0. \tag{50}$$

Proof of (49), (50). To prove (49), write $E_{d,n} = E\big(w_n; \widehat{S}[-5R_n + d, 5R_n - d]\big)$. Recall that $A_1, A_0 \in H_2(M, L)$ are the classes of the disks u_1, u_0 respectively. We have:

$$\begin{aligned} E_{d,n} &= \int_{A_1+A_0} \omega - E(w_n; \widehat{S}[-\infty, -5R_n + d]) - E(w_n; \widehat{S}[5R_n - d, \infty]) \\ &= \int_{A_1+A_0} \omega - E(w_n \circ \varphi_n^1; (\varphi_n^1)^{-1} \circ \rho^1_{R_n}(\widehat{S}[-\infty, d])) \\ &\qquad\qquad - E(w_n \circ \varphi_n^0; (\varphi_n^0)^{-1} \circ \rho^0_{R_n}(\widehat{S}[-d, \infty])) \end{aligned} \tag{51}$$

By (43) we have for $n \gg 1$:

$$(\varphi_n^1)^{-1} \circ \rho^1_{R_n}(\widehat{S}[-\infty, d]) \supset \widehat{S}[-\infty, d-1],$$
$$(\varphi_n^0)^{-1} \circ \rho^0_{R_n}(\widehat{S}[-d, \infty]) \supset \widehat{S}[-d+1, \infty].$$

Putting this together with (51) we obtain:

$$E_{d,n} \le \int_{A_1} \omega - E(w_n \circ \varphi_n^1; \widehat{S}[-\infty, d-1]) + \int_{A_0} \omega - E(w_n \circ \varphi_n^0; \widehat{S}[-d+1, \infty])$$

It follows from (w-1) that

$$\lim_{n\to\infty} E_{d,n} \le E(u_1; \widehat{S}[d-1, \infty]) + E(u_0; \widehat{S}[-\infty, -d+1]),$$

hence $\lim_{d\to\infty} \lim_{n\to\infty} E_{d,n} = 0$. This proves (49). The proof of (50) is similar. □

We shall now need the following estimate relating the "variation" of a pseudo-holomorphic strip to its energy:

Lemma 4.8.4 (See Lemma A.6 in [32], Lemma 4.7.3 in [44])**.** *There exist positive constants $\delta_{4.8.4} = \delta_{4.8.4}(M, \omega, J)$, $C_{4.8.4}$ such that every J-holomorphic curve $u : S[r_1, r_2] \to M$ with $u\big([r_1, r_2] \times \{0\} \cup [r_1, r_2] \times \{1\}\big) \subset L$ and with $E_{r_1,r_2} = E(u; S[r_1, r_2]) \le \delta_{4.8.4}$ satisfies:*

$$\sup_{z,z' \in S[r_1+1, r_2-1]} \operatorname{dist}\big(u(z), u(z')\big) \le C_{4.8.4} \sqrt{E_{r_1,r_2}}.$$

The final step in the proof of Lemma 4.8.3. In view of (46), (47), in order to prove Lemma 4.8.3 it is enough to prove the following: *For every $\epsilon > 0$ there exists $a_0 = a_0(\epsilon)$ and $N_0 = N_0(\epsilon)$ such that for every $n \geq N_0$ we have:*

$$\sup_{z \in \widehat{S}[-5R_n + a_0, 5R_n - a_0]} \operatorname{dist}\big(w_n(z), u_{R_n}(z)\big) < \epsilon. \tag{52}$$

We prove (52). Let $\epsilon > 0$. By (49), (50) there exists $a_0 = a_0(\epsilon)$ and $N_0 = N_0(\epsilon)$ such that for every $n \geq N_0$:

$$E\big(w_n; \widehat{S}[-5R_n + a_0 - 1, 5R_n - a_0 + 1]\big) \leq \min\Big\{\delta_{4.8.4}, \frac{\epsilon^2}{9C^2_{4.8.4}}\Big\}, \tag{53}$$

$$E\big(v_{R_n}; \widehat{S}[-5R_n + a_0 - 1, 5R_n - a_0 + 1]\big) \leq \min\Big\{\delta_{4.8.4}, \frac{\epsilon^2}{9C^2_{4.8.4}}\Big\}. \tag{54}$$

We have for every $z \in \widehat{S}[-5R_n + a_0, 5R_n - a_0]$:

$$\begin{aligned} \operatorname{dist}\big(w_n(z), u_{R_n}(z)\big) \leq & \operatorname{dist}\big(w_n(z), w_n(-5R_n + a_0, \tfrac{1}{2})\big) \\ & + \operatorname{dist}\big(w_n(-5R_n + a_0, \tfrac{1}{2}), u_{R_n}(-5R_n + a_0, \tfrac{1}{2})\big) \\ & + \operatorname{dist}\big(u_{R_n}(-5R_n + a_0, \tfrac{1}{2}), u_{R_n}(z)\big). \end{aligned} \tag{55}$$

By (53) and Lemma 4.8.4, the first term in the right-hand side of (55) satisfies:

$$\operatorname{dist}\big(w_n(z), w_n(-5R_n + a_0, \tfrac{1}{2})\big) < \frac{\epsilon}{3}, \quad \forall z \in \widehat{S}[-5R_n + a_0, 5R_n - a_0], \; \forall n \geq N_0. \tag{56}$$

Increasing N_0 if necessary[1] we have from (46) that the second term in the right-hand side of (55) satisfies:

$$\operatorname{dist}\big(w_n(-5R_n + a_0, \tfrac{1}{2}), u_{R_n}(-5R_n + a_0, \tfrac{1}{2})\big) < \frac{\epsilon}{3}, \quad \forall n \geq N_0. \tag{57}$$

As for the last term in the right-hand side of (55) we have:

$$\begin{aligned} & \operatorname{dist}\big(u_{R_n}(-5R_n + a_0, \tfrac{1}{2}), u_{R_n}(z)\big) \\ & \leq 2 \sup_{z \in \widehat{S}} \operatorname{dist}\big(v_{R_n}(z), u_{R_n}(z)\big) + \operatorname{dist}\big(v_{R_n}(-5R_n + a_0, \tfrac{1}{2}), v_{R_n}(z)\big). \end{aligned}$$

Recall that $v_{R_n} = \exp_{u_{R_n}}(\xi_{R_n})$ with $\|\xi_{R_n}\|_{L^\infty} \xrightarrow[n \to \infty]{} 0$. Thus increasing N_0 once more if necessary, and by (54) and Lemma 4.8.4 we obtain

$$\operatorname{dist}\big(u_{R_n}(-5R_n + a_0, \tfrac{1}{2}), u_{R_n}(z)\big) < \frac{\epsilon}{3}, \quad \forall z \in \widehat{S}[-5R_n + a_0, 5R_n - a_0], \; \forall n \geq N_0. \tag{58}$$

Substituting the estimates (56), (57), (58) in (55) proves (52), hence concludes the proof of Lemma 4.8.3. □

[1] This increase may depend on $a_0 = a_0(\epsilon)$ which however has already been fixed.

4.9. **An auxiliary inequality.** Fix $p > 2$. Let $E \to S$ be a vector bundle endowed with a Riemannian metric with norm $|\cdot|$ and let ∇ be a metric connection on E.

Proposition 4.9.1. *There exists a constant $C_{4.9.1}$ such that for every $W^{1,p}_{\mathrm{loc}}$-section $\eta : S \to E$ and every $x \in S$*

$$|\eta(x)| \leq C_{4.9.1}\Big(\int_{K_x} |\eta|^p + |\nabla\eta|^p\Big)^{1/p},$$

where K_x can be taken to be either $S[x, x+1]$ or $S[x-1, x]$.

Proof. The proof for the case of a trivial vector bundle with trivial connection and standard metric is essentially contained in the proof of Lemma B.1.16 in [44]. To adapt the proof to the case of a general connection use Remark 3.5.1 in [44]. □

5. Proof of the main algebraic statement.

This section contains the proofs of the results stated in §2. Each of the first eight subsections below is focused on a part of the relations to be verified. Each such verification is based on three steps. First, appropriate moduli spaces are defined. Then, some regularity properties are established for these spaces along the lines in §3. Finally, by making use of the gluing results in §4, the desired relations are deduced out of the description of the boundary of the compactification of the moduli spaces. As many types (more than a dozen) moduli spaces will appear in these verifications it is useful to shortly summarize here the basic idea in the construction of all of them. Very likely, this construction will appear quite familiar to the reader in view of the "bubble tree" description of bubbling as it appears in [44] and, even more so, given the cluster moduli spaces as introduced in [23]. It is obviously a construction that naturally extends that of the pearl moduli spaces already introduced in §3.

All the moduli spaces used below consist of configurations modeled on planar trees with oriented edges with, roughly, the following properties. Each of the edges corresponds to a negative gradient trajectory of a Morse function. This Morse function might be defined on L or, sometimes, might be a Morse function defined on the ambient manifold M and more than a single Morse function can be used in the same tree. Each edge carries a marking indicating which function is used along that edge. The orientation of the edge corresponds to that of the negative gradient flow. The tree might have several entries but has a single exit. The entries and the exit are the only vertices of valence one and each of them corresponds to a critical point of the function marking the edge ending there. Each internal vertex has a marking by a certain Maslov class and corresponds to either a J-holomorphic disk with boundary on L or to a J holomorphic sphere in that class together with a number of marked points (situated on the boundary or the interior) equal in number to the valence of the vertex. These marked points correspond to the ends of the edges reaching the corresponding vertex so that the boundary marked points are ends of edges labeled by functions defined on L and the interior marked points are ends of edges labeled by functions defined on M. Depending on the valence and on the particular moduli spaces in question, specific incidence relations describe at which explicit marked point on the disk arrives which edge. Finally, constant J spheres and disks are allowed as long as they are stable in the sense that, the valence of the corresponding vertex is at least three if it corresponds to a sphere and, if this valence is two, then the vertex corresponds to a disk with one marked point in the interior and one on the boundary. All the vertices in this paper have valence at most 4.

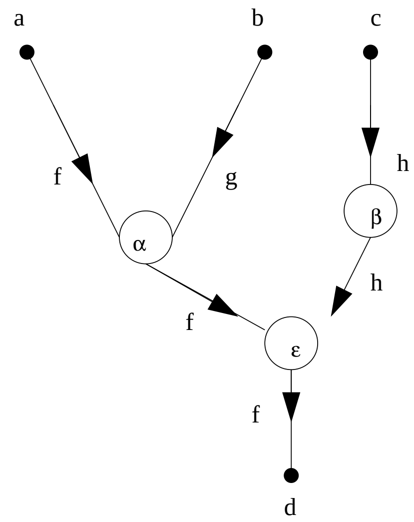

FIGURE 6. The functions $f, g, h : L \to \mathbb{R}$, $a, d \in \mathrm{Crit(f)}, \mathrm{b} \in \mathrm{Crit(g)}, \mathrm{c} \in \mathrm{Crit(h)}$; the three disks are of Maslov indexes, respectively, α, β and ϵ.

This general setup allows for a variety of moduli spaces but the basic rules are simple: a differential or a morphism is defined by using moduli spaces with one entry, an operation requires two entries, associativity requires three, the structures involving the ambient quantum homology require some internal marked points etc.

The proofs given below are very explicit in a number of cases, in the sense that the relevant moduli spaces are described in all detail as well as the related compactness arguments. This happens for the pearl moduli spaces in §5.1 as well as for the module structure in §5.3. In these cases the "tree" language summarized above is not explicitly used. However, this language does appear in the description of many of the other moduli spaces. In those cases, we only generally give the relevant special incidence relations needed and indicate what variants of the standard arguments used in §5.1, §5.3 are necessary in the proof.

5.1. **The pearl complex.** We recall that we have fixed an almost complex structure J which is compatible with ω in the sense that $\omega(X, JY)$ is a Riemannian metric We also fix a Morse function $f : L \to \mathbb{R}$ together with a metric ρ on L so that the pair (f, ρ) is Morse-Smale. The induced negative gradient flow of f is denoted by γ. We start by reformulating (in a slightly more general context) a definition from §3.1.

Definition 5.1.1. Given two points $a, b \in L$ we consider the moduli space $\mathcal{M}(a, b, \lambda; J, f, \rho)$ whose elements are objects $(t_0, u_0, t_1, u_1, t_2, \ldots u_k, t_{k+1}), k \in \mathbb{N}$ so that:

i. For $0 \le i \le k$, u_i is a non-constant J-holomorphic disk $u_i : (D, \partial D) \to (M, L)$.
ii. $\sum_i [u_i] = \lambda \in H_2(M, L; \mathbb{Z})$.
iii. For $1 \le i \le k$, $t_i \in (0, +\infty)$, $t_0, t_{k+1} \in (0, +\infty]$, so that if we put $a_i = u_i(-1)$, $b_i = u_i(1)$, then $\gamma_{t_i}(b_{i-1}) = a_i$ for all $i \le k$, moreover $\gamma_{t_{k+1}}(b_k) = b$ and $\gamma_{-t_0}(a_1) = a$ (of course, if t_0 is infinite this means that a is a critical point of f and similarly for t_{k+1}).

We then let $\mathcal{P}(x, y, \lambda; J, f, \rho)$ the moduli space obtained by dividing $\mathcal{M}(x, y, \lambda; J, f, \rho)$ by the action of the obvious reparametrization group. It is understood in the description above that k is not fixed. As a matter of convention we also want to include in our moduli spaces objects for $k = -1$: these are just the usual negative flow lines joining x to y. We will mostly use these moduli spaces when $x, y \in \mathrm{Crit(f)}$ and in this case this definition is coherent with that in §3.1. We will drop the decorations J, f, ρ when they are clear from the context.

We now define

$$\mathcal{C}^{+}(L; f, \rho, J) = (\mathbb{Z}/2\langle \mathrm{Crit(f)} \rangle \otimes \Lambda^{+}, \mathrm{d})$$

with the grading induced by $|x| = ind_f(x)$ for all $x \in \mathrm{Crit(f)}$ and with the differential defined by

$$dx = \sum_{|y| = |x| + \mu(\lambda) - 1} \#_{\mathbb{Z}_2} \mathcal{P}(x, y, \lambda) y t^{\bar{\mu}(\lambda)} .$$

The next proposition obviously implies the main part of point i. of Theorem 2.1.1 - we will discuss the augmentation issue at the end of the section.

Proposition 5.1.2. *For a generic choice of J the map d is well defined and is a differential. Two generic choices (f, ρ, J) and (f', ρ', J') produce chain complexes whose homologies are canonically isomorphic.*

The proof of this proposition occupies the rest of the section. The main ingredients are the transversality results in §3, the gluing results in §4 as well as the Gromov compactness theorem for holomorphic disks as described by Frauenfelder in [32].

We will use further the notation of section §3. In particular, we have:

$$\mathcal{P}(x, y, \lambda; J, f, \rho) = \bigcup_{\substack{k, \mathbf{A}=(A_0, \ldots, A_k) \\ \sum A_i = \lambda}} \mathcal{P}(x, y, \mathbf{A}; J, f, \rho) \ . \tag{59}$$

When $\delta = |x| - |y| + \mu(\lambda) - 1 \leq 1$ and for a generic J, we see from Proposition 3.1.3 that the spaces $\mathcal{P}(x, y, \mathbf{A}; J, f, \rho)$ are manifolds of dimension δ and, in case $\delta = 0$, these manifolds are already compact. By Gromov compactness there are only finitely many sequences $\mathbf{A} = (A_0, \ldots, A_k)$ of non zero homology classes so that A_i contains a J-holomorphic disk $0 \leq i \leq k$ and $\sum A_i = \lambda$. This means that the sum in the definition of the differential d is finite and so this sum is well defined. For further use we denote by $\mathcal{P}(x, y, \emptyset) = \mathcal{P}(x, y, 0; J, f, \rho)$ the moduli space of negative flow lines of f joining x to y. We will also use the notaion $\mathcal{P}(x, y, (\mathbf{A}', \mathbf{A}''); J, f, \rho)$ introduced in §3

5.1.1. *Verification of* $d^2 = 0$. Here is the main ingredient in showing that d is a differential:

Lemma 5.1.3. *Fix* $\delta = 1$ *and* $\mathbf{A} = (A_0, \ldots, A_k)$. *For a generic* J, *the natural Gromov compactification* $\overline{\mathcal{P}}(x, y, \mathbf{A})$ *is a topological* 1*-dimensional manifold whose boundary is described by:*

$$\partial \overline{\mathcal{P}}(x, y, \mathbf{A}) = \bigcup_{z, \mathbf{A}=(\mathbf{A}', \mathbf{A}'')} \mathcal{P}(x, z, \mathbf{A}') \times \mathcal{P}(z, y, \mathbf{A}'') \cup \mathcal{T} \cup \mathcal{T}' \ . \tag{60}$$

Where the terms $\mathcal{T}$ *and* $\mathcal{T}'$ *are given by:*

$$\mathcal{T} = \bigcup_{\mathbf{A}=(\mathbf{A}', \mathbf{A}''),\ \mathbf{A}', \mathbf{A}'' \neq \emptyset} \mathcal{P}(x, y, \mathbf{A}', \mathbf{A}'')$$

and

$$\mathcal{T}' = \bigcup_{i\ ;\ A_i = A_i' + A_i'',\ A_i', A_i'' \neq 0} \mathcal{P}(x, y, (A_0, \ldots A_{i-1}, A_i'), (A_i'', A_{i+1}, \ldots A_k)) \ .$$

We emphasize that in the first term of formula (60), in the decomposition $\mathbf{A} = (\mathbf{A}', \mathbf{A}'')$ it is possible that $\mathbf{A}' = \emptyset$ or $\mathbf{A}'' = \emptyset$.

We first remark that the lemma immediately implies $d^2 = 0$. Indeed, notice that, by the lemma, each term:

$$\mathcal{P}(x, y, \mathbf{A}', \mathbf{A}'')$$

with $\mathbf{A}' = (A_0, \ldots, A_i)$, $\mathbf{A}'' = (A_{i+1}, \ldots, A_k)$ appears as a boundary in precisely two moduli spaces: $\overline{\mathcal{P}}(x, y, (\mathbf{A}', \mathbf{A}''))$ and $\overline{\mathcal{P}}(x, y, (A_0 \ldots, A_{i-1}, A_i + A_{i+1}, A_{i+2}, \ldots, A_k))$.

This means that if we define:

$$\overline{\mathcal{P}}(x,y,\lambda) = \coprod_{\sum A_i=\lambda} \overline{\mathcal{P}}(x,y,\mathbf{A})\ ,$$

then

$$0 = \#_{\mathbb{Z}_2}(\partial\overline{\mathcal{P}}(x,y,\lambda)) = \#_{\mathbb{Z}_2}(\bigcup_{z,\lambda'+\lambda''=\lambda} \mathcal{P}(x,z,\lambda')\times\mathcal{P}(z,y,\lambda''))$$

which clearly means $d^2=0$.

Proof of Lemma 5.1.3. We start by making precise the compactification $\overline{\mathcal{P}}(x,y,\mathbf{A})$. We denote by $\mathcal{P}_L$ the space of continuous paths $\{\gamma:[0,b]\to L : b\geq 0\}$. We consider the subspace $\mathcal{P}_{f,\rho}\subset\mathcal{P}_L$ of those paths $\gamma:[0,b]\to L$ which are flow lines of $-\nabla_g f$ reparametrized so that $f(\gamma(t)) = f(\gamma(0))-t$. We allow that one or both of the ends of such of a flow line be a critical point of f. Exponential convergence close to critical points insures that this choice of parametrization is continuous when defined on the usual moduli space of negative flow lines (parametrized by time). The space $\mathcal{P}_{f,\rho}$ has a natural compactification $\overline{\mathcal{P}}_{f,\rho}$ formed by adding all broken flow lines to $\mathcal{P}_{f,\rho}$.

We now notice that $\mathcal{P}(x,y,\mathbf{A})$ is included in:

$$\mathcal{L} = \mathcal{P}_{f,\rho}\times\mathcal{M}(A_0,J)/G_{-1,1}\times\mathcal{P}_{f,\rho}\times\mathcal{M}(A_1,J)/G_{-1,1}\times\cdots\mathcal{P}_{f,\rho}\times\mathcal{M}(A_l,J)/G_{-1,1}\times\mathcal{P}_{f,\rho}\ .$$

There is a natural compactification $\overline{\mathcal{L}}$ of $\mathcal{L}$ obtained by compactifying each term $\mathcal{M}(A_i,J)$ in the sense of Gromov and by replacing $\mathcal{P}_{f,\rho}$ by $\overline{\mathcal{P}}_{f,\rho}$. We take on $\mathcal{P}(x,y,\mathbf{A})$ the induced topology and define its compactification to be the closure of this space inside $\overline{\mathcal{L}}$.

I. Structure of the compactification. The first part of the proof is to show that, for $\delta=1$ and a generic J, if $u\in\overline{\mathcal{P}}(x,y,\mathbf{A})\backslash\mathcal{P}(x,y,\mathbf{A})$ then u has the one of the forms given in the right hand side of (60). It is clear that $u=(\gamma_0,u_0,\gamma_1,\ldots u_k,\gamma_{k+1})$ with $u_i\in\overline{\mathcal{M}(A_i,J)/G_{-1,1}}$, $\gamma_i\in\overline{\mathcal{P}}_{f,\rho}$. To proceed with the proof it is useful to reinterpret in more combinatorial terms Propositions 3.1.3 and 3.1.6. Let $\mathcal{P}(x,y,\mathbf{A},s)\subset\overline{\mathcal{P}}(x,y,\mathbf{A})$ be the moduli space whose definition is identical with that of $\mathcal{P}(x,y,\mathbf{A})$ except that s of the flow lines γ_i are of length 0. Proposition 3.1.3 shows that, generically, if $\delta\leq 1$, then $\dim(\mathcal{P}(x,y,\mathbf{A},0))=\delta$ and Proposition 3.1.6 shows that $\dim(\mathcal{P}(x,y,\mathbf{A},1))=\delta-1$ (both spaces being void if the respective dimension is negative). The same argument also shows that $\mathcal{P}(x,y,\mathbf{A},s)=\emptyset$ whenever $s\geq 2$.

Returning to our $u\in\overline{\mathcal{P}}(x,y,\mathbf{A})$, denote by α_i the end of γ_i and by β_i the beginning of γ_{i+1}. Clearly, $\alpha_i,\beta_i\in Image(u_i)$. The structure of u_i is described by a bubbling tree (see [32] and [44]) which carries two marked points which are mapped to α_i and β_i. It is important to notice that these two marked points are distinct but they might both lie on a "ghost" (or constant) disk. We will denote these two marked points by -1 and $+1$ respectively (by a slight abuse in notation). In this tree let u_i^{-1} and u_i^{+1} be the two curves carrying -1 and $+1$ respectively, and consider the path of minimal length which joins the vertices u_i^{-1} and u_i^{+1}. In case $\alpha_i \neq \beta_i$ we denote by v_i the nodal curve formed by the union of all the vertices along this path. If $\alpha_i = \beta_i$ we let v_i be the constant map from the disk equal to α_i. We notice that v_i may be assumed not to contain any sphere component. Moreover $\omega(v_i) \leq \omega(u_i)$ with equality if and only if the tree is linear with u_i^{-1} and u_i^{+1} at its ends. We notice that $v = (\gamma_0, v_0, \ldots v_k, \gamma_{k+1}) \in \overline{\mathcal{P}}(x, y, \mathbf{B})$ where $\mathbf{B} = ([v_0], \ldots, [v_k])$. Let l_i be the number of components in v_i less one. Let b_i be the number of breaks in γ_i and let s be the number of flow lines γ_j which are of length 0. Our statement follows if we show, $\forall i \ \omega(v_i) = \omega(u_i)$ and $\sum l_i + \sum b_i + s = 1$.

The key observation is that, because the components of the v_i's are disks, we may rewrite in a unique way:

$$v \in \mathcal{P}(x, z_1, \mathbf{B}_1, s_1) \times \ldots \times \mathcal{P}(z_r, y, \mathbf{B}_{r+1}, s_{r+1})$$

and, in this case, $\sum s_r = \sum l_i + s$; $\sum b_i = r$.

We also have

$$2 = \delta + 1 \geq (|x| - |z_1| + \mu(B_1)) + (|z_1| - |z_2| + \mu(B_2)) + \ldots + (|z_r| - |y| + \mu(B_{r+1}))$$

with equality if and only if $[v] = [u]$. Now notice that it is not possible that $r > 1$ because in that case at least one of the parenthesis in the sum above would need to be ≤ 0 which implies that the respective space is void. We are thus left with two cases. The first is $r = 1$ and $(|x| - |z_1| + \mu(B_1)) = 1$, $(|z_1| - |y| + \mu(B_2)) = 1$. But this implies that $s_1 = 0$ and $s_2 = 0$ and $[v] = [u]$. The last case is $r = 0$. Then, as $N_L = 2$, we obtain $[v] = [u]$ and, as $u \notin \mathcal{P}(x, y, \mathbf{A})$ we get $s_1 = 1$.

II. Gluing. The fact that $\partial\overline{\mathcal{P}}(x, y, \mathbf{A})$ is contained in the union appearing on the right side of (60) follows from Gromov compactness combined with standard arguments from Morse theory.

Next, we need to show that each element appearing on the right side of (60) is a boundary point of $\overline{\mathcal{P}}(x, y, \mathbf{A})$ and moreover, that each such element corresponds to a *unique* end of the 1-dimensional manifold $\mathcal{P}(x, y, \mathbf{A})$ (hence $\overline{\mathcal{P}}(x, y, \mathbf{A})$ is indeed a compact manifold with boundary).

By the results in §3 we have that for a generic choice of J we have $\mathcal{P}(x, y, \mathbf{A}) = \mathcal{P}^{*,d}(x, y, \mathbf{A})$ for all x, y, $\mathbf{A}$ so that $|x| - |y| + \mu(\mathbf{A}) - 1 \leq 1$. This implies by standard Morse theory that the elements in the first term of the decomposition (60) can be glued in the obvious way. The uniqueness statement is also standard in this case.

The elements in $\mathcal{T}'$ are also seen as boundary points by first applying the results in §3 to show that the transversality necessary for the gluing described in §4 is achieved generically and then gluing as there. The uniqueness statement follows too from the results in §4. (See Corollary 4.2.1.)

Finally, we are left to deal with the elements in $\mathcal{T}$. Consider one such element $u = (\gamma_0, u_0, \gamma_1, \ldots, u_i, u_{i+1}, \ldots u_k, \gamma_{k+1}) \in \mathcal{P}(x, y, \mathbf{A}', \mathbf{A}'')$ with $\mathbf{A}' = (A_0, \ldots A_i)$ and $\mathbf{A}'' = (A_{i+1}, \ldots, A_k)$. The transversality proved in §3 implies immediately the result in this case if $p(u) = u_i(+1) = u_{i+1}(-1) \notin \mathrm{Crit(f)}$. But, again for a generic choice of J, it follows by inspecting the definition of $\mathcal{P}(x, y, \mathbf{A}', \mathbf{A}'')$ that $p(u)$ avoids the finite set $\mathrm{Crit(f)}$ whenever $\delta \leq 1$. □

5.1.2. *Invariance.* The purpose of this subsection is to show that for two generic sets of data (J, f, ρ) and (J', f', ρ') we can define a chain morphism

$$\phi : \mathcal{C}^+(L; f, \rho, J) \to \mathcal{C}^+(L; f', \rho', J')$$

which induces an isomorphism in homology. This morphism $\phi = \phi^{\overline{J}, F, G}$ depends on auxiliary data: a generic homotopy of almost complex structures $\overline{J}$ joining J to J' and a Morse cobordism (F, G) relating (f, ρ) to (f', ρ') (see [25] for the formal definition of Morse cobordisms). Of course, all this construction is typical for invariance proofs in Morse or Floer theories and, after defining the relevant moduli spaces, we will only sketch the rest of the proof as there are no new transversality issues compared with the last section and the gluing arguments are analogous to the ones before.

We now fix a smooth family of almost complex structures $\overline{J} = J_t,\ t \in [0,1]$ so that $J_0 = J$ and $J_1 = J'$. We also fix a Riemannian metric G on $L \times [0,1]$ so that $G|_{L \times \{0\}} = \rho$ and $G|_{L \times \{1\}} = \rho'$ as well as a Morse function $H : L \times [0,1] \to \mathbb{R}$ so that $H|_{L \times \{0\}}(x) = f(x) + h$ (for some constant h) $H|_{L \times \{1\}}(x) = f'(x)$, (H, G) is Morse-Smale with $\mathrm{Crit_i(H)} = \mathrm{Crit_{i-1}(f)} \times \{0\} \cup \mathrm{Crit_i(f')} \times \{1\}$ and, finally, $(\partial H / \partial t)(x, t) = 0$ for $t = 0, 1$, $(\partial H / \partial t)(x, t) < 0$ for $t \in (0, 1)$. It is easy to see how to construct such $(H, \overline{J})$ (we refer to [25] for an explicit such construction).

Given two critical points $x \in \mathrm{Crit(f)}$ and $y' \in \mathrm{Crit(f')}$ we define the moduli space $\mathcal{M}(x, y', \lambda; \overline{J}, H, G)$ by using a slightly modified version of Definition 5.1.1. The only changes are listed below:

- i'. The u_i's are so that for each u_i there is some $\tau_i \in [0,1]$ with $u_i : (D, \partial D) \to (M, L) \times \{\tau_i\} \subset (M, L) \times [0,1]$ and u_i is J_{τ_i}-holomorphic.
- iii'. The incidence relations at point iii. take place inside $L \times [0,1]$ with the flow of $-\nabla_G H$ on $L \times [0,1]$ taking the place of the flow $-\nabla_\rho(f)$.

We denote by $\mathcal{P}(x, y', \lambda; \overline{J}, H, G)$ the resulting moduli space after division by the reparametrization group. By inspecting the construction and proofs in §3 it is easy to see that all the arguments carry over when replacing the moduli spaces $\mathcal{M}(A, J)$ with moduli spaces

$\mathcal{M}(A, \overline{J})$ which are made of disks that are J_τ-holomorphic, $\tau \in [0,1]$ (and in the class $A \in \Lambda$) and, for each such disk, the relevant evaluation maps take values in $L \times \{\tau\}$. The key reason for this is that Lemmas 3.2.2 and 3.2.3 still apply when J is replaced with a generic one parametric family of almost complex structures $\overline{J}$ and this implies that the analogue of Proposition 3.1.3 remains true in this setting (the same remains true in fact even if for higher parametric families).

The conclusion is that for a generic choice of $\overline{J}$, H and G, if $\tilde{\delta} = |x| - |y'| + \mu(\lambda)$ then, for $\tilde{\delta} \leq 1$, the moduli space $\mathcal{P}(x, y', \lambda)$ is a manifold of dimension $\tilde{\delta}$, is compact if $\tilde{\delta} = 0$ and is void if $\tilde{\delta} \leq -1$. The compactification $\overline{\mathcal{P}}(x, y', \lambda)$ of $\mathcal{P}(x, y', \lambda)$ can then be defined as in the last section and, by the same method as there, we see that when $\tilde{\delta} = 1$, we have:

$$\begin{aligned} 0 = \#_{\mathbb{Z}_2}(\partial \overline{\mathcal{P}}(x, y', \lambda)) = \#_{\mathbb{Z}_2}(& \bigcup_{z \in \mathrm{Crit(f)}, \lambda' + \lambda'' = \lambda} \mathcal{P}(x, z, \lambda') \times \mathcal{P}(z, y', \lambda'')) + \\ & + \#_{\mathbb{Z}_2}(\bigcup_{z' \in \mathrm{Crit(f')}, \lambda' + \lambda'' = \lambda} \mathcal{P}(x, z', \lambda') \times \mathcal{P}(z', y', \lambda'')) \end{aligned} \tag{61}$$

where in both unions we take into account the cases when λ' or λ'' is null (in that case the relevant moduli spaces are just the usual moduli spaces of Morse trajectories). Of course, this identity also depends on a gluing argument in which J is no longer constant. However, this is a reasonably straightforward adaptation of the argument in §4 and so, for brevity, we will not make it explicit here.

For $x \in \mathrm{Crit(f)}$ we now put:

$$\phi^{\overline{J}, F, G}(x) = \sum_{y' \in \mathrm{Crit(f')},\ \lambda\ ;\ \mathrm{|x| - |y'|} + \mu(\lambda) = 0} \#_{\mathbb{Z}_2}(\mathcal{P}(x, y', \lambda; J, H, G)) y t^{\bar{\mu}(\lambda)} .$$

For generic choices of the data, formula (61) implies that $\phi^{\overline{J}, H, G}$ is a chain morphism.

It is very easy to see that this chain morphism induces an isomorphism in homology. Indeed, both the pearl complex differential and the morphism ϕ clearly respect the degree filtration. Therefore, ϕ induces a morphism between the spectral sequences $E(\phi) : E(J, f, \rho) \to (E'(J', f', \rho')$ associated to the two filtered complexes $\mathcal{C}^+(L; J, f, \rho)$ and $\mathcal{C}^+(L; J', f', \rho')$. It is clear that the E^0 terms in these spectral sequences only involve moduli spaces of Morse trajectories so that $E^1(\phi)$ is already an isomorphism. But this means that $H_*(\phi)$ is an isomorphism.

The argument needed to prove that this isomorphism is canonical - while essentially standard - is more complicated. First, we need to notice that if $f = f'$, $\overline{J}$ is the constant family $\overline{J}_\tau = J$, G is also the constant metric $G_\tau = \rho$ and H is a trivial homotopy in the sense that $H(x,t) = f(x) + h(t)$ for some appropriate function $h : [0,1] \to \mathbb{R}$), then $\phi^{\overline{J}, H, G} = id$. This is easily seen because on one hand these constant choices of homotopies are regular and, on the other hand, the only moduli spaces appearing in the definition of $\phi^{\overline{J}.H,G}$ which are 0-dimensional are the flow lines in $L \times [0,1]$ which project to L on constant paths (equal to critical points of f).

We now fix a second set of homotopies $(H', G', \overline{J}')$ relating (f, ρ, J) and (f', ρ', J') in the same way as above and we intend to show that the two morphisms $\phi^{\overline{J},H,G}$ and $\phi^{\overline{J}',H',G'}$ are chain homotopic. The construction of this chain homotopy is perfectly similar to the construction of the chain morphism $\phi^{\overline{J},H,G}$. We start with a Morse homotopy $F : L \times [0,1] \times [0,1] \to \mathbb{R}$ relating H to H' so that $F(x,t,0) = H(x,t) + k$, $F(x,t,1) = H'(x,t) + k'$ (with k and k' appropriate constants) , $F(x,0,\tau)$ and $F(x,1,\tau)$ are trivial homotopies (of f and of f', respectively). The critical points of F verify $\mathrm{Crit_k(F)} = \mathrm{Crit_{k-2}(f)} \times \{0,0\} \cup \mathrm{Crit_{k-1}(f)} \times \{1,0\} \cup \mathrm{Crit_{k-1}(g)} \times \{0,1\} \cup \mathrm{Crit_k(g)} \times \{1,1\}$ (see [25] for the construction of such an F). We also consider homotopies $\tilde{J}_{t,\tau}$ between $\overline{J}$ and $\overline{J}'$ as well as $\tilde{G}$ between G and G'. We now define pearl type moduli spaces in the same way as just above except that points i' and iii' are replaced by:

i". The u_i's are so that for each u_i there is some $(t_i, \tau_i) \in [0,1] \times [0,1]$ with $u_i : (D, \partial D) \to (M, L) \times \{(t_i, \tau_i)\} \subset (M, L) \times [0,1] \times [0,1]$ and u_i is $\tilde{J}_{t_i,\tau_i}$-holomorphic.

iii". The incidence relations at point iii. take place inside $L \times [0,1] \times [0,1]$ with the flow of $-\nabla_{\tilde{G}} F$ on $L \times [0,1] \times [0,1]$ taking the place of the flow $-\nabla_\rho(f)$.

It is easy to see that a generic choice of $\tilde{G}$ and $\tilde{J}$ suffice for the transversality required to give the resulting moduli spaces $\mathcal{P}(x, y'; \tilde{J}, F, \tilde{G})$ a structure of manifold of dimension $|k| - |q| + \mu(\lambda) - 1$ where $x \in \mathrm{Crit_k(F)}$, $y' \in \mathrm{Crit_q(F)}$ and λ is the total homotopy class in $\pi_2(M, L)$ of the configuration. Let ξ be the Λ^+-module morphism obtained by counting the elements in the 0-dimensional such moduli spaces when $x \in \mathrm{Crit_k(f)} = \mathrm{Crit_{k+2}(F)}|_{\mathrm{L}\times\{(0,0)\}}$ and $y \in \mathrm{Crit_{k+1}(f')} = \mathrm{Crit_{k+1}(F)}|_{\mathrm{L}\times\{(1,1)\}}$. We want to remark that ξ is precisely the wanted chain homotopy.

$$\begin{tikzcd}
f \arrow[rr, dash, "H"] \arrow[dr, dash, "id"'] \arrow[dd, dash] & & f' \arrow[dr, dash, "id"] \arrow[dd, dash, dotted] & \\
& f \arrow[rr, dash, "H'"] \arrow[dd, dash] & & f' \arrow[dd, dash] \\
(0,0) \arrow[rr, dash, dotted] \arrow[dr, dash] & & (1,0) \arrow[dr, dash, dotted] & \\
& (0,1) \arrow[rr, dash] & & (1,1)
\end{tikzcd} \tag{62}$$

The verticals in this picture correspond to $L\times$ the coordinate in the horizontal square; F is defined on the whole cube; trivial Morse homotopies are defined on the left and right vertical faces; the Morse homotopy H is defined on the back face and H' is defined on the front face.

For this we need to analyze the boundary of the compactification of the 1-dimensional moduli spaces $\mathcal{P}(x, y'; \tilde{J}, F, \tilde{G})$ (clearly, we use here both Gromov compactness and an appropriate variant of the gluing results in §4) and we see that the only boundary elements which count are of four types. Indeed, in one-dimensional such moduli spaces side bubbling is not possible and, by applying the exact same method as that used in §5.1.1 to show $d^2 = 0$, we see that, for generic choices of data, the only remaining terms correspond to a pearl configuration breaking at some critical point of F. Thus there are four possibilities: a break in $z \in \mathrm{Crit(f)} \times \{(0,0)\}$ - this corresponds to an element in $\mathcal{P}(x, z; J, f, \rho)$ followed by one counted in ξ; a break in $z' \in \mathrm{Crit(f')} \times \{(1,0)\}$- this corresponds to an element in $\mathcal{P}(x, z'; \overline{J}, H, G)$ followed by one associated to the trivial homotopy $F|_{L\times\{1\}\times[0,1]}$; a break in $z'' \in Crit(f) \times \{(0,1)\}$ - this corresponds to aan elelement associated to the trivial homotopy $F|_{L\times\{0\}\times[0,1]}$ followed by an element in $\mathcal{P}(z'', y'; \overline{J}', H', G')$; finally, an element $z''' \in \mathrm{Crit(f')} \times \{(1,1)\}$ which corresponds to an element counted in ξ followed by an element of $\mathcal{P}(z''', y'; J', f', \rho')$. Puting all of this together we obtain:

$$d\xi + \xi d = I_1 \circ \phi^{\overline{J},H,G} + \phi^{\overline{J}',H',G'} \circ I_2$$

where both I_1, I_2 are induced by trivial homotopies. Given that trivial homotopies induce the identity at the level of the pearl complexes we obtain the claim.

Remark 5.1.4. a. In the proof of the invariance we may avoid entirely the use of spectral sequences by proving - again by constructing an appropriate chain homotopy - that given two homotopies H from f to f' and H' from f' to f'' and assuming $H''=H\#H'$ is the concatenated homotopy from f to f'', then $\phi^{H''}$ is chain homotopic to $\phi^{H'}\circ\phi^{H''}$.

b. A useful feature of the pearl complex is the following. Let f be a Morse function on L with a single maximum, x_n. Then any non-void moduli space $\mathcal{M}(x_n,x;\lambda)$, $|x_n|-|x|+\mu(\lambda)\leq 2$, is of dimension at least 1 when $\lambda\neq 0$. This means that in each of the complexes $\mathcal{C}^+(L;J,f)$, $\mathcal{C}(L;J,f)$ we have $dx_n=0$ (similarly, it is easy to see that the minimum of f, if it is unique, can not be a boundary in this complex). This has the following interpretation: if L as before has the property that there exists a point $p\in L$ and some J so that no J-disk passes through p, then $QH_*(L)\neq 0$. Indeed, in this case, we may assume that p is the maximum x_n of some function f and that $\mathcal{C}(L;f,J)$ is defined so that we know $dx_n=0$. If no J-disk goes through x_n, then x_n can not be a boundary. Using point v. of Theorem 2.1.1 this means in particular that if $HF_*(L)=0$ (for example if L is displaceable by a Hamiltonian diffeomorphism), then through each point in L passes a J-holomorphic disk.

c. It is easy to see that $Q^+H_*(L)$ can never vanish. Indeed, assume that f is a function with a single maximum m. In that case, as before, $dm=0$ in the (positive) pearl complex but m can never be a boundary in $\mathcal{C}^+(L;f)$ (it obviously can be in $\mathcal{C}(L;f)$).

5.1.3. *Augmentation.* We now want to remark that there exists a chain morphism:

$$\epsilon_{L,f}:\mathcal{C}^+(L;f)\to\Lambda^+$$

where the differential in the target is trivial. Moreover, this morphism will be easily seen to commute with the comparison maps constructed above so that the induced map in homology is canonical and will be denoted by ϵ_L. The definition of $\epsilon_{L,f}$ is simple: it is a Λ^+-module map so that $\epsilon_{L,f}(x)=0$ for all $x\in\mathrm{Crit(f)},\ |\mathrm{x}|>0$ and $\epsilon_{L,f}(z)=1$ for all $z\in\mathrm{Crit(f)},\ |\mathrm{z}|=0$. The reason this definition produces a chain morphism is similar to the point b. in Remark 5.1.4. Indeed it is easy to see that for a minimum $x_0\in\mathrm{Crit_0(f)}$ if a moduli space $\mathcal{P}(x,x_0;\lambda)$ is non-void and $x\neq x_0$,$\lambda\neq 0$, then the dimension of this moduli space is at least 1. Given the form of our differential it follows that a minimum can only appear in the Morse part of the differential of a critical point. But, in that part minima always appear in pairs (because, for a critical point of index 1 the Morse differential is always an even sum of minima). This implies that ϵ_L is a chain map as claimed. To prove the invariance of $\epsilon_{L,f}$ it is enough to notice that if $\phi:\mathcal{C}^+(L;f)\to\mathcal{C}^+(L;f')$ is the morphism constructed in the last section, then we have $\epsilon_{L,f'}\circ\phi=\epsilon_{L,f}$. This happens because in a way similar as above we see that a "comparison" moduli space $\mathcal{P}(x,x'_0;\lambda)$ with $x'_0\in\mathrm{Crit_0(f')}$ $x\in\mathrm{Crit(f)}$ can be 0-dimensional and non-void only if $x\in\mathrm{Crit_0(f)}$ and $\lambda=0$.

5.2. **The quantum product.** We construct here an operation

$$* : \mathcal{C}^+_k(L; f, J) \otimes \mathcal{C}^+_l(L; f', J) \to \mathcal{C}^+_{k+l-n}(L; f'', J)$$

where f' and f'' are generic small deformations of the Morse function f. To simplify notation we will assume that the critical points of both f' and f'' coincide with those of f. This morphism of chain complexes is defined by:

$$x * y = \sum_{y,\lambda} \#_{\mathbb{Z}_2}(\mathcal{P}(x, y, z; \lambda, J)) z t^{\bar{\mu}(\lambda)}, \tag{63}$$

where the moduli space $\mathcal{P}(x, y, z; \lambda, J)$ is described as:

$$\begin{aligned}
&\mathcal{P}(x, y, z; \lambda, J) = \{(l_1, l_2, l_3, u) \ : \\
&(l_1, l_2, l_3) \in \mathcal{P}(x, a_1, \lambda_1; f, J) \times \mathcal{P}(y, a_3, \lambda_3; f', J) \times \mathcal{P}(a_2, z, \lambda_2; f'', J) \ , \\
&u : (D, \partial D) \to (M, L) \ , \bar{\partial}_J u = 0, \ u(e^{2ik\pi/3}) = a_k, \ k \in \{1, 2, 3\} \ , \\
&\lambda_1 + \lambda_2 + \lambda_3 + [u] = \lambda \ \} \ .
\end{aligned}$$

Here, of course, the spaces $\mathcal{P}(a, b, \lambda; f, J)$ are pearl moduli spaces which join the points a and b in L. The sum above is taken only for those elements so that $|x|+|y|+\mu(\lambda)-|z|-n = 0$. Notice that we do allow in this description that u be the constant map.

Proposition 5.2.1. *For generic choices of data, the operation defined in equation (63) is well defined and a chain map. It induces in homology an associative product*

$$Q^+H_k(L) \otimes QH^+_j(L) \to Q^+H_{k+j-n}(L)$$

which is independent of the choices made in the construction.

Proof. We start the proof by describing the moduli spaces involved in a more precise way. We will use the notation of §3. Fix three sequences of nonvanishing homology classes: $\mathbf{A} = (A_1, A_2, \ldots A_k)$, $\mathbf{A}' = (A'_1, A'_2, \ldots A'_{k'})$, $\mathbf{A}'' = (A''_1, A''_2, \ldots A''_{k'})$ together with another homology class U which can also be null. We now define:

$$\begin{aligned}
&\mathcal{P}(x, y, z, \mathbf{A}, \mathbf{A}', \mathbf{A}'', U; f, f', f'', J) = \{(u, u', u'', v) \in \mathcal{P}(x, a_1, \mathbf{A}) \times \\
&\times \mathcal{P}(y, a_3, \mathbf{A}') \times \mathcal{P}(a_2, z, \mathbf{A}'') \times \mathcal{M}(U, J) \ : \ a_k = v(e^{2\pi i k/3}), \ k = 1, 2, 3\}
\end{aligned} \tag{64}$$

It is clear that the moduli space $\mathcal{P}(x, y, z; \lambda, J)$ is the union of the one given above when $\sum A_i + \sum A'_i + \sum A''_i + U = \lambda$. As the notation for all these moduli spaces will rapidly become hard to manipulate we will sometimes denote by $\mathcal{P}_{(64)}$ a moduli space as defined in equation (64). We will denote by $\mathcal{P}^{*,d}_{(64)}$ the moduli space defined as in (64) but so that all the J-disks involved are simple and absolutely distinct. For an element $(u, u', u'', v) \in \mathcal{P}^{*,d}_{(64)}$ we will call the disk v the core of the configuration.

With these notation the first part of the proof is to show that:

Lemma 5.2.2. *For a generic set of almost complex structures J we have*

$$\mathcal{P}_{(64)} = \mathcal{P}^{*,d}_{(64)}$$

whenever $\delta' = |x| + |y| - |z| + \mu(\mathbf{A} + \mathbf{A}' + \mathbf{A}'' + U) - n \leq 1$.

If this is true it follows that $\mathcal{P}_{(64)}$ is a manifold of dimension δ' when $\delta' = 0, 1$ and is void when $\delta' < 0$. Moreover, by Gromov compactness it also follows that for $\delta' = 0$ this space is compact so that the sum in (63) is well defined.

Proof of Lemma 5.2.2. This is a straightforward adaptation of Proposition 3.1.3 and we will only indicate here the specific additional verifications that are needed in this case. Clearly, an analogue of Proposition 3.1.6 is also needed - obviously the relevant moduli space in this case is defined in the same way as in $\mathcal{P}_{(64)}$ except that precisely one of the edges (or flow lines of one of f, f' or f'') is required to have length 0. The dimension of these moduli spaces is controlled by $\delta' - 1$. To show them, Lemma 3.2.3 can be expanded in the sense that, in the statement, the roots of order 2 of unity, $+/-1$, may be replaced with the roots of order 3 of 1 - the reason for that is, of course, that any three points on the boundary of a pseudo-holomorphic disk may be carried by reparametrization to the roots of order 3. Compared with the proof of Lemma 3.2.3 the only additional remark is that the cyclic order of the three points z_1, z_2, z_3 on the disk u' so that $u'(z_k) = u(e^{2\pi i k})$ is the same as the cyclic order of the corresponding three roots of the unity on u. It is also needed to show that, generically, we may assume that the cores of the elements in our moduli spaces send the three roots of the unity to distinct points - this is again a simple exercise. This argument covers the reduction to simple disks in the proof of Proposition 3.1.3. To pursue with the reduction to absolutely distinct disks it turns out that it is convenient to work with moduli spaces more general that those defined in equation (64): they are again formed by (u, u', u'', v) except that the pearls u, u', u'' are more general in the sense that along each of their edges we may use any one of the negative gradient flows $-\nabla f, -\nabla f', -\nabla f''$. More precisely, for the definition of the (u, u', u'') in the incidence relation iii. in Definition 5.1.1 anyone of these three flows may be allowed with the assumption that, at the core, the two "entry" flow lines correspond to different flows. To see why these more general moduli spaces are needed recall the principle behind the reduction to absolutely distinct disks in Case 2 in §3.3 (for $n \geq 3$). We assume that some disks are not absolutely distinct for some element of our moduli space and we show that, in this case, there exists a configuration of Maslov index lower by at least two and with the same ends. To obtain this new configuration recall from Lemma 3.2.2 that, if the disks are not absolutely distinct, then the image of one disk is included in that of some other and we replace the smaller disk with the bigger one. We then argue that this configuration belongs to a moduli space of negative virtual dimension and strictly lower Maslov class than the initial one and, by induction, such a configuration can not exist which leads to a contradiction. Coming back to our moduli spaces $\mathcal{P}_{(64)}$ we see that if we assume that some disks in an element of this moduli space are not absolutely distinct, then the reduction described in Case 2 §3.3 leads to a configuration of lower Maslov class which does not necessarily lie in $\mathcal{P}_{(64)}$ but is an element of the more general moduli spaces introduced above. With these more general moduli spaces the reduction to simple disks mentioned above still works without problems and, additionally, the combinatorial argument in Case 2 in the proof of Proposition 3.1.3 also adapts in an obvious way and this proves the statement when $\dim(L) = n \geq 3$. In the case $n \leq 2$ it is important to note that the only Maslov indexes involved are still 2 and 4. This allows for the last part - valid for $n \leq 2$ - of the proof of Proposition 3.1.3 to adapt to this case. □

The second step of the proof of the proposition is to show by using the previous lemma that the operation $*$ provides a chain map. To do this we need to consider the compactification $\overline{\mathcal{P}}_{(64)}$ of our moduli spaces and we need to show the appropriate analogue of the Lemma 5.1.3. Here is the statement.

Lemma 5.2.3. *For $x, y, z \in \mathrm{Crit(f)}$, $\delta' = 1$ and a generic almost complex structure J, the space $\overline{\mathcal{P}}_{(64)}(x, y, z) = \overline{\mathcal{P}}(x, y, z, \mathbf{A}, \mathbf{A}', \mathbf{A}'', U; f, f', f'', J)$ is a compact, 1-dimensional manifold whose boundary verifies:*

$$\begin{aligned}(65)\qquad &\partial\overline{\mathcal{P}}_{(64)}(x, y, z) = \cup_{x'}\mathcal{P}(x, x') \times \mathcal{P}_{(64)}(x', y, z) \bigcup \\ &\bigcup \cup_{y'}\mathcal{P}(y, y') \times \mathcal{P}_{(64)}(x, y', z) \bigcup \cup_{z'}\mathcal{P}_{(64)}(x, y, z') \times \mathcal{P}(z', z) \cup \mathcal{R} \cup \mathcal{R}' \cup \mathcal{R}''\end{aligned}$$

Where $\mathcal{R}$ assembles the terms $\xi = (u, u', u'', v)$ so that one edge in one of u, u', u'' is of length 0; $\mathcal{R}'$ assembles the terms in which one of the disks in one of the classes A_i, A'_j, A''_k splits in two (each piece being a non-constant disk carrying two incidence points); $\mathcal{R}''$ assembles the terms in which the core splits in two (with possibly one of the pieces being a stable ghost disk and each piece carrying two incidence points). The unions in the first three terms are taken over all possible splittings such as to respect the homology classes $\mathbf{A}, \mathbf{A}', \mathbf{A}'', U$ (these classes have been omitted in the notation).

The proof of this lemma is very similar to that of Lemma 5.1.3 so that we will omit the details besides indicating the points where some differences occur. Of course, the condition $N_L \geq 2$ is crucial in insuring that if bubbling off of some disk occurs each piece will carry at least two incidence points. The main difference concerns the set $\mathcal{R}''$. This set consists of configurations associated to the bubbling off of the core. It is important to notice that as the core carries three marked points (which geometrically correspond to the three attachment points $a_i, i = 1, 2, 3$) the bubbling off of a "ghost" disk has to be allowed as long as this disk is stable.

Given this result we proceed to show that $*$ induces a chain map. For this it is enough to show that the number of elements in $\partial\overline{\mathcal{P}}(x, y, z; \lambda)$ is the same as the sum S of the number of elements in the first three terms in (65) when $\mathbf{A}, \mathbf{A}', \mathbf{A}'', U$ vary such that

$$\sum A_i + \sum A'_j + \sum A''_k + U = \lambda \ .$$

Indeed, this implies that $S = 0$ which gives precisely the algebraic identity equivalent to $*$ being a chain map. Thus the proof is reduced to showing that each element coming from the terms $\mathcal{R}$, $\mathcal{R}'$, $\mathcal{R}''$ appears twice when $\mathbf{A}, \mathbf{A}', \mathbf{A}'', U$ vary as above. To check this, again, the only case which is different from the proof of Proposition 5.1.2 concerns the terms of type $\mathcal{R}''$. But it is easy to see that the condition $N_L \geq 2$ insures that each element of type $\mathcal{R}''$ can also be viewed as an element in a set of type $\mathcal{R}$ obtained in a configuration with a trivial core when one of the flow lines attached to the core is of length zero. For this argument it is useful to notice that, generically, the elements of type $\mathcal{R}''$ have the property that if a ghost disk has bubbled off in the core then this ghost disk carries at most two of the marked points.

Therefore, $*$ is a chain map and thus induces an operation in homology. By using the same techniques as above combined with the invariance proof from §5.1.2 we obtain that at the homology level this product is independent of the choices made in its definition.

Lemma 5.2.4. *There exists a canonical element $w_L \in Q^+H_n(L)$ (resp. $QH_n(L)$) which is a unit with respect to the quantum cap product, i.e. $w_L * \alpha = \alpha$ for every $\alpha \in Q^+H(L)$ (resp. $\alpha \in QH(L)$).*

Proof. It is easy to see that we may take $f'' = f'$ in the definition of the product as before. Assume now that f has a single maximum, x_n, and take generic J so that the pearl complexes are defined. By Remark 5.1.4, x_n is always a cycle and we will denote its homology class by w_L. Under these assumptions it is easy to see that, at the chain level, we have $x_n * y = y$ for all critical points $y \in \mathrm{Crit(f')}$. This means that w_L is the unit for our product $*$.

By standard (essentially Morse-theoretic) arguments it follows that w_L is indeed canonical in the sense that when we change our Morse function the identification morphism ϕ (described in §5.1.2) preserves that homology class. $\square$

Remark 5.2.5. a. In view of the proof of Lemma 5.2.4 we will sometimes denote w_L by abuse of notation also as $[L]$.

b. Let f be a Morse function with a single maximum x_n. Since $w_L = [x_n]$ is the unit it follows that $QH_*(L) = 0$ iff x_n is a boundary in $\mathcal{C}(L; f, J)$. Indeed, suppose that $x_n = d(\eta)$ and let α be a cycle. Then we have $a = x_n * a = (d\eta) * a = d(\eta * a)$ (see also Remark 2.2.6 for the same statement in the context of the minimal models).

c. It is useful to note that the product described above is not commutative (even in the graded sense) in general. See Proposition 6.2.3 in §6.2.1 for a concrete example.

The only point left to conclude the proof of Proposition 5.2.1 is that, in homology, this product is associative.

Lemma 5.2.6. *The product*

$$* : Q^+H_k(L) \otimes Q^+H_i(L) \to Q^+H_{i+k-n}(L)$$

is associative.

Proof of Lemma 5.2.6. To prove this result some new moduli spaces need to be used. We remark that the pearl moduli spaces $\mathcal{P}(x, y, \lambda)$ may be viewed as modeled over linear trees (with oriented edges). Similarly, the moduli spaces $\mathcal{P}(x, y, z; \lambda)$ used to define the product are modeled over trees with two entries and one exit (and hence with a single vertex of valence three).

The moduli spaces needed to prove the associativity are modeled over more general trees and we describe them rigorously now.

We consider trees $\mathcal{T}$ with oriented edges embedded in $\mathbb{R} \times [0, 1] \subset \mathbb{R}^2$ with three entries lying on the line $\mathbb{R} \times \{1\}$ and one exit on the line $\mathbb{R} \times \{0\}$ and so that the edges strictly decrease the y-coordinate. The vertices of the tree - except for the entries and the exit - are labeled by elements of $H_2(M, L; \mathbb{Z})$ and the label of each such vertex will be called its *class*. The entries are labeled in order by the three critical points x, y, z and the exit is labeled by w. Each edge is labeled by an element of the set $\{1, 2, 3\}$. Clearly, such a tree $\mathcal{T}$ has either two vertices of valence three or one vertex of valence four and each internal vertex has a single exit. Fixing $x, y, z, w \in \mathrm{Crit(f)}$ and such a tree $\mathcal{T}$ we denote the associated moduli space by $\mathcal{P}_{\mathcal{T}}(x, y, z, w)$ (the rule here is that the last critical point is the exit; this is coherent with previous notation). An element of this moduli space consists of a family of J-holomorphic disks one for each vertex of the tree, in the class of that vertex, together with a family of strictly positive real numbers, one for each edge in the tree, so that for each edge we have an incidence relation like the one in Definition 5.1.1 iii. This relates the vertices joined by the respective edge but instead of the flow γ one might use any one of the negative gradient flows $\gamma_1 = \gamma$, γ_2 induced by $-\nabla f'$ or γ_3 induced by $-\nabla f''$ so that the flow γ_i is used precisely when the label of the edge is i. The incidence points are as follows: for the vertices of valence two they are the points

$-1, +1$ so that the entry corresponds to -1 and the exit to $+1$; for the vertices of valence three they are, in cyclic order, the roots of unity of order 3 so that $e^{2i\pi/3}$ corresponds to the the entry at the left, $e^{4i\pi/3}$ corresponds to the exit, 1 corresponds to the entry on the right; the incidence points z_1, z_2, z_3, z_4 on a disk of valence four are - in order - so that z_1, z_2, z_4 are the roots of unity of order 3, $(z_1 = e^{2i\pi/3})$, z_3 is a point strictly in between z_2 and z_4 and z_1, z_3, z_4 are, from left to right, the entries and z_2 is the exit. All the J-disks are stable in the sense that if a disk is trivial then it carries at least three incidence (or marked) points and, moreover, the labeling of the edges arriving at any vertex are pairwise distinct. Finally, a last condition is necessary in defining $\mathcal{P}_{\mathcal{T}}(x, y, z, w)$:

(66) the labeling of the edges arriving at each vertex respects the planar order; the label of the exiting edge equals the smallest of the labels of the arriving edges; the edge starting in x has label 1, the edge starting in y has label 2 and the edge starting in z has label 3.

This last condition implies that the labeling of the edges of a tree $\mathcal{T}$ is completely determined by its topological type. It will be useful in the arguments below to also consider moduli spaces defined exactly as above but without imposing condition (66). These more general moduli spaces will be denoted by $\mathcal{G}_{\mathcal{T}}(x, y, z, w)$.

Remark 5.2.7. Notice that these more general moduli spaces have already appeared - in the case of just two entries - in the proof of Lemma 5.2.2. Clearly, the definition described above easily extends to trees with more entries.

The way to proceed from this point is clear: we first define moduli spaces $\mathcal{P}^{*;d}_{\mathcal{T}}(x, y, z, w)$ which are as above except that all the J-disks involved are simple and absolutely distinct. Similarly we define moduli spaces $\mathcal{G}^{*;d}_{\mathcal{T}}(x, y, z, w)$. By arguments similar to those in §3 and in the proof of Lemma 5.2.2 it is not difficult to see that for $n \geq 3$ and a generic choice of almost complex structure J

$$\mathcal{G}_{\mathcal{T}}(x, y, z, w) = \mathcal{G}^{*;d}_{\mathcal{T}}(x, y, z, w)$$

whenever $\delta'' = |x| + |y| + |z| + \mu(\mathcal{T}) - |w| - 2n + 1 \leq 1$ where $\mu(\mathcal{T})$ is the sum of the Maslov classes of all the labels of the internal vertices in $\mathcal{T}$. Moreover, $\mathcal{G}^{*;d}_{\mathcal{T}}(x, y, z, w)$ is a manifold of dimension δ''. The same combinatorial arguments as in the proof of Proposition 3.1.3 case $n \leq 2$ are sufficient to also show the same statement for $n = 2$ when $N_L \geq 3$. For $n = 2$ and $N_L = 2$ a more involved argument is needed because we need to consider the case of $\mu(\mathcal{T}) = 6$. In all cases, we deduce that for a generic J and $\delta'' \leq 1$

$$\mathcal{P}_{\mathcal{T}}(x, y, z, w) = \mathcal{P}^{*;d}_{\mathcal{T}}(x, y, z, w) \tag{67}$$

and this moduli space is a manifold of dimension δ''.

It is then needed to consider the compactifications $\overline{\mathcal{P}}_{\mathcal{T}}(x,y,z,w)$ and establish a boundary formula as in Lemma 5.2.3. We will not state this formula explicitly as it is very similar to the ones before but we will describe the boundary terms. They are defined as the usual elements in $\mathcal{P}_{\mathcal{T}}(x,y,z,w)$ except for precisely one modification which fits into one of the following categories:

$i_{\mathcal{T}}$. One of the edges in $\mathcal{T}$ corresponds to a flow line of 0 length.

$ii_{\mathcal{T}}$. One of the disks corresponding to a vertex in T is replaced by a cusp curve with two components (due to bubbling off); the bubbling off of a ghost disk is possible if the vertex in question is of valence at least 3. Each of the pieces appearing in a cusp curve carries at least two incidence points.

$iii_{\mathcal{T}}$. One of the edges in $\mathcal{T}$ corresponds to a flow line which is broken once.

Notice that, in this description, the coincidence of two marked points (which clearly may occur on a vertex of valence four) corresponds to the bubbling off of a ghost disk and, clearly, the reason why each component in a cusp curve as in $ii_{\mathcal{T}}$ carries at least two incidence points is that $N_L \geq 2$ and $\delta'' \leq 1$.

The purpose of this construction is, of course, to define a chain homotopy:

$$\xi : \mathcal{C}^+(L; f, J) \otimes \mathcal{C}^+(L; f', J) \otimes \mathcal{C}^+(L; f'', J) \to \mathcal{C}^+(L; f, J)$$

so that $\xi : ((- \ * \ -) \ * \ -) \simeq (- \ * \ (- \ * \ -))$. The definition of ξ is clear:

$$\xi(x \otimes y \otimes z) = \sum_{\lambda,\ |\mathcal{T}|=\lambda,\ w} \#_{\mathbb{Z}_2}(\mathcal{P}_{\mathcal{T}}(x,y,z,w)) w e^{\lambda}$$

where the sum is over all those terms so that $\delta'' = |x| + |y| + |z| - |w| + 1 - 2n + \mu(\lambda) = 0$ and the sum of all the labels of the internal vertices of $\mathcal{T}$ is denoted by $|\mathcal{T}|$. We now use the description of the boundary of $\overline{\mathcal{P}}_{\mathcal{T}}(x,y,z,w)$ given above to justify

$$(d\xi + \xi d)(x \otimes y \otimes z) = (x * y) * z - x * (y * z)\ . \tag{68}$$

We let

$$\overline{\mathcal{P}}(x,y,z,w;\lambda)$$

be the (disjoint) union of all the spaces $\overline{\mathcal{P}}_{\mathcal{T}}(x,y,z,w)$ where $\mathcal{T}$ runs over all the topological types of trees so that $|\mathcal{T}| = \lambda$. To prove (68) we start by noticing that the coefficients of $wt^{\bar{\mu}(\lambda)}$ in this equation are in bijection with the union of the terms $iii_{\mathcal{T}}$. Indeed, the terms on the left in (68) correspond to those elements in which the break in the flow line disconnects the tree so that one of the remaining pieces is a linear tree (in the sense that it does not contain any vertex of valence three or more) and the terms on the right correspond precisely to those elements in which the tree is broken in two parts none of which is linear. Thus, to finish the proof it is enough to notice that the terms of type $i_{\mathcal{T}}$ and $ii_{\mathcal{T}}$ all cancel out when $\mathcal{T}$ varies.

For this we look at a configuration of type $i_{\mathcal{T}}$ and remark that if one of the edges of $\mathcal{T}$ becomes of zero length, then, generically, this configuration also appears as a configuration of type $ii_{\mathcal{T}'}$ corresponding to a tree $\mathcal{T}'$ so that in $\mathcal{T}'$ is obtained from $\mathcal{T}$ by replacing the edge in question together with the two bounding vertices by a single vertex. The fact that the labeling of the edges of each topological type of tree is unique is essential here as it implies that this tree $\mathcal{T}'$ is unique. Therefore, all elements of type $i_{\mathcal{T}}$ and $ii_{\mathcal{T}}$ cancel and this proves the lemma and concludes the proof of the proposition.

□

5.3. **The quantum module structure.** In this section we define an operation

$$* : Q^+H_k(M) \otimes Q^+H_s(L) \to Q^+H_{k+s-2n}(L), \quad a \otimes x \longmapsto a * x,$$

which will make $Q^+H_*(L)$ an algebra over $Q^+H_*(M)$.

Let $h : M \to \mathbb{R}$, $f : L \to \mathbb{R}$ be Morse functions and ρ_M, ρ_L Riemannian metrics on M, L. From now on we will make the following assumptions on (f, ρ_L) and (h, ρ_M):

Assumption 5.3.1. Each of the pairs (f, ρ_L) and (h, ρ_M) is Morse-Smale and h has a single maximum. Furthermore:

(1) For every $a \in \text{Crit}(h)$ the unstable submanifold W^u_a is transverse to L.
(2) For every $a \in \text{Crit}(h)$, $x, y \in \text{Crit}(f)$, W^u_a is transverse to W^u_x and W^s_y.

5.3.1. *The external operation.* We will describe here the relevant moduli spaces very explicitly and without using the "tree" model which has been used at the end of the previous section.

Let $J \in \mathcal{J}(M, \omega)$ and $A \in H_2(M, L; \mathbb{Z})$. Define the following evaluation maps:

$$ev_{-1,1} : \mathcal{M}(A, J) \to L \times L, \quad ev_{-1,0,1} : \mathcal{M}(A, J) \to L \times M \times L,$$
$$ev_{-1,1}(u) = (u(-1), u(1)), \quad ev_{-1,0,1}(u) = (u(-1), u(0), u(1)).$$

Let $\mathbf{A} = (A_1, \ldots, A_l)$ be a vector of *non-zero* classes in $H_2(M, L; \mathbb{Z})$. Put

$$\mathcal{M}(\mathbf{A}, J) = \mathcal{M}(A_1, J) \times \cdots \times \mathcal{M}(A_l, J),$$
$$ev : \mathcal{M}(\mathbf{A}, J) \to L^{\times 2l}, \quad ev(u_1, \ldots, u_l) = (ev_{-1,1}(u_1), \ldots, ev_{-1,1}(u_l)).$$

Given $x, y \in \text{Crit}(f)$ recall from previous sections the notation for the pearl moduli-spaces:

$$\mathcal{M}(x, y; \mathbf{A}, J) = ev^{-1}(W^u_x \times Q^{\times(l-1)}_{f,\rho_L} \times W^s_y), \quad \mathcal{P}(x, y; \mathbf{A}, J) = \mathcal{M}(x, y; \mathbf{A}, J)/\mathbf{G},$$

where $\mathbf{G} = G_{-1,1} \times \cdots \times G_{-1,1}$ (see figure 1).

Now, given $1 \leq k \leq l$ define the evaluation map:

$$ev_{(k)} : \mathcal{M}(\mathbf{A}, J) \to L^{\times(2k-1)} \times M \times L^{\times(2l-2k+1)},$$

$$ev_{(k)}(u_1, \ldots, u_l) = (ev_{-1,1}(u_1), \ldots, ev_{-1,1}(u_{k-1}), ev_{-1,0,1}(u_k), ev_{-1,1}(u_{k+1}), \ldots, ev_{-1,1}(u_l)).$$

Given $x, y \in \text{Crit}(f)$ and $a \in \text{Crit}(h)$ put

$$\mathcal{P}_I(a, x, y; (\mathbf{A}, k), J) = ev_{(k)}^{-1}\big(W_x^u \times Q_{f,\rho_L}^{\times(k-1)} \times W_a^u \times Q_{f,\rho_L}^{\times(l-k)} \times W_y^s\big)/\mathbf{G}_k, \tag{69}$$

$$\mathcal{P}_I(a, x, y; \mathbf{A}, J) = \bigcup_{k=1}^{l} \mathcal{P}_I(a, x, y; (\mathbf{A}, k), J), \tag{70}$$

where $\mathbf{G}_k$ is taken here to be $(G_{-1,1})^{\times k}$ with the k'th factor replaced by the trivial group. See figure 7. For each element $u \in \mathcal{P}_I(a, x, y; (\mathbf{A}, k), J)$ we will call the disk u_k the *center* of u.

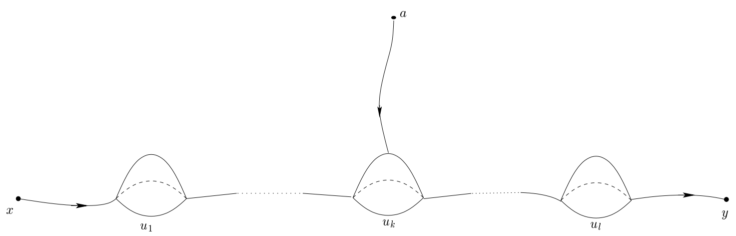

FIGURE 7. An element of $\mathcal{P}_I(a, x, y; (\mathbf{A}, k), J)$

Denote by $\Phi_t : L \to L$, $t \in \mathbb{R}$, the *negative* gradient flow of (f, ρ_L) (i.e. the flow of the vector field $-\text{grad}_{\rho_L} f$). Consider the (non-proper) embedding

$$(L \setminus \text{Crit}(f)) \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \hookrightarrow L \times L \times L, \quad (x, t, s) \longmapsto (x, \Phi_t(x), \Phi_{t+s}(x)).$$

Denote the image of this embedding by $Q'_{f,\rho_L} \subset L \times L \times L$. We will also need the subset $Q_{f,\rho_L} \subset L \times L$ defined by (5) in §3.1.

Let $\mathbf{A} = (A_1, \ldots, A_l)$ be a vector of non-zero classes, $0 \leq k \leq l$, $x, y \in \text{Crit}(f)$ and $a \in \text{Crit}(h)$. Define the following space (see figure 8):

$$\mathcal{P}_{I'}(a, x, y; (\mathbf{A}, k), J) =$$

$$\begin{cases} \big\{(\mathbf{u}, p) \in \mathcal{P}(x, y; \mathbf{A}, J) \times (W_a^u \cap L) \mid (u_k(1), p, u_{k+1}(-1)) \in Q'_{f,\rho_L}\big\} & \text{if } 0 < k < l, \\ \big\{(\mathbf{u}, p) \in \mathcal{P}(x, y; \mathbf{A}, J) \times (W_a^u \cap W_x^u) \mid (p, u_1(-1)) \in Q_{f,\rho_L}\big\} & \text{if } k = 0, \\ \big\{(\mathbf{u}, p) \in \mathcal{P}(x, y; \mathbf{A}, J) \times (W_a^u \cap W_y^s) \mid (u_l(1), p) \in Q_{f,\rho_L}\big\} & \text{if } k = l. \end{cases}$$

$$\mathcal{P}_{I'}(a, x, y; \mathbf{A}, J) = \bigcup_{k=0}^{l} \mathcal{P}_{I'}(a, x, y; (\mathbf{A}, k), J).$$

Note that the p in $(\mathbf{u}, p)$ cannot be neither x nor y due to the definition of Q_{f,ρ_L}.

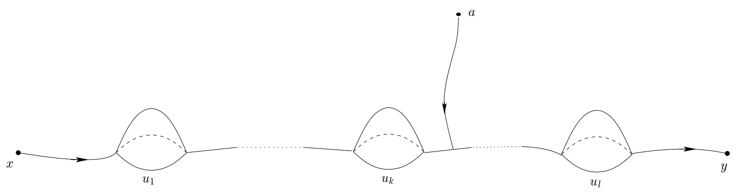

FIGURE 8. An element of $\mathcal{P}_{I'}(a, x, y; (\mathbf{A}, k), J)$

Remark 5.3.2. It is clear that we could as well define the space $\mathcal{P}_{I'}(--)$ as a subspace of $\mathcal{P}_I(--)$ consisting of those configurations so that their center is the constant disk.

Proposition 5.3.3. *Let (f, ρ_L), (h, ρ_M) be as above. There exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ with the following properties. For every $J \in \mathcal{J}_{\text{reg}}$, $\mathbf{A}$ as above, and every $x, y \in \text{Crit}(f)$, $a \in \text{Crit}(h)$ with $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n = 0$ each of the spaces $\mathcal{P}_I(a, x, y; \mathbf{A}, J)$, $\mathcal{P}_{I'}(a, x, y; \mathbf{A}, J)$ is a compact 0-dimensional manifold, hence a finite set. Furthermore, each element of these spaces consists of simple and absolutely distinct disks.*

The proof is given in Section 5.3.5 below.

In case $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n = 0$ put:

$$n_I(a, x, y; \mathbf{A}, J) = \#_{\mathbb{Z}_2}\mathcal{P}_I(a, x, y; \mathbf{A}, J), \quad n_{I'}(a, x, y; \mathbf{A}, J) = \#_{\mathbb{Z}_2}\mathcal{P}_{I'}(a, x, y; \mathbf{A}, J),$$
$$n(a, x, y; \mathbf{A}, J) = n_I(a, x, y; \mathbf{A}, J) + n_{I'}(a, x, y; \mathbf{A}, J).$$

Finally, in case $|a| + |x| - |y| - 2n = 0$ put $n(a, x, y) = \#_{\mathbb{Z}_2}(W_a^u \cap W_x^u \cap W_y^s)$.

Denote by $(C_*(h), \partial^h)$ the Morse complex of $(M; h, \rho_M)$ and by $(\mathcal{C}_*^+, d^f)$ the pearl complex associated to $(L; f, \rho_L, J)$. Define an operation:

$$* : C_k(h) \otimes \mathcal{C}_q^+ \longrightarrow \mathcal{C}_{q+k-2n}^+, \tag{71}$$

by the formula

$$a * x = \sum_y n(a, x, y)y + \sum_{y, \mathbf{A}} n(a, x, y; \mathbf{A}, J)yt^{\bar{\mu}(\mathbf{A})}, \tag{72}$$

where the first sum is taken over all $y \in \text{Crit}(f)$ with $|a| + |x| - |y| - 2n = 0$ and the second sum over all $y, \mathbf{A}$ with $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n = 0$. Note that the first sum is the Morse theoretic interpretation of the classical cap product operation of $H_*(M)$ on $H_*(L)$.

Proposition 5.3.4. *The homomorphism (71) is a chain map, namely*

$$d^f(a * x) = \partial^h(a) * x + a * d^f(x).$$

In particular it induces a well defined operation:

$$* : Q^+H_k(M) \otimes Q^+H_s(L) \to Q^+H_{k+s-2n}(L).$$

The proof of Proposition 5.3.4 will occupy Sections 5.3.2 – 5.3.11 below.

5.3.2. *Moduli spaces related to the external operation.* In order to prove Proposition 5.3.4 we will need to introduce several types of moduli spaces.

Type II. Let $\mathbf{A}$ be a vector of non-zero classes and $1 \leq k \leq l$. Define the following spaces (see figure 9):

$$\mathcal{P}_{II_1}(a, x, y; (\mathbf{A}, k), J) = \big\{\mathbf{u} \in \mathcal{P}(x, y; \mathbf{A}, J) \mid u_k(-1) \in W^u_a\big\},$$
$$\mathcal{P}_{II_2}(a, x, y; (\mathbf{A}, k), J) = \big\{\mathbf{u} \in \mathcal{P}(x, y; \mathbf{A}, J) \mid u_k(1) \in W^u_a\big\},$$
$$\mathcal{P}_{II_1}(a, x, y; \mathbf{A}, J) = \bigcup_{k=1}^{l} \mathcal{P}_{II_1}(a, x, y; (\mathbf{A}, k), J), \quad \mathcal{P}_{II_2}(a, x, y; \mathbf{A}, J) = \bigcup_{k=1}^{l} \mathcal{P}_{II_2}(a, x, y; (\mathbf{A}, k), J).$$

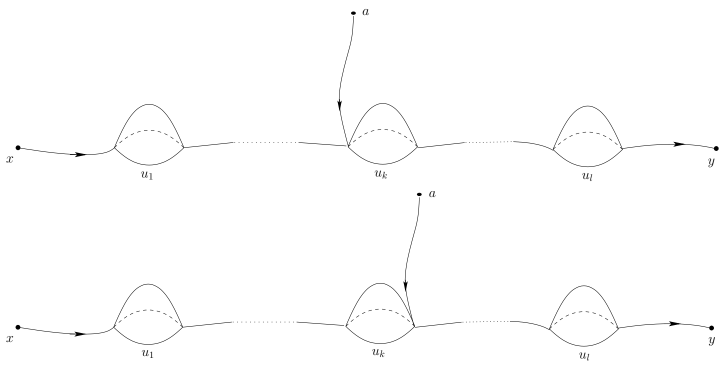

FIGURE 9. Elements of the spaces $\mathcal{P}_{II_1}(a, x, y; (\mathbf{A}, k), J)$, $\mathcal{P}_{II_2}(a, x, y; (\mathbf{A}, k), J)$.

Type III. Let $\mathbf{B}' = (B'_1, \ldots, B'_{l'})$, $\mathbf{B}'' = (B''_1, \ldots, B''_{l''})$ be two vectors of *non-zero* classes in $H_2(M, L; \mathbb{Z})$. Let $1 \leq k' \leq l'$ and $1 \leq k'' \leq l''$. Put $\mathbf{A} = (\mathbf{B}', \mathbf{B}'') = (B'_1, \ldots, B'_{l'}, B''_1, \ldots, B''_{l''})$. Define

$$\begin{aligned} &\mathcal{P}_{III_1}(a, x, y; (\mathbf{B}', k'), \mathbf{B}'', J) = \\ &ev^{-1}_{(k')}\big(W^u_x \times Q^{\times(k'-1)}_{f,\rho_L} \times W^u_a \times Q^{\times(l'-k')}_{f,\rho_L} \times \mathrm{diag(L)} \times Q^{\times(l''-1)}_{f,\rho_L} \times W^s_y\big)/\mathbf{G}_{III_1}, \\ &\mathcal{P}_{III_2}(a, x, y; \mathbf{B}', (\mathbf{B}'', k''), J) = \\ &ev^{-1}_{(l'+k'')}\big(W^u_x \times Q^{\times(l'-1)}_{f,\rho_L} \times \mathrm{diag(L)} \times Q^{\times(k''-1)}_{f,\rho_L} \times W^u_a \times Q^{\times(l''-k'')}_{f,\rho_L} \times W^s_y\big)/\mathbf{G}_{III_2}. \end{aligned} \tag{73}$$

Here $\mathbf{G}_{III_1}$ (respectively $\mathbf{G}_{III_2}$) is the group $G_{-1,1}^{\times(l'+l'')}$ with the k''th (respectively $(l'+k'')$'th) component replaced by the trivial group. Finally put

$$\mathcal{P}_{III_1}(a,x,y;\mathbf{B}',\mathbf{B}'',J) = \bigcup_{k'=1}^{l'} \mathcal{P}_{III_1}(a,x,y;(\mathbf{B}',k'),\mathbf{B}'',J),$$
$$\mathcal{P}_{III_2}(a,x,y;\mathbf{B}',\mathbf{B}'',J) = \bigcup_{k''=1}^{l''} \mathcal{P}_{III_2}(a,x,y;\mathbf{B}',(\mathbf{B}'',k''),J).$$

See figure 10.

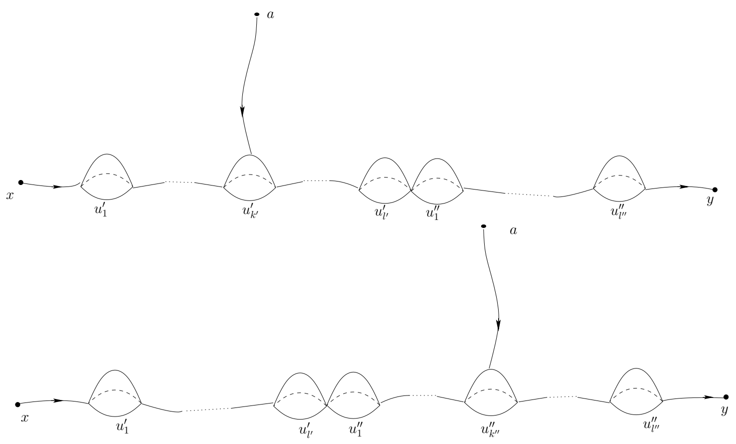

FIGURE 10. Elements of the spaces $\mathcal{P}_{III_1}(a,x,y;(\mathbf{B}',k'),\mathbf{B}'',J)$, $\mathcal{P}_{III_2}(a,x,y;\mathbf{B}',(\mathbf{B}'',k''),J)$.

Type III'. Let $\mathbf{B}'=(B'_1,\ldots,B'_{l'})$, $\mathbf{B}''=(B''_1,\ldots,B''_{l''})$ be two vectors of non-zero classes. Given $x,y\in \mathrm{Crit}(f)$ put

$$\mathcal{P}(x,y;\mathbf{B}',\mathbf{B}'',J) = ev^{-1}\big(W_x^u \times Q_{f,\rho_L}^{\times(l'-1)} \times \mathrm{diag}(L) \times Q_{f,\rho_L}^{\times(l''-1)} \times W_y^s\big)/\mathbf{G},$$

where $ev : \mathcal{M}((\mathbf{B}',\mathbf{B}''),J) \to L^{\times(2l'+2l'')}$ is the evaluation map from Section 5.3.1 and $\mathbf{G}=G_{-1,1}^{\times(l'+l'')}$.

Let $0\leq k' < l'$. Define the following space:

$$\mathcal{P}_{III'_1}(a,x,y;(\mathbf{B}',k'),\mathbf{B}'',J) =$$
$$\begin{cases} \big\{(\mathbf{u}',\mathbf{u}'',p)\in \mathcal{P}(x,y;\mathbf{B}',\mathbf{B}'',J)\times (W_a^u\cap L) \mid (u'_{k'}(1),p,u'_{k'+1})\in Q'_{f,\rho_L}\big\} & \text{if } k'>0, \\ \big\{(\mathbf{u}',\mathbf{u}'',p)\in \mathcal{P}(x,y;\mathbf{B}',\mathbf{B}'',J)\times (W_a^u\cap W_x^u) \mid (p,u_1(-1))\in Q_{f,\rho_L}\big\} & \text{if } k'=0. \end{cases}$$

Let $0 < k'' \leq l''$. Define:

$\mathcal{P}_{III'_2}(a, x, y; \mathbf{B}', (\mathbf{B}'', k''), J) =$

$$\begin{cases} \{(\mathbf{u}', \mathbf{u}'', p) \in \mathcal{P}(x, y; \mathbf{B}', \mathbf{B}'', J) \times (W^u_a \cap L) \mid (u'_{k''}(1), p, u'_{k''+1}) \in Q'_{f,\rho_L}\} & \text{if } k'' < l'', \\ \{(\mathbf{u}', \mathbf{u}'', p) \in \mathcal{P}(x, y; \mathbf{B}', \mathbf{B}'', J) \times (W^u_a \cap W^s_y) \mid (u_{l''}(1), p) \in Q_{f,\rho_L}\} & \text{if } k'' = l''. \end{cases}$$

Finally put:

$$\mathcal{P}_{III'_1}(a, x, y; \mathbf{B}', \mathbf{B}'', J) = \bigcup_{k'=0}^{l'-1} \mathcal{P}_{III'_1}(a, x, y; (\mathbf{B}', k'), \mathbf{B}'', J),$$

$$\mathcal{P}_{III'_2}(a, x, y; \mathbf{B}', \mathbf{B}'', J) = \bigcup_{k''=1}^{l''} \mathcal{P}_{III'_2}(a, x, y; \mathbf{B}', (\mathbf{B}'', k''), J).$$

See figure 11.

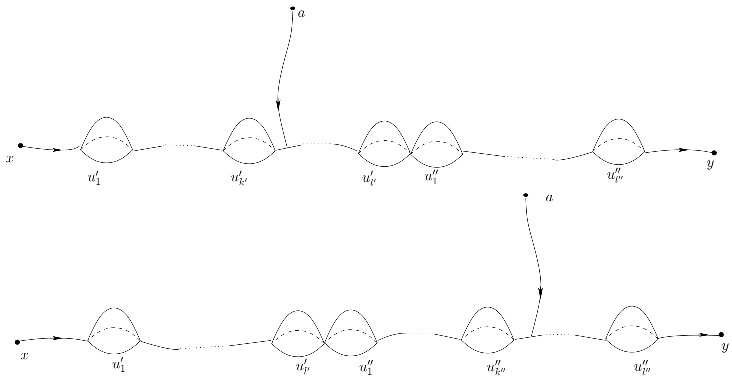

FIGURE 11. Elements of the spaces $\mathcal{P}_{III'_1}(a, x, y; (\mathbf{B}', k'), \mathbf{B}'', J)$, $\mathcal{P}_{III'_2}(a, x, y; \mathbf{B}', (\mathbf{B}'', k''), J)$.

Proposition 5.3.5. *There exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$ the following holds:*

1. *For every a, x, y and every vector $\mathbf{A}$ of non-zero classes with $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n = 1$ each of the spaces $\mathcal{P}_{II_i}(a, x, y; \mathbf{A}, J)$, $i = 1, 2$, is a compact 0-dimensional manifold, hence a finite set. Moreover, every $(u_1, \ldots, u_l)$ in this space consists of simple and absolutely distinct disks.*
2. *For every a, x, y and every two vectors $\mathbf{B}'$, $\mathbf{B}''$ of non-zero classes with $|a| + |x| - |y| + \mu(\mathbf{B}') + \mu(\mathbf{B}'') - 2n = 1$ each of the spaces $\mathcal{P}_{III_i}(a, x, y; \mathbf{B}', \mathbf{B}'', J)$, $\mathcal{P}_{III'_i}(a, x, y; \mathbf{B}', \mathbf{B}'', J)$, $i = 1, 2$, is a compact 0-dimensional manifold, hence a finite set. Moreover, every $(u'_1, \ldots, u'_{l'}, u''_1, \ldots, u''_{l''})$ in this space consists of simple and absolutely distinct disks.*

We put

$$\begin{aligned}
&n_{II_i}(a,x,y;\mathbf{A},J) = \#_{\mathbb{Z}_2}\mathcal{P}_{II_i}(a,x,y;\mathbf{A},J),\\
&n_{III_i}(a,x,y;\mathbf{B}',\mathbf{B}'',J) = \#_{\mathbb{Z}_2}\mathcal{P}_{III_i}(a,x,y;\mathbf{B}',\mathbf{B}'',J),\\
&n_{III_i}(a,x,y;\mathbf{B}',\mathbf{B}'',J) = \#_{\mathbb{Z}_2}\mathcal{P}_{III_i'}(a,x,y;\mathbf{B}',\mathbf{B}'',J),
\end{aligned}$$

whenever $|a|+|x|-|y|+\mu(\mathbf{A})-2n=1$ or $|a|+|x|-|y|+\mu(\mathbf{B}')+\mu(\mathbf{B}'')-2n=1$.

5.3.3. *Identities.* Given $a,a'\in\mathrm{Crit}(h)$ with $|a'|=|a|-1$, set $n(a,a')=\#_{\mathbb{Z}_2}(W^u_a\cap W^s_{a'})/\mathbb{R}$, i.e. the number modulo 2 of (negative gradient) trajectories of h going from a to a'. Similarly for $x,x'\in\mathrm{Crit}(f)$ we have $n(x,x')$. In order to simplify the notation we will omit the J's from $n_I(a,x,y;\mathbf{A},J)$'s and from the n_{II}'s, n_{III}'s etc. Given two vectors of non-zero classes $\mathbf{B}'=(B'_1,\ldots,B'_{l'})$, $\mathbf{B}''=(B''_1,\ldots,B''_{l''})$ write

$$\mathbf{B}'\#\mathbf{B}''=(B'_1,\ldots,B'_{l'-1},B'_{l'}+B''_1,B''_2,\ldots,B''_{l''}).$$

Proposition 5.3.6. *Let (f,ρ_L), (h,ρ_M) be as above. There exists a second category subset $\mathcal{J}_{\mathrm{reg}}\subset\mathcal{J}(M,\omega)$ such that for every $J\in\mathcal{J}_{\mathrm{reg}}$, every $x,y\in\mathrm{Crit}(f)$, $a\in\mathrm{Crit}(h)$ and $\mathbf{A}$ with $|a|+|x|-|y|+\mu(\mathbf{A})-2n=1$ the following two identities hold:*

$$\begin{aligned}
&\sum_{|x'|=|x|-1} n(x,x')n_I(a,x',y;\mathbf{A}) + \sum_{|y'|=|y|+1} n_I(a,x,y';\mathbf{A})n(y',y)+\\
&\sum_{|a'|=|a|-1} n(a,a')n_I(a',x,y;\mathbf{A}) + \sum_{\substack{(\mathbf{A}',\mathbf{A}'')=\mathbf{A}\\ |x'|=|x|+\mu(\mathbf{A}')-1}} n(x,x';\mathbf{A}')n_I(a,x',y;\mathbf{A}'')+\\
&\sum_{\substack{(\mathbf{A}',\mathbf{A}'')=\mathbf{A}\\ |y'|=|y|-\mu(\mathbf{A}'')+1}} n_I(a,x,y';\mathbf{A}')n(y',y;\mathbf{A}'') + n_{II_1}(a,x,y;\mathbf{A})+n_{II_2}(a,x,y;\mathbf{A})+\\
&\sum_{(\mathbf{B}',\mathbf{B}'')=\mathbf{A}} \big(n_{III_1}(a,x,y;\mathbf{B}',\mathbf{B}'')+n_{III_2}(a,x,y;\mathbf{B}',\mathbf{B}'')\big)+\\
&\sum_{\mathbf{C}'\#\mathbf{C}''=\mathbf{A}} \big(n_{III_1}(a,x,y;\mathbf{C}',\mathbf{C}'') + n_{III_2}(a,x,y;\mathbf{C}',\mathbf{C}'')\big)=0.
\end{aligned}\tag{74}$$

$$
\begin{aligned}
&\sum_{|x'|=|x|-1} n(x,x')n_{I'}(a,x',y;\mathbf{A}) + \sum_{|y'|=|y|+1} n_{I'}(a,x,y';\mathbf{A})n(y',y) + \\
&\sum_{|a'|=|a|-1} n(a,a')n_{I'}(a',x,y;\mathbf{A}) + \sum_{\substack{(\mathbf{A}',\mathbf{A}'')=\mathbf{A} \\ |x'|=|x|+\mu(\mathbf{A}')-1}} n(x,x';\mathbf{A}')n_{I'}(a,x',y;\mathbf{A}'') + \\
&\sum_{\substack{(\mathbf{A}',\mathbf{A}'')=\mathbf{A} \\ |y'|=|y|-\mu(\mathbf{A}'')+1}} n_{I'}(a,x,y';\mathbf{A}')n(y',y;\mathbf{A}'') + n_{II_1}(a,x,y;\mathbf{A}) + n_{II_2}(a,x,y;\mathbf{A}) + \\
&\sum_{(\mathbf{B}',\mathbf{B}'')=\mathbf{A}} \big(n_{III'_1}(a,x,y;\mathbf{B}',\mathbf{B}'') + n_{III'_2}(a,x,y;\mathbf{B}',\mathbf{B}'')\big) + \\
&\sum_{\mathbf{C}'\#\mathbf{C}''=\mathbf{A}} \big(n_{III'_1}(a,x,y;\mathbf{C}',\mathbf{C}'') + n_{III'_2}(a,x,y;\mathbf{C}',\mathbf{C}'')\big) = 0.
\end{aligned}
\tag{75}
$$

Proof of Proposition 5.3.4. Let $a \in \mathrm{Crit}(h)$, $x \in \mathrm{Crit}(f)$. As we work with $\mathbb{Z}_2$-coefficients we have to show that $d^f(a*x) + \partial^h(a)*x + a*d^f(x) = 0$. Write

$$
d^f(a*x) + \partial^h(a)*x + a*d^f(x) = \sum_{|y|-\mu(\lambda)=|a|+|x|-2n-1} m_{y,\lambda} y t^{\bar{\mu}(\lambda)}, \quad m_{y,\lambda} \in \mathbb{Z}_2.
$$

Fix y, λ with $|y| - \mu(\lambda) = |a| + |x| - 2n - 1$. We will show below that the coefficient $m_{y,\lambda}$ of $yt^{\bar{\mu}(\lambda)}$ in this sum is 0. Note that for $\lambda = 0$ this follows from standard arguments from Morse theory (see e.g. [58]). Therefore we assume from now on that $\lambda \neq 0$.

Take the sum of identities (74) and (75), then sum up the result over all possible vectors $\mathbf{A} = (A_1, \ldots, A_l)$ (of all possible lengths l) with $\sum A_i = \lambda$.

First note that the summands $n_{II_1}(a,x,y;\mathbf{A}) + n_{II_2}(a,x,y;\mathbf{A})$ being present in both identities (74) and (75) cancel out. Next, note that when summing over all possible $\mathbf{A}$'s the summands of the type $n_{III_1}(a,x,y;\mathbf{B}',\mathbf{B}'')$ and $n_{III_1}(a,x,y;\mathbf{C}',\mathbf{C}'')$ are in $1-1$ correspondence: whenever the first one appears for $(\mathbf{B}',\mathbf{B}'') = \mathbf{A}$ the second one appears when summing over $\widetilde{\mathbf{A}} = \mathbf{B}'\#\mathbf{B}''$ and vice versa. The same holds for the summands of the type n_{III_2}, $n_{III'_1}$, $n_{III''_2}$. Thus after summing over all $\mathbf{A}$'s these summands cancel out and we obtain:

$$
\begin{aligned}
\sum_{\mathbf{A},\sum A_i=\lambda} \Big(&\sum_{|x'|=|x|-1} n(x,x')n(a,x',y;\mathbf{A}) + \sum_{|y'|=|y|+1} n(a,x,y';\mathbf{A})n(y',y) + \\
&\sum_{|a'|=|a|-1} n(a,a')n(a',x,y;\mathbf{A}) + \sum_{\substack{(\mathbf{A}',\mathbf{A}'')=\mathbf{A} \\ |x'|=|x|+\mu(\mathbf{A}')-1}} n(x,x';\mathbf{A}')n(a,x',y;\mathbf{A}'') + \\
&\sum_{\substack{(\mathbf{A}',\mathbf{A}'')=\mathbf{A} \\ |y'|=|y|-\mu(\mathbf{A}'')+1}} n(a,x,y';\mathbf{A}')n(y',y;\mathbf{A}'') \Big) = 0.
\end{aligned}
\tag{76}
$$

Note that the right-hand side of identity (76) is exactly the coefficient $m_{y,\lambda}$ of the term $yt^{\bar{\mu}(\lambda)}$ in

$$d^f(a * x) + \partial^h(a) * x + a * d^f(x).$$

□

5.3.4. *Proof of Proposition 5.3.6.* We will give two different proofs for identity (74). The first proof is based on a natural compactification of the 1-dimensional manifolds $\mathcal{P}_I(a, x, y; \mathbf{A}, J)$ when $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n = 1$. This approach is quite straightforward but has the drawback that it proves identity (74) only under the assumption that $N_L \geq 3$. The reason for this restriction comes from transversality issues. (As for identity (75), this proof works well for every $N_L \geq 2$). In Sections 5.3.6 – 5.3.11 we will give an alternative proof for identity (74) that works for every $N_L \geq 2$. This approach is based on *Hamiltonian perturbations*. Although the second proof is more general we found it worth presenting the first proof too since it is more geometric and is closer in spirit to the theory of J-holomorphic disks.

We start with the first proof. An important ingredient in this proof is the next proposition.

Proposition 5.3.7. *There exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$, $a \in \text{Crit}(h)$, $x, y \in \text{Crit}(f)$ and every vector $\mathbf{A}$ of non-zero classes with $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n = 1$ the following holds:*

1. *If $N_L \geq 3$ then the space $\mathcal{P}_I(a, x, y; J, \mathbf{A})$ is a smooth 1-dimensional manifold. Moreover every $(u_1, \ldots, u_l)$ in this space consists of simple and absolutely distinct disks.*
2. *If $N_L \geq 2$ then the space $\mathcal{P}_{I'}(a, x, y; J, \mathbf{A})$ is a smooth 1-dimensional manifold. Moreover every $(u_1, \ldots, u_l)$ in this space consists of simple and absolutely distinct disks.*

We defer the proof of this proposition to Section 5.3.5. Note that statement (1) of Proposition 5.3.7 assumes $N_L \geq 3$.

Proof of Proposition 5.3.6. We start with identity (74). For brevity we omit the J from the notation of the moduli spaces $\mathcal{P}_I$ etc.

By Proposition 5.3.7 the space $\mathcal{P}_I(a, x, y; \mathbf{A})$ is a 1-dimensional manifold. We claim that it admits a compactification, in the Gromov topology, into a *compact* 1-*dimensional manifold with boundary* $\overline{\mathcal{P}_I(a, x, y; \mathbf{A})}$ whose boundary points consists of the following disjoint union:

$$\begin{aligned}
&\partial\overline{\mathcal{P}_I(a,x,y;\mathbf{A})} = \overline{\mathcal{P}_I(a,x,y;\mathbf{A})} \setminus \mathcal{P}_I(a,x,y;\mathbf{A}) = \\
&\Big(\bigcup_{|x'|=|x|-1} \mathcal{P}(x',y)\times\mathcal{P}_I(a,x',y;\mathbf{A})\Big) \coprod \Big(\bigcup_{|y'|=|y|+1} \mathcal{P}_I(a,x,y';\mathbf{A})\times\mathcal{P}(y',y)\Big) \coprod \\
&\Big(\bigcup_{|a'|=|a|-1} \mathcal{P}(a,a')\times\mathcal{P}_I(a',x,y;\mathbf{A})\Big) \coprod \\
&\Big(\bigcup_{\substack{(\mathbf{A}',\mathbf{A}'')=\mathbf{A}\\ |x'|=|x|+\mu(\mathbf{A}')-1}} \mathcal{P}(x,x';\mathbf{A}')\times\mathcal{P}_I(a,x',y;\mathbf{A}'')\Big) \coprod \\
&\Big(\bigcup_{\substack{(\mathbf{A}',\mathbf{A}'')=\mathbf{A}\\ |y'|=|y|-\mu(\mathbf{A}'')+1}} \mathcal{P}_I(a,x,y';\mathbf{A}')\times\mathcal{P}(y',y;\mathbf{A}'')\Big) \coprod \\
&\mathcal{P}_{II_1}(a,x,y;\mathbf{A}) \coprod \mathcal{P}_{II_2}(a,x,y;\mathbf{A}) \coprod \\
&\bigcup_{(\mathbf{B}',\mathbf{B}'')=\mathbf{A}} \Big(\mathcal{P}_{III_1}(a,x,y;\mathbf{B}',\mathbf{B}'') \coprod \mathcal{P}_{III_2}(a,x,y;\mathbf{B}',\mathbf{B}'')\Big) \coprod \\
&\bigcup_{\mathbf{C}'\#\mathbf{C}''=\mathbf{A}} \Big(\mathcal{P}_{III_1}(a,x,y;\mathbf{C}',\mathbf{C}'') \coprod \mathcal{P}_{III_2}(a,x,y;\mathbf{C}',\mathbf{C}'')\Big).
\end{aligned} \tag{77}$$

Note that this immediately proves identity (74) since the number of points in $\partial\overline{\mathcal{P}_I(a,x,y;\mathbf{A})}$ is on the one hand exactly the left-hand side of the identity (74) while on the other hand the number of boundary points of a compact 1-dimensional manifold is always even.

We turn to proving (77). We claim that the boundary $\partial\overline{\mathcal{P}_I(a,x,y;\mathbf{A})}$ is included in the union that appears in (77). To see this let $\mathbf{u}^{(n)} = (u_1^{(n)},\ldots,u_l^{(n)})$ be a sequence of elements in $\mathcal{P}_I(a,x,y;\mathbf{A})$. By compactness there exists a subsequence, still denoted by $\mathbf{u}^{(n)}$ that converges in the Gromov topology. Assume by contradiction that the limit of this subsequence is not an element of the spaces in the right-hand side of (77). Without loss of generality we may assume that the whole sequence $\mathbf{u}^{(n)}$ lies in the space $\mathcal{P}_I(a,x,y;(\mathbf{A},k),J)$ for the same $1\le k\le l$. (See (69) for the definition of this space). As $\mathbf{u}^{(n)}$ does not converge to an element of the right-hand side of (77) one of the following occurs for the limit of $\mathbf{u}^{(n)}$:

(1) More than one breaking occurs at the gradient trajectories involved in the definition of $\mathcal{P}_I(a,x,y;\mathbf{A})$.
(2) More than two gradient trajectories connecting two consecutive disks in $\mathbf{u}^{(n)}$ shrink to a point.
(3) Bubbling of a J-holomorphic sphere occurs at a point on one of the disks in $\mathbf{u}^{(n)}$.

(4) Bubbling of a holomorphic disk occurs at a point $p \neq -1, 1$ on the boundary of one of the disks $u_j^{(n)}$. See figures 12, 13.
(5) Bubbling of a holomorphic disk occurs at a point $p = -1$ or $p = 1$ but the marked point p corresponds in the limit to the attaching point of the two bubble disks.
(6) $\mathbf{u}^{(n)}$ converges in the Gromov topology to a configuration of disks with at least *two* disks bubbling.
(7) A combination of the above.

Possibilities (1),(2),(3) and (6) can be ruled out by a dimension count. Indeed, using the techniques of section 3 (See also Section 5.3.5 below) it follows that for generic J each of the configurations in (1),(2),(3) and (6) must have negative dimension, hence cannot occur. We turn to possibility (4). Assume first that $j \neq k$, i.e. the bubbling occurs in one of the disks that is not connected to the gradient trajectory going from $a \in \mathrm{Crit}(h)$ (See figure 12.) It follows that one of the components of the limit is an element of $\mathcal{P}_I(a, x, y; \mathbf{A}')$ for some $\mathbf{A}'$ with $\mu(\mathbf{A}') \leq \mu(\mathbf{A}) - N_L$. Again, the techniques of section 3 and a dimension computation show that this configuration has negative dimension, hence impossible for generic J. Finally, assume that $j = k$. Recall that $u_k^{(n)}(0) \in W_a^u$ for every n. Note that the points $-1, 0, 1 \in D$ lie on the same hyperbolic geodesic. Since the conformal structure of the disks is preserved when passing to limits in the Gromov topology, the marked points $-1, 0, 1$ on $u_k^{(n)}$ must converge to the same bubble disk. (Thus the right-hand side of figure 13 is impossible.) It follows that one of the components of the limit of $\mathbf{u}^{(n)}$ is an element of $\mathcal{P}_I(a, x, y; \mathbf{A}')$ for some $\mathbf{A}'$ with $\mu(\mathbf{A}') \leq \mu(\mathbf{A}) - N_L$ and we arrive at contradiction as before (See the left-hand side of picture 13.) This rules out possibility (4). Possibility (5) is ruled out in a similar way. Finally, possibility (6) is ruled out by a combination of the above arguments.

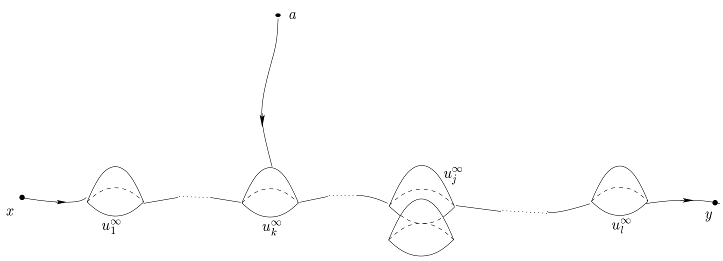

FIGURE 12. Limit $\mathbf{u}^\infty$ of $\mathbf{u}^{(n)}$ in possibility (4) with $j \neq k$.

Finally we have to prove the following statement:

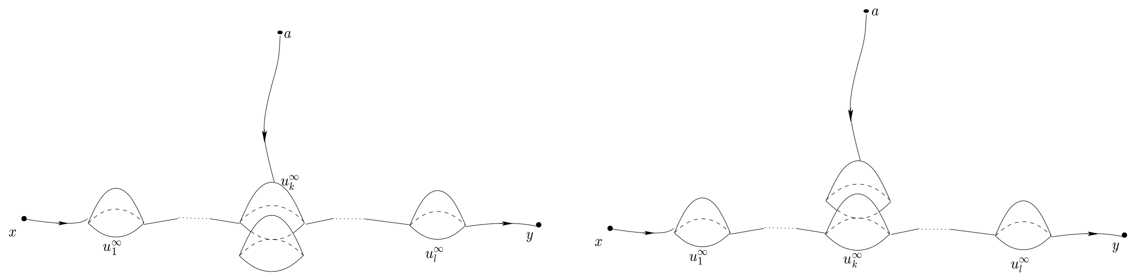

FIGURE 13. Possibility (4) for $j = k$.

5.3.8. *Every point in the spaces from the right-hand side of* (77) *is indeed a limit of a sequence of elements from* $\mathcal{P}_I(a, x, y; \mathbf{A})$. *Moreover every such element corresponds to* exactly *one end of the* 1*-dimensional manifold* $\mathcal{P}_I(a, x, y; \mathbf{A})$.

We start with the spaces appearing in the first 6 lines of the union on the right-hand side of (77). In each of these 6 cases, Statement 5.3.8 follows from general transversality arguments and gluing theorems in the framework of Morse theory. (In each of these cases one has to add boundary components to the submanifold Q_{f,ρ_L} to include either a breaking of a gradient trajectory or to add parts of $\mathrm{diag}(L \setminus \mathrm{Crit})(f)$ to Q_{f,ρ_L} and then make the suitable evaluation maps transverse to this enlarged manifold with boundary).

Finally, it remains to prove Statement 5.3.8 for elements from $\mathcal{P}_{III_i}(a, x, y; \mathbf{C}', \mathbf{C}'')$, $i = 1, 2$, where $\mathbf{C}' \# \mathbf{C}'' = \mathbf{A}$. This follows from the gluing procedure described in Section 4.2.2 (see Corollary 4.2.2 there). Note that we need here both the existence and uniqueness results of Corollary 4.2.2.

We now turn to the proof of identity (75). The proof is very similar to the proof of the identity (74), but there are two slight changes. First of all, we are using now statement (2) of Proposition 5.3.7 which assumes only $N_L \geq 2$ and therefore the proof works well for every $N_L \geq 2$. The second change is when $|a| = 2n$. In this case the spaces $\mathcal{P}_{I'}(a, x, y; (\mathbf{A}, k))$, for $k = 0$ and $k = l$ have additional boundary points. Indeed, we may have a sequence $(\mathbf{u}^{(n)}, p_n) \in \mathcal{P}_{I'}(a, x, y; (\mathbf{A}, 0))$ with $p_n \to x$ as $n \to \infty$. Similarly we may have $(\mathbf{u}^{(n)}, p_n) \in \mathcal{P}_{I'}(a, x, y; (\mathbf{A}, l))$ with $p_n \to y$ as $n \to \infty$. Therefore the boundary of $\overline{\mathcal{P}_{I'}(a, x, y; \mathbf{A})}$ contains the following additional points:

$$\Big(\mathcal{P}(x, y; \mathbf{A}) \times \{x\}\Big) \coprod \Big(\mathcal{P}(x, y; \mathbf{A}) \times \{y\}\Big). \tag{78}$$

As before, standard transversality arguments show that all the points in (78) can indeed be realized as boundary points of $\mathcal{P}_{I'}(a, x, y; \mathbf{A})$ and that each of them corresponds to exactly one end of this 1-dimensional manifold.

Note that since $|a| = 2n$ we have $|x| - |y| + \mu(\mathbf{A}) = 1$ hence $\mathcal{P}(x, y; \mathbf{A})$ is a finite set. Finally note that the number of points appearing in (78) is even (due to our assumption that h has a single maximum) hence their appearance will not change our identity. $\square$

5.3.5. *Proof of Propositions 5.3.3, 5.3.5 and 5.3.7.*

Proof of Proposition 5.3.3. Denote by $\mathcal{P}_I^{*,\mathrm{d}}(a, x, y; \mathbf{A}, J) \subset \mathcal{P}_I(a, x, y; \mathbf{A}, J)$ the subspace of sequences of disks that are simple and absolutely distinct. By standard arguments, for generic J this space is a smooth manifold of dimension 0. We claim that $\mathcal{P}_I^{*,\mathrm{d}}(a, x, y; \mathbf{A}, J) = \mathcal{P}_I(a, x, y; \mathbf{A}, J)$.

The main idea is that the dimension of a configuration of disks in $\mathcal{P}_I(a, x, y; \mathbf{A}, J)$ that is not simple or not absolutely distinct is negative, hence such configurations cannot occur for generic J. Most of the arguments are very similar to those appearing in Section 3 therefore we will only give an outline of the proof explaining how to adjust the arguments from Section 3 to work in the present situation.

As in Section 3 we separate the proof to the cases $n = \dim L \geq 3$ and $n = \dim L \leq 2$. We start with $n \geq 3$. The proof in this case can be carried out in a similar way to the proof of Proposition 3.1.3 from Section 3, with Lemmas 3.2.2 and 3.2.3 being the main tools. The only adjustment needed is when applying Lemma 3.2.3 to the k'th disk in $\mathbf{u} = (u_1, \ldots, u_l)$ where $\mathbf{u} \in \mathcal{P}_I(a, x, y; ((A), k), J)$. Indeed, suppose (by contradiction) that u_k is not simple. Then by Lemma 3.2.3 we obtain a new disk u'_k with the same image as u, with $u'_k(-1) = u_k(-1)$, $u'_k(1) = u_k(1)$ and $\mu[u'_k] \leq \mu([u_k]) - N_L$, but we cannot assume that $u'_k(0) = u_k(0)$ anymore. Therefore the sequence of disks $\mathbf{u}'$ obtained from $\mathbf{u}$ by replacing u_k by u'_k is not necessarily an element of $\mathcal{P}_I(a, x, y; \mathbf{A}', J)$, where $\mathbf{A}'$ is the vector of classes of $\mathbf{u}'$. However, Lemma 3.2.3 still implies that there exists a point $p \in \mathrm{Int}\,(D)$ such that $u'_k(p) = u_k(0) \in W_a^u$. Consider the space of disks $\mathbf{v} = (v_1, \ldots, v_l)$ in $\mathcal{P}(x, y; \mathbf{A}', J)$ with the additional condition that there exists a point $q \in \mathrm{Int}\,(D)$ with $v_k(q) \in W_a^u$. A simple computation shows that the dimension of this space is

$$\begin{aligned} &|a| + |x| - |y| - 2n + \mu(\mathbf{A}') + 1 \leq \\ &|a| + |x| - |y| - 2n + \mu(\mathbf{A}) - N_L + 1 = -N_L + 1 < 0. \end{aligned} \tag{79}$$

On the other hand $\mathbf{u}'$ belongs to this space, a contradiction. It remains to show that the disks present in the configurations of $\mathcal{P}_I(a, x, y; \mathbf{A}, J)$ are also absolutely distinct. Under the assumption $n \geq 3$ we may apply Lemma 3.2.2. Assuming that two disks u_1 and u_2 in a configuration $u \in \mathcal{P}_I(a, x, y; \mathbf{A}, J)$ are not absolutely distinct, this result allows us to eliminate the "smallest" of the two disks - say u_1 - by u_2 thus reducing the total energy of the configuration. It is easy to see that this replacement can always be done in such a way that the resulting object u' is still modeled on a tree. However, it is also easy to notice that u' might not be in any of the spaces $\mathcal{P}_-(a,x,y;\mathbf{A}',J)$ but only in a more general type of moduli space. These moduli spaces consist of configurations modeled on a tree with two entries and one exit - as before - so that one of the entering edges corresponds to a flow line of h, all the other edges are flow lines of f and the vertex of valence three can be the end of the flow line of h or, in contrast to the cases described before, it is also allowed to have its two entering edges correspond to flow lines of f. To achieve transversality, an important point is to notice that if the attaching points of the two entering edges happen to coincide in this last case, then the respective configuration can be rewritten as an element of the same type of moduli space only modeled on a tree in which one of these edges (the shortest one) is eliminated and a new vertex of valence three is introduced. This vertex corresponds to a ghost (or constant) disk to which one of the branches of the tree is attached by an edge of length 0. Now we apply the proof before to first show that the disks present in these configurations are all simple for those moduli spaces whose virtual dimension is 0 or 1 and then we notice that the replacement argument before allows to show by recurrence on total energy that the disks are also absolutely distinct for a second category subset of J's in the same range of dimensions. The condition $N_L \geq 3$ is used again in the "replacement" argument as in this process the incidence condition for the flow line of h with the center of the disk at its end might not be preserved.

We now turn to the case $n = \dim L = 2$. Since $|a| + |x| - |y| - 2n + \mu(\mathbf{A}) = 0$ we must have $\mu(\mathbf{A}) \leq 6$. Therefore the number of disks l in an element of $\mathcal{P}_I(a,x,y;\mathbf{A},J)$ is ≤ 3. Consider the case $l = 1$ first. We have to show that for generic J every J-holomorphic disk u with $u(-1) \in W^u_x$, $u(1) \in W^s_y$, $u(0) \in W^u_a$ and with $|a| + |x| - |y| - 2n + \mu(A) = 0$ must be simple.

Suppose by contradiction that u is a J-holomorphic disk as above which is not simple. Applying Theorem 3.2.1 we obtain domains $\mathfrak{D}_1, \ldots, \mathfrak{D}_r \subset D$ and J-holomorphic disks $v_1, \ldots, v_r$ through which u factors on each of the domains $\mathfrak{D}_i$. Denote by m_i the degree of the map $\pi_i : \overline{\mathfrak{D}}_i \to D$ for which $u_i|_{\overline{\mathfrak{D}}_i} = v_i \circ \pi_i$. As $N_L \geq 2$ and $\mu(A) \leq 6$ we have $r \leq 3$.

Case 1. $l = 1$, $r = 3$. In this case we have three simple disks v_1, v_2, v_3 each of them with Maslov number 2. In this case $\mu(A) = 6$ and $m_1 = m_2 = m_3 = 1$. By assumption we have $|x| - |y| = -2 - |a| \leq -2$. We may assume that $u(-1) \neq u(1)$ (for otherwise by omitting u we obtain a negative gradient trajectory of f going from x to y which is impossible).

Assume first that the three disks v_1, v_2, v_3 are absolutely distinct. Note that there are several possibilities as to how the three points $-1, 0, 1$ are distributed among the domains $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3$. In case the points -1, 1 lie in the same domain, say $\mathfrak{D}_1$ we can omit the other two disks v_2, v_3 and after reparametrizing v_1 obtain an element of the space $\mathcal{P}_I^{*,\mathrm{d}}(x,y;B)$ where $\mu(B)=2$. Note that this is impossible since the dimension of this space is $|x|-|y|+\mu(B)-1\leq -1$. Therefore we assume that -1, 1 lie in different domains, say $-1\in\mathfrak{D}_1$, $1\in\mathfrak{D}_2$. If the point 0 lies in interior of $\mathfrak{D}_1$ then after reparametrizing v_1 we may assume that $v_1(-1)\in W_x^u$, $v_1(0)\in W_a^u$. On the other hand, a simple computation shows that the space of such disks (with $\mu=2$) has dimension $|x|+|a|-2$. But $|x|+|a|-2=-4+|y|<0$. A contradiction. A similar argument shows that it is impossible for 0 to lie on the boundary of $\mathfrak{D}_1$. The same arguments applied to v_2 show that it is impossible for 0 to lie neither in $\operatorname{Int}\mathfrak{D}_2$ nor on the boundary of this domain. We are left with the case $0\in\operatorname{Int}\mathfrak{D}_3$. Note that the boundary of $\mathfrak{D}_3$ must have a non-trivial intersection with the boundary of at least one of the domains $\mathfrak{D}_1$, $\mathfrak{D}_2$. (In fact this intersection must be at least 1-dimensional.) Without loss of generality assume that such an intersection occurs with the boundary of $\mathfrak{D}_1$.

Reparametrizing v_1,v_3 we may assume that $v_1(-1)\in W_x^u$, $v_3(0)\in W_a^u$ and $v_1(1)=v_3(-1)$. Put $B'=[v_1]$, $B''=[v_3]$. It follows that the space

$$\begin{aligned}\Big\{(w',w'')\in\mathcal{M}(B',J)/G_{-1,1}\times\mathcal{M}(B'',J)\,\Big|\,&w'(-1)\in W_x^u, w''(0)\in W_a^u,\\ &w'(1)=w''(-1),(w',w'')\text{ are simple and absolutely distinct}\Big\}\end{aligned} \tag{80}$$

is not empty. On the other hand a simple computation shows that the dimension of this space is $|x|+|a|-1=-3+|y|<0$, a contradiction.

Remark 5.3.9. It is worth noting that the pair (v_1,v_3) in fact gives rise to a 1-parametric family of elements in the space in (80) since the boundary of at least one of the domains $\mathfrak{D}_1$, $\mathfrak{D}_2$ must intersect the boundary of $\mathfrak{D}_3$ along a 1-dimensional piece. Therefore there is a 1-parametric family of reparametrizations of (v_1,v_3) which lies in the space in (80). Here we have not used this fact as we got contradiction by showing that the dimension of this space is negative. However, this observation can used to rule out other configurations below.

This concludes the proof of case 1 when v_1,v_2,v_3 are absolutely distinct. It remains to deal with the case when these three disks are not absolutely distinct. Again, there are several subcases to be considered:

- $v_1(D)\subset v_2(D)\cup v_3(D)$ but v_2,v_3 are absolutely distinct.
- $v_1(D),v_2(D)\subset v_3(D)$.

In each of these two subcases one has to deal with all possible distribution of the three points $-1,0,1$. Ruling out these possibilities is made by similar arguments to the above and we will not give the details here.

Case 2. $l = 1$**,** $r = 2$**.** Here we have two simple disks v_1, v_2 and two multiplicities m_1, m_2. As $\mu(\mathbf{A}) \leq 6$ we may assume that one of the following holds:

- $\mu([v_1]) = 2$, $m_1 = 1$, $\mu_([v_2]) = 2$, $m_2 = 1$; $\mu(\mathbf{A}) = 4$, $|a| + |x| - |y| = 0$.
- $\mu([v_1]) = 2$, $m_1 = 1$, $\mu_([v_2]) = 2$, $m_2 = 2$; $\mu(\mathbf{A}) = 6$, $|a| + |x| - |y| = -2$.)
- $\mu([v_1]) = 2$, $m_1 = 1$, $\mu([v_2]) = 4$, $m_2 = 1$; $\mu(\mathbf{A}) = 6$, $|a| + |x| - |y| = -2$.)

Again, by arguments similar to those used above one can extract in each of these three cases a configuration which has negative dimension, hence deduce that it cannot occur for generic J.

Case 3. $l = 1$**,** $r = 1$**.** In this case the disk u is multiply covered, say with multiplicity $m \geq 2$. We denote by v its reduction. By assumption $|a| + |x| - |y| = 4 - m\mu([v]) \leq 0$. First note that $u(-1) \neq u(1)$ for otherwise we would get a negative gradient trajectory of f going from x to y which is impossible since $|x| \leq |y|$. Let $p', p'' \in \partial D$ be points such that $v(p') = u(-1)$, $v(p'') = u(1)$. As $p' \neq p''$ we can reparametrize v so that $v(-1) \in W^u_x$, $v(1) \in W^s_y$. Let $q \in D$ a point such that $v(q) \in W^u_a$ (note that we cannot assume anymore that $q = 0$). Put $B = [v]$. Assume first that $q \in \operatorname{Int} D$. Then (v, q) belongs to the space

$$\{(w, p) \in \big(\mathcal{M}^*(B, J) \times \operatorname{Int} D\big)/G_{-1,1} \mid w(-1) \in W^u_x, w(p) \in W^u_a, w(1) \in W^s_y\}.$$

On the other hand a simple computation shows that the dimension of this space is:

$$-3 + |x| + |a| - |y| + \mu(B) = 1 - (m-1)\mu(B) < 0,$$

where the last inequality follows from $m \geq 2$, $\mu(B) \geq 2$. A contradiction. This concludes the proof for the case $l = 1$.

The remaining cases are $l = 2, 3$. The proofs for these cases are again similar to the preceding ones hence we omit the details. Let us only list the variety of configurations needed to be ruled out. In the case $l = 2$ we have two disks u_1, u_2 and there are four possibilities to be ruled out:

(1) u_1 is simple with $\mu([u_1]) = 2$. $\mu([u_2]) = 4$ and u_2 is double covered.
(2) u_1 is simple with $\mu([u_1]) = 2$. $\mu([u_2]) = 4$ and u_2 can be decomposed into two simple disks v'_2, v''_2 each with $\mu = 2$.
(3) u_1, u_2 are both simple with $\mu([u_1]) = 2$, $\mu([u_2]) = 4$, but $u_1(D) \subset u_2(D)$.
(4) u_1, u_2 are both simple with $\mu(u_1) = \mu(u_2) = 2$ but u_1, u_2 are not absolutely distinct, i.e. $u_1(D) \subset u_2(D)$ or $u_2(D) \subset u_1(D)$.

Finally, in case $l = 3$ we have three disks with $\mu([u_1]) = \mu([u_2]) = \mu([u_3]) = 2$ hence they are all simple. In this case one has to rule out the possibility that they are not absolutely distinct. This concludes the proof for the case $n = \dim L = 2$.

Finally, in case $n = \dim L = 1$ there is nothing to prove. Indeed, the assumption $|a| + |x| - |y| - 2n + \mu(\mathbf{A}) = 0$ implies that we have $\mu(\mathbf{A}) \leq 3$ hence every element $\mathbf{u} \in \mathcal{P}_I(a, x, y; \mathbf{A}, J)$ consists of exactly one simple disk.

Next, we turn to compactness of the space $\mathcal{P}_I(a, x, y; \mathbf{A}, J)$ (under the assumption $|a| + |x| - |y| - 2n + \mu(\mathbf{A}) = 0$). The main argument is again a combination of Gromov compactness theorem with a dimension count. One lists all the possible limits, in the Gromov topology, of sequences $\mathbf{u}^{(n)} \in \mathcal{P}_I(a, x, y; \mathbf{A}, J)$ as in (77). (As in the proof of Proposition 5.3.6 one first has to show that none of the possibilities (1)-(7) listed there can appear.) Then using the same methods as above one shows that under the assumption $|a|+|x|-|y|-2n+\mu(\mathbf{A}) = 0$ all the elements in the spaces appearing in the spaces in (77) are simple and absolutely distinct. Finally, a dimension computation shows that when $|a|+|x|-|y|-2n+\mu(\mathbf{A}) = 0$ all these spaces have negative dimension hence empty. This completes the proof of the statement on compactness, hence the proof Proposition 5.3.3 for the space $\mathcal{P}_I(a, x, y; \mathbf{A}, J)$.

The proof for the space $\mathcal{P}_{I'}(a, x, y; \mathbf{A}, J)$ is similar (and in fact simpler since all disks in this space have only two marked points). □

The proof of Proposition 5.3.5 is similar to the proof of Proposition 5.3.3, hence we omit it.

Outline of proof of Proposition 5.3.7. The proof goes along the same lines as the proof of Proposition 5.3.3 with one main difference for the space $\mathcal{P}_I(a, x, y; \mathbf{A}, J)$. While in Proposition 5.3.3 we had the assumption $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n = 0$ now we have $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n = 1$. This is the reason that we have to assume that $N_L \geq 3$ rather than just $N_L \geq 2$. The point is that when $N_L = 2$ non-simple disks might appear in $\mathcal{P}_I(a, x, y; \mathbf{A}, J)$ but not for $N_L \geq 3$. For example, the computation of dimension in (79) gives us in the present situation:

$$
\begin{aligned}
&|a| + |x| - |y| - 2n + \mu(\mathbf{A}') + 1 \leq \\
&|a| + |x| - |y| - 2n + \mu(\mathbf{A}) - N_L + 1 = -N_L + 2.
\end{aligned}
$$

In order for this number to be negative we need $N_L \geq 3$.

The rest of the proof continues in an analogous way to the proof of Proposition 5.3.3. □

5.3.6. *A more general proof for identity* (74) *of Proposition 5.3.6.* Below is an alternative proof of identity 5.3.6 which works under the more general assumption $N_L \geq 2$.

This approach consists of the following 3 steps.

Step 1. We perturb the Cauchy-Riemann equation to a non-homogeneous equation by adding a perturbation term H (generated by Hamiltonian functions). The perturbation procedure is applied to elements of the spaces $\mathcal{P}_I, \mathcal{P}_{II_i}, \mathcal{P}_{III_i}$ in the following way. In each element of these spaces, which is a chain of disks, we perturb only the single disk in the chain which has 3 marked points. All the other disks in the chain are left unperturbed hence remain J-holomorphic. The result of this procedure is "perturbations" $\mathcal{P}_I(-, H), \mathcal{P}_{II_i}(-, H), \mathcal{P}_{III_i}(-, H)$ of the original spaces $\mathcal{P}_I, \mathcal{P}_{II_i}, \mathcal{P}_{III_i}$.

Step 2. By counting the number of points in these spaces we obtain "perturbed" versions $n_I(-, H), n_{II_i}(-, H), n_{III_i}(-, H)$ of the numbers $n_I(-), n_{II_i}(-), n_{III_i}(-)$. The advantage of the above perturbations is that now it is easy to achieve transversality for the perturbed disk (without any simplicity requirements). Therefore, by similar arguments to those from the older proof of identity (74) we conclude that identity (74) holds (for $N_L \geq 2$) with the n_I, n_{II_i}, n_{III_i}'s replaced by their perturbed analogues.

Step 3. Finally, we prove that for small enough perturbations H the numbers $n_I(-, H)$, $n_{II_i}(-, H)$, $n_{III_i}(-, H)$ coincide with the numbers $n_I(-), n_{II_i}(-), n_{III_i}(-)$ hence identity (74) in fact holds in its original form.

We now turn to the implementation of the proof.

5.3.7. *Hamiltonian perturbations.* Here we briefly summarize the necessary ingredients from the theory of Hamiltonian perturbations that will be needed later. The material of this section is mostly based on [3, 44]. We refer the reader to these texts for the foundations and more details.

Let $H \in \Omega^1(D, C_0^\infty(M))$ be a 1-form on the disk D with values in $C_0^\infty(M)$. If (s, t) are coordinates on D we can write H as

$$H = F ds + G dt$$

for some compactly supported smooth functions $F, G : D \times M \to \mathbb{R}$. Denote by $pr_M : D \times M \to M$, $pr_D : D \times M \to D$ the projections. We write $F_{s,t}(x) = F(s, t, x)$ and $G_{s,t}(x) = G(s, t, x)$. Define the following 2-form on $D \times M$:

$$\widetilde{\omega}_H = pr_M^* \omega - dH = pr_M^* \omega - d'F \wedge ds - d'G \wedge dt + (\partial_t F - \partial_s G) ds \wedge dt,$$

where d' stands for exterior derivative in the M-direction. Henceforth we will work with H's that have the following additional property:

$$\widetilde{\omega}_H|_{T(\partial D \times L)} = 0. \tag{81}$$

We denote by $\mathcal{H}$ the space of all H satisfying (81).

Note that the form $\widetilde{\omega}_H$ may be degenerate however for $\kappa > 0$ large enough the form

$$\widetilde{\omega}_{H,\kappa} = \widetilde{\omega}_H + pr_D^*(\kappa\, ds \wedge dt) \tag{82}$$

is symplectic and $\partial D \times L$ is still Lagrangian with respect to it. How large should κ be taken for this purpose is determined by the *curvature* function $R_H : D \times M \to \mathbb{R}$:

$$R_H(s,t,x) = \partial_s G - \partial_t F + \{F_{s,t}, G_{s,t}\},$$

where $\{\cdot,\cdot\}$ is the Poisson bracket on (M,ω). A simple calculation (see [44, 3]) shows that if $\kappa > R_H(s,t,x)$ for every $(s,t,x) \in D \times M$ then $\widetilde{\omega}_{H,\kappa}$ is symplectic.

Let $X_{F_{s,t}}$ be the Hamiltonian vector field corresponding to the function $F_{s,t}$. Put $X_F(s,t,x) = X_{F_{s,t}}(x)$, $(s,t,x) \in D \times M$. Similarly we have $G_{s,t}$, $X_{G_{s,t}}$ and $X_G(s,t,x)$. Given an ω-compatible almost complex structure J on M, consider the following elliptic boundary value problem:

$$\begin{cases} u : (D, \partial D) \to (M, L) \\ \partial_s u + J(u)\partial_t u = -X_F(s,t,u) - J(u)X_G(s,t,u) \end{cases} \tag{83}$$

Solutions u of this equation will be called (J,H)-*holomorphic disks*. We denote by $\mathcal{M}(A,J,H)$ the space of (J,H)-holomorphic disks u with $u_*([D]) = A \in H_2(M,L;\mathbb{Z})$.

As noted by Gromov [36] solutions of (83) are in 1-1 correspondence with $\widetilde{J}$-holomorphic disks in $D \times M$ for some $\widetilde{J}$. To describe this correspondence define the following endomorphism $\widetilde{J}_H : T(D \times M) \to T(D \times M)$:

$$\widetilde{J}_H = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ JX_F - X_G & JX_G + X_F & J \end{pmatrix}. \tag{84}$$

Proposition 5.3.10 (See [44]). *(1) $\widetilde{J}_H$ is an almost complex structure on $D \times M$. Moreover, if $\kappa > R_H(s,t,x)$ for every $(s,t,x) \in D \times M$ then $\widetilde{J}_H$ is compatible with $\widetilde{\omega}_{H,\kappa}$.*

(2) $u : (D, \partial D) \to (M, L)$ is a solution of (83) if and only if its graph $\widetilde{u} : (D, \partial D) \to (D \times M, \partial D \times L)$ defined by $\widetilde{u}(z) = (z, u(z))$ is $\widetilde{J}_H$-holomorphic.

(3) There exists a subset $\mathcal{H}_{\mathrm{reg}} \subset \mathcal{H}$ of second category such that for every $H \in \mathcal{H}$ and every $A \in H_2(M,L;\mathbb{Z})$ the space $\mathcal{M}(A,J,H)$ is either empty or a smooth manifold of dimension $n + \mu(A)$.

Remark 5.3.11. (1) In contrast to the case of genuine J-holomorphic disks, (J,H)-holomorphic disks in the class $0 \in H_2(M,L;\mathbb{Z})$ are not constant for general H. Thus we may have a (non-constant) (J,H)-holomorphic disk u with $\omega([u]) = 0$.

(2) In contrast to the case of genuinely J-holomorphic disks we do not have to require the (J, H)-holomorphic disks to be simple in order for $\mathcal{M}(A, J, H)$ to be a smooth manifold. See [44] for more details.

Given $c > 0$ put

$$\mathcal{H}_c = \{H \in \mathcal{H} \mid R_H(s,t,x) \leq c, \, \forall\, (s,t,x) \in D \times M\}. \tag{85}$$

Define

$$A_L = \inf\{\omega(A) \mid A \in \pi_2(M, L), \, \omega(A) > 0\}.$$

Note that when L is monotone the inf is attained and $A_L > 0$.

Proposition 5.3.12. *Suppose $L \subset (M, \omega)$ is monotone. Let $c < \frac{A_L}{\pi}$ and $H \in \mathcal{H}_c$. Then for every (J, H)-holomorphic disk u we have $\omega([u]) \geq 0$ and $\mu([u]) \geq 0$.*

Proof. Pick $c < \kappa < \frac{A_L}{\pi}$. Then the form $\widetilde{\omega}_{H,\kappa}$ is symplectic and $\widetilde{J}_H$ is compatible with it. Let $u : (D, \partial D) \to (M, L)$ be a (J, H)-holomorphic disk and let $\widetilde{u} = (z, u(z))$ be its graph. Then $\widetilde{u}$ is a *non-constant* $\widetilde{J}_H$-holomorphic disk with boundary on $\partial D \times L$ hence

$$0 < \widetilde{\omega}_{H,\kappa}([\widetilde{u}]) = \omega([u]) + \pi\kappa.$$

It follows that $\omega([u]) > -\pi\kappa > -A_L$, hence $\omega([u]) \geq 0$. The statement on μ follows from monotonicity. □

5.3.8. *Compactness.* Let u_ν be a sequence in $\mathcal{M}(A, J, H)$. Then either there exists a C^∞-convergent subsequence or there exists a subsequence u_{ν_k} that converges in the sense of Gromov to a bubble tree consisting of maps $v_0, v_1, \ldots, v_l : (D, \partial D) \to (M, L)$, $w_1, \ldots, w_r : \mathbb{C}P^1 \to M$, where v_0 is (J, H)-holomorphic, $v_1, \ldots, v_l$, $w_1, \ldots, w_r$ are J-holomorphic, at least one of l, r is ≥ 1 and $[v_0] + [v_1] + \ldots + [v_l] + j_*[w_1] + \ldots + j_*[w_r] = A \in H_2(M, L; \mathbb{Z})$. Here $j_* : H_2(M; \mathbb{Z}) \to H_2(M, L; \mathbb{Z})$ is the natural homomorphism. The root of the tree can be thought of as v_0.

The relation to Gromov's compactness theorem for genuine pseudo-holomorphic disks can be understood as follows. Let $\widetilde{u}_\nu : (D, \partial D) \to (D \times M, \partial D \times L)$ be the sequence of $\widetilde{J}_H$-holomorphic disks obtained as graphs of the u_ν's. Note that $[\widetilde{u}_\nu] = [D] + A \in H_2(D \times M, \partial D \times L; \mathbb{Z})$ for every ν. If u_ν does not have a C^∞-convergent subsequence then by Gromov compactness theorem there exists a subsequence $\widetilde{u}_{\nu_k}$ which converges in the Gromov topology to a bubble tree consisting of $\widetilde{J}_H$-holomorphic disks $\widehat{v}_0, \ldots, \widehat{v}_l$ and $\widetilde{J}_H$-holomorphic spheres $\widehat{w}_1, \ldots, \widehat{w}_r$ such that $[\widehat{v}_0] + [\widehat{v}_1] + \cdots + [\widehat{v}_l] + j'_*[\widehat{w}_1] + \ldots + j'_*[\widehat{w}_r] = [D] + A$, where $j' : H_2(D \times M; \mathbb{Z}) \to H_2(D \times M, \partial D \times L; \mathbb{Z})$ is the natural map. As $pr_D : (D \times M, \widetilde{J}_H) \to (D, i)$ is holomorphic it follows that *exactly* one of the $\widehat{v}_i$'s, say $\widehat{v}_0$, projects surjectively to D while each of $\widehat{v}_1, \ldots, \widehat{v}_l, \widehat{w}_1, \ldots, \widehat{w}_r$ have constant projection to

D. It follows that all the disks and spheres but $\widehat{v}_0$ are in fact J-holomorphic maps lying in fibres of pr_D. Moreover it is easy to see that $\widehat{v}_0$ is, after a suitable reparametrization, of the form $\widetilde{v}_0$ with v_0 being (J, H)-holomorphic.

5.3.9. *Perturbation of the spaces $\mathcal{P}_I, \mathcal{P}_{II}, \mathcal{P}_{III}$.* Let $J \in \mathcal{J}(M, \omega)$ and $H \in \mathcal{H}$. We will perturb each of the spaces $\mathcal{P}_I$, $\mathcal{P}_{II_i}$, $\mathcal{P}_{III_i}$, $i = 1, 2$ by requiring that in every chain of disks forming an element of these spaces the (single) disk that has 3 marked points is (J, H)-holomorphic. All the other disks in each chain will remain J-holomorphic.

We start with the spaces $\mathcal{P}_I$. Let $\mathbf{A} = (A_1, \ldots, A_l)$ be a vector of non-zero classes and $1 \leq k \leq l$. Let $x, y \in \text{Crit}(f)$, $a \in \text{Crit}(h)$. Define $\mathcal{P}_I(a, x, y; (\mathbf{A}, k), J, H)$ to be the space of all $(u_1, \ldots, u_l)$ such that:

(1) $u_i \in \mathcal{M}(A_i, J)/G_{-1,1}$ for every $i \neq k$.
(2) $u_k \in \mathcal{M}(A_k, J, H)$.
(3) $u_1(-1) \in W^u_x$, $u_l(1) \in W^s_y$.
(4) $(u_i(1), u_{i+1}(-1)) \in Q_{f,\rho_L}$ for every $1 \leq i \leq l-1$, $u_k(0) \in W^u_a$.

In other words elements $(u_1, \ldots, u_l)$ of $\mathcal{P}_I(a, x, y; (\mathbf{A}, k), J, H)$ are the same as those of $\mathcal{P}_I(a, x, y; (\mathbf{A}, k), J)$ only that now u_k is (J, H)-holomorphic. Put

$$\mathcal{P}_I(a, x, y; \mathbf{A}, J, H) = \bigcup_{k=1}^{l} \mathcal{P}_I(a, x, y; (\mathbf{A}, k), J, H).$$

Next define $\mathcal{P}_{II_1}(a, x, y; (\mathbf{A}, k), J, H)$ to be the space of all $(u_1, \ldots, u_{k-1}, v, u_k, \ldots, u_l)$ such that:

(1) $u_i \in \mathcal{M}(A_i, J)/G_{-1,1}$ for every $1 \leq i \leq l$.
(2) $v \in \mathcal{M}(0, J, H)$.
(3) $ev_{(k)}(u_1, \ldots, u_{k-1}, v, u_k, \ldots, u_l) \in W^u_x \times Q^{\times(k-1)}_{f,\rho_L} \times W^u_a \times \text{diag(L)} \times Q^{\times(l-k)}_{f,\rho_L} \times W^y_s$, where $ev_{(k)}$ is the evaluation map defined in Section 5.3.1.

Note that there are $(l+1)$ disks in every element of this space and the disk v which has 3 marked points is in the class 0. The relation to the unperturbed space $\mathcal{P}_{II_1}(a, x, y; (\mathbf{A}, k), J)$ is the following. In the unperturbed space the disk v corresponds to a constant disks, namely a point, where this point lies on the boundary of the disk u_k and is connected by a gradient trajectory coming from a. (See figure 9.)

Similarly define $\mathcal{P}_{II_2}(a, x, y; (\mathbf{A}, k), J, H)$ to be the space of all $(u_1, \ldots, u_k, v, u_{k+1}, \ldots, u_l)$ such that:

(1) $u_i \in \mathcal{M}(A_i, J)/G_{-1,1}$ for every $1 \leq i \leq l$.
(2) $v \in \mathcal{M}(0, J, H)$.
(3) $ev_{(k+1)}(u_1, \ldots, u_k, v, u_{k+1}, \ldots, u_l) \in W^u_x \times Q^{\times k}_{f,\rho_L} \times \text{diag}(L) \times W^u_a \times Q^{\times(l-k-1)}_{f,\rho_L} \times W^s_y$.

Finally for $i = 1, 2$ put

$$\mathcal{P}_{II_i}(a, x, y; \mathbf{A}, J, H) = \bigcup_{k=1}^{l} \mathcal{P}_{II_i}(a, x, y; (\mathbf{A}, k), J, H).$$

We now turn to the spaces $\mathcal{P}_{III}$. Let $\mathbf{B}' = (B'_1, \ldots, B'_l)$, $\mathbf{B}'' = (B''_1, \ldots, B''_{l''})$ two vectors of non-zero classes. Let $1 \leq k' \leq l'$, $1 \leq k'' \leq l''$. We define $\mathcal{P}_{III_1}(a, x, y; (\mathbf{B}', k'), \mathbf{B}'', J, H)$ (resp. $\mathcal{P}_{III_1}(a, x, y; (\mathbf{B}', k'), \mathbf{B}'', J, H)$) in the same way as in (73) only that now $u'_{k'}$ (resp. $u''_{k''}$) is (J, H)-holomorphic. Put

$$\mathcal{P}_{III_1}(a, x, y; \mathbf{B}', \mathbf{B}'', J, H) = \bigcup_{k'=1}^{l'} \mathcal{P}_{III_1}(a, x, y; (\mathbf{B}', k'), \mathbf{B}'', J, H)$$
$$\mathcal{P}_{III_2}(a, x, y; \mathbf{B}', \mathbf{B}'', J, H) = \bigcup_{k''=1}^{l''} \mathcal{P}_{III_2}(a, x, y; \mathbf{B}', (\mathbf{B}'', k''), J, H).$$

Note that the space $\mathcal{P}_{II_1}(a, x, y; (\mathbf{A}, k), J, H)$ could be viewed as a special case of the space $\mathcal{P}_{III_1}(a, x, y; (\mathbf{B}', l'), \mathbf{B}'', J, H)$ if we would have allowed the l''th class $B'_{l'}$ in $\mathbf{B}'$ to be 0. Namely the space $\mathcal{P}_{II_1}(a, x, y; (\mathbf{A}, k), J, H)$ is the same as $\mathcal{P}_{III_1}(a, x, y; (\mathbf{B}', l'), \mathbf{B}'', J, H)$ with $\mathbf{B}' = (A_1, \ldots, A_{k-1}, 0)$, $\mathbf{B}'' = (A_k, \ldots, A_l)$. A similar remark holds for the perturbations of $\mathcal{P}_{II_2}$ and $\mathcal{P}_{III_2}$. Nevertheless it will be more convenient for us to work with vectors $\mathbf{B}'$, $\mathbf{B}''$ of *non-zero classes*, hence separate the perturbations of $\mathcal{P}_{II_i}$ from those of $\mathcal{P}_{III_i}$.

The following proposition is analogous to Propositions 5.3.3 and 5.3.5.

Proposition 5.3.13. *Let (f, ρ_L), (h, ρ_M) be as in assumption 5.3.1. Then there exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$ there exists a second category subset $\mathcal{H}_{\text{reg}}(J) \subset \mathcal{H}$ with the following properties. For every $x, y \in \mathrm{Crit}(f)$, $a \in \mathrm{Crit}(h)$ and every vectors of non-zero classes $\mathbf{A}$, $\mathbf{B}'$, $\mathbf{B}''$ the following holds:*

1. *If $|a|+|x|-|y|+\mu(\mathbf{A})-2n = 0$ then $\mathcal{P}_I(a, x, y; \mathbf{A}, J, H)$ is a compact 0-dimensional manifold.*
2. *If $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n = 1$ then each of $\mathcal{P}_{II_i}(a, x, y; \mathbf{A}, J, H)$, $i = 1, 2$, is a compact 0-dimensional manifold.*
3. *if $|a| + |x| - |y| + \mu(\mathbf{B}') + \mu(\mathbf{B}'') - 2n = 1$ then each of $\mathcal{P}_{III_i}(a, x, y; \mathbf{B}', \mathbf{B}'', J, H)$, $i = 1, 2$, is a compact 0-dimensional manifold.*

Moreover, in each of the above three cases the J-holomorphic part of every element of these spaces consist of simple and absolutely distinct disks.

The proof is very similar to that of Propositions 5.3.3, 5.3.5 hence we omit it.

In view of Proposition 5.3.13, for $J \in \mathcal{J}_{\text{reg}}$, $H \in \mathcal{H}_{\text{reg}}(J)$ define:

$$n_I(a,x,y;\mathbf{A},H) = \#_{\mathbb{Z}_2}\mathcal{P}_I(a,x,y;\mathbf{A},J,H), \quad \text{when } |a|+|x|-|y|+\mu(\mathbf{A})-2n=0,$$
$$n_{II_i}(a,x,y;\mathbf{A},H) = \#_{\mathbb{Z}_2}\mathcal{P}_{II_i}(a,x,y;\mathbf{A},J,H), \quad \text{when } |a|+|x|-|y|+\mu(\mathbf{A})-2n=1,$$
$$n_{III_i}(a,x,y;\mathbf{B}',\mathbf{B}'',H) = \#_{\mathbb{Z}_2}\mathcal{P}_{III_i}(a,x,y;\mathbf{B}',\mathbf{B}'',J,H),$$
$$\text{when } |a|+|x|-|y|+\mu(\mathbf{B}')+\mu(\mathbf{B}'')-2n=1.$$

(For simplicity we have suppressed here the J's from the notation.)

We will also need the following analogue of statement (1) of Proposition 5.3.7. Note however that now we do not require N_L to be ≥ 3 anymore.

Proposition 5.3.14. *Let (f,ρ_L), (h,ρ_M) be as in assumption 5.3.1. Then there exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M,\omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$ the following holds. There exists a second category subset $\mathcal{H}_{\text{reg}}(J) \subset \mathcal{H}$ such that for every $H \in \mathcal{H}_{\text{reg}}(J)$, every vector of non-zero classes $\mathbf{A}$ and every $x,y \in \mathrm{Crit}(f)$, $a \in \mathrm{Crit}(h)$ with $|a|+|x|-|y|+\mu(\mathbf{A})-2n=1$ the space $\mathcal{P}_I(a,x,y;\mathbf{A},J,H)$ is a smooth 1-dimensional manifold. Moreover, the J-holomorphic part of every element of this space consist of simple and absolutely distinct disks.*

Proof. The proof is very similar to that of Propositions 5.3.7 and 5.3.3 however in contrast to Proposition 5.3.7 we do not need to assume here that $N_L \geq 3$ anymore (i.e. $N_L \geq 2$ is enough). The point is that in order to assure that the space $\mathcal{P}_I(a,x,y;(\mathbf{A},k),J,H)$ is a smooth manifold it is enough to require that the disks $(u_1,\ldots,u_{k-1},u_{k+1},\ldots,u_l)$ are simple and absolutely distinct without any requirement on the (perturbed) disk u_k. Recall that it was due to *this* disk that we had to assume in Proposition 5.3.7 that $N_L \geq 3$ in order to assure simplicity. Finally note that absolutely distinctiveness of the J-holomorphic part of each element of $\mathcal{P}_I(a,x,y;\mathbf{A},J,H)$ does not require $N_L \geq 3$ either and can be proved in a similar way as in Proposition 5.3.7. □

5.3.10. *Identities for the perturbed spaces.*

Proposition 5.3.15. *Let (f,ρ_L), (h,ρ_M) be as in assumption 5.3.1. Then there exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M,\omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$ there exists a second category subset $\mathcal{H}_{\text{reg}}(J) \subset \mathcal{H}$ with the following property. For every $H \in \mathcal{H}_{\text{reg}}(J)$, every $x,y \in \mathrm{Crit}(f)$, $a \in \mathrm{Crit}(h)$ and $\mathbf{A}$ with $|a|+|x|-|y|+\mu(\mathbf{A})-2n=1$ identity (74) holds with $n_I(-)$ replaced by $n_I(-,H)$ $n_{II_i}(-)$ by $n_{II_i}(-,H)$ and $n_{III_i}(-)$ by $n_{III_i}(-,H)$.*

Proof. The proof is very similar to that of Proposition 5.3.6 with the following two slight changes. Now we use the compactness statement from Section 5.3.8 rather than compactness for genuinely pseudo-holomorphic disks. The second difference is that the gluing procedure between a J-holomorphic disk u' and a (J, H)-holomorphic disks u'' with $u'(1) = u''(-1)$ is now carried out in $D \times M$. Namely we consider u' as a $\widetilde{J}_H$-holomorphic disk in the fibre $\{-1\} \times M$ and glue it to the $\widetilde{J}_H$-holomorphic section $\widetilde{u}''$ corresponding to u''. Standard arguments imply that the result of the gluing is (after a suitable reparametrization) again a ($\widetilde{J}_H$-holomorphic) *section* hence descends to a (J, H)-holomorphic disk. The precise details of this type of gluing are given in Section 4.2.3.

The rest of the proof is a straightforward adaption of the proof of Proposition 5.3.6 $\square$

Proposition 5.3.16. *Let (f, ρ_L), (h, ρ_M) be as in assumption 5.3.1. Then there exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ and a neighbourhood (in the C^∞-topology) $\mathcal{U}$ of $0 \in \mathcal{H}$ such that for every $J \in \mathcal{J}_{\text{reg}}$ there exists a second category subset $\mathcal{H}_{\text{reg}}(J) \subset \mathcal{H}$ with the following properties. For every $H \in \mathcal{H}_{\text{reg}}(J) \cap \mathcal{U}$, every $x, y \in \text{Crit}(f)$, $a \in \text{Crit}(h)$ and every vectors $\mathbf{A}$, $\mathbf{B}'$, $\mathbf{B}''$ of non-zero classes the following holds:*

1. *If $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n = 0$ then $n_I(a, x, y; \mathbf{A}, H) = n_I(a, x, y; \mathbf{A})$.*
2. *If $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n = 1$ then $n_{II_i}(a, x, y; \mathbf{A}, H) = n_{II_i}(a, x, y; \mathbf{A})$, $i = 1, 2$.*
3. *If $|a| + |x| - |y| + \mu(\mathbf{B}') + \mu(\mathbf{B}'') - 2n = 1$ then $n_{III_i}(a, x, y; \mathbf{B}', \mathbf{B}'', H) = n_{II_i}(a, x, y; \mathbf{B}', \mathbf{B}'')$, $i = 1, 2$.*

Proof. The proof is based on a standard cobordism argument. We outline the main ideas. Suppose that we are under the assumption appearing in statement (1) of the Proposition. We define the set $\mathcal{J}_{\text{reg}}$ to be the intersection of all the $\mathcal{J}_{\text{reg}}$'s from the previous propositions and require in addition that for every $J \in \mathcal{J}_{\text{reg}}$ the following holds:

(i) For every two vectors $\mathbf{C}'$, $\mathbf{C}''$ of non-zero classes and every $x', y' \in \text{Crit}(f)$, $a' \in \text{Crit}(h)$ with $|a'|+|x'|-|y'|+\mu(\mathbf{C}')+\mu(\mathbf{C}'')-2n \leq 0$ the spaces $\mathcal{P}_{III_i}(a', x', y'; \mathbf{C}', \mathbf{C}'', J)$, $i = 1, 2$, are *empty*.

(ii) For every vector of non-zero classes $\mathbf{A}'$ and every $x', y' \in \text{Crit}(f)$, $a \in \text{Crit}(h)$ with $|a'|+|x'|-|y'|+\mu(\mathbf{A}')-2n \leq 0$ the spaces $\mathcal{P}_{II_i}(a', x', y'; \mathbf{A}', J)$, $i = 1, 2$, are *empty*.

Standard arguments imply that the resulting set $\mathcal{J}_{\text{reg}}$ is of second category. We now claim the following:

5.3.17. *There exists a neighbourhood $\mathcal{U}$ of $0 \in \mathcal{H}$ such that for every $H \in \mathcal{U}$ and every a, x, y, $\mathbf{A}$ with $|a| + |x| - |y| + \mu(\mathbf{A}) - 2n \leq 0$ the following holds:*

1. *For every splitting $\mathbf{A} = (\mathbf{C}', \mathbf{C}'')$ of $\mathbf{A}$ into two vectors $\mathbf{C}'$, $\mathbf{C}''$ of non-zero classes we have $\mathcal{P}_{III_i}(a, x, y; \mathbf{C}', \mathbf{C}'', J, H) = \emptyset$ for $i = 1, 2$.*
2. *$\mathcal{P}_{II_i}(a, x, y; \mathbf{A}, J, H) = \emptyset$, for $i = 1, 2$.*

We prove 5.3.17. Indeed, if the contrary to statement 5.3.17 happens then there exists a sequence $H_\nu \in \mathcal{H}$ with $H_\nu \longrightarrow 0$ in the C^∞-topology and elements $\mathbf{u}_\nu$ in either $\mathcal{P}_{III_i}(a_\nu, x_\nu, y_\nu; \mathbf{C}'_\nu, \mathbf{C}''_\nu, J, H_\nu)$ or $\mathcal{P}_{II_i}(a_\nu, x_\nu, y_\nu; \mathbf{A}_\nu, J, H_\nu)$, where x_ν, y_ν, a_ν, $\mathbf{A}_\nu$, $\mathbf{C}'_\nu$, $\mathbf{C}''_\nu$ satisfy $|a_\nu| + |x_\nu| - |y_\nu| + \mu(\mathbf{A}_\nu) - 2n \leq 0$ or $|a_\nu| + |x_\nu| - |y_\nu| + \mu(\mathbf{C}'_\nu) + \mu(\mathbf{C}''_\nu) - 2n \leq 0$. Viewing the disks in $\mathbf{u}_\nu$ as $\widetilde{J}_{H_\nu}$-holomorphic disks in $D \times M$ (one as a section and the others lying in the fibres of $D \times M \to D$) it is easy to see that their energy is uniformly bounded for symplectic forms of the type $\widetilde{\omega}_{H_\nu, \kappa}$ with fixed κ. (See (82) and Proposition 5.3.12). Therefore by Gromov compactness theorem, after passing to a subsequence, the sequence converges to a chain of bubble trees $\mathbf{u}_\infty$ all of whose elements are $(J, 0)$-holomorphic (i.e. genuinely J-holomorphic) disks and spheres. As $J \in \mathcal{J}_{\text{reg}}$, by a dimension count argument we can rule out all possible configurations for $\mathbf{u}_\infty$ except of maybe the cases $\mathbf{u}_\infty \in \mathcal{P}_{III_i}(a, x, y; \mathbf{C}', \mathbf{C}'', J)$ or $\mathbf{u}_\infty \in \mathcal{P}_{II_i}(a, x, y; \mathbf{A}', J)$ for some a', x', y', $\mathbf{A}$ or $\mathbf{C}', \mathbf{C}''$ that satisfy $|a'| + |x'| - |y'| + \mu(\mathbf{C}') + \mu(\mathbf{C}'') - 2n \leq 0$ or $|a'| + |x'| - |y'| + \mu(\mathbf{A}') - 2n \leq 0$. However these two last cases are again impossible by the definition of $\mathcal{J}_{\text{reg}}$ in (i) and (ii) above. This proves statement 5.3.17.

We continue with the proof of Proposition 5.3.16. Fix $J \in \mathcal{J}_{\text{reg}}$ and let $\mathcal{H}_{\text{reg}}(J)$ be the intersection of all the $\mathcal{H}_{\text{reg}}(J)$'s from the previous Propositions. Let $\mathcal{U}$ be the neighbourhood of $0 \in \mathcal{H}$ defined by statement 5.3.17. Replace $\mathcal{U}$ if needed by its connected component containing 0. Let $H \in \mathcal{H}_{\text{reg}}(J) \cap \mathcal{U}$. Take a generic path $\{H_\lambda\}_{0 \leq \lambda \leq 1}$ in $\mathcal{U}$ with $H_0 = H$ and $H_1 = 0$. Then

$$\mathcal{P}_I(a, x, y; \mathbf{A}, J, \{H_\lambda\}) = \{(\lambda, \mathbf{u}) \mid 0 \leq \lambda \leq 1, \, \mathbf{u} \in \mathcal{P}_I(a, x, y; \mathbf{A}, J, H_\lambda)\}$$

is a 1-dimensional smooth manifold whose boundary is

$$\partial \mathcal{P}_I(a, x, y; \mathbf{A}, J, \{H_\lambda\}) = \mathcal{P}_I(a, x, y; \mathbf{A}, J, H_0) \coprod \mathcal{P}_I(a, x, y; \mathbf{A}, J). \tag{86}$$

It is important to note here that the space lying over $\lambda = 1$, namely $\mathcal{P}_I(a, x, y; \mathbf{A}, J)$ consists of chains of disks that are simple and absolutely distinct. This is assured by Proposition 5.3.3 (which holds for $N_L \geq 2$).

We claim that the 1-dimensional manifold $\mathcal{P}_I(a, x, y; \mathbf{A}, J, \{H_\lambda\})$ is compact. This follows form a straightforward dimension count. Indeed occurrence of bubbling of spheres reduces the dimension by at least 2 hence cannot appear in a generic 1-dimensional family. Similarly, bubbling of a J-holomorphic disk at a point $p \in \partial D$ with $p \neq -1, 1$ also yields negative dimension since $N_L \geq 2$. It remains to rule out the following three cases:

- Shrinking to a point of a gradient trajectory connecting two consecutive disks in $\mathcal{P}_I(a, x, y; \mathbf{A}, J, \{H_\lambda\})$.
- Bubbling of a J-holomorphic disk at $p = -1$ or $p = 1$ in one of the disks participating in $\mathcal{P}_I(a, x, y; \mathbf{A}, J, \{H_\lambda\})$.

- Breaking of a gradient trajectories (of f or h) involved in $\mathcal{P}_I(a, x, y; \mathbf{A}, J, \{H_\lambda\})$.

The first possibility is ruled out by (1) of statement 5.3.17. The second possibility is ruled out by (1) and (2) of statement 5.3.17. Note that we have to apply statement 5.3.17 for a different $\mathbf{A}$ here. Namely we have to take the $\mathbf{C}'$, $\mathbf{C}''$ coming from the bubbling and change $\mathbf{A}$ to be $(\mathbf{C}', \mathbf{C}'')$. Also note that when bubbling of a disk occurs in the perturbed disk, the remaining perturbed component might have now $\mu = 0$. This is why we needed also (2) in statement 5.3.17. The third possibility can be ruled out in a similar way to the first one. This completes the proof that $\mathcal{P}_I(a, x, y; \mathbf{A}, J, \{H_\lambda\})$ is compact.

To conclude the proof of Proposition 5.3.16 note that since $\mathcal{P}_I(a, x, y; \mathbf{A}, J, \{H_\lambda\})$ is compact it follows from (86) that $n_I(a, x, y; \mathbf{A}, H) = n_I(a, x, y; \mathbf{A})$.

The proofs of statements (2) and (3) are similar. □

5.3.11. *Conclusion of the Proof of identity* (74) *for every* $N_L \geq 2$. Identity (74) follows now immediately by combining Propositions 5.3.15 and 5.3.16.

By similar methods, it is not difficult to see that, in homology, the operation defined by formula (72) is independent of the choices made in its definition.

5.3.12. *Module structure.* The purpose of this section is to show the following identities:

Proposition 5.3.18. *For any* $\alpha \in Q^+H_*(L)$, $a, b \in Q^+H_*(M)$ *we have*

(1) $(a * b) * \alpha = a * (b * \alpha)$.
(2) $[M] * \alpha = \alpha$. *(Recall that* $[M] \in Q^+H_{2n}(M)$ *is the unit with respect to the quantum cap product).*

We will present first the proof of point (1) in a particular case: when $N_L \geq 4$. The proof in this case is the cleanest as it does not require the use of perturbations. The case when $N_L \leq 4$ will be treated later. It is also useful to note that if using the full Novikov ring instead of $Q^+H(-)$, the formula in point (1) of the Proposition follows (for any $N_L \geq 2$) from the comparison with Floer homology described in the next section.

Proof. We start with the proof of point (1). The proof is very similar to the one for the associativity of the quantum product as described in the proof of Lemma §5.2.6. One important point which is useful to note before going further is that it is essential for this formula that the product used on $Q^+H_*(M)$ is the quantum cap product (rather than the classical cap-product).

We now proceed with the description of the necessary moduli spaces (this is, obviously, independent of our restriction on N_L).

It is again useful to use the language of trees as we already did in the section on the quantum product §5.2. We first notice that the moduli spaces in the first subsections of §5.3 can all be described in these terms. For example $\mathcal{P}_I(a,x,y;\mathbf{A},J)$ is modeled on a tree with two entries so that one of the entries is connected by an edge to the unique vertex of valence three which is labeled by a non-zero element of $H_2(M,L;\mathbb{Z})$. This edge corresponds to a flow line of the function h. It is useful to consider the edges of the tree being labeled by either the function f or the function h. The same type of description applies to the other moduli spaces each of which is obtained by adding some auxiliary conditions: $\mathcal{P}_{I'}(a,x,y;\mathbf{A},J)$ has the same description as $\mathcal{P}_I(a,x,y;\mathbf{A},J)$ except that the vertex of valence three is a constant disk (in other words the labeling of this vertex is by the element $0 \in H_2(M,L;\mathbb{Z})$. Further, the two moduli spaces $\mathcal{P}_{II_1}(a,x,y;\mathbf{A},J)$ and $\mathcal{P}_{II_2}(a,x,y;\mathbf{A},J)$ are both obtained as before but with the additional conditions that one of the edges labeled by f is of length 0 and one of the two adjacent vertices of this edge is a constant disk of valence three. For the moduli spaces $\mathcal{P}_{III_1}(a,x,y;\mathbf{A},J)$, $\mathcal{P}_{III_2}(a,x,y;\mathbf{A},J)$ the requirement is that one of the edges labeled by f is of length 0 and, finally, the spaces $\mathcal{P}_{III'_1}(a,x,y;\mathbf{A},J)$ and $\mathcal{P}_{III'_2}(a,x,y;\mathbf{A},J)$ are characterized by the fact that one edge labeled by f is of length 0 and the vertex of valence three is a constant disk.

We now start to discuss the specific moduli spaces required in the proof. These moduli spaces are similar to the one used to show the associativity of the quantum product in §5.2. As there, they are modeled on trees with three entries and one exit. We will also need to use two Morse-Smale functions on M, h_1 and h_2, together with the Morse function f on L. We will assume that h_1 and h_2 have the same critical points and are in generic position. We consider planar trees $\mathcal{T}$ with three entries and one exit and which are labeled as follows: each edge is labeled by a number from $\{1,2,3\}$ and each vertex is labeled by an element of $H_2(M,L;\mathbb{Z})$.

For each topological type of such a tree, $\mathcal{T}$, and for $a,b \in \mathrm{Crit(h_1)}$, $x,y \in \mathrm{Crit(f)}$ we denote by $\mathcal{P}_{\mathcal{T}}(a,b,x,y;J)$ the moduli space of configurations which satisfy:

(1) There are four vertices of valence 1, the three entries and the single exit. The graph is planar with the three entry points on the line $\mathbb{R}\times\{1\}$ the i-th entry point (along this line) is the origin of an edge labeled by i, the edge arriving in the exit point is labeled by 3.
(2) Each vertex corresponds to a pseudo-holomorphic disk with boundary on L or to a J-holomorphic sphere.

(3) Each edge corresponds to a flow line (of non-zero length) so that if the label of that edge is 1, the corresponding function is h_1, if the label is 2 the function is h_2 and if the label is 3 then the function is f.
(4) A vertex of valence 2 corresponds to a pseudo-holomorphic disk of non-vanishing class.
(5) For each vertex of valence 2 the incidence relations are as in the definition of the moduli spaces used in the pearl complex, in particular, both the entry and exit edges are labeled by 3.
(6) Any vertex of trivial class is at least of valence 3
(7) If a vertex is of valence 3 and is a pseudo-holomorphic disk, then one of the entering edges and the exit edge are labeled by 3 and the incidence relations at this vertex are as in the definition of the moduli space $\mathcal{P}_I(- - --)$.
(8) If a vertex of valence 3 corresponds to a pseudo-holomorphic sphere, then the two entry edges are labeled by 1 and 2 and the exit edge by 1 so that the incidence points are 1 - which is the attaching point for the entering edge labeled by 1, $e^{2\pi i/3}$ - which corresponds to the exit edge and $e^{4\pi i/3}$ - which is the attaching point of the edge labeled by 3 (we view here S^2 as $\mathbb{C} \cup \{\infty\}$).
(9) If a vertex is of valence 4, then it corresponds to a J-holomorphic disk with boundary on L and the incidence relations are so that the point -1 is the attaching point of an entering flow line of f, the point $+1$ is the attaching point of the exiting edge which is again a flow line of f, the point 0 is the attaching point of an entering flow line corresponding to h_2, there is a real point of coordinates $(x, 0)$ with $0 < x < 1$ which is the attaching point of a flow line of h_1.

Notice that, with this definition, there can not be more than a single J-holomorphic sphere in such a configuration and, moreover, if such a sphere appears there are a h_1-flow line starting from a and a h_2-flow line starting from b which both arrive at the sphere as as well as an exiting h_1-flow line which starts from the sphere and ends at the center of a disk with boundary on L (this disk might be constant).

These moduli spaces will be used to define a chain homotopy ξ so that we have:

$$(d\xi + \xi d)(a \otimes b \otimes x) = (a * b) * x + a * (b * x) \ . \tag{87}$$

For this we need to first study the regularity properties of our moduli spaces, $\mathcal{P}_{\mathcal{T}}(a, b, x, y)$. To do so we let $\mathcal{P}^{*,d}_{\mathcal{T}}(a, b, x, y)$ be the subset of $\mathcal{P}_{\mathcal{T}}(a, b, x, y)$ which is constituted by the configurations in which each disk and sphere is simple and they are all absolutely distinct. It is clear, by the standard transversality arguments used earlier in the paper that, for a generic almost complex structure J, $\mathcal{P}^{*,d}_{\mathcal{T}}(a, b, x, y)$ is a manifold of dimension $|a| + |b| +$

$|x|-|y|-4n+1+\mu(\mathcal{T})$. We denote $\delta''' = \delta'''(a,b,x,y,\mathcal{T}) = |a|+|b|+|x|-|y|-4n+1+\mu(\mathcal{T})$ and we intend to show:

Lemma 5.3.19. *For $\delta''' \leq 1$ and if $N_L \geq 4$, and a generic choice of almost complex structure, we have:*

$$\mathcal{P}_{\mathcal{T}}(a,b,x,y) = \mathcal{P}^{*;d}_{\mathcal{T}}(a,b,x,y)\ .$$

Proof of Lemma 5.3.19. The first step is to reduce to the case of simple disks and spheres. The argument is very similar to the proof of the associativity for the quantum product combined with some arguments already used to show that the external product is a chain map. First we notice that as spheres are either simple or multiple covered it is easy to see that we may assume that the sphere appearing in a configuration $u \in \mathcal{P}_{\mathcal{T}}(a,b,x,y)$ for $\delta''' \leq 1$ is simple. So we are left with proving that all the disks can also be assumed to be simple and that all the objects involved are absolutely distinct. In the proof we also need a type of slightly more general moduli spaces than the ones described before. They will be denoted by $\mathcal{P}'_{\mathcal{T}}(a,b,x,y)$ and they are characterized by the the fact that in condition (7) above we allow for the interior marked point (inside the disk) to be different from 0 and, similarly, in condition (9) the two marked points are again allowed to be arbitrary. The virtual dimension of this moduli space is $\delta'''+2$. The proof proceeds by recurrence: we intend to show by induction over $\mu(\mathcal{T})$ that if $\delta''' = 1$, then the claim is true. For this we will need to also show that if $\delta'''-2<0$, then $\mathcal{P}'_{\mathcal{T}}(a,b,x,y)$ can only contain configurations made out of simple, absolutely distinct curves and thus $\mathcal{P}'_{\mathcal{T}}(a,b,x,y) = \emptyset$ in this case. As mentioned before, we may already assume that the spheres appearing in the configurations described above are simple. To shorten notation we will denote in the paragraph below: $\mathcal{P} = \mathcal{P}_{\mathcal{T}}(a,b,x,y)$ and $\mathcal{P}' = \mathcal{P}'_{\mathcal{T}}(a,b,x,y)$. To prove our claim we will actually need to work with a type of even more general moduli spaces (this is similar to the first part of the proof of Proposition). We will denote these moduli spaces by $\mathcal{G} = \mathcal{G}_{\mathcal{T}}(a,b,x,y)$, $\mathcal{G}' = \mathcal{G}'_{\mathcal{T}}(a,b,x,y)$ they consist of configurations as those in $\mathcal{P}$ and, respectively, in $\mathcal{P}'$ except that: condition (5) is modified so as to allow vertices of valence two so that the entering edge is labeled by 1 or 2 and the exit one by 3; condition (7) and (8) are modified so that vertices of valence three or four (which correspond to disks) are allowed to have more than one (possibly even three) entering edges labeled by 3 (the incidence points on such a disk are assumed distinct, two of them being $+1$ and -1). We intend to show that, for a generic J, these moduli spaces are formed by configurations containing only simple absolutely distinct curves. This obviously implies the statement for $\mathcal{P}$ and $\mathcal{P}'$.

We now consider a configuration u that might appear either in $\mathcal{G}$ or in $\mathcal{G}'$, we assume that $\delta''' \leq 1$ if it is in $\mathcal{G}$ and that $\delta'''-2<0$ in the second case. We also assume the statement proved for the moduli spaces with $\mu(\mathcal{T}) < k$ so that we take here $\mu(\mathcal{T}) = k$. Suppose that one disk u_0 appearing in u is not simple. There are two cases to consider. The first is when $n = \dim L \geq 3$ so that we may use Lemma 3.2.3. Assuming now that u is in $\mathcal{G}'$ we see that we may replace u_0 by another disk u_0' which is simple so that $u_0(-1/+1) = u_0'(-1/+1)$, $u_0(D) = u_0'(D)$ and $\mu([u_0']) < \mu([u_0])$. and so, by replacing u_0 by u_0', we obtain a configuration of lower Maslov class than that of u and which belongs to $\mathcal{G}'$ which contradicts the induction hypothesis. Suppose now that $u \in \mathcal{G}$. It is only here that the hypothesis $N_L \geq 4$ intervenes. Indeed, we suppose again that the disk u_0 appearing in u is not simple and we let u_0' be the simple disk provided as above by the Lemma 3.2.3. We consider the configuration u' obtained by replacing u_0 by u_0'. We obviously have $\mu(u') \leq \mu(u) - N_L \leq \mu(u) - 4$. The key remark is that, in general, u' is not an element in $\mathcal{G}$ but is an element of $\mathcal{G}'_{\mathcal{T}'}$ (for a suitable tree $\mathcal{T}'$). The virtual dimension of this last moduli space is

$$|a|+|b|+|x|-|y|-4n+3+\mu(u') \leq |a|+|b|+|x|-|y|-4n+3+\mu(u)-4 = \delta'''(a,b,x,y,\mathcal{T})-2 \leq -1.$$

As $\mu(\mathcal{T}') < k$ we again see that this leads to a contradiction and so all disks may be assumed simple at least for $n \geq 3$. We pursue under this assumption to show that all the curves involved may also be assumed to be absolutely distinct. Lemma 3.2.2 plays an important role here. Indeed, it implies that if the curves are not absolutely distinct, then there are two disks u_1 and u_2 so that $u_1(D) \subset u_2(D)$ and the same relation is valid for the respective boundaries. We intend to use this fact to eliminate u_1 from the tree. It is not difficult to see that by appropriately replacing u_1 by u_2 in the tree and possibly eliminating the chain connecting u_1 to u_2 in $\mathcal{T}$ we obtain a configuration which still belongs to $\mathcal{G}'_{\mathcal{T}'}$ (for some other appropriate tree $\mathcal{T}'$) but the total Maslov index of this configuration is at least smaller by 4 compared to the initial one so that, by induction, generically this is not possible.

We are now left to treat the case $n = \dim L = 2$. In this case the condition $N_L \geq 4$ implies that at most two disk can appear in the configurations that are relevant to us and the combinatorial arguments used before (in §3) for this type of configuration suffice to conclude. □

We will return to the case $N_L \leq 3$ later but will now pursue with the proof of the identity (87) for $N_L \geq 4$.

We define

$$\xi(a \otimes b \otimes x) = \sum_{y,\mathcal{T}} \#_{\mathbb{Z}_2}(\mathcal{P}_{\mathcal{T}}(a,b,x,y)) y t^{\bar{\mu}(\mathcal{T})},$$

where we recall that $\bar{\mu}(\mathcal{T})$ is $\frac{1}{N_L}$ of the total Maslov class of the tree $\mathcal{T}$ and the sum is taken over all elements so that $|a| + |b| + |x| - |y| + \mu(\mathcal{T}) - 4n + 1 = 0$.

As always, we will deduce formula (87) from the study of the boundary of the Gromov compactification of the space $\mathcal{P}_{\mathcal{T}}(a,b,x,y)$. Let $\overline{\mathcal{P}}_{\mathcal{T}}(a,b,x,y)$ denote this compactification. We now describe the types of configurations to $u \in \overline{\mathcal{P}}_{\mathcal{T}}(a,b,x,y) \backslash \mathcal{P}_{\mathcal{T}}(a,b,x,y)$ when $|a|+|b|+|x|-|y|+\mu(\mathcal{T})-4n+1=1$. These configurations are described in the same way as in the points (1) - (9) before except that, additionally one of the following conditions is satisfied:

(1') Precisely one edge in $\mathcal{T}$ can be represented by a flow line of 0 length.

(2') Precisely one vertex in the tree $\mathcal{T}$ is represented by a cusp curve with two components - one a disk an the other a sphere. In this case, the sphere carries two marked points corresponding to the incidence points of a flow lines of h_1 and one of h_2 and the disk carries an internal marked point (which is its junction with the sphere) which is situated on the real line joining -1 to $+1$. It is possible for one of the disk or the sphere to be a constant one as long as it is stable.

(3') Precisely one vertex in the tree is represented by a cusp curve with two components both of which are disks. In this case each of the two components carry two boundary marked points (one of which is their junction point) and each of them might also carry some internal marked points situated on the real line joining -1 to $+1$. It is possible for one of the disks to be a constant one as long as it is stable.

(4') Precisely, one edge in the tree is represented by a broken flow line of either h_1, h_2 or f.

In view of the descriptions of the compactifications of the various moduli spaces described earlier in the paper, this statement is rather obvious. The only point worth explicit mention here is that if a sphere bubbles off, then for dimension reasons, it necessarily has to carry three marked points (one of them being a junction point with a disk). In view of this, as the internal marked points on each disk are along the line joining -1 to $+1$ the only such possible bubbling fits in case (2').

The techniques described earlier in the paper (in particular, the gluing results) insure that each of the configurations described at (1'), (3'), (4') is in fact a boundary point of the 1-dimensional manifold with boundary $\overline{\mathcal{P}}_{\mathcal{T}}(a,b,x,y)$. An additional gluing argument applies for the configurations at (2') which is concerned with the gluing of a sphere to a disk in an internal point of the disk. However, this type of gluing is already well understood as it coincides, essentially, with the closed case (the gluing of two J-holomorphic spheres). See [44] for example.

To prove relation (87) we now consider the sum S of all these boundary points when $\mathcal{T}$ takes all possible values (and a,b,x,y are fixed). We first notice that each of the terms described by (1'), (2') and (3') appears twice in this sum. Indeed, each cusp configuration also appears from the "collapsing" of an edge in a tree $\mathcal{T}'$ with one more vertex and one more edge than $\mathcal{T}$. Therefore the sum S, which vanishes, equals the sum of the terms of type (4'). We now consider the terms appearing in S and notice that they are of the following types:

(i) the broken flow line is associated to f and is broken below the vertices of valence strictly greater than two.
(ii) the broken flow line is associated to h_1 and is broken above all vertices of valence two.
(iii) the broken flow line is associated to h_2 and is broken above all the vertices of valence two.
(iv) the broken flow line is associated to h_1 and is broken below a vertex of valence three (which, obviously, corresponds to a J-sphere).
(v) the broken flow line is associated to f and is broken above all the vertices of valence strictly greater than two.
(vi) the broken flow line is associated to f and it is broken above some vertex of valence three and below some other vertex of valence three.

We now observe that the terms counted in (i) are precisely those appearing in $d\xi(a\otimes b\otimes x)$; the terms in (ii) correspond to those in $\xi((da)\otimes b\otimes x)$; the terms in (iii) correspond to those in $(\xi a\otimes(db)\otimes x)$; the terms in (iv) correspond to those in $(a*b)*x$; the terms in (v) correspond to those in $\xi(a\otimes b\otimes(dx))$; finally, the terms in (vi) correspond to those appearing in $a*(b*x)$. This concludes our proof when $N_L\geq 4$.

We now give the argument for the case $2\leq N_L\leq 3$ under the assumption that that $n=\dim L\geq 3$. We will not give an explicit proof of the formula (87) in the case $N_L\leq 3$ and $n=2$. However, we mention that if coefficients are taken in Λ instead of Λ^+ then the module axiom for our exterior operation follows from the comparison with Floer homology which is explained in §5.6.

Returning to the case $N_L\leq 3$ and $n\geq 3$ the general argument follows the lines of the proof above except that the perturbation techniques described in §5.3.7 are also required. Indeed, in the definitions of the various moduli spaces - which are perfectly similar to the ones given before - we need that the disks carrying one or two marked points in their interior be perturbed disks in the sense of that section. All the arguments work in the same way as in the case treated above (for $N_L\geq 4$) and are in fact simpler because in all replacement arguments as well as in the reduction to simple disks, the disks carrying marked points in their interior do not intervene (because transversality is insured for these perturbed disks by a generic choice of perturbation). There is only one point where this proof requires a new argument. Indeed, we consider the sum S' which is defined as S

above except that the disks of valence strictly greater than 2 are (J, H)-disks (for a small, generic H). This sum S' again vanishes. However, besides the terms of type (4') which appear in S' there are some other terms which contribute to this sum. These are all the configurations obtained by the bubbling off of a (perturbed) disk of valence four giving as result a cusp curve made of a (J, H)-disk of valence three which is joined to a genuine pseudo-holomorphic disk of valence three together with all the configurations obtained when an edge joining two (J, H)-disks of valence three reduces to 0. One point is crucial here: in the first type of configuration described here - which is obtained by bubbling off - the (J, H)-disk appears before the J-disk in the tree (this happens because, the first internal incidence point is the center of the disk and the second incidence point is in the interval $(0, +1) \subset D$) !

Denote by S'' the number of configurations of these two types. Clearly, if $H = 0$ these two types of configurations coincide and, by the usual gluing results, $S'' = 0$. In general, another argument is needed to show that $S'' = 0$. For this, one considers the moduli space $\mathcal{P}_{tan,H,H'}(a, b, x, y)$ consisting of configurations of the usual type but with two disks of valence three which share an incidence point and so that the first disk (in the order of the tree) is a (J, H)-disk and the second disk is a (J, H')-disk. The result follows if we can show that, when the virtual dimension of the space $\mathcal{P}_{tan,H,H'}(a, b, x, y)$ is equal to 0, for all sufficiently small H and H' these moduli spaces are constituted by simple, absolutely distinct disks and the number of elements in $\mathcal{P}_{tan,H,H'}(a, b, x, y)$ equals that of $\mathcal{P}_{tan,0,0}(a, b, x, y)$. It is not difficult to check that this result follows by a usual cobordism argument similar to Proposition 5.3.16 if we prove that $\mathcal{P}_{tan,0,0}(a, b, x, y)$ is made of simple, absolutely distinct disks for generic almost complex structures. In turn the key point for this is that, as we only need this result when the virtual dimension of the respective moduli space is 0, we can reason as in Proposition 5.3.3. In other, words assuming that the disks are not all simple and absolutely distinct we may construct a new configuration of strictly lower Maslov index (which leads to a drop of virtual dimension of at least 2 because $N_L \geq 2$) but the condition on one of the marked points lying in the center of one of the disks of valence three might be lost which leads to a potential increase in virtual dimension by 1. As a whole, we still obtain a configuration belonging to a moduli space of virtual dimension at most -1 which leads to a contradiction because, inductively, this moduli space is assumed to be void.

It remains to prove point (2) of Proposition 5.3.18. Let $h : M \to \mathbb{R}$ be a Morse function with a single maximum Θ. Let $f : L \to \mathbb{R}$ be a Morse function and $x \in \mathrm{Crit}(f)$. Then we have (at the chain level): $\Theta * x = x \in \mathcal{C}^+(L; f, J)$. The reason for this is that, because

Θ is a maximum, the moduli spaces $\mathcal{P}(\Theta, x, y; \lambda)$ which define the module operation as described in §5.3.1 can only be 0-dimensional and non-void if $x = y$, $\lambda = 0$ and, in fact, the "pearls" appearing in these configurations are constant equal to x.

Since $[M] = [\Theta]$ point (2) follows immediately. □

5.3.13. *Algebra structure.* In this section we will show the following identity:

Proposition 5.3.20. *For any $x, y \in Q^+H_*(L)$, $a \in Q^+H(M)$ we have*

$$a * (x * y) = (a * x) * y \ .$$

Proof. The argument is very similar to the one used for Proposition 5.3.18. We will again make use of moduli spaces modeled on trees with three entries and one exit so that the first entry is a critical point of a Morse function $h : M \to \mathbb{R}$ but the second and third entries are both critical points of Morse functions, f_1 and f_2, on L. We will also assume that h, f_1, f_2 are all in generic position and that f_1 and f_2 have the same critical points. The moduli spaces needed for our argument are similar to those used to prove the associativity of the quantum product, in particular, in the proof of Lemma 5.2.6. For $a \in \mathrm{Crit(h)}$, $y, z, w \in \mathrm{Crit(f_1)}$ and $\mathcal{T}$ a planar tree as there we will denote by $\mathcal{P}_{\mathcal{T}}(a, y, z, w)$ the relevant moduli spaces. They are defined as the moduli spaces $\mathcal{P}_{\mathcal{T}}(-, -, -, -)$ in that Lemma except that the labeling of the edges is by the symbols 0, 1, 2 and is so that there is a single edge labeled by 0 which corresponds to a flow line of $-\nabla h$, the edges labeled by 1 and 2 correspond, respectively, to flow lines of $-\nabla f_1$ and $-\nabla f_2$. The various incidence conditions are as follows. At the vertices of valence 2 the incidence points are $+1$ and -1 and the entry edge is labeled by the same symbol as the exiting edge; at a vertex of valence three if all the incidence points are on the boundary, then they are the roots of order three of the unity (as in Lemma 5.2.6), the entrance edges are labeled by 1 and 2 and the exit edge is labeled by 1; a vertex of valence three might also have only two incidence points on the boundary and one in the interior - in this case the edge arriving in the interior is labeled by 0 and the other entering edge is labeled by 1 as well as the exiting edge, the incidence relations in this case are as in §5.3.1; finally a vertex of valence four has three marked points on the boundary so that two correspond to -1 and $+1$ and there is an entering edge labeled by 1 arriving at -1 and the exiting edge is again labeled by 1 and is attached at $+1$, there is an additional incidence point $\theta \in (0, \pi) \subset S^1$ where is attached an entering edge labeled by 2, the fourth incidence point is $0 \in D$ and is the arrival point of the edge labeled by 0.

The virtual dimension of these moduli spaces is $|a| + |y| + |z| - |w| + \mu(\mathcal{T}) - 3n + 1$ and they are used to define a chain homotopy

$$\xi' : C_k(h) \otimes \mathcal{C}^+_q(f_1) \otimes \mathcal{C}^+(f_2)_p \to \mathcal{C}^+_{k+q+p-3n+1}(f_1)$$

between $(-*-)*-$ and $-*(-*-)$.

The argument goes again in two stages: first we need to show that the necessary transversality is satisfied for such moduli spaces of virtual dimension at most 1 and, as a second step, an appropriate boundary formula needs to be justified. This second step is essentially identical with the one used to prove formula (87) so we will leave it to the reader. The first step reduces to showing that

$$\mathcal{P}_{\mathcal{T}}(a,y,z,w)=\mathcal{P}_{\mathcal{T}}^{*,d}(a,y,z,w)$$

where, as always, the superscript $*, d$ indicates those configurations which consist of simple absolutely disjoint disks. To show this equality the procedure is again as before in the paper. Without the use of any perturbations it is easy to treat the case $N_L \geq 3$ and $n \geq 3$. In case $N_L = 2$, $n \geq 3$ the use of perturbations is necessary. Finally, for $N_L = 2$, $n = 2$ the argument is much more combinatorial and we will not give it - rather we refer to the section concerning the comparison with Floer homology: the relation for $QH(-)$ (but not for $Q^+H(-)$) follows in general from there. $\square$

5.3.14. *Two-sided algebra structure.* The purpose in this paragraph is to prove:

Proposition 5.3.21. *For any $a \in H^+Q(M)$ and $x, y \in H^+Q(L)$ we have:*

$$a*(x*y)=x*(a*y)\ .$$

Proof. The proof is similar to the one in the last subsection. We will use the same type of moduli spaces - now denoted by $\mathcal{P}_{\mathcal{T}}(y,a,z,w)$ - except for a couple of very simple modification.

i. We require the vertices of valence three with three incidence points on the boundary to correspond to edges which, in the cyclic order, verify: the first is labeled by 2 and is an entry edge, the second is an exit edge and is labeled by 1, the third in an entry edge and is labeled by 1.
i. For a vertex of valence four we require the incidence points to be -1, $+1$ and 0 and to satisfy the same properties as in the proof of Proposition 5.3.20 but the fourth incidence point which again corresponds to an entry edge labeled by 2 is now a point in $(\pi, 2\pi) \subset S^1$.

Using these moduli spaces, the usual arguments, applied as in the last section produce a chain homotopy

$$\xi'' : C(h)_k \otimes \mathcal{C}^+(f_2)_q \otimes \mathcal{C}^+(f_1)_p \to \mathcal{C}^+(f_1)_{k+q+p-3n+1}$$

between the two order three products which appear in the statement. $\square$

5.4. **Quantum inclusion.** There is a canonical map:

$$i_L : Q^+H_*(L) \to Q^+H_*(M) \tag{88}$$

which is defined at the level of chain complexes by:

$$\begin{gathered} i_L : \mathcal{C}^+_k(L; f, J) \to (\mathbb{Z}_2\langle \mathrm{Crit(h)}\rangle \otimes \Lambda^+)_{\mathrm{k}}, \\ i_L(x) = \sum_{\substack{a\in \mathrm{Crit}(h) \\ \lambda,\mathcal{T}}} \#_{\mathbb{Z}_2}\mathcal{P}_{\mathcal{T}}(x,a)at^{\bar{\mu}(\mathcal{T})}, \end{gathered} \tag{89}$$

where $\mathcal{P}_{\mathcal{T}}(x,a)$ is again a moduli space modeled on a tree which will be linear here (that is a tree with one entry and one exit) which we will describe in more detail below. The sum is taken over all such trees $\mathcal{T}$ with $|x| - |a| + \mu(\mathcal{T}) = 0$.

The first part of point iii. of Theorem 2.1.1 comes down to:

Proposition 5.4.1. *The map in equation* (88) *is well defined and it verifies:*

$$i_L(a \ * \ x) = a * i_L(x) \ ,\forall \ a \in Q^+H_*(M) \ , \ x \in Q^+H_*(L) \ .$$

Here is now the more explicit description of the moduli space $\mathcal{P}_{\mathcal{T}}(x,a)$. It consists of configurations that are similar to those lying in the moduli spaces of pearls - as in Definition 5.1.1 - except that the condition imposed to the last disk in the string, u_k, is modified so that $\gamma'_{+\infty}(b_k) = a$, where $b_k = u_k(0)$ and γ' is the negative gradient flow of h, and the disk u_k may also be constant. In other words, the "exit" incidence point on the last disk is the interior point $0 \in D$ (instead of $+1 \in \partial D$) and the exit edge is a negative gradient line of h which ends in $a \in \mathrm{Crit(h)}$. When $n \geq 3$, the methods described before immediately show that with the previous definition i_L is a chain map. When $n = 2$ a simple combinatorial argument suffices. It is also easy to see that, in homology, this map does not depend of the choices made in its construction. It is useful to note that, as on the last disk u_k there are just two marked points - one interior and one on the boundary, the use of perturbations is not necessary in these arguments.

We now justify the equation in Proposition 5.4.1. This is based on the construction of an appropriate chain homotopy. In turn, this depends on defining yet other moduli spaces $\mathcal{P}_{\mathcal{T}}(a,y,b)$. Here $a, b \in \mathrm{Crit(h)}$ and $y \in \mathrm{Crit(f)}$ and $\mathcal{T}$ is a tree with two entries and one exit. As for the moduli spaces $\mathcal{P}_{-}(x,a)$ above, the key point is that the last edge in the tree is again a flow line of the negative gradient of h which arrives in b. There is also an edge which corresponds to a negative gradient flow line of h which leaves from a. All the vertices of valence at least two correspond to J-disks with the possible exception of a single one which is of valence three and may correspond to a J-sphere. We now describe the incidence relations. There are two types of vertices of valence two: J-disks with the two marked points on their boundary - in this case the incidence points are -1 which is the entry point of a flow line of $-\nabla f$ and $+1$ which is the exiting point of a flow line of $-\nabla f$; a J-disk with one marked point on the boundary -1 which is an entry point of a flow line of $-\nabla f$ and an interior marked point 0 which is the exiting point of a flow line of $-\nabla h$. There also are three possibilities for the vertex of valence 3. In the first, this vertex corresponds to a J-disk and the incidence points are -1, 0, $+1$ so that -1, 0 are entry points for flow lines of respectively, $-\nabla f$, $-\nabla h$ and $+1$ is an exiting point of a flow line of $-\nabla f$. In the second case, again the vertex corresponds to a J-disk but this time the marked points are $-1, p, 0$ where $p \in (-1, 0)$ and -1 is again an entry point for a flow line of $-f$, p is an entry point for a flow line of $-h$ and 0 is an exit point for a flow line of $-h$. Finally, in the third case the vertex in question corresponds to a J-sphere the marked points in that case are roots of order three of the unity so that the first root is an entrance point of a flow line of $-h$ originating in a, the second is an exiting flow line of $-h$ arriving in b and the third is an entering flow line of $-h$ (whose origin is, necessarily, at the center of a J-disk). Trivial disks and spheres can appear in these configurations as long as they are stable. It is easy to see that the virtual dimension of these moduli spaces is $|x| + |a| - 2n + \mu(\mathcal{T}) - |b| + 1$. As always we will only need to use those moduli spaces of virtual dimension 0 and 1 and the chain homotopy in question is defined by

$$\xi''(a \otimes x) = \sum \#_{\mathbb{Z}_2}(\mathcal{P}_{\mathcal{T}}(a, x, b)) b t^{\bar{\mu}(\mathcal{T})}$$

where the sum is over all the trees $\mathcal{T}$ so that the virtual dimension is 0. Of course, this definition is only valid generically: in that case, the methods described earlier in the paper can be easily applied here to show that $\mathcal{P}_{\mathcal{T}}(a, x, b)$ is a 0-dimensional compact manifold. Finally, to show that:

$$(d\xi'' + \xi'' d)(a \otimes x) = i_L(a * x) - a * i_L(x)$$

the regularity of the moduli spaces of dimension 1 is needed together with a boundary description. This follows again by the methods described earlier in the paper. However, notice that the use of perturbations is necessary in this case.

The last step to conclude point iii of Theorem 2.1.1 is to show relation (4):

$$< h^*, i_L(x) >= \epsilon_L(h * x)$$

for all $h \in H_*(M)$ and $x \in Q^+H_*(L)$. First denote by m the minimum of f (we assume it is unique to simplify the discussion). Notice that there is a bijection

$$b : \mathcal{P}_{\mathcal{T}}(x, a) \to \mathcal{P}_{\mathcal{T}'}(a, x, m)$$

where $\mathcal{P}_{\mathcal{T}'}(a, x, m)$ are the moduli spaces of the type used in the definition of the module operation in §5.3 associated to the function f and its critical points x and m and *to the* *function* $-h$ together with its critical point a; the tree $\mathcal{T}'$ is obtained by inverting the last edge in the tree $\mathcal{T}$ and adding one edge going from the last disk in $\mathcal{T}$ to the critical point m. This also shows how to define the bijection b: for each configuration v in $\mathcal{P}_{\mathcal{T}}(x, a)$ we consider the last disk u_k of v and the point $u_k(+1)$. This point belongs to the unstable manifold of m (generically, as always) and so, by replacing the last edge in v by a negative gradient flow line of $-h$ which goes from a to $u_k(0)$ and adding to v one edge given by a negative gradient flow line of f which joins $u_k(+1)$ to m we obtain $b(v)$. It is then clear that this map b so defined is bijective. Combining this with the definition of ϵ_L the relation (4) follows. □

5.5. **Spectral sequences.** We now notice that all the structures defined above are compatible with the degree filtration so that the point iv. of Theorem 2.1.1 is trivial.

5.6. **Comparison with Floer homology.** We deal here with the last point of Theorem 2.1.1. Recall from the statement of Theorem 2.1.1 that we denote by

$$\mathcal{C}(L; f, \rho, J) = \mathcal{C}^{+}(L; f, \rho, J) \otimes_{\Lambda^{+}} \Lambda \ .$$

5.6.1. *Comparing complexes.* The version of Floer homology which we need is defined with the help of an auxiliary Hamiltonian $H : M \times [0, 1] \to M$ and its construction is standard (see [48]).

Put $H_t(x) = H(x, t)$. Denote

$$\mathcal{P}_0(L, L) = \{\gamma : [0, 1] \to M : \gamma(0) \in L,\ \gamma(1) \in L \ \ \gamma \simeq *\} \ .$$

The generators of the Floer complex, $CF(L; H, J)$, are elements of $\mathcal{P}_0(L, L)$ which are orbits of the Hamiltonian vector field X_t^H (defined by $\omega(X_t^H, Y) = -dH_t(Y)$). We denote this set of orbits by I_H. There is a natural map $\pi_2(M, L) \to \pi_1(\mathcal{P}_0(M, L), *)$ and extensions to $\pi_1(\mathcal{P}_0(M, L), *)$ of each of the maps ω and μ (it is easy to see that these extensions continue to be proportional in this case). We need to fix a base point, $\bar{\eta}$, in the space $\bar{\mathcal{P}}_0(L, L)$ which is the abelian cover of $\mathcal{P}_0(L, L)$ which is associated to the kernel of μ. Once this choice is made the Floer complex is defined by

$$CF(L; H, J) = \mathbb{Z}_2\langle I_H\rangle \otimes \Lambda \ .$$

The differential is defined by first fixing one lift $\bar{x} \in \bar{\mathcal{P}}_0(L, L)$ for each of the elements $x \in I_H$ and there is a Maslov index $\mu(\bar{x}), 2\mu(\bar{x}) \in \mathbb{Z}$ which is well defined. This determines a grading on $CF(L; H, J)$ by the formula $|x \otimes t^r| = \mu(\bar{x}) - rN_L$. The differential is defined by

$$dx = \sum_{y,\lambda} \#_{\mathbb{Z}_2}\mathcal{M}(\bar{x}, \bar{y}; \lambda) y t^{\bar{\mu}(\lambda)}, \tag{90}$$

where $x, y \in I_H$, $\lambda \in \pi_1(\mathcal{P}_0(M, L), *)/\ker \mu$ and $\mathcal{M}(\bar{x}, \bar{y}; \lambda)$ is described as follows. First let $\mathcal{M}'(\bar{x}, \bar{y}; \lambda)$ be the moduli space of paths $\bar{u} : \mathbb{R} \to \bar{\mathcal{P}}_0(L, L)$ so that $\bar{u}(-\infty) = \bar{x}$, $\bar{u}(+\infty) = \lambda\bar{y}$ and the projection u of $\bar{u}$ to $\mathcal{P}_0(L, L)$ verifies

$$\partial u/\partial s + J\partial u/\partial t + \nabla H_t(u) = 0, \tag{91}$$

where ∇H_t is defined with respect to the metric g_J associated to (ω, J). There is an obvious action of $\mathbb{R}$ on this moduli space and we let

$$\mathcal{M}(\bar{x}, \bar{y}; \lambda) = \mathcal{M}'(\bar{x}, \bar{y}; \lambda)/\mathbb{R}\ .$$

Generically, this is a manifold of dimension

$$\mu(\bar{x}) - \mu(\bar{y}) + \mu(\lambda) - 1$$

and the sum in (90) is taken over all of those $\bar{y}, \lambda$ so that the respective moduli space is 0-dimensional. The homology of this complex is the Floer homology of L, $HF(L)$, and does not depend on H and J. It depends on the choice of the base point $\bar{\eta}$ up to translation.

Remark 5.6.1. We leave it to the reader to verify that using the positive Novikov ring in this case is not possible.

The point ii. of Theorem 2.1.1 is a consequence of the following:

Proposition 5.6.2. *For generic (f, ρ, H, J) there are chain morphisms*

$$\psi : \mathcal{C}(L; f, \rho, J) \to CF(L; H, J)$$

and

$$\phi : CF(L; H, J) \to \mathcal{C}(L; f, \rho, J)$$

which induce canonical isomorphisms in homology. These induced maps are inverse one to the other.

Remark 5.6.3. These morphisms are constructed by the Piunikin-Salamon-Schwarz method and, indeed, they are the exact Lagrangian counterpart of the PSS morphisms. Such morphisms have been discussed in the Lagrangian setting - when bubbling is avoided - in [8, 4, 38] and, in the general cluster setting, in [23].

Proof of Proposition 5.6.2. Notice that, given $\gamma \in I_H$ we may view each element $\bar{\gamma} \in \bar{\mathcal{P}}_0(L, L)$ which covers γ as a pair (γ, u) where u is a "half-disk" capping γ. For our fixed lifts $\bar{\gamma}$ of the orbits $\gamma \in I_H$ we denote $\bar{\gamma} = (\gamma, u_\gamma)$.

To define the morphisms ψ and ϕ some new moduli spaces are necessary. Given $x \in \mathrm{Crit(f)}$ and $\gamma \in I_H$ we will define next the moduli spaces $\mathcal{P}_{\mathcal{I}}(x, \gamma)$ and $\mathcal{P}_{\mathcal{I}}(\gamma, x)$. In both cases $\mathcal{T}$ is a linear tree as those used in the definition of the pearl moduli spaces. Compared to the definition of the pearl moduli spaces - Definition 5.1.1 - the configurations assembled in $\mathcal{P}_{\mathcal{T}}(x,\gamma)$ have the property that the last vertex in the chain does not correspond to a J-disk but rather to an element $u_k : \mathbb{R}\times[0,1]\to M$ so that we have:

$$u_k(\mathbb{R}\times\{0,1\})\subset L\ ,\ \partial_s(u_k)+J(u_k)\partial_t(u_k)+\beta(s)\nabla H_t(u)=0\ ,u_k(+\infty)=\gamma$$

and $r_{t_k}(u_{k-1}(+1))=u_k(-\infty)$ where r_t is the negative gradient flow of f and β is a smooth cut-off function which is increasing and vanishes for $s\leq 1/2$ and equals 1 for $s\geq 1$. The virtual dimension of this moduli space is $|x|-|\bar{\gamma}|+\mu(\mathcal{T})-1$ where $|\bar{\gamma}|=\mu(\bar{\gamma})$ and $\mu(\mathcal{T})$ is by definition the sum of the Maslov indices of the disks corresponding to the vertices appearing in the tree $\mathcal{T}$ to which we add the Maslov class of the disk obtained by gluing u_k to u_γ along γ. Similarly, the moduli space $\mathcal{P}_{\mathcal{T}}(\gamma,x)$ has an analogue definition except that the end conditions are reversed. More precisely, the first disk u_0 is the solution of an equation:

$$u_0(R\times\{0,1\})\subset L\ ,\ \partial_s(u_0)+J(u_0)\partial_t(u_0)+\xi(s)\nabla H(u,t)=0\ ,u_0(-\infty)=\gamma$$

, $r_{-t_1}(u_1(-1))=u_0(+\infty)$ where ξ is a smooth curt-off function which is decreasing and vanishes for $s\geq 1/2$ and equals 1 for $s\leq 0$. The same methods as those used earlier in the paper show that for a generic J we have an equality

$$\mathcal{P}_{\mathcal{T}}(x,\gamma)=\mathcal{P}^{*;d}_{\mathcal{T}}(x,\gamma)$$

(and similarly for the moduli spaces $\mathcal{P}_{\mathcal{T}}(\gamma,x)$) where the moduli space on the left consists of configurations containing only simple, absolutely distinct disks. For H generic the appropriate transversality of the evaluation maps can be achieved when the virtual dimension is at most 1. The definition of the morphism ψ is now as follows:

$$\psi(x)=\sum_{\gamma,\mathcal{T}}\#(\mathcal{P}_{\mathcal{T}}(x,\gamma))\gamma t^{\bar{\mu}(\mathcal{T})}$$

where the sum is taken over all trees so that the dimension of the respective moduli spaces is 0. Similarly,

$$\phi(\gamma)=\sum_{x,\mathcal{T}}\#(\mathcal{P}_{\mathcal{T}}(\gamma,x))xt^{\bar{\mu}(\mathcal{T})}$$

where again we only take into account moduli spaces of dimension 0. By analyzing the boundary of the compactifications of the moduli spaces $\mathcal{P}_{\mathcal{T}}(x,\gamma)$ and $\mathcal{P}_{\mathcal{T}}(\gamma,x)$ it is easy to show that both ϕ and ψ are chain morphisms. We then need to show that the compositions $\phi\circ\psi$ and $\psi\circ\phi$ are both chain homotopic with the respective identities. As in the non-bubbling case, this proof is based on a gluing argument - which allows to view the product $\mathcal{P}_{\mathcal{T}}(x,\gamma)\times\mathcal{P}_{\mathcal{T}'}(\gamma,y)$ as part of the boundary of a moduli space $\mathcal{P}_{\mathcal{T}\#\mathcal{T}'}(x,y,k,H)$ where this last moduli space is modeled on the tree $\mathcal{T}\#\mathcal{T}'$ which is obtained by gluing $\mathcal{T}'$ at the end of $\mathcal{T}$ and it consists of pearl-like objects except that the k-th disk verifies a perturbed Floer type equation of the form $\partial_s(u_0) + J(u_0)\partial_t(u_0) + \nu_R(s)\nabla H(u,t) = 0$ where ν_R is a family of smooth functions so that $\nu_R(s) = 0$ for $|s| > R$, is increasing for $s < 0$ and decreasing for $s > 0$ and, for R sufficiently big, it is equal to 1 for $|s| < R - 1$.

□

5.6.2. *Module action and internal product on $HF(L)$.* In this subsection we want to notice that there exists a natural action of $QH(M)$ on $HF(L)$ which is identified via the PSS maps with the action discussed in §5.3. The definition of this module structure is completely similar to the $\cap$- action of singular homology on Hamiltonian Floer homology as it is described for example in [31] or in [59]. Similarly, we also have the "half"-pair of pants product on $HF(L)$.

Given a Morse function $h : M \to \mathbb{R}$ together with a Riemannian metric ρ_M so that the pair (h, ρ_M) is Morse-Smale as in §5.3 we define an operation:

$$* : C_k(h) \otimes CF_l(L; H, J) \to CF_{l+k-2n}(L; H, J) \tag{92}$$

as follows

$$a * x = \sum_{y,\lambda} \#\mathcal{M}(a; \bar{x}, \bar{y}; \lambda) y t^{\bar{\mu}(\lambda)} \ .$$

Here

$$\mathcal{M}(a; \bar{x}, \bar{y}; \lambda) = \{(u, p) \in \mathcal{M}'(\bar{x}, \bar{y}; \lambda) \times W_a^u \ : \ u(0, 1/2) = p\} \tag{93}$$

and W_a^u is the unstable submanifold of the critical point a for the flow of $-\nabla h$. As always, the sum above is understood to be taken only when the moduli spaces in question are finite.

The Floer intersection product is an associative operation with unit

$$HF_p(L) \otimes HF_q(L) \to HF_{p+q-n}(L) \ .$$

We first recall that this product is defined by a chain map:

$$* : CF_k(L; H, J) \otimes CF_l(L; H', J) \to CF_{k+l-n}(L; H'', J)$$

where H' and H'' are small deformations of H and

$$x * y = \sum_{z,\lambda} \#(\mathcal{M}_z^{x,y}(\lambda)) z t^{\bar{\mu}(\lambda)},$$

where the moduli spaces $\mathcal{M}^{x,y}_{z}(\lambda)$ consist of semi-pants (or half pants) with their boundaries on L and which are otherwise similar to the usual pair of pants used to define the standard product in Hamiltonian Floer homology.

Again, both structures are well defined for generic choices of data and independent of these choices and the PSS maps identify them in homology with the ones described in §5.3 and §5.2. We leave this verification to the reader. It is also easy to verify that analogue statements are valid for the map i_L and the spectral sequence compatibility.

5.7. **Duality.** The purpose of this section is to prove the Corollary 2.2.1. We start by recalling the conventions and the notation in §2.2.1. Fix generic f, ρ, J so that the pearl complex $\mathcal{C}^{+}(L; f, \rho, J)$ is defined. Let $\mathcal{C}^{\odot} = \hom_{\Lambda}(\mathcal{C}(L; f, \rho, J), \Lambda)$ endowed with the differential which is the adjoint of the differential of $\mathcal{C}^{+}(L; f, \rho, J)$ and with the grading $|x^{*}| = -|x|$. Recall also that $s^{k}\mathcal{C}$ indicates the k-th suspension of the complex $\mathcal{C}$.

Proof of Corollary 2.2.1. The first step is to notice that the pearl complex $\mathcal{C}^{+}(L; -f, \rho, J)$ is also defined and there is a basis preserving isomorphism i between $\mathcal{C}^{+}(L; -f, \rho, J)$ and $s^{n}\mathcal{C}^{\odot}$ defined in the same way as in the Morse case (see Remark 2.2.2 a): it sends each generator represented by a critical point x of f, $ind_{f}(x) = k$ to a generator of degree $n-k$ represented by the same critical point x only viewed as critical point of $-f$.

At the same time, there is a comparison chain morphism

$$\phi : \mathcal{C}^{+}(L; f, \rho, J) \to \mathcal{C}^{+}(L; -f, \rho, J)$$

as in §5.1.2 which induces a (canonical) isomorphism in homology. Therefore we get a morphism

$$\eta : \mathcal{C}^{+}(L; f, \rho, J) \to s^{n}\mathcal{C}^{\odot} \ , \ \eta = i \circ \phi$$

which induces an isomorphism in homology and this concludes the first part of the Corollary. For the second part we need to show that the bilinear map $\tilde{\eta}$ associated to η coincides with the reduced quantum product $Q^{+}H(L) \otimes Q^{+}H(L) \xrightarrow{*} Q^{+}H(L) \xrightarrow{\epsilon_L} \Lambda$.

The first step for this is to notice that it is enough to work with a comparison morphism $\phi : \mathcal{C}^{+}(L; f', J) \to \mathcal{C}^{+}(L; -f, J)$ where f' and f are in generic position. In fact, it is clear that we may even replace ϕ with any other chain morphism which is chain homotopic to it. There is a specific such chain morphism which will be useful in the proof. Its definition is quite general so we now take h another generic Morse function and we describe this new comparison morphism $\phi' : \mathcal{C}^{+}(L; f'J) \to \mathcal{C}^{+}(L; h, J)$. It will be used in our proof for $h = -f$. The construction of ϕ' is based on counting the elements of certain moduli spaces $\mathcal{P}^{!}(f', h; x, y; J, \lambda)$ which are modeled on linear trees, as the pearl moduli spaces, except that there is an additional marked point on the tree which is placed in the interior of an edge in the tree. The key property of these moduli spaces is that all the edges (or segments of edges) above this marked point correspond to negative flow lines of f' and all the edges (or segments) below this marked point - in the tree - correspond to negative flow lines of h. The virtual dimension of these moduli spaces is $|x| - |y| + \mu(\lambda)$ where λ is the total homotopy class of the configuration. The same methods used earlier in

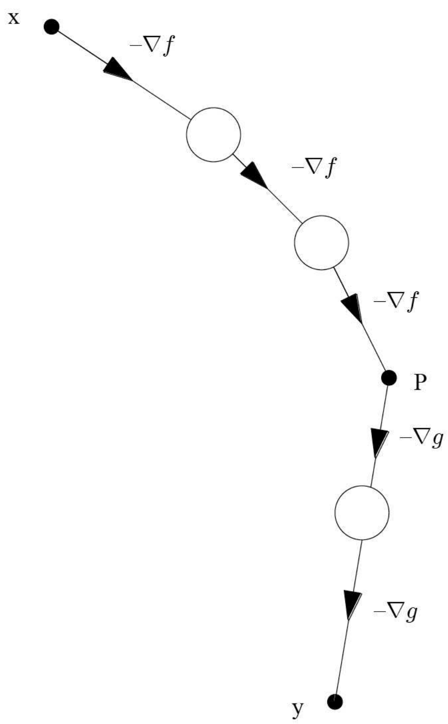

FIGURE 14. $x \in \mathrm{Crit}(f')$, $y \in \mathrm{Crit}(h)$, P is the new marked point.

the paper show that, with generic choices of defining data, counting the elements in the 0-dimensional such moduli spaces does indeed define a chain morphism. The proof now consists of two steps. The first, is to remark that ϕ and ϕ' are chain homotopic. We will postpone this argument and proceed to describe the last step in the proof. For this we first fix a third Morse function $f'' : L \to \mathbb{R}$ which has a single minimum m and is in generic position with respect to f' and $-f$. We now notice, by reviewing the definition of the moduli spaces $\mathcal{P}(x, y, m; f', f, f'')$ from §5.2 that they coincide in dimension 0 with the moduli spaces $\mathcal{P}^!(f', -f; x, y; J)$. Indeed, if $\mathcal{P}(x, y, m; f', f, f'')$ is of dimension 0, then the elements of this moduli space have the property that their unique vertex of valence three is a constant disk and, moreover, the chain of pearls associated to f'' and arriving in m is a single flow line of $-\nabla f''$ joining the vertex of valence three to m. But as $\mathcal{P}(x, y, m; f', f, f'')$ are precisely the moduli spaces which compute $\epsilon_L \circ (- * -)$ we obtain $\epsilon_L(x * y) =< \phi'(x), y >= (i \circ \phi'(x))(y)$ which implies our claim.

To conclude our proof, we need to justify that ϕ' and ϕ are chain homotopic. Recall that ϕ is defined by making use of a Morse homotopy between f' and h. Let $H : L \times [0,1] \to \mathbb{R}$ be such a Morse homotopy. We intend to apply a method similar to the proof of the fact that the comparison morphism is canonical in homoloy as in §5.1.2. However, we apply that method in a more general situation: the place of the second Morse homotopy H' will be taken by the discontinous function $H'(x,t) = f'(x)$ for $t \in [0, 1/2)$ and $H'(x,t) = h(x)$ for $t \in [1/2, 1]$. The reason that this method still works is that, despite the discontinuity of H', the vector field $\nabla H'$ is still well defined on $L \times [0,1]$ and its negative flow lines are still well defined and continuous - they follow the negative gradient of f' for $t \in [0, 1/2)$ and then follow along the negative gradient of h for $t \in [1/2, 1]$. The only additional ingredient with respect to the method described in §5.1.2 is a gluing statement. In essence, this is already seen in the case of the purely Morse theoretic version of our statement (that is if no pseudo-holomorphic disks are present): each trajectory of $-\nabla H'$ has to be shown to be precisely the end of a one parametric family of flow lines of the Morse homotopy relating H to H'. We leave this last step as exercise for the reader. □

5.8. **Action of the symplectomorphism group.** The purpose of this section is to prove Corollary 2.2.3.

Proof of Corollary 2.2.3. We assume that $\phi : L \to L$ is a diffeomorphism which is the restriction to L of the symplectomorphism $\bar{\phi}$ and f, ρ, J are such that the chain complex $\mathcal{C}^+(L; f, \rho, J)$ is defined. Let $f^\phi = f \circ \phi^{-1}$. There exists a basis preserving isomorphism

$$h^\phi : \mathcal{C}^+(L; f, \rho, J) \to \mathcal{C}^+(L; f^\phi, \rho^*, J^*)$$

induced by $x \to \phi(x)$ for all $x \in \mathrm{Crit(f)}$ where ρ^*, J^* are obtained by the push-forward of ρ, J by means of ϕ and the symplectomorphism $\bar{\phi}$. The isomorphism h^ϕ acts in fact as an identification of the two complexes.

Finally, there is also the standard comparison chain morphism

$$c : \mathcal{C}^+(L; f^\phi, \rho^*, J^*) \to \mathcal{C}^+(L; f, \rho, J) \ .$$

We now consider de composition $k = c \circ h^\phi$. It is clear that this map induces an isomorphism in homology and that it preserves the augmented ring structure (as each of its factors does so). We now inspect the Morse theoretic analogue of these morphisms - in the sense that we consider instead of the complexes $\mathcal{C}(L, f, -)$ the respective Morse complexes $C(f, -)$. It is easy to see that, by possibly redescribing $H_*(c)$ as a morphism induced by a chain morphism given in the same way as ϕ' in the proof of Corollary 2.2.1, the Morse theoretic version of k induces in Morse homology precisely $H_*(\phi)$. But this means that at the E^2 stage of the degree filtration the morphism induced by k has the form $H_*(\phi) \otimes id_{\Lambda^+}$.

We now denote $k = \hbar(\bar{\phi})$ and we need to verify that for any two elements $\bar{\phi}, \bar{\psi} \in Symp(M)$ we have $\hbar(\bar{\phi} \circ \bar{\psi}) = \hbar(\phi) \circ \hbar(\psi)$. It is easy to see that this is implied by the commutativity up to homotopy of the following diagram:

$$\begin{tikzcd} \mathcal{C}^+(L; f') \arrow[r, "h^\phi"] \arrow[d, "c"'] & \mathcal{C}^+(L; (f')^\phi) \arrow[d, "c'"] \\ \mathcal{C}^+(L; f) \arrow[r, "h^\phi"'] & \mathcal{C}^+(L, f^\phi) \end{tikzcd}$$

for any two Morse function f and f' so that the respective complexes are defined. To see this, first we use some homotopy H, joining f to f', to provide the comparison morphism c and we then use the homotopy $H \circ \phi^{-1}$ to define c'.

Finally, recall that the module structure of $Q^+H_*(L)$ over $Q^+H(M)$ is defined by using an additional Morse function $F : M \to \mathbb{R}$. If we put $F^\phi = F \circ \phi^{-1}$ we see easily that the external operations defined by using f, F, ρ, J and $f^\phi, F^\phi, \rho^*, J^*$ are identified one to the other via the application h^ϕ (extended in the obvious way to the critical points of F). The usual comparison maps then are used, as before, to compare (by using appropriate homotopies) $f^\phi, F^\phi, \rho^*, J^*$ to f, F, ρ, J. At the level of the Morse quantum homology on M the composition of these two maps induces $H_*(\bar{\phi}) \otimes id_{\Lambda^+}$. Therefore, if $\bar{\phi} \in Symp_0(M)$, it follows that this last map is the identity and proves the claim. □

Remark 5.8.1. It results from the proof above that for $\hbar(\phi)$ to be an algebra automorphism it is sufficient that $\bar{\phi}$ induce the identity at the level of the singular homology of M, e.g. ϕ is homotopic to the identity.

5.9. **Minimality for the pearl complex.** This subsection is purely algebraic and its purpose is to show the statements in §2.2.3.

Proof of Proposition 2.2.4. We start by choosing generators for the complex (G, d_0) as follows: $G = \mathbb{Z}_2 < x_i : i \in I > \oplus \mathbb{Z}_2 < y_j : j \in J > \oplus \mathbb{Z}_2 < y'_j : j \in J >$ so that $d_0 x_i = 0$, $d_0(y_j) = 0$, $d_0 y'_j = y_j$, $\forall j \in J$. The index families I and J are finite. Clearly, $\mathcal{H} \cong \mathbb{Z}_2 < x_i >$ and we will identify further these two vector spaces and denote $\mathcal{C}_{min} = \mathbb{Z}_2 < \tilde{x}_i > \otimes \Lambda^+$ where $\tilde{x}_i, i \in I$ are of the same degree as the x_i's (obviously, the differential on $\mathcal{C}_{min}$ remains to be defined). We will construct ϕ and ψ and δ so that $\phi_0(x_i) = \tilde{x}_i$, $\phi_0(y_j) = \phi_0(y'_j) = 0$ and $\psi_0(\tilde{x}_i) = x_i$. The construction is by induction. We fix the following notation: $\mathcal{C}^k = \mathbb{Z}_2 < x_i,\ y'_j,\ y_j\ :\ \ |x_i| \geq k, |y'_j| \geq k\ \ > \otimes \Lambda^+$. Similarly, we put $\mathcal{C}^k_{min} = \mathbb{Z}/2 < \tilde{x}_i\ :\ |x_i| \geq k\ \ > \otimes \Lambda^+$.

Notice that there are some generators in $\mathcal{C}^k$ which are of degree $k-1$, namely the y_j's of that degree. With this notation we also see that $\mathcal{C}^k$ is a sub-chain complex of $\mathcal{C}$. To simplify notation we will identify the generators of these complexes by their type - x, y', y, $\tilde{x}$ and their degree. Assume that n is the maximal degree of the generators in G. For the generators of $\mathcal{C}^n$ we let ϕ be equal to ϕ_0, we put $\delta = 0$ on $\mathcal{C}^n_{min}$ and we also let $\psi = \psi_0$ on $\mathcal{C}^n_{min}$. To see that $\psi : \mathcal{C}^n \to \mathcal{C}^n_{min}$ is a chain morphism with these definitions it suffices to remark that if a generator of type y has degree $n-1$, then $y = d_0 y' = dy'$ and so $dy = 0$. We now assume ϕ, δ, ψ defined on $\mathcal{C}^{n-s+1}$, $\mathcal{C}^{n-s+1}_{min}$ so that ϕ, ψ are chain morphisms, they induce isomorphisms in homology and $\phi \circ \psi = id$.

We now indend to extend these maps to $\mathcal{C}^{n-s}$, $\mathcal{C}^{n-s}_{min}$. We first define ϕ on the generators of type x and y' which are of degree $n-s$: $\phi(x) = \tilde{x}$, $\phi(y') = 0$. We let $\delta(\tilde{x}) = \phi^{n-s+1}(dx)$ (when needed, we use the superscript $(-)^{n-s+1}$ to indicate the maps previously constructed by induction). Here it is important to note that, as $d_0 x = 0$, we have that $dx \in \mathcal{C}^{n-s+1}$. We consider now the generators of type $y \in \mathcal{C}^{n-s}$ which are of degree $n-s-1$ and we put $\phi(y) = \phi^{n-s+1}(dy' - y)$. This makes sense because $dy' - y \in \mathcal{C}^{n-s+1}$. If we write $dy' = y + y''$ we see $\phi(dy') = \phi(y) + \phi(y'') = 0 = \delta(\phi(y'))$ so that, to make sure that ϕ^{n-s} is a chain morphism with these definitions, it remains to check that $\delta\phi(y) = \phi(dy)$ for all generators of type y and of degree $n-s-1$. But $\delta\phi(y) = \delta\phi^{n-s+1}(y'')$ and as ϕ^{n-s+1} is a chain morphism, we have $\delta\phi^{n-s+1}(y'') = \phi^{n-s+1}d(y'')$ which implies our identity because $dy'' + dy = d^2 y' = 0$. It is clear that ϕ so defined induces an isomorphism on the d-homology of $\mathcal{C}^{n-s}$ because the kernel of ϕ is generated by couples (y', dy') so that it is acyclic. To conclude our induction step it remains to construct the map ψ on the generators $\tilde{x}$ of degree $n-s$. We now consider the difference $dx - \psi^{n-s+1}(\delta\tilde{x})$ and we want to show that there exists $\tau \in \mathcal{C}^{n-s+1}$ so that $d\tau = dx - \psi^{n-s+1}(\delta\tilde{x})$ and $\tau \in \ker(\psi^{n-s+1})$. Assuming the existence of this τ we will put $\psi(\tilde{x}) = x - \tau$ and we see that ψ is a chain map and $\phi \circ \psi = id$. To see that such a τ exists remark that first $w = dx - \psi^{n-s+1}(\delta\tilde{x}) \in \mathcal{C}^{n-s+1}$ and $dw = d(\psi^{n-s+1}(\delta\tilde{x})) = \psi^{n-s+1}(\delta \circ \delta\tilde{x}) = 0$ (because ψ^{n-s+1} is a chain map). Moreover, $\phi(w) = \phi^{n-s+1}(dx) - \delta\tilde{x} = 0$ because $\phi^{n-s+1} \circ \psi^{n-s+1} = id$. Therefore w is a cycle belonging to $\ker(\phi^{n-s+1})$. But ϕ^{n-s+1} is a chain morphism which induces an isomorphism in homology and which is surjective. Therefore $H_*(\ker(\phi^{n-s+1})) = 0$. Thus there exists $\tau \in \ker(\phi^{n-s+1})$ so that $d\tau = w$ and this concludes the induction step. With this construction it is clear that ϕ_0 induces an isomorphism in d_0-homology and as $\phi_0 \circ \psi_0 = id$ we deduce that so does ψ_0.

This construction concludes the first part of the statement and to finish the proof of the proposition we only need to prove the uniqueness result. The following lemma is useful.

Lemma 5.9.1. *Let G, G' be finite dimensional, graded $\mathbb{Z}_2$-vector spaces. A morphism $\xi : G \otimes \Lambda^+ \to G' \otimes \Lambda^+$ is an isomorphism iff ξ_0 is an isomorphism.*

Proof. Indeed, it is immediate to see that if ξ is an isomorphism, then ξ_0 is one: the surjectivity of ξ implies that of ξ_0 and a dimension count concludes this direction. Conversely, if ξ_0 is surjective a simple induction argument shows that ξ is surjective too. Assume that the maximal degree in G is k. Obviously $\xi = \xi_0|_{G_k \otimes \Lambda^+}$. We now suppose that ξ is surjective when restricted to $G_{k-s+1} \otimes \Lambda^+ \to G'_{k-s+1} \otimes \Lambda^+$. Take $g' \in G'_{k-s}$. Then $g' = \xi_0(g)$ for some $g \in G_{k-s}$ so that we may write $\xi(g) = g' + g''t$ for some $g'' \in G'_{k-s+1} \otimes \Lambda^+$. But, by the induction hypothesis, $g'' \in Im(\xi|_{G_{k-s+1} \otimes \Lambda^+})$ so that $g' = \xi(g) - \xi(g''')t$ with $\xi(g''') = g''$, $g''' \in G_{k-s+1} \otimes \Lambda^+$. A dimension count again shows the injectivity of ξ. □

To end the proof of the proposition, suppose $\phi' : \mathcal{C} \to \mathcal{C}'$ and $\psi' : \mathcal{C}' \to \mathcal{C}$ are chain morphisms so that $\phi' \circ \psi' = id$ with $\mathcal{C}' = (H \otimes \Lambda^+, \delta')$, $\delta'_0 = 0$ and H some graded, $\mathbb{Z}_2$-vector space and ϕ', ψ', ϕ'_0, ψ'_0 induce isomorphisms in the (respective) homology. We want to show that there exists a chain map $c : \mathcal{C}_{min} \to \mathcal{C}'$ so that c is an isomorphism. This is quite easy: we define $c(u) = \phi' \circ \psi(u)$, $\forall u \in \mathcal{C}_{min}$. Now $H_*(\phi_0)$ and $H_*(\phi'_0)$, $H_*(\psi_0)$, $H_*(\psi'_0)$ are all isomorphisms (in d_0-homology). So $H(c_0)$ is an isomorphism but as $\delta_0 = 0 = \delta'_0$ we deduce that c_0 is an isomorphism. □

Proof of Corollary 2.2.5. Suppose that $\mathcal{C}(L; f, \rho, J)$ is defined and apply the Proposition 2.2.4 to it. Denote by $(\mathcal{C}_{min}, \phi, \psi)$ the resulting minimal complex and the chain morphisms as in the statement of 2.2.4. The only part of the statement which remains to be shown is that given a different set of data $(f', \rho'.J')$ so that $\mathcal{C}(L; f'\rho', J')$ is defined, there are appropriate morphisms ϕ', ψ' as in the statement. There are comparison morphisms: $h : \mathcal{C}(L; f'\rho', J') \to \mathcal{C}(L; f\rho, J)$ as well as $h' : \mathcal{C}(L; f\rho, J) \to \mathcal{C}(L; f'\rho', J')$ so that, by construction, both h and h' are inverse in homology and both induce an isomorphism in Morse homology and again these two isomorphisms are inverse (see §5.1.2). Define $\phi' : \mathcal{C}(L; f', \rho', J') \to \mathcal{C}_{min}$, $\psi'' : \mathcal{C}_{min} \to \mathcal{C}(L; f', \rho', J')$ by $\phi' = \phi \circ h$ and $\psi'' = h' \circ \psi$. It is clear that ϕ', ψ'', ϕ'_0 and ψ''_0 induce isomorphism in homology. Moreover, as h_0 and h'_0 are inverse in homology and $\delta_0 = 0$ in $\mathcal{C}_{min}$ it follows that $\phi'_0 \circ \psi''_0 = id$. This means by the Lemma 5.9.1 that $v = \phi' \circ \psi''$ is a chain isomorphism so that v_0 is the identity. We now put $\psi' = \psi'' \circ v^{-1}$ and this verifies all the needed properties. The uniqueness of $\mathcal{C}_{min}(L)$ now follows from the uniqueness part in the Proposition 2.2.4. □

5.10. **Proof of the action estimates.** We first recall the definition of the two spectral invariants involved. Fix a generic pair (H, J) consisting of a 1-periodic Hamiltonian $H : M \times S^1 \to \mathbb{R}$ and an almost complex structure J so that the Floer complex $CF_*(H, J)$ is well defined. We will assume the coefficients of this complex to be in the usual Novikov ring Λ. We recall that the generators of $CF_*(H, J)$ as a module over Λ are pairs formed by contractible orbits of X^H together with fixed cappings. Fix also a Morse function

$f : L \to \mathbb{R}$ as well as a Riemannian metric g on L so that the pearl complex $\mathcal{C}^+(L; f, \rho, J)$ is well defined.

We first need to provide a description of our module external operation which involves the two complexes above. This is based on moduli spaces $\mathcal{P}(\gamma, x, y; \lambda)$ similar to the ones used in §5.3 except that the vertex of valence three in the string of pearls is now replaced by a half-tube with boundary on L and with the $-\infty$ end on γ. The total homotopy class λ of the configuration obtained in this way is computed by using the capping associated to γ to close the semi-tube to a disk and adding up the homotopy class of this disk to the homotopy classes of the other disks in the string of pearls. More explicitly, a half tube as before is a solution

$$u : (-\infty, 0] \times S^1 \to M$$

of Floer's equation $\partial u/\partial s + J\partial u/\partial t + \nabla H(u, t) = 0$ with the boundary conditions

$$u(\{0\} \times S^1) \subset L \quad \lim_{t\to-\infty} u(s, t) = \gamma(t) \ .$$

The incidence points on the "exceptional" vertex which corresponds to u are so that the point $u(0, 1)$ is an exit point for a flow line and $u(0, -1)$ is the entry point. Both compactification and bubbling analysis for these moduli spaces are similar to what has been discussed before to which is added the study of transversality and bubbling for the spaces of half-tubes as described by Albers in [5] and, as described in [5], an additional assumption is needed for this part: H is assumed to be such that no periodic orbit of X^H is completely included in L.

Counting elements in these moduli spaces defines an operation:

$$CF(H, J) \otimes \mathcal{C}(L; f, \rho, J) \to \mathcal{C}(L; f, \rho, J)$$

and, by using the Hamiltonian version of the Piunikin-Salamon-Schwarz homomorphism, it is easy to see that, in homology, this operation is canonically identified with the module action as described in §5.3.

The Floer complex $CF_*(H, J)$ is filtered by the values of the action functional

$$\mathcal{A}_H(\overline{x}) = \int_0^1 H(x(t), t)dt - \int_D \hat{x}^*\omega$$

where $\overline{x} = (x, \hat{x})$ with x a C^∞ loop in M and $\hat{x}$ a cap of this loop. This action is compatible with the Novikov ring in the sense that: $\mathcal{A}_H(\gamma \otimes t^k) = \mathcal{A}_H(\gamma) - k\tau$ (where τ is the monotonicity constant). The filtration of order $\nu \in \mathbb{R}$ of the the Floer complex, CF^ν, is the graded $\mathbb{Z}_2$-vector space generated by all the elements $\gamma \otimes \lambda$ of action at most ν. This is a sub-complex because the differential decreases action.

We now fix $\alpha \in H_*(M;\mathbb{Z}_2)$ and define $\sigma(\alpha, H)$ by:

$$\sigma(\alpha, H) = \inf\{\nu : PSS(\alpha) \in \text{Image}(\ H(CF^\nu) \to HF(H,J)\ )\}. \tag{94}$$

Here $PSS : H_*(M;\mathbb{Z}_2) \to HF(H,J)$ is the Piunikin-Salamon-Schwarz homomorphism. We also need a similar definition for a cohomology class $\beta \in H^*(M;\mathbb{Z}_2)$. A little more notation is needed for this. We recall that we work here over the Novikov ring Λ. We also recall the algebraic notation from §2.2.1: Λ^* is be the ring Λ with reverse grading (in short the element t has now degree N_L) and given a free chain complex $\mathcal{C} = G \otimes \Lambda$ with G a $\mathbb{Z}_2$ vector space, $\mathcal{C}^* = \hom_{\mathbb{Z}_2}(G, \mathbb{Z}_2) \otimes \Lambda^*$ where the grading of the dual x^* of a basis element $x \in G$ is $|x^*| = |x|$ with the differential given as the adjoint of the differential in $\mathcal{C}$.

From the Floer complex $CF(H,J)$ we define the associated Floer co-homology by $H^kF(H,J) = H^k(CF(H,J)^*)$ and similarly for all the various subcomplexes involved. In particular, we have $H^k(CF^\nu) = H^k((CF^\nu)^*)$ as well as morphisms

$$p_\nu : CF(H,J)^* \to (CF^\nu)^*$$

which are induced by the inclusions $CF^\nu \hookrightarrow CF(H,J)$. Moreover, the inverse PSS map induces also a comparison morphism $H^*(M;\mathbb{Z}_2) \to H^*F(H,J)$ which we will denote by PSS'. In view of this we may now define:

$$\sigma(\beta, H) = \sup\{\nu :\ PSS'(\beta) \in \ker(H^*F(H,J) \to H^*(CF^\nu))\}. \tag{95}$$

Assuming that H is normalized, it is well known that $\sigma(\alpha, H)$, the spectral invariant of α, only depends on the class $[\phi^H] \in \widetilde{Ham}(M)$. The same holds for the spectral invariant of the co-homology class β.

Remark 5.10.1. The spectral invariant of a co-homology class β satisfies te following property which will be useful in the following. For any $\epsilon > 0$ there exists a co-chain $c = \sum_i \gamma_i^*$ so that $[c] = \beta$, γ_i^* are dual to orbits γ_i (the Λ-coefficients are integrated in γ_i by a possible change of capping - in other words we view $\mathcal{C}$ as a $\mathbb{Z}_2$ vector space) and $\mathcal{A}_{H'}(\gamma_i) \geq \sigma(\beta, H) - \epsilon$ for all i. For this, first notice that the map $p_\nu : CF(H,J)^* \to (CF^\nu)^*$ is surjective and that its kernel is generated by those γ^* with $\mathcal{A}_H(\gamma) > \nu$. Fix $\nu = \sigma(\beta, H) - \epsilon$ and let $c' = \sum \gamma_i^*$ be so that $[c'] = \beta$. Let $p_\nu(c') = \delta$. Then as $\nu < \sigma(\beta, H)$ we deduce $\delta = \partial^* h$ and as p_ν is surjective, we may view h as an element of $CF(H,J)^*$ so that $p_\nu(c' - \partial^* h) = 0$. But $\beta = [c' - \partial^* h]$ so that our claim follows.

Remark 5.10.2. To avoid possible confusion with various other conventions used in the literature, notice that, with the definitions above, we have in general $\sigma(\alpha^*, H) \neq \sigma(\alpha, H)$ where α^* is the co-homology class Poincaré dual to α.

5.10.1. *Proof of Corollary 2.3.1.* We recall that $\alpha \in H_*(M;\mathbb{Z}_2)$ is fixed as well as $x, y \in Q^+H(L)$ so that $y \neq 0$ and $\alpha * x = yt^k +$ higher order terms. We also fix $\phi \in \widetilde{Ham}(M)$. We first intend to show that: $\sigma(\alpha,\phi) - \text{depth}_L(\phi) + k\tau \geq 0$. By inspecting the definition of *depth* in §2.3 we see that this reduces to showing that for every normalized Hamiltonian H with $[H] = \phi$, we have

$$\sigma(\alpha, H) - \int_0^1 H(\gamma(t),t)dt + k\tau \geq 0$$

for some loop $\gamma : S^1 \to L$. By a small perturbation of H we may assume that no closed orbit of H is contained in L.

Now assume that $\eta = \sum \gamma_i \otimes \lambda_i$ is a cycle in $CF(H,J)$ so that $[\eta] = PSS(\alpha)$ where γ_i are generators of $CF(H,J)$ and $\lambda_i \in \Lambda$. The relation $\alpha * x = yt^k + \ldots$ implies that there exits some periodic orbit γ_i, say γ_1, and critical points $x_1, y_1 \in \text{Crit(f)}$, $|x_1| = |x|$, $|y_1| = |y|$ so that the moduli space $\mathcal{P}(\gamma_1', x_1, y_1; t^k) \neq \emptyset$ where $\gamma_1' = \gamma_1 \# \lambda_1$ is the same orbit of X^H as γ_1 but with the capping changed by λ_1. We now consider an element $v \in \mathcal{P}(\gamma_1', x_1, y_1; t^k)$ and we focus on the corresponding half-tube u (which is part of v). The usual energy estimate for this half-tube gives:

$$0 \leq \int_{-\infty}^0 \int_0^1 ||\partial u/\partial s||^2 dt ds = \int_{[-\infty,0]\times S^1} u^*\omega + \int_{S^1} H(\gamma_1(t),t) - \int_{S^1} H(u(0,t),t)\ . \tag{96}$$

We now want to remark that:

$$\mathcal{A}_H(\gamma_1') + k\tau \geq \int_{[-\infty,0]\times S^1} u^*\omega + \int_{S^1} H(\gamma_1(t),t)\ .$$

Indeed, this is obvious in view of the definition of the action and given that $k\tau$ equals the symplectic area of all the disks in v + the area of the tube u + the area of the cap corresponding to γ_1' (we have equality here iff no J-disks appear in v). Given any $\epsilon > 0$, in view of the definition of $\sigma(\alpha, H)$, it follows that we may find in $CF^{\sigma(\alpha,H)+\epsilon}$ a cycle η with $[\eta] = PSS(\alpha) \in HF(H,J)$. Applying the discussion above to this η means $\mathcal{A}_H(\gamma_1') \leq \sigma(\alpha, H) + \epsilon$ and this implies the claimed inequality.

We now want to show the second inequality in Corollary 2.3.1: $\text{height}_L(\phi) - \sigma(\alpha^*, \phi) + k\tau \geq 0$. Again by taking a look at the definition of $\text{height}_L(\phi)$ we see that it is enough to show that for some normalized Hamiltonian H' we have $\int_0^1 H'(\gamma(t),t) - \sigma(\alpha^*, H') + k\tau \geq 0$ for some loop γ in L. We now return to the choices of $H, v, u, \gamma_1', x_1, y_1$ used when establishing formula (96). We now define $H'(x,t) = -H(x,t)$. We notice that the the periodic orbits of H' are related to those of H by the formula $\gamma(t) \to \gamma(1-t)$ and, moreover, the complex $CF(H',J)$ is in fact identified with $s^n(CF(H,J))^{\odot}$ (to recall the algebraic notation $(-)^{\odot}$ etc see §2.2.1 and §5.7). The equation (96) can be interpreted by looking at u as a Floer half tube for H' parametrized by $[0,+\infty)\times S^1$ with the 0 end on L and the $+\infty$ end on $\tilde{\gamma}_1'$ (where for a loop γ, $\tilde{\gamma}$ is the loop $\gamma(1-t)$). We obtain:

$$0 \leq \int u^*\omega - \int_{S^1} H'(\tilde{\gamma}_1(t),t)dt + \int_{S^1} H'(u(0,-t),t)dt \ . \tag{97}$$

So that is suffices to show $k\tau - \sigma(\alpha^*, H') \geq \int u^*\omega - \int_{S^1} H'(\tilde{\gamma}_1(t),t)dt$. By noticing that the capping corresponding to $\tilde{\gamma}_1'$ is changed with respect to that of γ_1' by $(s,t)\to(s,1-t)$, it is now easy to see that:

$$\mathcal{A}_{H'}(\tilde{\gamma}_1') \leq \int_{S^1} H'(\tilde{\gamma}_1(t),t)dt - \int u^*\omega + k\tau \ .$$

And the proof of the desired inequality reduces to showing that, for any $\epsilon > 0$ there exists γ_1 as before so that $\sigma(\alpha^*, H') - \epsilon \leq \mathcal{A}_{H'}(\tilde{\gamma}_1')$. But this follows immediately from Remark 5.10.1. $\square$

5.11. **Replacing $\mathcal{C}^+$, Q^+H by $\mathcal{C}$, QH.** At this stage we indicate that everything proved in §5.1 – 5.5 as well as in §5.7, 5.8, 5.9 continues to hold (with the same proofs) if we replace $\mathcal{C}^+$ and Q^+H by $\mathcal{C}$ and QH respectively everywhere. The same is true for §5.10 if we replace $\mathcal{C}^+(L)$ by $\mathcal{C}(L)$ but in that section we still have to work with the full version of $HF(M)$.

Note however, that in contrast to the above, in §5.6 it is essential to work with $\mathcal{C}$ and QH rather than their positive versions.

6. Applications and examples.

Our applications are grouped in three categories. Ths first has to do with the algebraic constraints coming from the interplay between singular and quantum structures. There are many such examples in the paper. In §6.1 we see, for example, that when the singular homology of L is generated as an algebra (with the intersection product) by classes of sufficiently high degree (depending on the minimal Maslov number), then, either, the quantum homology of L is additively just singular homology with Novikov coefficients or it vanishes. Other examples come from the interplay of the topology of the ambient manifold and that of the Lagrangian. Thus, in §6.2, we see that if the ambient manifold is $\mathbb{C}P^n$, then the resulting restrictions on the Floer homology of the Lagrangian are stringent. As a concrete example we discuss the Clifford torus in detail in §6.2.1. Additional assumptions on the Lagrangian - for example, $2H_1(L;\mathbb{Z}) = 0$ as in §6.2.2 - lead to more homological rigidity as such Lagrangians are seen to have a homology algebra very much like that of $\mathbb{R}P^n$. We also discuss Lagrangian submanifolds of the quadric in §6.3, as well as in complete intersections in §6.4. In §6.5 we also describe some examples that go in the opposite direction: the existence of certain Lagrangian submanifolds is seen to imply restrictions on the quantum homology of the ambient manifold.

The second type of application is related to ways of measuring the size of Lagrangians as well as that of the space surrounding them inside the ambient manifold. Indeed, as introduced in [8] and in [24] there is a natural notion of Gromov width of a Lagrangian and obviously one can also consider the width of the complement of that Lagrangian. Moreover, one can also define packing numbers for the Lagrangian, its complement as well as mixed numbers. Our techniques allow us to give estimates - in §6.6 - for many of these numbers in all the cases mentioned above.

The third type of application - presented in §6.11 - is concerned with the fact that the properties of our machinery can be used to define certain numerical invariants roughly of Gromov-Witten type which are associated to configurations different from the usual ones. We discuss this construction in a very explicit way for two dimensional monotone tori - this turns out to be a surprisingly rich case. The invariants in question are associated to triangles lying on the tori and are expressed as polynomials involving numbers of J-holomorphic disks passing through the vertexes and/or the edges of the triangle.

There is an underlying unifying idea for all of these diverse applications. Lagrangian submanifolds exhibit considerable rigidity: topological (or algebraic) for our first class of applications, geometric for the second and arithmetic for the third.

6.1. **Full Floer homology.**

Proposition 6.1.1. *Let $L^n \subset (M^{2n}, \omega)$ be a monotone Lagrangian with $N_L \geq 2$. Assume that its singular homology $H_*(M; \mathbb{Z}_2)$ is generated as an algebra by $H_{\geq n-k}(L; \mathbb{Z}_2)$. Suppose further that $N_L > k$. Then:*

1. *either $QH_*(L) = 0$; or*
2. *there exist isomorphisms of graded vector spaces $QH_*(L) \cong (H(L; \mathbb{Z}_2) \otimes \Lambda)_*$ and $Q^+H_*(L) \cong (H(L; \mathbb{Z}_2) \otimes \Lambda^+)_*$. These isomorphisms are in general not canonical. Moreover, these isomorphisms, in general, do not respect the ring structures.*

If $N_L > k+1$ only the second alternative occurs. Furthermore, when the second alternative occurs (whether $N_L > k+1$ or $N_L = k+1$) there exist canonical injections of $H_{\geq n-k}(L; \mathbb{Z}_2)$ into $QH_{\geq n-k}(L)$ and into $Q^+H_{\geq n-k}(L)$ which generate these algebras over Λ and over Λ^+ with respect to the quantum product.

Example 6.1.2. Tori with minimal Maslov class at least 2 furnish a nice example. Another immediate example is $\mathbb{R}P^n \subset \mathbb{C}P^n$. Other examples will appear later in this section.

Proof of Proposition 6.1.1. We will provide two proofs for this proposition. The first one is based on a spectral sequence argument while the second uses the minimal model machinery described in §2.2.3. We include both arguments precisely to illustrate the role of these minimal models.

A. We will use here an argument involving spectral sequences. Choose a generic $J \in \mathcal{J}(M, \omega)$. Let $f : L \to \mathbb{R}$ be a Morse function with exactly one maximum x_n and fix a generic Riemannian metric on L. Denote by $(CM_*(f), \partial_0)$, $(\mathcal{C}_*(f, J), d)$ the Morse and pearl complexes associated to f, J and the Riemannian metric.

Recall that $\mathcal{C}(f, J)$ is filtered by the degree filtration $\mathcal{F}^p\mathcal{C}$ (which is a decreasing filtration). It will be more convenient to work here with an increasing version of the same filtration. Put

$$\mathcal{F}_p\mathcal{C}_i(f, J) = \bigoplus_{j \geq -p} \mathcal{C}_{i+jN_L}(f, J)t^j.$$

Clearly this is a bounded increasing filtration. It gives rise to a spectral sequence $\{E^r_{p,q}, d_r\}_{r\geq 0}$ which converges to $QH_*(L) = H_*(\mathcal{C}(f, J), d)$. A simple computation shows that:

(1) $E^0_{p,q} = CM_{p+q-pN_L}t^{-p}$, $d_0 = \partial_0$.
(2) $E^1_{p,q} = H_{p+q-pN_L}(L; \mathbb{Z}_2)t^{-p}$.
(3) The sequence collapses after a finite number of steps. In fact this number of steps is $\leq [\frac{n+1}{N_L}] + 1$.

Let $f' : L \to \mathbb{R}$ be a small perturbation of f. Note that the filtration $\mathcal{F}_p$ is compatible with the quantum product $*$ in the sense that

$$* : \mathcal{F}_{p_1}\mathcal{C}_{i_1}(f, J) \otimes \mathcal{F}_{p_2}\mathcal{C}_{i_2}(f', J) \longrightarrow \mathcal{F}_{p_1+p_2}\mathcal{C}_{i_1+i_2-n}(f, J).$$

By taking f' close enough to f we may assume that the canonical quasi-isomorphism between $\mathcal{C}_*(f, J)$ and $\mathcal{C}_*(f', J)$ is in fact a base preserving isomorphism (see §5.1.2). Thus we may identify the spectral sequences $\{E^r_{p,q}(f, J), d_r\}$ and $\{E^r_{p,q}(f', J), d_r\}$ and denote both of them by $\{E^r_{p,q}, d_r\}$. It follows that this spectral sequence is multiplicative.

Denote by $*_r$ the product induced by $*$ on $\{E^r_{*,*}\}$ and by $*_\infty$ the product induced by $*$ on $\{E^\infty_{*,*}\}$. Note that although there exists an isomorphism $H_l(\mathcal{C}, d) \cong \oplus_{p+q=l} E^\infty_{p,q}$ for every $l \in \mathbb{Z}$, the products $*$ on $H_*(\mathcal{C}, d)$ and $*_\infty$ on $\oplus_{p+q=l} E^\infty_{p,q}$ are in general not isomorphic.

A direct computation shows that the product $*$ induces on the E^1 level the classical cap product, namely the product

$$*_1 : E^1_{p_1,q_1} \otimes E^1_{p_2,q_2} \to E^1_{p_1+p_2,q_1+q_2-n}$$

is the cap product

$$\cap : H_{p_1+q_1-p_1N_L}(L; \mathbb{Z}_2)t^{-p_1} \otimes H_{p_2+q_2-p_2N_L}(L; \mathbb{Z}_2)t^{-p_2} \longrightarrow H_{p_1+p_2+q_1+q_2-(p_1+p_2)N_L-n}(L; \mathbb{Z}_2)t^{-p_1-p_2}$$

and d_1 satisfies Leibniz rule with respect to $\cap$. It follows that $\{E^1_{p,q}\}_{p+q-pN_L \geq n-k}$ generate with respect to $*_1 = \cap$ the whole $\{E^1_{*,*}\}$.

Assume now that $N_L > k+1$. For degree reasons d_1 vanishes on $E^1_{p,q}$ whenever $p + q - pN_L \geq n-k$. Since d_1 satisfies Leibniz rule with respect to $*_1$ it follows that $d_1 = 0$ everywhere. Therefore $E^2_{*,*} = E^1_{*,*}$ and $*_2 = *_1 = \cap$. The same argument applied to d_2 shows that $d_2 = 0$, hence $E^3_{*,*} = E^2_{*,*} = E^1_{*,*}$. Proceeding by induction we obtain that $d_r = 0$ for every $r \geq 1$ hence $E^\infty_{*,*} = \cdots = E^1_{*,*}$ and $*_\infty = \cdots = *_1 = \cap$.

It follows that there exists an isomorphism

$$H_*(\mathcal{C}, d) \cong \bigoplus_{p+q=*} E^1_{p,q} = \bigoplus_{p\in\mathbb{Z}} H_{*-pN_L}(L;\mathbb{Z}_2)t^{-p} = \big(H(L;\mathbb{Z}_2)\otimes\Lambda\big)_*.$$

We now turn to the case $N_L = k+1$. First note that exactly as in the in the case $N_L > k+1$ we have that $d_1 = 0$ on all $E^1_{p,q}$ with $p+q-pN_L \geq n-k+1$. Let us examine now the behavior of d_1 on $E^1_{p,q}$ for $p+q-pN_L = n-k$. For such p, q we have

$$E^1_{p,q} = H_{n-k}(L;\mathbb{Z}_2)t^{-p}, \quad E^1_{p-1,q} = H_n(L;\mathbb{Z}_2)t^{-p+1}.$$

Moreover, it is easy to see that

$$d_1 : H_{n-k}(L;\mathbb{Z}_2)t^{-p} \to H_n(L;\mathbb{Z}_2)t^{-p+1}$$

takes the form $d_1 = \delta_1 t$ where $\delta_1 : H_{n-k}(L;\mathbb{Z}_2) \to H_n(L;\mathbb{Z}_2)$ does not depend on p. There are now two cases to consider:

Case I. $\delta_1 \neq 0$**.** Since $H_n(L;\mathbb{Z}_2) = \mathbb{Z}_2[L]$ is 1-dimensional (and we work here over $\mathbb{Z}_2$ which is a *field*) it follows that $[L]$ is in the image of δ_1. Therefore $[L] \in E^1_{0,n} = H_n(L;\mathbb{Z}_2)$ is the image under d_1 of some element in $E^1_{1,n} = H_{n-k}(L;\mathbb{Z}_2)t^{-1}$. It follows that the homology class of $[L]$ in $E^2_{0,n}$ is zero. But $[L]$ is the unit of $E^1_{*,*}$ with respect to $*_1 = \cap$ and so its homology class in $E^2_{0,n}$ is the unit of $E^2_{*,*}$ with respect to $*_2$. As this class is zero we have $E^2_{*,*} = 0$. It follows that $H_*(\mathcal{C}, d) = QH_*(L) = 0$.

Case II. $\delta_1 = 0$**.** In this case $d_1 = 0$ on $E^1_{p,q}$ for $p+q-pN_L = n-k$. The proof now continues exactly as in the case $N_L > k+1$ discussed above, showing that the spectral sequence degenerates at the E^1 level. It follows that $QH_*(L) = H_*(\mathcal{C}, d) \cong \big(H(L;\mathbb{Z}_2)\otimes\Lambda\big)_*$. This concludes the proof of the two alternatives for $QH(L)$.

We now prove that $H_{\geq n-k}(L;\mathbb{Z}_2)$ canonically injects into $QH_{\geq n-k}(L)$ under the assumption that either $N_L > k+1$, or $N_L = k+1$ and $QH(L) \neq 0$. To see this note first that we have a *canonical* homomorphism $\sigma : H_l(L;\mathbb{Z}_2) \to QH_l(L)$ induced by the inclusion $CM_l(f) \subset \mathcal{C}_l(f, J)$ for $l \geq n-k$. Indeed, assume first that $N_L > k+1$ and let $x \in CM_l(f)$, $l \geq n-k$, be a ∂_0-cycle. For degree reasons $dx = \partial_0 x = 0$. Similarly, if x is a ∂_0-boundary, say $x = \partial_0 y$, $y \in CM_{l+1}(f)$, then again by degree reasons

$x = dy$. This shows that the inclusion $CM_l(f) \subset \mathcal{C}_l(f, J)$ induces a homomorphism $\sigma : H_l(L; \mathbb{Z}_2) \to H_l(\mathcal{C}, d) = QH_l(L)$. Suppose now that $N_L = k + 1$ and $QH(L) \neq 0$. For degree reasons $d = \partial_0$ on $CM_l(f)$ for every $l \geq n - k + 1$. As for $CM_{n-k}(f)$ we can write $d = \partial_0 + \partial_1 t$, where $\partial_1 : CM_{n-k}(f) \to CM_n(f)$. It follows that ∂_1 vanishes on all ∂_0-cycles, for otherwise the maximum $x_n \in CM_n(f)$ would be a d-boundary implying that $QH(L) = 0$. It follows that every ∂_0-cycle $x \in CM_{n-k}(f)$ is also a d-cycle. For degree reasons every ∂_0-boundary in $CM_{n-k}(f)$ is also a d-boundary. Thus in this case too we have the homomorphism σ induced by the inclusion $CM_l(f) \subset \mathcal{C}_l(f, J)$, $l \geq n - k$.

That σ is canonical follows from the definition of the canonical identifications on QH described in §5.1.2. The point is that, for degree reasons, the chain morphism $\phi : \mathcal{C}_l(f, J) \to \mathcal{C}_l(f', J')$ defined in §5.1.2 coincides with the analogous chain morphism in Morse theory $CM_l(f) \to CM_l(f')$ for for $l \geq n - k$ (because ϕ involves only non-negative powers of t). This completes the proof that the homomorphism σ is well defined and canonical.

Next, we prove that $\sigma : H_l(L; \mathbb{Z}_2) \to QH_l(L)$ is injective for $l \geq n - k$. Again, we assume that either $N_L > k + 1$, or $N_L = k + 1$ and $QH(L) \neq 0$. To prove this first note that for degree reasons $CM_l(f) = \mathcal{F}_0\mathcal{C}_l(f, J)$ and in fact d coincides with ∂_0 here. Therefore this equality induces an isomorphism $\sigma' : H_l(L; \mathbb{Z}_2) \to H_l(\mathcal{F}_0\mathcal{C}, d)$. Denote by $\mathcal{F}_pH_l(\mathcal{C}, d)$ the p-level of the associated filtration on the homology $H_l(\mathcal{C}, d)$, i.e. the image of the map $\iota : H_l(\mathcal{F}_p\mathcal{C}, d) \to H_l(\mathcal{C}, d)$ induced from the inclusion. For degree reasons we have $\mathcal{F}_{-1}H_l(\mathcal{C}, d) = 0$, hence $E^\infty_{0,l} = \mathcal{F}_0H_l(\mathcal{C}, d)/\mathcal{F}_{-1}H_l(\mathcal{C}, d) = \mathcal{F}_0H_l(\mathcal{C}, d)$. On the other hand by what we have previously proved we know that $E^\infty_{0,l} = E^1_{0,l} = H_l(L; \mathbb{Z}_2)$. Putting all these together we obtain:

$$H_l(L; \mathbb{Z}_2) \xrightarrow[\cong]{\sigma'} H_l(\mathcal{F}_0\mathcal{C}, d) \xrightarrow[\text{surjective}]{\iota} \mathcal{F}_0H_l(\mathcal{C}, d) = E^\infty_{0,l} = H_l(L; \mathbb{Z}_2). \tag{98}$$

It follows that $\iota \circ \sigma'$ is surjective. By dimensions reasons $\iota \circ \sigma'$ is an isomorphism. But the image of $\iota \circ \sigma'$ is the same as the image of σ, hence σ is injective.

It remains to show that the image of σ in $QH_{\geq n-l}(L)$ generates $QH_*(L)$ with respect to the quantum product. Fix $i \in \mathbb{Z}$. Denote by $\mathcal{A}_* \subset H_*(\mathcal{C}, d)$ the subalgebra generated (over Λ) by $\sigma\big(H_{\geq n-k}(L; \mathbb{Z}_2)\big)$ with respect to the quantum product $*$. We want to prove that $\mathcal{A}_* = H_*(\mathcal{C}, d)$.

Recall that $H_i(\mathcal{C}, d)$ is filtered by the induced increasing filtration $\{\mathcal{F}_pH_i(\mathcal{C}, d)\}_{p\in\mathbb{Z}}$. This filtration is bounded so that $\mathcal{F}_pH_i(\mathcal{C}, d) = 0$ for every $p \leq p_0$ for some p_0, and $\mathcal{F}_pH_i(\mathcal{C}, d) = H_i(\mathcal{C}, d)$ for $p \gg 0$. Therefore it is enough to prove that

$$\mathcal{F}_pH_i(\mathcal{C}, d) \subset \mathcal{A}_i, \quad \forall\, p. \tag{99}$$

We will prove (99) by induction on p. Put

$$(100) \qquad \mathcal{G}_i = \bigoplus_{p \geq p_0} \mathcal{F}_p H_i(\mathcal{C}, d)/\mathcal{F}_{p-1} H_i(\mathcal{C}, d) = \bigoplus_{p+q=i} E^{\infty}_{p,q} = \bigoplus_{p \geq p_0} H_{i-pN_L}(L; \mathbb{Z}_2) t^{-p}.$$

By what we have proved the quantum product $*$ descends to a product $[*]$ on $\mathcal{G}_*$ which is identified with the classical cap product $\cap$ on the right-hand side of (100) (here we extend $\cap$ in an obvious way over Λ). By the assumption of the proposition $E^{\infty}_{0, \geq n-k}$ generates $\mathcal{G}_*$ (over Λ) with respect to $[*] = \cap$.

Obviously (99) holds for $p \leq p_0$ since $\mathcal{F}_p H_i(\mathcal{C}, d) = 0$ for such p's. Assume the statement is true for every $p \leq p'$. Let $x \in \mathcal{F}_{p'+1} H_i(\mathcal{C}, d)$. We want to prove that $x \in \mathcal{A}$. The corresponding element in (100), $[x] = x\big(\mathrm{mod} \mathcal{F}_{p'} H_i(\mathcal{C}, d)\big)$, can be identified with an element in $H_{i-p'+1N_L}(L; \mathbb{Z}_2) t^{-p'-1}$. By the assumption of the proposition $[x]$ can be expressed as a linear combination (over Λ) of $[*]$-products of elements in $E^{\infty}_{0, \geq n-k} = H_{\geq n-k}(L; \mathbb{Z}_2)$. Keeping in mind that $[*]$ is induced from $*$ this means that

$$x = a + x_{(p')}$$

for some elements $a \in \mathcal{A}$ and $x_{(p')} \in \mathcal{F}_{p'} H_i(\mathcal{C}, d)$. By the induction hypothesis $x_{(p')} \in \mathcal{A}$ hence $x \in \mathcal{A}$ too. The desired statement follows now by induction.

The analogous statements of the proposition for $Q^+H(L)$ are proved essentially in the same way. However, it is important to note that alternative 1 holds only for $QH(L)$ (in fact $Q^+H(L)$ cannot vanish). The reason for this lies in "**Case I.**" in the proof above where we had to use negative powers of t.

B. Here is now the second argument based on the minimal model machinery from §2.2.3. Consider the pearl complex $\mathcal{C}(f, J)$ and recall from §2.2.3 that there exists a chain complex $(\mathcal{C}_{min}, \delta)$, unique up to isomorphism, and chain morphisms $\phi : \mathcal{C}(f, J) \to \mathcal{C}_{min}$, $\psi : \mathcal{C}_{min} \to \mathcal{C}(f, J)$ so that $\phi \circ \psi = id$, $\mathcal{C}_{min} = H_*(L; \mathbb{Z}_2) \otimes \Lambda^+$, $\delta_0 = 0$ (where δ_0 is obtained from δ by putting $t = 0$) and ϕ (ϕ_0), ψ (ψ_0) induce isomorphisms in quantum (respectively, Morse) homology. By Remark 2.2.6 the quantum product in $\mathcal{C}(f, J)$ can be transported by the morphisms ϕ and ψ to a product $* : \mathcal{C}_{min} \otimes \mathcal{C}_{min} \to \mathcal{C}_{min}$ which is a chain map and a quantum deformation of the singular intersection product (notice though that, as the maps ϕ and ψ are not canonical, this product is not canonical either at the chain level).

We will now show by induction that either $QH(L) = 0$ or $\delta = 0$. Let $x \in H_{n-k+s}(L)$ with $s \geq 0$. We indentify $H_*(L; \mathbb{Z}_2)$ with the generators of $\mathcal{C}_{min}$ and we notice that $N_L > k$ implies for degree reasons that $\delta x = 0$ when $|x| > n-k$ and $\delta x = \epsilon_x [L] t$ when $|x| = n-k$, $N_L = k+1$ with $\epsilon_x \in \{0, \}$. If for some such x we have $\epsilon_x = 1$, then, by Remark 2.2.6, we deduce $QH(L) = 0$. Thus, we now assume $QH(L) \neq 0$ and we assume, by induction, that $\delta y = 0$ for all $y \in H_*(L; \mathbb{Z}_2)$ such that $|y| > n-k-s$, $s \geq 1$. Consider $x \in H_{n-k-s}(L; \mathbb{Z}_2)$ so that $x = x_1 \cdot x_2 \cdot \ldots \cdot x_r$ with $x_i \in H_{\geq n-k}(L; \mathbb{Z}_2)$. We then have $\delta(x_i) = 0$ and we write $\delta(x_1 * x_2 * \ldots * x_r) = \sum_i x_1 * \ldots \delta(x_i) * \ldots * x_r = 0$. At the same time

$$x_1 * x_2 * \ldots * x_r = x + \sum_j z_j t^j \tag{101}$$

with $z_j \in H_{>n-k-s}(L)$ so that, by the induction hypothesis, $\delta(z_j) = 0$. We conclude $\delta x = 0$. But given our assumption on the structure of the singular homology of L, this implies that $\delta = 0$. This means that $Q^+H(L) \cong H(L) \otimes \Lambda^+$. The equation (101) immediately implies the rest of the statement at (2). □

Remark 6.1.3. a. It is important to notice that the dichotomy that we have proved when $N_L = k + 1$ depends on the fact that we work over a field. This appears in "case **I**" in the proof above.

b. As shown by Cho [20], the Clifford torus has the property that for a certain choice of spin structure the associated Floer homology with rational coefficients vanishes. In [22, 21] there are some examples of non-monotone tori which satisfy similar dichotomy type properties (the argument used there is different from the one here).

c. The proof above extends in obvious ways to other examples of Lagrangians with particular singular homology.

d. Some partial results of the type above have also been obtained by Buhovsky in [16].

6.1.1. *Criteria for vanishing and non-vanishing of Floer homology.* Here is a related but slightly different, and possibly more explicit point of view on the same phenomenon from Proposition 6.1.1 (see also Remark 2.2.6). Let $L^n \subset (M^{2n}, \omega)$ be a monotone Lagrangian submanifold with $N_L \geq 2$. Denote by $H_2^D \subset H_2(M, L; \mathbb{Z})$ the image of the Hurewicz homomorphism $\pi_2(M, L) \to H_2(M, L; \mathbb{Z})$. Denote by $\partial : H_2(M, L; \mathbb{Z}) \to H_1(L; \mathbb{Z})$ the boundary homomorphism and by $\partial_{\mathbb{Z}_2} : H_2(M, L; \mathbb{Z}) \to H_1(L; \mathbb{Z}_2)$ the composition of ∂ with the reduction mod 2, $H_1(L; \mathbb{Z}) \to H_1(L; \mathbb{Z}_2)$. Given $A \in H_2^D$ and $J \in \mathcal{J}(M, \omega)$ consider the evaluation map

$$ev_{A,J} : (\mathcal{M}(A, J) \times \partial D)/G \longrightarrow L, \quad ev_{A,J}(u, p) = u(p),$$

where $G = \mathrm{Aut}(D) \cong PSL(2, \mathbb{R})$ is the group of biholomorphisms of the disk.

For every $J \in \mathcal{J}(M, \omega)$ let $\mathcal{E}_2(J)$ be the set of all classes $A \in H_2^D$ with $\mu(A) = 2$ for which there exist J-holomorphic disks with boundaries on L in the class A:

$$\mathcal{E}_2(J) = \{A \in H_2^D \mid \mu(A) = 2, \quad \mathcal{M}(A, J) \neq \emptyset\}.$$

Define:

$$\mathcal{E}_2 = \bigcap_{J \in \mathcal{J}(M,\omega)} \mathcal{E}_2(J).$$

Standard arguments show that:

(1) $\mathcal{E}_2(J)$ is a finite set for every J.
(2) There exists a second category subset $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M,\omega)$ such that for every $J \in \mathcal{J}_{\text{reg}}$, $\mathcal{E}_2(J) = \mathcal{E}_2$. In other words, for generic J, $\mathcal{E}_2(J)$ is independent of J.
(3) For every $J \in \mathcal{J}$ and every $A \in \mathcal{E}_2(J)$ the space $\mathcal{M}(A, J)$ is compact and all disks $u \in \mathcal{M}(A, J)$ are simple.
(4) For $J \in \mathcal{J}_{\text{reg}}$ and $A \in \mathcal{E}_2$, the space $(\mathcal{M}(A, J) \times \partial D)/G$ is a compact smooth manifold without boundary. Its dimension is $n = \dim L$. In particular, for generic $x \in L$, the number of J-holomorphic disks $u \in \mathcal{M}(A, J)$ with $u(\partial D) \ni x$ is finite.
(5) For every $A \in \mathcal{E}_2$ and $J_0, J_1 \in \mathcal{J}_{\text{reg}}$ the manifolds $(\mathcal{M}(A, J_0) \times \partial D)/G$ and $(\mathcal{M}(A, J_1) \times \partial D)/G$ are cobordant via a compact cobordism. Moreover, the evaluation maps ev_{A,J_0}, ev_{A,J_1} extend to this cobordism, hence $\deg_{\mathbb{Z}_2} ev_{A,J_0} = \deg_{\mathbb{Z}_2} ev_{A,J_1}$. In other words $\deg_{\mathbb{Z}_2} ev_{A,J}$ depends only on $A \in \mathcal{E}_2$.
(6) In fact, the set $\mathcal{J}_{\text{reg}}$ above can be taken to be the set of all $J \in \mathcal{J}(M,\omega)$ which are regular for all classes $A \in H_2^D$ in the sense that the linearization of the $\overline{\partial}_J$ operator is surjective at every $u \in \mathcal{M}(A, J)$.

Let $J \in \mathcal{J}_{\text{reg}}$ and let $x \in L$ be a generic point. Define a one dimensional $\mathbb{Z}_2$-cycle $\delta_x(J)$ to be the sum of the boundaries of all J-holomorphic disks with $\mu = 2$ whose boundaries pass through x. Of course, if a disk meets x along its boundary several times we take its boundary in the sum with appropriate multiplicity. Thus the precise definition is:

$$\delta_x(J) = \sum_{A \in \mathcal{E}_2} \sum_{(u,p) \in ev_{A,J}^{-1}(x)} u(\partial D). \tag{102}$$

By the preceding discussion the homology class $D_1 = [\delta_x(J)] \in H_1(L; \mathbb{Z}_2)$ is independent of J and x. In fact

$$D_1 = \sum_{A \in \mathcal{E}_2} (\deg_{\mathbb{Z}_2} ev_{A,J}) \partial_{\mathbb{Z}_2} A. \tag{103}$$

In view of the proof of Proposition 6.1.1 the next result follows easily.

Proposition 6.1.4. *Let $L \subset (M,\omega)$ be a monotone Lagrangian submanifold with $N_L \geq 2$.*

(1) If $D_1 \neq 0$ then $QH_*(L) = 0$.
(2) Suppose that $D_1 = 0$ and $H_*(L; \mathbb{Z}_2)$ is generated as an algebra by $H_{n-1}(L; \mathbb{Z}_2)$ with respect to the classical cap product. Then $QH_*(L) \cong (H(L; \mathbb{Z}_2) \otimes \Lambda)_*$. This

isomorphism is neither canonical nor multiplicative. However for $* \geq n-1$*, there is a canonical injection* $H_{\geq n-1}(L;\mathbb{Z}_2) \hookrightarrow QH_{\geq n-1}(L)$*.*

In particular, if $H_*(L;\mathbb{Z}_2)$ *is generated as an algebra by* $H_{n-1}(L;\mathbb{Z}_2)$ *then* $QH_*(L)$ *can be either* $(H(L;\mathbb{Z}_2)\otimes\Lambda)_*$ *or* 0 *according to whether* D_1 *vanishes or not. When* $D_1 = 0$ *all the above continues to hold for* $QH(L)$ *replaced by* $Q^+H(L)$ *and* Λ *replaced by* Λ^+*.*

Remark 6.1.5. When $D_1 = 0$ but $H_*(L;\mathbb{Z}_2)$ is not generated as an algebra by $H_{n-1}(L;\mathbb{Z}_2)$, the theorem does not say anything on $QH(L)$. In this case it is possible to define in a similar way higher classes $D_j \in H_j(L;\mathbb{Z}_2)$, $1 \leq j \leq N_L - 1$, which sometimes give more information.

Proof of Proposition 6.1.4. Choose a generic $J \in \mathcal{J}(M,\omega)$. Let $f : L \to \mathbb{R}$ be a generic Morse function with precisely one local minimum $x \in L$ and fix a generic Riemannian metric on L. Denote by $(CM_*(f), \partial_0)$, $(\mathcal{C}_*(f,J), d)$ the Morse and pearl complexes associated to f, J and the chosen Riemannian metric.

For degree reasons the restriction of d to $CM_{n-1}(f) \subset \mathcal{C}_{n-1}(f,J)$ is given by $d = \partial_0 + \partial_1 t$, where $\partial_1 : CM_{n-1}(f) \to CM_n(f) = \mathbb{Z}_2 x$ counts pearly trajectories with holomorphic disks of Maslov index 2 (of course, if $N_L > 2$ then $\partial_1 = 0$ and also $\delta_x(J) = 0$, $D_1 = 0$). Since x is a maximum of f no $-\mathrm{grad}(f)$ trajectories can enter x (i.e. $W^s_x = \{x\}$). Therefore for every $y \in \mathrm{Crit}_{n-1}(f)$ we have

$$\partial_1 y = \#_{\mathbb{Z}_2}\big(W^u_y \cap \delta_x(J)\big)x. \tag{104}$$

We prove statement 1. Suppose that $D_1 \neq 0$. By Poincaré duality there exists an $(n-1)$-dimensional cycle C in L such that

$$\#_{\mathbb{Z}_2} C \cap \delta_x(J) \neq 0.$$

Let $z \in CM_{n-1}(f)$ be a ∂_0-cycle representing $[C] \in H_{n-1}(L;\mathbb{Z}_2)$. Then

$$d(z) = \partial_1(z)\otimes t = \#_{\mathbb{Z}_2}\big(W^u_z \cap \delta_x(J)\big)x \otimes t = \#_{\mathbb{Z}_2}\big(C \cap \delta_x(J)\big)x \otimes t = ax \otimes t$$

for some non-zero scalar a. (Of course, $a \neq 0$ is the same as $a = 1$ here, since we work over $\mathbb{Z}_2$. However we wrote ax to emphasize that the argument works well over every field.) It follows that $[x] = 0 \in QH_n(L)$. But $[x]$ is the unity of $QH_*(L)$, hence $QH_*(L) = 0$.

We prove statement 2.We will use here an argument involving spectral sequences, similar to the proof of Proposition 6.1.1. Recall from the proof of Proposition 6.1.1 that the degree filtration gives rise to a spectral sequence $\{E^r_{p,q}, d_r\}_{p,q}$ that converges to $QH_*(L)$ and $E^1_{p,q} = H_{p+q-pN_L}(L;\mathbb{Z}_2)t^{-p}$. A simple computation shows that the differential d_1 is induced from the operator ∂_1 mentioned earlier in the proof. Moreover, as explained in the proof of Proposition 6.1.1, the quantum product $*$ endows $\{E^r_{p,q}, d_r\}_{r\geq 1}$ with a multiplicative structure which coincides for $r = 1$ with the classical cap product $\cap$. In particular d_1 satisfies Leibniz rule with respect to $\cap$, and the d_r, $r \geq 2$, satisfy too Leibniz rule with respect to the products induced on E^r. Since $H_*(L;\mathbb{Z}_2)$ is generated by $H_{n-1}(L;\mathbb{Z}_2)$ we conclude that $\{E_1^{p,q}\}_{p+q-pN_L=n-1}$ generate with respect to the cap product the whole of $E^1_{*,*}$. As $D_1 = 0$ we obtain from formula (104) that d_1 vanishes on $E^1_{p,q}$ whenever $p + q - pN_L = n - 1$. It follows that $d_1 = 0$ on all $E_1^{p,q}$.

The above implies that $E^2_{p,q} = E^1_{p,q}$ and the product induced on E^2 is still the cap product. In particular $E^2_{*,*}$ is generated with respect to $\cap$ by $\{E^2_{p,q}\}_{p+q-pN_L=n-1}$. For degree reasons d_2 vanishes on $\{E^2_{p,q}\}_{p+q-pN_L=n-1}$. As d_2 satisfies Leibniz rule too it follows that $d_2 = 0$ everywhere. Thus $E^3_{*,*} = E^2_{*,*} = E^1_{*,*}$. The same argument shows that $d_r = 0$ for every $r \geq 2$ hence $E^r_{*,*} = E^1_{*,*}$. In other words the spectral sequence collapses at level $r = 1$. Since this spectral sequence converges to $H_*(\mathcal{C}(f, J), d) = QH_*(L)$, we conclude that $QH_*(L) \cong (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$. □

Let us turn to some examples. First of all, if $N_L \geq 3$ then $\mathcal{E}_2 = \emptyset$ hence $D_1 = 0$. Therefore if $H_*(L;\mathbb{Z}_2)$ is generated as an algebra by $H_{n-1}(L;\mathbb{Z}_2)$ we must have $QH_*(L) \cong (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$. An example of such a Lagrangian is $\mathbb{R}P^n \subset \mathbb{C}P^n$, $n \geq 2$.

Example 6.1.6. Let $\mathbb{T}_{\text{clif}} = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid |z_0| = \cdots = |z_n|\}$ be the n-dimensional Clifford torus. This is a monotone Lagrangian torus with $N_L = 2$. The Floer homology of $\mathbb{T}_{\text{clif}}$ was computed by Cho [20] by a direct computation of the Floer complex. Below we will review this computation from the perspective of Proposition 6.1.4.

A simple computation shows that $H_2^D \cong \pi_2(\mathbb{C}P^n, \mathbb{T}_{\text{clif}}) \cong \mathbb{Z}A_0 \oplus \cdots \oplus \mathbb{Z}A_n$, where A_i, is represented by the map $v_i : (D, \partial D) \to (\mathbb{C}P^n, \mathbb{T}_{\text{clif}})$ given by $v_i(z) = [1 : \cdots : z : \cdots : 1]$ (here the z stands in the i'th entry). A straightforward computation shows that $\mu(A_i) = 2$ for every i.

Let J_0 be the standard complex structure of $\mathbb{C}P^n$. We will use the following facts proved by Cho [20]:

(1) $\mathcal{E}_2(J_0) = \{A_0, \dots, A_n\}$.
(2) J_0 is regular for each of the classes A_i.
(3) $ev_{A_i,J_0} : (\mathcal{M}(A_i, J_0) \times \partial D)/G \to \mathbb{T}_{\text{clif}}$ is a diffeomorphism, hence $\deg_{\mathbb{Z}_2} ev_{A_i,J_0} = 1$. In fact, given a point $\xi = [\xi_0 : \cdots : \xi_n] \in \mathbb{T}_{\text{clif}}$, the unique disk (up to reparametrization) $u : (D, \partial D) \to (\mathbb{C}P^n, \mathbb{T}_{\text{clif}})$ in the class A_i with $u(\partial D) \ni \xi$ is given by $u(z) = [\xi_0 : \cdots : \xi_{i-1} : \xi_i z : \xi_{i+1} : \cdots : \xi_n]$.

It follows from the discussion in §6.1.1 that $\mathcal{E}_2 = \{A_0, \dots, A_n\}$ and that for every $J \in \mathcal{J}_{\text{reg}}$, $\deg_{\mathbb{Z}_2} ev_{A_i,J} = 1$. A simple computation shows that $\partial A_0 + \cdots + \partial A_n = 0 \in H_1(L;\mathbb{Z})$ hence we have:

$$D_1 = \sum_{i=0}^{n} (\deg_{\mathbb{Z}_2} ev_{A_i,J}) \partial_{\mathbb{Z}_2} A_i = 0.$$

By Proposition 6.1.4 we have

$$QH_*(\mathbb{T}_{\text{clif}}) \cong (H(\mathbb{T}_{\text{clif}}; \mathbb{Z}_2) \otimes \Lambda)_*, \quad Q^+H_*(\mathbb{T}_{\text{clif}}) \cong (H(\mathbb{T}_{\text{clif}}; \mathbb{Z}_2) \otimes \Lambda^+)_*, \tag{105}$$

and there are canonical injections

$$H_{n-1}(\mathbb{T}_{\text{clif}}; \mathbb{Z}_2) \hookrightarrow QH_{n-1}(\mathbb{T}_{\text{clif}}), \quad H_n(\mathbb{T}_{\text{clif}}; \mathbb{Z}_2) \hookrightarrow QH_n(\mathbb{T}_{\text{clif}}) \tag{106}$$

and similarly for $Q^+H(\mathbb{T}_{\text{clif}})$.

6.2. **Lagrangian submanifolds of $\mathbb{C}P^n$.** Endow $\mathbb{C}P^n$ with the standard Kähler symplectic structure ω_{FS}, normalized so that $\int_{\mathbb{C}P^1} \omega_{\text{FS}} = \pi$. Let $L \subset \mathbb{C}P^n$ be a monotone Lagrangian submanifold with minimal Maslov number $N_L \geq 2$. Below we will carry out computations involving the quantum homology $QH(\mathbb{C}P^n)$ and the Floer homology $QH(L)$. We will work with the following (simplified) version of the Novikov ring $\Lambda = \mathbb{Z}_2[t, t^{-1}]$ where $\deg t = -N_L$. Put $QH(\mathbb{C}P^n) = H(\mathbb{C}P^n; \mathbb{Z}_2) \otimes \Lambda$ with the grading induced from both factors. Denote by $h \in H_{2n-2}(\mathbb{C}P^n; \mathbb{Z}_2)$ the class of the hyperplane, and by $u \in H_{2n}(\mathbb{C}P^n; \mathbb{Z}_2)$ the fundamental class. It is well known that (see e.g. [44]):

$$h^{*j} = \begin{cases} h^{\cap j}, & 0 \leq j \leq n \\ u \otimes t^{\frac{2(n+1)}{N_L}}, & j = n+1 \end{cases} \tag{107}$$

Note that our choice of grading is somewhat different than the convention usually taken in quantum homology theory. For example, if $N_L = n+1$ then $\deg t = -(n+1)$ and we get from (107) that $h^{*(n+1)} = u \otimes t^2$ (*not* $u \otimes t$!). Usually in the theory of quantum homology the degree of t is taken to be $-2N_M$ where N_M is the minimal Chern number of (M, ω). Here we have defined $\deg t = -N_L$ in order to keep compatibility with the Novikov ring used for Floer homology. Note however that we have $N_L | 2N_M$ thus our ring Λ_* is obtained from the "conventional" Novikov ring by a variable change.

It follows from (107) that h is an invertible element. Therefore by Theorem 2.1.1 we have:

Corollary 6.2.1. *Let $L \subset \mathbb{C}P^n$ be a monotone Lagrangian with $N_L \geq 2$. Then $QH_*(L)$ is 2-periodic, i.e. $QH_i(L) \cong QH_{i-2}(L)$ for every $i \in \mathbb{Z}$. In fact the homomorphism $QH_i(L) \to QH_{i-2}(L)$ given by $\alpha \mapsto h * \alpha$ is an isomorphism for every $i \in \mathbb{Z}$.*

Remark 6.2.2. (1) The first part of Theorem 6.2.1 was proved before by Seidel using the theory of graded Lagrangian submanifolds [62]. The 2-periodicity in [62] follows from the fact that $\mathbb{C}P^n$ admits a Hamiltonian circle action which induces a shift by 2 on graded Lagrangian submanifolds. Note that this is compatible with our perspective since that S^1-action gives rise to an invertible element in $QH(\mathbb{C}P^n)$ (the Seidel element [61, 44]) whose degree is exactly $2n$ minus the shift induced by the S^1-action. In our case the Seidel element turns out to be h.

(2) Let $\widetilde{\Lambda} = \mathbb{Z}_2[t]]$ be the ring of formal Laurent series with finitely many negative terms, i.e. elements of $\widetilde{\Lambda}$ are of the form $p(t) = \sum_{i=N}^{\infty} a_i t^i$, $N \in \mathbb{Z}$. Note that $\widetilde{\Lambda}$ is a field. If we define QH with coefficients in $\widetilde{\Lambda}$ then $QH(\mathbb{C}P^n; \widetilde{\Lambda})$ is isomorphic to the ring $\widetilde{\Lambda}[x]/\{x^{n+1} = t\}$. It is easy to see that this ring is in fact a field. Thus if we define $QH(L; \widetilde{\Lambda})$ to be QH with coefficients in $\widetilde{\Lambda}$ we obtain for every $0 \neq \alpha \in QH(L; \widetilde{\Lambda})$ an *injective* homomorphism $QH(\mathbb{C}P^n; \widetilde{\Lambda}) \hookrightarrow QH(L; \widetilde{\Lambda})$, defined by $a \mapsto a * \alpha$.

6.2.1. *The Clifford torus.* We now consider the 2-dimensional Clifford torus $\mathbb{T}^2_{\text{clif}} \subset \mathbb{C}P^2$ and compute all our structures in this case. We denote $\Lambda = \mathbb{Z}_2[t, t^{-1}]$, $\Lambda^+ = \mathbb{Z}_2[t]$ where $deg(t) = -2$. We denote by $h \in H_2(\mathbb{C}P^2; \mathbb{Z}_2)$ the generator. Recall from (105) that $QH_*(\mathbb{T}^2_{\text{clif}}) \cong (H(\mathbb{T}^2_{\text{clif}}; \mathbb{Z}_2) \otimes \Lambda)_*$. In particular:

$$QH_0(\mathbb{T}^2_{\text{clif}}) \cong H_0(\mathbb{T}^2_{\text{clif}}; \mathbb{Z}_2) \oplus H_2(\mathbb{T}^2_{\text{clif}}; \mathbb{Z}_2)t, \tag{108}$$

$$QH_1(\mathbb{T}_{\text{clif}}) \cong H_1(\mathbb{T}_{\text{clif}}; \mathbb{Z}_2). \tag{109}$$

Recall from (106) that the isomorphism in (109) is canonical and the second summand in (108) is canonical too. (Note however that the first summand in (108) is *not* canonical. See §6.11 for more details on that).

Proposition 6.2.3. *Let $w \in H_2(\mathbb{T}^2_{\text{clif}}; \mathbb{Z}_2)$ be the fundamental class. There are generators $a, b \in H_1(\mathbb{T}^2_{\text{clif}}, \mathbb{Z}_2)$, and $m \in QH_0(\mathbb{T}^2_{\text{clif}}) \cong (H_*(\mathbb{T}^2_{\text{clif}}; \mathbb{Z}_2) \otimes \Lambda)_0$ which together with w generate $QH(\mathbb{T}^2_{\text{clif}})$ as a Λ-module and verify the following relations:*

i. $a * b = m + wt$, $b * a = m$, $a * a = b * b = wt$, $m * m = mt + wt^2$.
ii. $h * a = at$, $h * b = bt$, $h * w = wt$, $h * m = mt$.
iii. $i_L(m) = [pt] + ht + [\mathbb{C}P^2]t^2$, $i_L(a) = i_L(b) = i_L(w) = 0$.

All the above continues to hold for the positive version of QH, namely with Λ replaced by Λ^+ and $QH(\mathbb{T}^2_{\text{clif}})$ replaced by $Q^+H(\mathbb{T}^2_{\text{clif}})$.

Remarks. (1) As the formulae in i clearly show, the Lagrangian quantum product is *not commutative* (even when working over $\mathbb{Z}_2$).

(2) Point i of Proposition 6.2.3 has been obtained before by Cho [21] by a different approach. From the perspective of that paper the Clifford torus is a special case of a torus which appears as a fibre of the moment map defined on a toric variety. See also [22] for related results in this direction.

Proof of Proposition 6.2.3. We will use the following two geometric properties of the Clifford torus. The first is that, through each point of the Clifford torus, there are three different pseudo-holomorphic disks of Maslov index two. They belong to three families that we denote by γ_1, γ_2 and γ_3. Up to a possible change of basis, we may assume that the homotopy class of the boundaries of the elements in γ_1 is a, for γ_2 the same class is b and for γ_3 this class is $-a-b$. See figure 15. The second geometric fact is that there is a

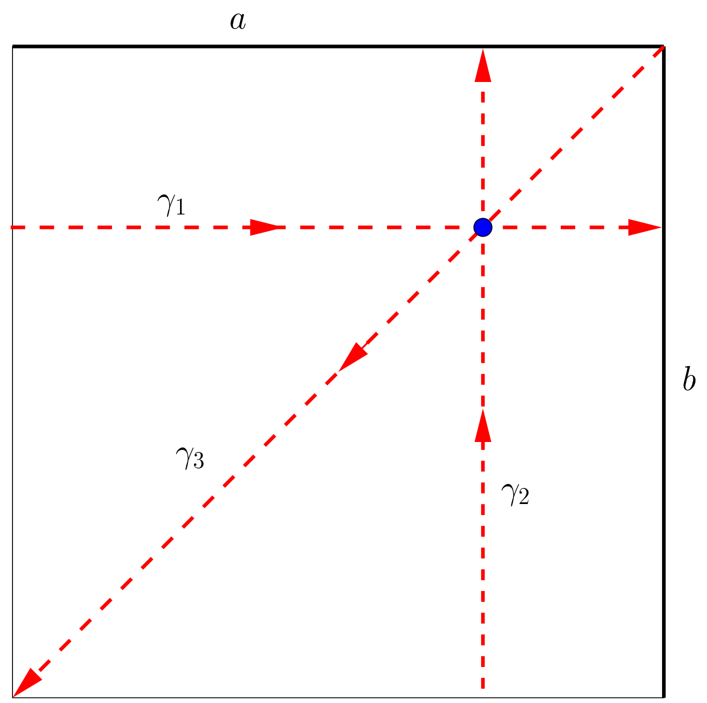

FIGURE 15. The boundaries of the 3 holomorphic disks with $\mu = 2$ through every point on $\mathbb{T}^2_{\text{clif}}$

symplectomorphism homotopic to the identity, $\bar{\phi} : \mathbb{C}P^2 \to \mathbb{C}P^2$, whose restriction to $\mathbb{T}^2_{\text{clif}}$ is the permutation of the two factors in $\mathbb{T}^2_{\text{clif}} \approx S^1 \times S^1$. We now consider a perfect Morse function $f : \mathbb{T}^2_{\text{clif}} \to \mathbb{R}$ and, by a slight abuse in notation, we let its minimum be m, we let the maximum be w and we let the two critical points of index 1 be denoted by a' and b' so that the unstable manifold of a' has the homotopy type $a \in H_1(\mathbb{T}^2_{\text{clif}}; \mathbb{Z}_2)$ and, similarly, the unstable manifold of the critical point b' has homotopy type b. We denote the disk in the family γ_i that passes through w by d_i. See figure 16. By possibly perturbing the function f slightly we may assume that the unstable manifold of a' intersects d_2 and d_3 in a single point and is disjoint from d_1. Similarly, we may assume that the unstable manifold of b' intersect d_1 and d_3 in a single point and is disjoint from d_2. With these choices the pearl complex $(\mathcal{C}(f, J, \rho), d)$ is well defined. Here we take J to be the standard complex

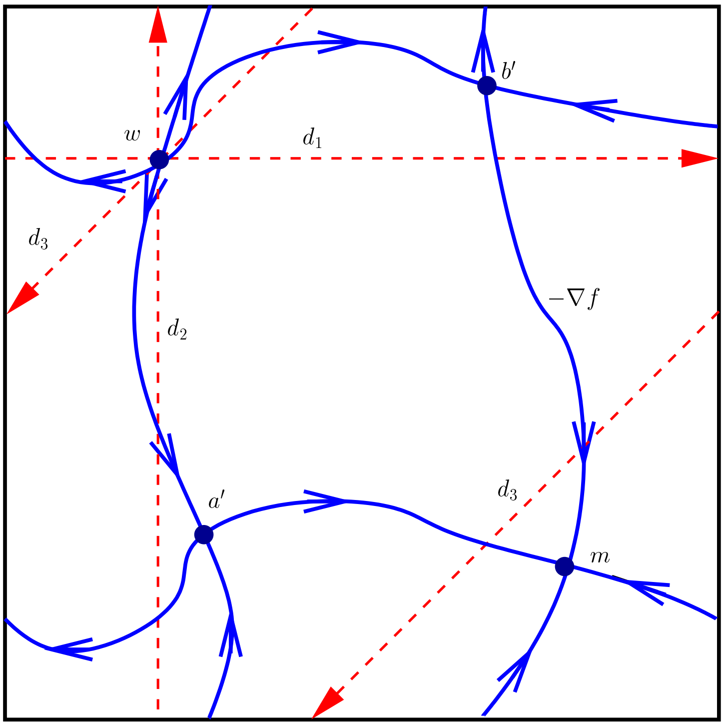

FIGURE 16. Trajectories of $-\nabla f$ and holomorphic disks on $\mathbb{T}^2_{\text{clif}}$.

structure of $\mathbb{C}P^2$ or a small perturbation of it and ρ a generic Riemannian metric on $\mathbb{T}^2_{\text{clif}}$. We claim that the differential d of the pearl complex vanishes. Indeed for dimension reasons we can write $d = \partial_0 + \partial_1 t$, where ∂_0 is the Morse differential and $\partial_1 : \mathcal{C}_* \to \mathcal{C}_{*+1}$ is an operator that counts the contribution of the pearly trajectories involving J-holomorphic disks with Maslov index 2. Since f is perfect $\partial_0 = 0$ hence $d = \partial_1 t$. As we have already seen before $QH_*(\mathbb{T}^2_{\text{clif}}) \cong (H(\mathbb{T}^2_{\text{clif}}; \mathbb{Z}_2) \otimes \Lambda)_*$. It follows that $\partial_1 = 0$ too since otherwise we would have $\dim_{\mathbb{Z}_2} QH_i(\mathbb{T}^2_{\text{clif}}) < \dim_{\mathbb{Z}_2}(H(\mathbb{T}^2_{\text{clif}}; \mathbb{Z}_2) \otimes \Lambda)_i$ for some i, a contradiction. This proves that $d = 0$.

It is instructive to give a more direct proof of the fact that $d = 0$ based on the specific knowledge of the $\mu = 2$ – holomorphic disks. For this purpose we first note that $da' = 0 = db'$. This is because the only two possibilities for da' are $da' = 0$ and $da' = wt$ and, as there are precisely two disks that go through w and intersect the unstable manifold of a' and each of them intersects it in exactly one point, we see that we are in the first case. The same argument applies to b'. A similar computation shows that $dm = 0$. Finally, $dw = 0$ for degree reasons.

Summarizing the above, $d = 0$ hence $QH_*(f, J, \rho) = H_*(\mathcal{C}(f, J, \rho), d) = \mathcal{C}_*(f, J, \rho)$. From now on we will view m, a', b', w as generators (over Λ) of $QH_*(\mathbb{T}^2_{\text{clif}})$. Note that m

depends on the choice of f in the sense that if we take another perfect Morse function g with minimum $\tilde{m}$ then $\tilde{m}$ might give an element of $QH_0(\mathbb{T}^2_{\text{clif}})$ which is different than m. On the other hand $a', b', w \in QH$ are canonical.

We now discuss the product. For degree reasons we have $a' * b' = m + \epsilon wt$, $b' * a' = m + \epsilon' wt$ with $\epsilon, \epsilon' \in \mathbb{Z}_2$. Of course, ϵ is the number modulo 2 of disks going - in order ! - through the following points: one point in the unstable manifold of a' then w and, finally one point in the unstable manifold of b'. Similarly, ϵ' is the number modulo 2 of disks going in order through a point in the unstable manifold of b', w and then a point in the unstable manifold of a'. There is a single disk through w which also intersects both the unstable manifolds of a' and b' - the disk d_3. However, the order in which the three types of points appear on the boundary of this disk implies that precisely one of ϵ and ϵ' is non-zero. Which one of the two is non-zero is, obviously, a matter of convention and we will take here $\epsilon \neq 0$. Notice that we also have $a * a = \delta wt$ with $\delta \in \{0, 1\}$. To estimate this product we need to use a second Morse function on $\mathbb{T}^2_{\text{clif}}$, $g : \mathbb{T}^2_{\text{clif}} to \mathbb{R}$. We will take this function to be perfect also and in such a way that the critical points of index one - denoted by a'' and b'' - have unstable and stable manifolds that are "parallel" copies of the respective stable and unstable manifolds of f. See figure 17. Now, there are precisely

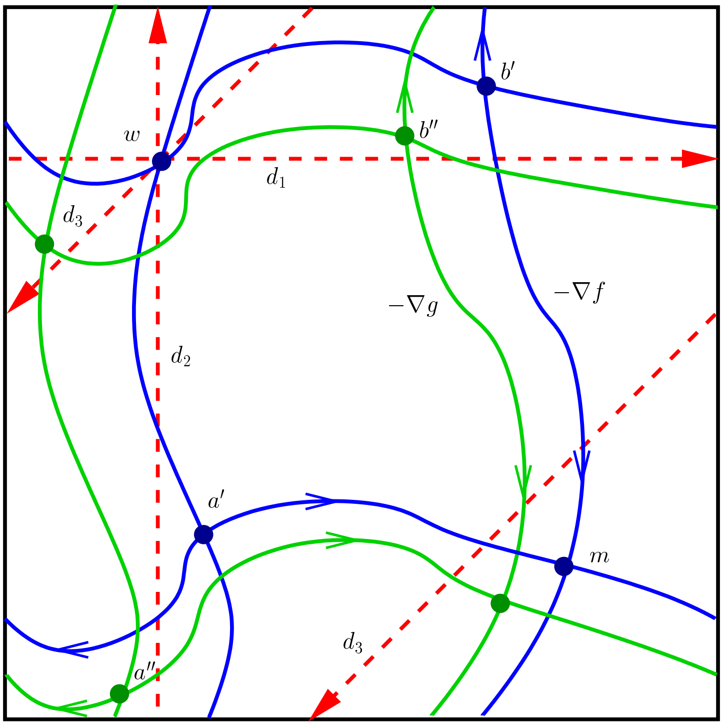

FIGURE 17. Trajectories of $-\nabla f$, $-\nabla g$ and holomorphic disks on $\mathbb{T}^2_{\text{clif}}$.

two pseudo-holomorphic disks that go through w as well as both the unstable manifolds of a' and a'': the disks d_2 and d_3. It is at this point that we use the fact that $[d_2] = b$, $[d_3] = -a - b$. Indeed, this means that the order in which these two disks pass through these three points is opposite. Thus, exactly one of these disks will contribute to δ and so $\delta = 1$. A similar argument shows $b * b = wt$. The formula for $m * m$ follows now from the associativity of the product. Indeed:

$$m * m = (a * b + wt) * (b * a) = a * (b * b) * a + b * at = mt + wt^2.$$

(Recall that we are working over $\mathbb{Z}_2$.)

We now determine what is the map

$$\tilde{\phi} : QH_*(\mathbb{T}^2_{\text{clif}}) \to QH_*(\mathbb{T}^2_{\text{clif}})$$

which is induced by $\bar{\phi}$. For degree reasons we have $\tilde{\phi}(w) = w$, $\tilde{\phi}(a) = b$, $\tilde{\phi}(b) = a$ and, by Corollary 2.2.3, we know that $\tilde{\phi}$ is a morphism of algebras over $QH(M)$ (from this also follows immediately that $\tilde{\phi}(m) = m + wt$).

Finally, we compute $h * a$ and $h * b$. We have, $h * a = h * \tilde{\phi}(b) = \tilde{\phi}(h * b)$. Now $h * a = (u_1 a + u_2 b)t$ with $u_1, u_2 \in \mathbb{Z}_2$ which implies that $h * b = (u_1 b + u_2 a)t$. As in Corollary 6.2.1 we also have that $h * (-) : H_1(\mathbb{T}^2_{\text{clif}}; \mathbb{Z}_2) \to H_1(\mathbb{T}^2_{\text{clif}}; \mathbb{Z}_2)t$ is an isomorphism. This implies that precisely one of u_1, u_2 is non zero. Assume that $u_1 = 0$ and $u_2 = 1$. Then $h * a = bt$, $h * (h * a) = at^2$ and $h * (h * (h * a)) = bt^3$ which is not possible because $h^{*3} = [\mathbb{C}P^2]t^3$ (where, $[\mathbb{C}P^2]$ denotes the fundamental class of $\mathbb{C}P^2$) and $[\mathbb{C}P^2] * a = a$. Thus we are left with $u_1 = 1$, $u_0 = 0$ as claimed. It is now easy to estimate $h * w$. Indeed, $h*wt = h*(a*a) = (h*a)*a = (a*a)t = wt^2$. Similarly $h*m = h*(b*a) = (h*b)*a = mt$. Finally, point iii is an immediate consequence of the first two points and of formula (4) from Theorem 2.1.1 in §2.

We turn to the proof for the positive quantum homology $Q^+H(\mathbb{T}^2_{\text{clif}})$. Recall that there is a canonical homomorphism $Q^+H(L) \to QH(L)$ induced from the inclusion $\mathcal{C}^+(f, J) \subset \mathcal{C}(f, J)$. But in our case ($L = \mathbb{T}^2_{\text{clif}}$) this homomorphism is actually an injection. This is because for the choices of f and J made above the differential d vanishes. As all the quantum operations (the product as well as the external product) involve only positive powers of t the statement on $Q^+H(\mathbb{T}^2_{\text{clif}})$ follows. □

Remark 6.2.4.

(1) It is easy to see that the augmentation in the case of the Clifford torus in $\mathbb{C}P^2$ is the obvious one. However, the duality map $\tilde{\eta} : Q^+H_*(L) \to Q^+H^*(L)$ (as defined in 2.2.1) verifies $\tilde{\eta}(m) = w^* + m^*t$, $\tilde{\eta}(a) = b^*$, $\tilde{\eta}(b) = a^*$ and $\tilde{\eta}(w) = m^*$. Here, by a slight abuse in notation, $Q^+H^*(L) = H_*((\mathcal{C}^+(L))^*)$.

(2) A careful inspection of the proof of Proposition 6.2.3 above shows that essentially the same argument can be applied for any monotone 2-dimensional Lagrangian

torus L, as long as we have enough information on the holomorphic disks of Maslov index 2 passing through a generic point in L. This will be further explored in §6.11.1 below.

6.2.2. *Lagrangians that look like $\mathbb{R}P^n \subset \mathbb{C}P^n$.* The real projective space $\mathbb{R}P^n$ viewed as a submanifold of $\mathbb{C}P^n$ is a monotone Lagrangian minimal Maslov number $= n+1$. Note that when $n \geq 2$, $H_1(\mathbb{R}P^n;\mathbb{Z}) = \mathbb{Z}_2$ hence $2H_1(\mathbb{R}P^n;\mathbb{Z}) = 0$. It turns out that the latter condition on its own imposes very strong restrictions on Lagrangians in $\mathbb{C}P^n$. As we will see below Lagrangians L with $2H_1(L;\mathbb{Z}) = 0$ have strong topological similarities to $\mathbb{R}P^n$. Moreover, as we will see in §6.2.3 their quantum structures are are almost determined by that condition. We start with topological restrictions.

Proposition 6.2.5. *Let $L \subset \mathbb{C}P^n$ be a Lagrangian submanifold with $2H_1(L;\mathbb{Z}) = 0$. Then:*

1. *$N_L = n+1$.*
2. *$H_i(L;\mathbb{Z}_2) \cong \mathbb{Z}_2$ for every $0 \leq i \leq n$, and $H_1(L;\mathbb{Z}) \cong \mathbb{Z}_2$.*
3. *There exists a canonical isomorphisms of graded vector spaces $QH_*(L) \cong (H(L;\mathbb{Z}_2)\otimes \Lambda)_*$. Hence by 1, 2 we have $QH_j(L) \cong \mathbb{Z}_2$ for every $j \in \mathbb{Z}$.*
4. *Let $\alpha_i \in H_i(L;\mathbb{Z}_2)$ be the generator. Then $\alpha_{n-2} \cap (-) : H_i(L:\mathbb{Z}_2) \to H_{i-2}(L;\mathbb{Z}_2)$ is an isomorphism for every $2 \leq i \leq n$. Thus we have $\alpha_i \cap \alpha_{n-2} = \alpha_{i-2}$ for every $2 \leq i \leq n$. Moreover $\alpha_{n-2} = h \cap_L [L]$, where $[L] = \alpha_n \in H_n(L;\mathbb{Z}_2)$ is the fundamental class and $\cap_L$ stands for the exterior cap product between elements of $H_*(\mathbb{C}P^n;\mathbb{Z}_2)$ and $H_*(L;\mathbb{Z}_2)$.*
5. *When $n =$ even, $\alpha_{n-1} \cap (-) : H_i(L;\mathbb{Z}_2) \to H_{i-1}(L;\mathbb{Z}_2)$ is an isomorphism for every $1 \leq i \leq n$. In particular $H_*(L;\mathbb{Z}_2)$ is generated by $\alpha_{n-1} \in H_{n-1}(L;\mathbb{Z}_2)$.*
6. *Let $\mathrm{inc}_* : H_i(L;\mathbb{Z}_2) \to H_i(\mathbb{C}P^n;\mathbb{Z}_2)$ be the homomorphism induced by the inclusion $L \subset \mathbb{C}P^n$. Then inc_* is an isomorphism for every $0 \leq i =$ even $\leq n$.*

Remark 6.2.6.
1. Proposition 6.2.5 has already been established in the past. Statements 2, 3 have been proved by Seidel [62] using the theory of graded Lagrangian submanifolds. An alternative approach which also proves statements 4, 5 has been given by Biran [15]. Below we give a different proof based on our theory.
2. Other than $\mathbb{R}P^n$ we are not aware of other Lagrangian submanifolds $L \subset \mathbb{C}P^n$ with $2H_1(L;\mathbb{Z}) = 0$. Note however that in $\mathbb{C}P^3$ there exists a Lagrangian submanifold L^3, not diffeomorphic to $\mathbb{R}P^3$, with $H_i(L;\mathbb{Z}_2) = \mathbb{Z}_2$ for every i. This Lagrangian is the quotient of $\mathbb{R}P^3$ by the dihedral group D_3. It has $H_1(L;\mathbb{Z}) \cong \mathbb{Z}_4$. This example is due to Chiang [18].

Proof of Proposition 6.2.5. Since $2H_1(L;\mathbb{Z}) = 0$ it is easy to see that $L \subset \mathbb{C}P^n$ is monotone. Moreover, a simple computation shows that the minimal Maslov number of L is $N_L = k(n+1)$ for some $k \geq 1$.

Let $f : L \to \mathbb{R}$ be a Morse function with exactly one local minimum x_0 and one local maximum x_n. Let $CM_*(f) = \mathbb{Z}_2\langle \mathrm{Crit}(f)\rangle$ be graded by Morse indices. Denote by $\mathcal{C}_* = (CM(f) \otimes \Lambda)_*$ the string of pearls complex. The differential $d : \mathcal{C}_* \to \mathcal{C}_{*-1}$ can be written as $d = \sum_{j\geq 0} \partial_j \otimes t^j$, where

$$\partial_j : CM_*(f) \longrightarrow CM_{*-1+jN_L}(f) \tag{110}$$

counts trajectories of pearls with total Maslov number $= jN_L$. Note that since L is monotone ∂_0 is just the Morse-homology differential.

We now prove statement 1. If $k \geq 2$ then $N_L = k(n+1) > n+1$ hence by formula (110) we have $\partial_j = 0$ for every $j \geq 1$ and we obtain:

$$QH_i(L) = \begin{cases} H_i(L;\mathbb{Z}_2), & 0 \leq i \leq n \\ 0, & n+1 \leq i \leq N_L - 1 \end{cases}$$

But this contradicts the 2-periodicity asserted by Corollary 6.2.1. Thus $k = 1$ and $N_L = n+1$. This proves statement 110. Note that this also implies that $H_1(L;\mathbb{Z}) \neq 0$ (for otherwise $N_L = 2(n+1)$). Since $2H_1(L;\mathbb{Z}) = 0$ we have $H_1(L;\mathbb{Z}_2) = H_1(L;\mathbb{Z}) \otimes \mathbb{Z}_2 \neq 0$. We will use this below.

We prove statements 2, 3. Consider the operator $\partial_1 : C_*(f) \to C_{*+n}(f)$. Clearly $\partial_1(x) = 0$ for every $x \in \mathrm{Crit}(f)$ with $|x| \neq 0$. Thus $d = \partial_0$ on $\mathcal{C}_j$ for every j that satisfies $j \not\equiv 0 (\mathrm{mod} n+1)$ and $j \not\equiv n(\mathrm{mod} n+1)$. It follows that

$$QH_{i+l(n+1)}(L) \cong H_i(L)t^{-l}, \quad \forall\, 0 < i < n, l \in \mathbb{Z}. \tag{111}$$

Next, consider the value of $\partial_1(x_0)$. There are two possibilities:

(i) $\partial_1(x_0) = x_n$.
(ii) $\partial_1(x_0) = 0$.

We claim that possibility i is impossible. Indeed by standard Morse theory $\partial_0(CM_1(f)) = 0$ and $\partial_0(CM_n(f)) = 0$, therefore if $\partial_1(x_0) = x_n$ then

$$d : \mathcal{C}_0 = \mathbb{Z}_2 x_0 \longrightarrow \mathcal{C}_{-1} = \mathbb{Z}_2 x_n t$$

is an isomorphism hence $QH_0(L) = 0$, $QH_{-1}(L) = 0$. By Corollary 6.2.1 we obtain $QH_j(L) = 0$ for every $j \in \mathbb{Z}$. On the other hand by (111) $QH_1(L) \cong H_1(L;\mathbb{Z}_2)$ and we have just seen that $H_1(L;\mathbb{Z}_2) \neq 0$. A contradiction. This proves that $\partial_1(x_0) = 0$. It follows that $d = \partial_0$ hence $QH_*(L) \cong (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$. In particular $QH_0(L) \cong \mathbb{Z}_2$ and $QH_{-1}(L) \cong QH_n(L) \cong \mathbb{Z}_2$. By Corollary 6.2.1 $QH_j(L) \cong \mathbb{Z}_2$ for every $j \in \mathbb{Z}$. We also conclude that $H_i(L;\mathbb{Z}_2) \cong QH_i(L) \cong \mathbb{Z}_2$ for every $0 \leq i \leq n$. Finally note that since $2H_1(L;\mathbb{Z}) = 0$ we have $H_1(L;\mathbb{Z}) \cong \mathbb{Z}_2^{\oplus r}$ for some $r \geq 0$, hence $H_1(L;\mathbb{Z}_2) = \mathbb{Z}_2^{\oplus r}$. But we have seen that $H_1(L;\mathbb{Z}_2) \cong \mathbb{Z}_2$ hence $r = 1$ and $H_1(L;\mathbb{Z}) \cong \mathbb{Z}_2$. This completes the proof of statements 2, 3.

To prove statement 4 recall formula (72) of §5.3 by which the quantum module operation $QH_l(M) \otimes QH_j(L) \to QH_{l+j-2n}(L)$ is defined. As $N_L = n+1$ and the degree of the hyperplane class h is $2n-2$, it follows from that formula that $h * \alpha = h \cap_L \alpha$ for every $\alpha \in QH_i(L) \cong H_i(L;\mathbb{Z}_2)$ for $2 \leq i \leq n$. Denote by $\alpha_i \in H_i(L;\mathbb{Z}_2)$ the generator. By Corollary 6.2.1 it follows that $h * \alpha_n = h \cap_L \alpha_n \neq 0$ hence $h * \alpha_n = \alpha_{n-2}$. Next, recall that $\alpha_n \in QH_n(L)$ is the unity hence

$$\alpha_{n-2} * \alpha_i = (h * \alpha_n) * \alpha_i = h * (\alpha_n * \alpha_i) = h * \alpha_i = \alpha_{i-2}, \quad \forall\, 2 \leq i \leq n.$$

But by formula (63) of §5.2 $\alpha_{n-2} * \alpha_i = \alpha_{n-2} \cap \alpha_i$ for every $2 \leq i \leq n$. This proves statement 4.

To prove statement 5 note that by Corollary 6.2.1

$$h^{*(\frac{n}{2}+1)} * \alpha_n = \alpha_{n-1} t.$$

Therefore

$$\alpha_{n-1} * \alpha_i = (h^{*(\frac{n}{2}+1)} * \alpha_n) * \alpha_i t^{-1} = (h^{*(\frac{n}{2}+1)} * \alpha_i) t^{-1} = \alpha_{i-1}, \quad \forall\, 1 \leq i \leq n,$$

where the last equality also follows from Corollary 6.2.1. But by looking at the Morse indices in formula (63) of §5.2 we conclude again that $\alpha_{n-1} * \alpha_i = \alpha_{n-1} \cap \alpha_i$ for every $1 \leq i \leq n$. Thus $\alpha_{n-1} \cap \alpha_i = \alpha_{i-1}$.

Finally, statement 6 will follow immediately from Proposition 6.2.8 below. We therefore postpone the proof. □

6.2.3. *Quantum structures.* Let $L \subset \mathbb{C}P^n$ be a Lagrangian with $2H_1(L;\mathbb{Z}) = 0$. In view of Proposition 6.2.5 denote by $\alpha_i \in QH_i(L)$ the generator for every $i \in \mathbb{Z}$. According to this notation we have $\alpha_{i+l(n+1)} = \alpha_i t^{-l}$ for every $i, l \in \mathbb{Z}$ and by Proposition 6.2.5, $h * \alpha_i = \alpha_{i-2}$ for every $i \in \mathbb{Z}$.

Proposition 6.2.7. *Let $L \subset \mathbb{C}P^n$ be as above. Let $k, j \in \mathbb{Z}$. If one of k, j is odd then:*

$$\alpha_k * \alpha_j = \alpha_{j+k-n}. \tag{112}$$

The same formula holds for every $k, j \in \mathbb{Z}$ (regardless of their parity) in each of the following cases:

1. *When $n =$ even.*
2. *When L is diffeomorphic to $\mathbb{R}P^n$.*
3. *More generally, when $\alpha_{n-1} \cap \alpha_{n-1} \neq 0$. (c.f. statement 5 of Theorem 6.2.5.)*

Proof. **Assume** $k =$ **odd.** By Corollary 6.2.1:

$$\begin{aligned}\alpha_k * \alpha_j &= (h^{*(-\frac{k+1}{2})} * \alpha_{-1}) * \alpha_j = (h^{*(-\frac{k+1}{2})} * \alpha_n T) * \alpha_j = (h^{*(-\frac{k+1}{2})} * \alpha_j)t \\ &= \alpha_{j+k+1} t = \alpha_{j+k-n}.\end{aligned}$$

Assume $j =$ **odd.** The proof is similar to the case $k =$ odd since for every $a \in QH(\mathbb{C}P^n)$ we have $a * (\alpha_k * \alpha_j) = \alpha_k * (a * \alpha_j)$.

Assume $n =$ **even.** We may assume that $k =$ even. Then we have $\alpha_k = h^{*\frac{n-k}{2}} * \alpha_n$ hence:

$$\alpha_k * \alpha_j = (h^{*\frac{n-k}{2}} * \alpha_n) * \alpha_j = h^{*\frac{n-k}{2}} * \alpha_j = \alpha_{j+k-n}.$$

Assume $n =$ **odd and** $\alpha_{n-1} \cap \alpha_{n-1} \neq 0$**.** In view of the above we may assume that k, j are both even. By the definition of the quantum product we have $\alpha_{n-1} * \alpha_{n-1} = \alpha_{n-1} \cap \alpha_{n-1} = \alpha_{n-2}$, where the last equality follows from the fact that $H_{n-2}(L; \mathbb{Z}_2) = \mathbb{Z}_2 \alpha_{n-2}$. Since $k+1$ and $j+1$ are both odd then by what we have proved above:

$$\alpha_k * \alpha_j = (\alpha_{k+1} * \alpha_{n-1}) * (\alpha_{n-1} * \alpha_{j+1}) = \alpha_{k+1} * \alpha_{n-2} * \alpha_{j+1} = \alpha_{k-1} * \alpha_{j+1} = \alpha_{k+j-n}.$$

□

The next result describes the quantum inclusion map $i_L : QH_*(L) \to QH_*(\mathbb{C}P^n)$. Denote by $a_j \in H_j(\mathbb{C}P^n; \mathbb{Z}_2)$ the generator, $0 \leq j \leq 2n$. Thus

$$a_j = \begin{cases} 0, & j = \text{odd} \\ h^{\cap(n-\frac{j}{2})}, & j = \text{even} \end{cases}$$

Proposition 6.2.8. *Let $L \subset \mathbb{C}P^n$ be as above.*

(1) If $n =$ even then:

$$\begin{aligned} i_L(\alpha_{2k}) &= a_{2k}, \quad \forall\, 0 \leq 2k \leq n, \\ i_L(\alpha_{2k+1}) &= a_{2k+n+2} t, \quad \forall\, 1 \leq 2k+1 \leq n-1. \end{aligned}$$

(2) If $n =$ odd then:

$$\begin{aligned} i_L(\alpha_{2k}) &= a_{2k} + a_{2k+n+1} t, \quad \forall\, 0 \leq 2k \leq n, \\ i_L(\alpha_{2k+1}) &= 0, \quad \forall\, k. \end{aligned}$$

Proof. By our notation $\alpha_0 = [\text{point}] \in H_0(L; \mathbb{Z}_2) \cong QH_0(L)$, $a_0 = [\text{point}] \in H_0(\mathbb{C}P^n; \mathbb{Z}_2)$. Recall also that $h^{*n} = a_0$ and $h^{*(n+1)} = ut^2$.

From the definition of i_L (see §5.4) it follows by a simple computation that:

$$i_L(\alpha_0) = a_0 + bt, \quad \text{for some } b \in H_{n+1}(\mathbb{C}P^n; \mathbb{Z}_2). \tag{113}$$

We claim that:

$$i_L(\alpha_{2k}) = a_{2k} + h^{*(-k)} * bt, \quad \forall\, 0 \leq 2k \leq n. \tag{114}$$

Indeed, by (113) we have:

$$\begin{aligned} i_L(\alpha_{2k}) &= i_L(h^{*(-k)} * \alpha_0) = h^{*(-k)} * i_L(\alpha_0) = h^{*(-k)} * (a_0 + bt) \\ &= h^{*(-k)} * (h^{*n} + bt) = h^{*(n-k)} + h^{*(-k)} * bt = a_{2k} + h^{*(-k)} * bt. \end{aligned}$$

Next we claim that :

$$i_L(\alpha_{n-1-2r}) = a_{2n-2r}t + h^{*(r+1)} * b, \quad \forall\, 0 \leq 2r \leq n-1. \tag{115}$$

Indeed by (113) we have:

$$\begin{aligned} i_L(\alpha_{n-1-2r}) &= i_L(h^{*(r+1)} * \alpha_{n+1}) = i_L(h^{*(r+1)} * \alpha_0 t^{-1}) = h^{*(r+1)} * (a_0 + bt)t^{-1} \\ &= (h^{*(r+1)} * h^{*n} + h^{*(r+1)} * bt)t^{-1} = (h^{*r} * ut^2 + h^{*(r+1)} * bt)t^{-1} \\ &= a_{2n-2r}t + h^{*(r+1)} * b. \end{aligned}$$

Suppose that n =even. Then $b = 0$ since $\deg b = n + 1$ =odd. Statement 1 of the theorem follows immediately from (114) and (115).

Assume now that n =odd. Comparing (114) to (115) with $2k = n-1$, $r = 0$, we obtain that $b = a_{n+1}$. The statement about $i_L(\alpha_{2k})$ follows now from (114). As for $i_L(\alpha_{2k+1})$, it is 0 since for n =odd, $QH_{2k+1}(\mathbb{C}P^n) = 0$ for every k.

Finally note that statement 6 of Proposition 6.2.5 follows from the above since in our case we have $\mathrm{inc}_* = i_{L|_{t=0}}$. □

6.2.4. *Existence of holomorphic disks satisfying constrains.* Denote by $\mathcal{J} = \mathcal{J}(\mathbb{C}P^n, \omega_{\mathrm{FS}})$ the space of ω_{FS}-compatible almost complex structures on $\mathbb{C}P^n$.

Proposition 6.2.9. *Let $L \subset \mathbb{C}P^n$ be a Lagrangian submanifold with $2H_1(L;\mathbb{Z}) = 0$. Assume that one of the following conditions is satisfied:*

(1) n = even.
(2) L is diffeomorphic to $\mathbb{R}P^n$.
(3) More generally, $\alpha_{n-1} \cap \alpha_{n-1} \neq 0$.

Let $x', x'' \in L$ two distinct points. Then for every $J \in \mathcal{J}$ there exists a J-holomorphic disk $u : (D, \partial D) \to (\mathbb{C}P^n, L)$ with $\mu([u]) = n+1$ and $u(\partial D) \ni x', x''$. For a generic choice of $J \in \mathcal{J}$ the number of such disks with $u(-1) = x'$, $u(1) = x''$, up to parametrizations fixing $-1, 1 \in D$, is ≥ 2 and even.

Remark 6.2.10. We will see in the proof that if $C^{n-1} \subset L$ is an $(n-1)$-dimensional $\mathbb{Z}_2$-cycle with $[C^{n-1}] = \alpha_{n-1}$ and such that $x', x'' \notin C^{n-1}$, then for generic $J \in \mathcal{J}$ there exist two J-holomorphic disks $u_1, u_2 : (D, \partial D) \to (\mathbb{C}P^n, L)$ with the following properties:

(i) $\mu([u_1]) = \mu([u_2]) = n+1$.
(ii) $u_1(-1) = x', \quad u_1(1) = x'', \quad u_1(i) \in C^{n-1}$.
(iii) $u_2(-1) = x', \quad u_2(1) = x'', \quad u_2(-i) \in C^{n-1}$.
(iv) u_2 is not obtained from u_1 by a reparametrization fixing $-1, 1 \in D$.

Moreover, the number of disks u_1 (resp. u_2) as above is odd.

Proof of Proposition 6.2.9. By Proposition 6.2.7 we have:

$$\alpha_0 * \alpha_{n-1} = \alpha_{-1} = \alpha_n t. \tag{116}$$

Let $f_1, f_2, f_3 : L \to \mathbb{R}$ be a generic triple of Morse functions with the same critical points (and the same indices at each critical point) and such that the f_i's have exactly one local minimum x' and one local maximum x''. Let ρ_1, ρ_2, ρ_3 be a generic triple of Riemannian metrics on L. Choose a generic $J \in \mathcal{J}$.

Denote by $CM_*(f_i)$ the Morse complex of f_i (with respect to ρ_i). Choose a cycle $y \in CM_{n-1}(f_3)$ that represents $\alpha_{n-1} \in H_{n-1}(L; \mathbb{Z}_2)$. Recall from the proof of Theorem 6.2.5 that the Floer differential coincides with the Morse differential thus $\alpha_0 = [x']$, $\alpha_n = [x'']$, $\alpha_{n-1} = [y]$ in Floer homology. It follows from (116) that $x' * y = x''t$. (Note that due to dimension we cannot have additional boundary terms on the right-hand side.) Therefore, in the notation of (63) (see §5.2) there exists a class $A \in H_2(\mathbb{C}P^n, L; \mathbb{Z})$ with $\mu(A) = n+1$ and a critical point y_0 participating in $y \in CM_{n-1}(f_3)$ such that $\mathcal{P}(x', y_0, x''; A, J) \neq \emptyset$. Let $(l_1, l_2, l_3, u) \in \mathcal{P}(x', y_0, x''; A, J)$. As x' is a minimum and x'' a maximum their unstable and stable manifolds are $W^u_{x'} = \{x'\}$, $W^s_{x''} = \{x''\}$ respectively. Moreover since $\mu(A) = n+1$ which is the minimal Maslov number it follows that the only possible configuration that (l_1, l_2, l_3, u) can take is the following (see figure 18):

- l_1 is the constant trajectory at x'.
- l_2 is a (negative) gradient trajectory (without J-holomorphic disks) emanating from y_0.
- l_3 is the constant trajectory at x''.
- The J-holomorphic disk u is not constant, hence it has $\mu([u]) = n+1$ and $u(e^{2\pi i/3}) = x'$, $u(1) \in l_2$, $u(e^{4\pi i/3}) = x''$.

We have proved that for generic $J \in \mathcal{J}$ there exists a J-holomorphic disk u with $\mu([u]) = n+1$ and $u(\partial D) \ni x'', x'$. As $N_L = n+1$ it follows from Gromov compactness theorem that there exists such a disk for every $J \in \mathcal{J}$.

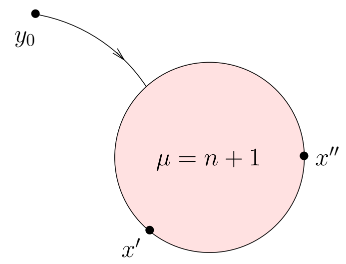

FIGURE 18. A J-holomorphic disk from $\mathcal{P}(x', y_0, x''; A, J)$.

Note that after a suitable reparametrization we may assume that $u(-1) = x'$, $u(1) = x''$. It remains to prove that for generic J the number of such disks is even. For this purpose we use the notation from the proof of Proposition 6.2.5. Recall the operator $\partial_1 : CM_*(f) \to CM_{*+n}(f)$. Since x' is a minimum and x'' a maximum, $\partial_1(x')$ counts the number (mod 2) of J-holomorphic disks u, up to parametrization, with $\mu([u]) = n + 1$ and $u(\partial D) \ni x'', x'$. As we have seen in the proof of Proposition 6.2.5, $\partial_1(x') = 0$, hence this number is even.

Finally, we prove the statement of Remark 6.2.10. First we claim that for every J-holomorphic disk u with $\mu[u] = n+1$ we have $[u(\partial D)] = \alpha_1 \in H_1(L; \mathbb{Z}_2)$. To see this note that $[u(\partial D)] \neq 0 \in H_1(L; \mathbb{Z})$ for otherwise $\mu([u]) = 2(n + 1)$. But by Proposition 6.2.5 $H_1(L; \mathbb{Z}) \cong \mathbb{Z}_2$. Thus $[u(\partial D)] \neq 0$ also in $H_1(L; \mathbb{Z}_2)$.

Consider the space $\mathcal{M}(x', x'')$ of J-holomorphic disks $u : (D, \partial D) \to (\mathbb{C}P^n, L)$ with $\mu([u]) = n+1$ and $u(-1) = x'$, $u(1) = x''$. The group $G_{-1,1} \subset \mathrm{Aut}(D)$ acts on this space. Denote by $\mathcal{P}(x', x'')$ its quotient. Note that for generic J, $\mathcal{P}(x', x'')$ is a finite set.

Denote by $\gamma_1, \gamma_2 \subset \partial D$ the arcs $\gamma_1 = \{e^{\pi i t}\}_{0 \leq t \leq \pi}$ and $\gamma_2 = \{e^{\pi i t}\}_{\pi \leq t \leq 2\pi}$. Let $y \in CM_{n-1}(f_3)$ be a cycle representing α_{n-1}. The union of the unstable submanifolds corresponding to the critical points in y is a pseudo cycle homologous to C^{n-1}. By choosing $J \in \mathcal{J}$ generic we may assume that C^{n-1} is in general position with respect to the evaluation map $\mathcal{M}(x', x'') \to L$ evaluating $\tilde{v} \in \mathcal{M}(x', x'')$ at a marked point $p_0 \in \partial D \setminus \{\pm 1\}$. For every $v \in \mathcal{P}(x', x'')$ put

$$n_1(v) = \#_{\mathbb{Z}_2}\big(\tilde{v}(\gamma_1) \cap C^{n-1}\big), \quad n_2(v) = \#_{\mathbb{Z}_2}\big(\tilde{v}(\gamma_2) \cap C^{n-1}\big),$$

where $\tilde{v} \in \mathcal{M}(x', x'')$ parametrizes v.

With this notation the coefficient of $x''t$ in $x' * y$ is $\sum_{v \in \mathcal{P}(x', x'')} n_1(v)$. Similarly, the coefficient of $x''t$ in $y * x'$ is $\sum_{v \in \mathcal{P}(x', x'')} n_2(v)$. But $x' * y = y * x' = x''t$ hence:

$$\sum_{v \in \mathcal{P}(x', x'')} n_1(v) = 1, \qquad \sum_{v \in \mathcal{P}(x', x'')} n_2(v) = 1.$$

Next note that for every $v \in \mathcal{P}(x', x'')$

$$n_1(v) + n_2(v) = \#_{\mathbb{Z}_2}\big(v(\partial D) \cap C^{n-1}\big) = \alpha_1 \cap \alpha_{n-1} = 1.$$

Let $u_1 \in \mathcal{P}(x', x'')$ with $n_1(u_1) = 1$. Then $n_2(u_1) = 0$ hence there exists $u_2 \neq u_1$ with $n_2(u_2) = 1$. After suitable reparametrizations of u_1, u_2 we obtain two disks with the properties claimed in Remark 6.2.10. $\square$

In case L does not satisfy one of the 3 conditions from Proposition 6.2.9 we still have the following weaker statement. (Note that this is theoretical, since we do not know Lagrangians in $\mathbb{C}P^n$ with $2H_1(L;\mathbb{Z}) = 0$ other than $\mathbb{R}P^n$.)

Proposition 6.2.11. *Let $L \subset \mathbb{C}P^n$ be a Lagrangian submanifold with $2H_1(L;\mathbb{Z}) = 0$. Then for every $x \in L$ and every $J \in \mathcal{J}$ there exists a J-holomorphic disk $u : (D, \partial D) \to (\mathbb{C}P^n, L)$ with $\mu([u]) = n+1$ and $u(\partial D) \ni x$.*

Proof. The proof is similar to that of Proposition 6.2.11, only that now we use the identity $\alpha_0 * \alpha_1 = \alpha_{1-n} = \alpha_2 t$. $\square$

The next result deals with existence of holomorphic disks satisfying mixed constrains, i.e. some of the marked points are on the boundary and some in the interior.

Proposition 6.2.12. *Let $L \subset \mathbb{C}P^n$ be a Lagrangian with $2H_1(L;\mathbb{Z}) = 0$.*

1. *For every $p \in \mathbb{C}P^n \setminus L$ and every $J \in \mathcal{J}$ there exists a simple J-holomorphic disk $u : (D, \partial D) \to (\mathbb{C}P^n, L)$ with $\mu([u]) = n+1$ and $u(0) = p$.*
2. *Let $x \in L$, $p \in \mathbb{C}P^n \setminus L$. Then for generic $J \in \mathcal{J}$ there exists a simple J-holomorphic disk $u : (D, \partial D) \to (\mathbb{C}P^n, L)$ with $\mu([u]) = 2n+2$ and $u(0) = p$, $u(-1) = x$.*
3. *Suppose $n = 2$ and L is diffeomorphic to $\mathbb{R}P^2$. Let $x', x'' \in L$ be two distinct points and $p \in \mathbb{C}P^2 \setminus L$. Then for generic J there exists a simple J-holomorphic disk $u : (D, \partial D) \to (\mathbb{C}P^2, L)$ with $\mu([u]) = 6$ and $u(-1) = x'$, $u(1) = x''$ and $u(0) = p$. Moreover the number of such disks is odd.*

Proof. Statement 1 follows from Proposition 6.2.8 since

$$i_L(\alpha_{n-1}) = \begin{cases} ut, & n = \text{even} \\ a_{n-1} + ut, & n = \text{odd}, \end{cases}$$

where $u = a_{2n} \in QH_{2n}(\mathbb{C}P^n)$ is the fundamental class.

We turn to statement 2. We start with the following identity:

$$a_0 * \alpha_0 = h^{*n} * \alpha_0 = \alpha_{-2n} = \alpha_2 t^2. \tag{117}$$

Let $f : L \to \mathbb{R}$ be a Morse function with exactly one local minimum at the point x. Let $g : \mathbb{C}P^n \to \mathbb{R}$ be a Morse function with exactly one local minimum at the point p. Choose a cycle $y \in C_2(f)$ representing α_2. Choose a generic $J \in \mathcal{J}$.

From formula (117) it follows that $p * x = yt^2 +$ boundary terms. By the definition of the $*$ operation (see formula (71) in §5.3.1) it follows that there exists a vector of non-zero classes $\mathbf{A} = (A_1, \dots, A_l)$ with $\mu(\mathbf{A}) = 2(n+1)$ such that one of the spaces $\mathcal{P}_I(p, x, y'; \mathbf{A}, J)$, $\mathcal{P}_{I'}(p, x, y'; \mathbf{A}, J)$ is not empty, where y' is a critical point participating in y. Note that since p is a minimum the only trajectory of $-\mathrm{grad} g$ emanating from p is constant. Since $p \notin L$ we must have $\mathcal{P}_{I'}(p, x, y'; \mathbf{A}, J) = \emptyset$, thus $\mathcal{P}_I(p, x, y'; \mathbf{A}, J) \neq \emptyset$. As $N_L = 2(n+1)$ we have $l \leq 2$.

We claim that $l = 1$. To see this first note that since x is a minimum the trajectories of $-\mathrm{grad} f$ emanating from x are constant. Therefore an element $\mathcal{P}_I(p, x, y'; \mathbf{A}, J)$ looks like one of the three types in figure 19. In the first case we have a (simple) disk $u_1 : (D, \partial D) \to (\mathbb{C}P^n, L)$ with $\mu([u_1]) = n+1$ and such that $u(-1) = x$, $u(0) = p$. A simple computation shows that the dimension of this configuration is negative hence cannot occur for generic J. In the second case we have a J-holomorphic disk u_2 with $u_2(0) = p$, $u_2(1) \in W^s_{y'}$ which again is a configuration of dimension -1 hence cannot occur for generic J. We are thus left with the last possibility ($l = 1$) in which we have a J-holomorphic disk u with $u(0) = p$ and $u(-1) = x$. Note that the disk u is simple since by Proposition 5.3.3 all the J-holomorphic disks coming from $\mathcal{P}_I(p, x, y'; \mathbf{A}, J)$ are simple.

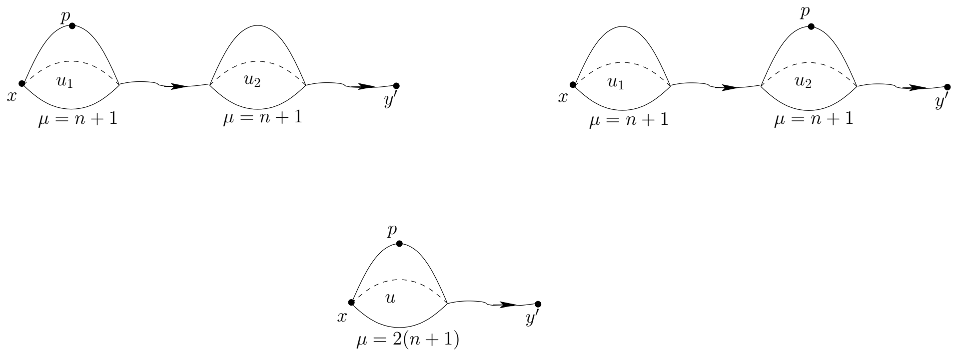

FIGURE 19. Possible elements of $\mathcal{P}_I(p, x, y'; \mathbf{A}, J)$.

To prove statement 3 we argue as above but we require in addition the function f to have a single local maximum. To keep compatibility with the notation we have already used denote the point x' by x and x'' by y'. Then the proof of statement 2 gives us a simple J-holomorphic disk with $\mu([u]) = 2(2+1) = 6$ and $u(-1) = x$, $u(1) \in W^s_{y'}$ and

$u(0) = p$. Note that now y' is the maximum (since $n = 2$), hence $W^s_{y'} = \{y'\}$ and we obtain $u(1) = y'$. □

Remark 6.2.13. In case L is the standard embedding $\mathbb{R}P^n \subset \mathbb{C}P^n$ some of the statements of Proposition 6.2.9-6.2.12 can be proved by more direct methods. This is typically done by degenerating to the standard almost complex structure J_{std} (for which the statement is obviously true) and passing to generic J using Gromov compactness theorem.

6.2.5. *Further questions and remarks.* In view of the results above one is led to ask the following questions. *Do there exist Lagrangian submanifolds $L \subset \mathbb{C}P^n$ with $2H_1(L;\mathbb{Z}) = 0$ different than $\mathbb{R}P^n$? Are all Lagrangian embeddings of $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$ Hamiltonianly isotopic to the standard embedding?*

It is obvious that using the Fukaya A_∞-category theory [34] or alternatively the theory of cluster homology [23, 24] one can get more enumerative results on J-holomorphic disks with boundary on Lagrangians as above. It would be interesting to test this even in dimension $n = 2$. For example, do these techniques imply that given 5 points $x_1, \ldots, x_5 \in L$ then for generic J there exists a J-holomorphic disk $u : (D, \partial D) \to (\mathbb{C}P^2, L)$ with $\mu([u]) = 6$ and $u(\partial D) \ni x_1, \ldots, x_5$? It would of course be also interesting to understand the relation of the above to the works of Welschinger [68, 69, 70] and of Solomon [65].

6.3. **Lagrangian submanifolds of the quadric.** Let $Q \subset \mathbb{C}P^{n+1}$ be a smooth complex n-dimensional quadric, where $n \geq 2$. More specifically we can write Q as the zero locus $Q = \{z \in \mathbb{C}P^{n+1} \mid q(z) = 0\}$ of a homogeneous quadratic polynomial q in the variables $[z_0 : \cdots : z_{n+1}] \in \mathbb{C}P^{n+1}$, where q defines a quadratic form of maximal rank. We endow Q with the symplectic structure induced from $\mathbb{C}P^{n+1}$. (Recall that we use the normalization that the symplectic structure ω_{FS} of $\mathbb{C}P^{n+1}$ satisfies $\int_{\mathbb{C}P^1} \omega_{\text{FS}} = \pi$.) When $n \geq 3$ we have by Lefschetz theorem $H^2(Q;\mathbb{R}) \cong \mathbb{R}$, therefore by Moser argument all Kähler forms on Q are symplectically equivalent up to a constant factor. When $n = 2$, $Q \subset \mathbb{C}P^3$ is symplectomorphic to $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} \oplus \omega_{FS})$. Also note that the symplectic structure on Q (in any dimension) does not depend (up to symplectomorphism) on the specific choice of the defining polynomial q (this follows from Moser argument too since the space of smooth quadrics is connected).

6.3.1. *Topology of the quadric.* The quadric has the following homology:

$$H_i(Q;\mathbb{Z}) \cong \begin{cases} 0 & \text{if } i = \text{odd} \\ \mathbb{Z} & \text{if } i = \text{even} \neq n \end{cases}$$

Moreover, when $n =$even, $H_n(Q;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. To see the generators of $H_n(Q;\mathbb{Z})$, write $n = 2k$. There exist two families $\mathcal{F}, \mathcal{F}'$ of complex k-dimensional planes lying on Q (see [35]). Let $P \in \mathcal{F}$, $P' \in \mathcal{F}'$ be two such planes belonging to different families. Put $a = [P]$, $b = [P']$. Then $H_n(Q;\mathbb{Z}) = \mathbb{Z}a \oplus \mathbb{Z}b$ and $h^{\cap k} = a + b$. Moreover, we have:

$$\begin{aligned} &\text{for } k = \text{odd}: \quad a \cap b = 1,\ a \cap a = b \cap b = 0, \\ &\text{for } k = \text{even}: \quad a \cap b = 0,\ a \cap a = b \cap b = 1. \end{aligned} \tag{118}$$

6.3.2. *Quantum homology of the quadric.* Let $h \in H_{2n-2}(Q;\mathbb{Z})$ be the class of a hyperplane section, $p \in H_0(Q;\mathbb{Z})$ the class of a point and $u \in H_{2n}(Q;\mathbb{Z})$ the fundamental class. We will first describe the quantum cohomology over $\mathbb{Z}$. Define $\Lambda_*^{\mathbb{Z}} = \mathbb{Z}[t, t^{-1}]$ where $\deg t = -N_L$. Here N_L is the minimal Maslov number of a Lagrangian submanifold that will appear later on. Note that that $c_1(Q) = nPD(h)$, hence $N_L | 2n$. Let $QH_*(Q;\Lambda^{\mathbb{Z}}) = (H(Q;\mathbb{Z}) \otimes \Lambda^{\mathbb{Z}})_*$ be the quantum homology endowed with the quantum cap product $*$.

Proposition 6.3.1 (See [9]). *The quantum product satisfies the following identities:*

$$\begin{aligned} &h^{*j} = h^{\cap j}\ \forall\, 0 \le j \le n-1, \quad h^{*n} = 2p + 2ut^{2n/N_L}, \\ &h^{*(n+1)} = 4ht^{2n/N_L}, \quad p * p = ut^{4n/N_L}. \end{aligned}$$

When $n =$ even we have the following additional identities:

(1) $h * a = h * b$.
(2) If $n/2 =$ odd then $a * b = p$, $a * a = b * b = ut^{2n/N_L}$.
(3) If $n/2 =$ even then $a * a = b * b = p$, $a * b = ut^{2n/N_L}$.

Proof. The first three identities and the fact that $h * a = h * b$ are proved in [9]. To prove the remaining two identities write $n = 2k$. Recall from [9] that

$$(a-b)*(a-b) = (a-b)\cap(a-b)\frac{1}{2}(h^{*n} - 4ut^{2n/NL}) = (a-b)\cap(a-b)(p - ut^{2n/N_L}).$$

Substituting (118) in this we obtain:

$$(a-b)*(a-b) = (-1)^k 2(p - ut^{2n/N_L}). \tag{119}$$

On the other hand we have $h^{*k} = h^{\cap k} = a + b$, hence

$$(a+b)*(a+b) = h^{*n} = 2p + 2ut^{2n/N_L}. \tag{120}$$

Next we claim that $a*a = b*b$. Indeed $a*a - b*b = (a+b)*(a-b) = h^{*k} * (a-b) = 0$. The desired identities follow from this together with (119), (120). $\square$

6.3.3. *Topological restrictions on Lagrangian submanifolds.* The quadric Q has Lagrangian spheres. To see this write Q as $Q = \{z_0^2 + \cdots + z_n^2 = z_{n+1}^2\} \subset \mathbb{C}P^{n+1}$. Then $L = \{[z_0 : \cdots : z_{n+1}] \in Q \mid z_i \in \mathbb{R}, \forall\, i\}$ is a Lagrangian sphere. The following theorem shows that for $n =$even, at least homologically, this is the only type of Lagrangian with $H_1(L;\mathbb{Z}) = 0$.

Proposition 6.3.2. *Assume* $\dim_{\mathbb{C}} Q =$ *even. Let* $L \subset Q$ *be a Lagrangian submanifold with* $H_1(L;\mathbb{Z}) = 0$. *Then* $H_*(L;\mathbb{Z}_2) \cong H_*(S^n;\mathbb{Z}_2)$.

Proof. Since $H_1(L;\mathbb{Z}) = 0$ we have $N_L = 2n$. Let $\Lambda = \mathbb{Z}_2[t, t^{-1}]$, where $\deg t = -2n$. Since $N_L = 2n > n+1$ we have $QH_*(L) \cong (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$, hence for every $q \in \mathbb{Z}$, $0 \leq r < 2n$ we have:

$$QH_{2nq+r}(L) \cong \begin{cases} H_r(L;\mathbb{Z}_2) & \text{if } 0 \leq r \leq n \\ 0 & \text{if } n+1 \leq r \leq 2n-1 \end{cases} \tag{121}$$

Denote by $QH_*(Q) = (H(Q;\mathbb{Z}_2) \otimes \Lambda)_*$ the quantum homology of the quadric (over $\mathbb{Z}_2$). Reducing modulo 2 the identities from Proposition 6.3.1 it follows that $a \in QH_n(Q)$ is an invertible element. Therefore $a* : QH_i(L) \to QH_{i-n}$ is an isomorphism for every $i \in \mathbb{Z}$. It now easily follows from (121) that $H_i(L;\mathbb{Z}_2) = 0$ for every $0 < i < n$. □

We are not aware of existence of any Lagrangian submanifolds in Q with $H_1(L;\mathbb{Z}) = 0$ which are not diffeomorphic to a sphere, and it is tempting to conjecture that spheres are indeed the only example.

Theorem 6.3.2 can be also proved by Seidel's method of graded Lagrangian submanifolds [62]. Indeed for $n =$even the quadric has a Hamiltonian S^1-action which induces a shift by n on $QH_*(L)$. To see this write $n = 2k$ and write Q as $Q = \{\sum_{j=0}^k z_j z_{j+1+k} = 0\}$. Then S^1 acts by $t \cdot [z_0 : \cdots : z_{2k+1}] = [tz_0 : \cdots : tz_k : z_{k+1} : \cdots : z_{2k+1}]$. A simple computation of the weights of the action at a fixed point gives a shift of n on graded Lagrangian submanifolds in the sense of [62].

When $n =$odd our methods (as well as those of [62]) do not seem to yield a similar result to Theorem 6.3.2. However the works of Buhovsky [17] and of Seidel [64] may be an evidence that such a result should hold.

6.3.4. *Quantum structures for Lagrangians of the quadric.* Let $L \subset Q$ be a Lagrangian submanifold with $H_1(L;\mathbb{Z}) = 0$. Assume that $n = \dim_{\mathbb{C}} Q \geq 2$. As $N_L = 2n > n+1$ we have a canonical isomorphism $QH_*(L) \cong (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$, where $\Lambda_* = \mathbb{Z}_2[t, t^{-1}]$ with $\deg t = -2n$. Denote by $\alpha_0 \in QH_0(L)$ the class of a point and by $\alpha_n \in QH_n(L)$ the fundamental class. Denote by $p \in QH_0(Q)$ the class of a point and by $u \in QH_{2n}(Q)$ the fundamental class. Denote by $i_L : QH_*(L) \to QH_*(L)$ the quantum inclusion map. With this notation we have:

Proposition 6.3.3. *Let $L \subset Q$ be as above. Then:*

(1) $p * \alpha_0 = \alpha_0 t$, $p * \alpha_n = \alpha_n t$.
(2) $i_L(\alpha_0) = p - ut$.
(3) If n is even then $\alpha_0 * \alpha_0 = \alpha_n t$.

Proof. By Proposition 6.3.1 $p \in QH_0(Q)$ is an invertible element, hence $p* : QH_i(L) \to QH_{i-2n}(L)$ is an isomorphism for every i. But $QH_0(L) \cong \mathbb{Z}_2\alpha_0$ and $QH_{-2n}(L) \cong \mathbb{Z}_2\alpha_0 t$. Therefore $p * \alpha_0 = \alpha_0 t$. The statement on $p * \alpha_n$ is similar. This proves 1.

We prove 2. It easily follows from the definition of the quantum inclusion map that

$$i_L(\alpha_0) = p + eut$$

for some $e \in \mathbb{Z}_2$. Clearly $h * \alpha_0 = 0$ since $h * \alpha_0$ belongs to $QH_{-2}(L) \cong QH_{2n-2}(L) = 0$. Therefore we have

$$0 = i_L(h * \alpha_0) = h * (p + eut) = h * p + eht.$$

On the other hand a simple computation based on the identities of Proposition 6.3.1 gives $h * p = ht$. It follows that $e = -1$. This proves point 2 of the theorem.

We prove 3. By Proposition 6.3.1 when $n =$even the element $a \in QH_n(Q)$ is invertible (even if we work with coefficients in $\mathbb{Z}_2$). Therefore $a * \alpha_n = \alpha_0$ and $a * \alpha_0 = \alpha_n t$. It follows that

$$\alpha_0 * \alpha_0 = (a * \alpha_n) * \alpha_0 = a * (\alpha_n * \alpha_0) = a * \alpha_0 = \alpha_n t.$$

□

Denote $\mathcal{J}$ the space of almost complex structures compatible with the symplectic structure of Q. The following corollary is a straightforward consequence of Theorem 6.3.3.

Corollary 6.3.4. *Let $L \subset Q$ be a Lagrangian submanifold with $H_1(L;\mathbb{Z}) = 0$. Assume $n = \dim_{\mathbb{C}} Q \geq 2$. Then the following holds:*

(1) Let $x \in L$ and $z \in Q \setminus L$. Then for every $J \in \mathcal{J}$ there exists a J-holomorphic disk $u : (D, \partial D) \to (Q, L)$ with $u(-1) = x$, $u(0) = z$ and $\mu([u]) = 2n$.
(2) Assume that $n =$ even. Let $x', x'', x'' \in L$. Then for every $J \in \mathcal{J}$ there exists a J-holomorphic disk $u : (D, \partial D) \to (Q, L)$ with $u(e^{2\pi i/3}) = x'$, $u(1) = x''$, $u(e^{4\pi i/3}) = x'''$ and $\mu([u]) = 2n$.

6.3.5. *Lagrangians with $2H_1(L;\mathbb{Z}) = 0$.* Let $L \subset Q$ be a Lagrangian submanifold with $H_1(L;\mathbb{Z}) \neq 0$ but $2H_1(L;\mathbb{Z}) = 0$. Assume that $n > 2$. The quadric contains several types of such Lagrangians. To see this write the quadric as $Q = \{z_0^2 + \cdots + z_{n+1}^2 = 0\}$. For every $0 \leq r \leq n$ put $L_r = \{[z_0 : \ldots : z_{n+1}] \in Q \mid z_0, \ldots, z_r \in \mathbb{R}, \quad z_{r+1}, \ldots, z_{n+1} \in i\mathbb{R}\}$. An easy computation shows that the L_r's are Lagrangian submanifolds of Q (so called "real quadrics") and that L_r is diffeomorphic to $(S^r \times S^{n-r})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on both factors by the antipode map. Note that when $1 < r < n-1$ we have $H_1(L_r;\mathbb{Z}) \cong \mathbb{Z}_2$ and $H_2(L_r;\mathbb{Z}_2) \neq 0$.

A Lagrangian L as above must be monotone. Moreover, when $n =$even its minimal Maslov number is $N_L = n$. To see this note that since the minimal Chern number of Q is n and $2H_1(L;\mathbb{Z}) = 0$ we have $2N_L = 2kn$ for some $k \in \mathbb{N}$. We claim that $k = 1$. Indeed if $k > 1$ then $N_L \geq 2n > n+1$ hence $QH_*(L) \cong H_*(L;\mathbb{Z}_2) \otimes \mathbb{Z}_2[t,t^{-1}]$ where $\deg t = -N_L \leq -2n$. In particular $QH_1(L) \cong H_1(L;\mathbb{Z}_2) \neq 0$ and $QH_{n+1}(L) = 0$. On the other hand this is impossible since $a* : QH_{n+1}(L) \to QH_1(L)$ is an isomorphism. A contradiction. This proves that $k = 1$, hence $N_L = n$.

As $N_L = n$ we will work now with the ring $\Lambda_* = \mathbb{Z}_2[t,t^{-1}]$ where $\deg t = -n$. Put $QH_*(Q) = (H(Q;\mathbb{Z}_2) \otimes \Lambda)_*$. Note that with this grading the variable t will appear in the identities of Proposition 6.3.1 in power 2.

Proposition 6.3.5. *Let $L^n \subset Q$ be a Lagrangian submanifold with $H_1(L;\mathbb{Z}) \neq 0$ but $2H_1(L;\mathbb{Z}) = 0$. Assume that $n = \dim_{\mathbb{C}} Q \geq 4$ and if $n =$ odd assume also that $N_L = n$. Suppose there exists $2 \leq j \leq n-2$ such that $H_j(L;\mathbb{Z}_2) \neq 0$. Then there exists an isomorphism of graded vector spaces $QH_*(L) \cong (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$. This isomorphism is canonical for general degree $*$ except perhaps for $* \equiv 0 (\mathrm{mod} n)$, but there is a canonical injection $H_n(L;\mathbb{Z}_2) \hookrightarrow QH_n(L)$ identifying the fundamental class $\alpha_n \in H_n(L)$ with the unit of $QH(L)$. Moreover, let $\alpha_0 \in QH_0(L)$ be an element such that $\{\alpha_0, \alpha_n t\}$ form a basis for $QH_0(L) \cong H_0(L;\mathbb{Z}_2) \oplus H_n(L;\mathbb{Z}_2)t$. Then one of the following two possibilities occurs:*

(1) $p * \alpha_0 = \alpha_n t^3$, $\;p * \alpha_n = \alpha_0 t$.
(2) $p * \alpha_0 = \alpha_0 t^2 + s\alpha_n t^3$, $\;p * \alpha_n = r\alpha_0 t + \alpha_n t^2$, for some $s, r \in \mathbb{Z}_2$ with $sr = 0$.

In case $n =$ odd we also have:

(1') Either $i_L(\alpha_0) = p$, $\quad i_L(\alpha_n) = ut$; or
(2') $i_L(\alpha_0) = p + ut^2$, $\quad i_L(\alpha_n) = rut$,

Where 1' corresponds to possibility 1 above and 2' to possibility 2.

Before proving Proposition 6.3.5, here is an immediate corollary of it:

Corollary 6.3.6. *Let $L \subset Q$ be a Lagrangian submanifold as in Theorem 6.3.5. Let $x \in L$ and $z \in Q \setminus L$. Then for every $J \in \mathcal{J}$ there exists a J-holomorphic disk $u : (D, \partial D) \to (Q, L)$ with $u(-1) = x$, $u(0) = z$ and $\mu([u]) = n$.*

Proof of Proposition 6.3.5. We first show that $QH_*(L) \cong (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$. Let $f : L \to \mathbb{R}$ be a Morse function and denote by $CM_*(f) = \mathbb{Z}_2\langle \mathrm{Crit}(f) \rangle$ the Morse complex of f. Put

$CF_* = (C(f) \otimes \Lambda)_*$. Since $N_L = n$ the Floer differential d can be written as $d = \partial_0 + \partial_1 t$, where $\partial_0 : CM_*(f) \to CM_{*-1}(f)$ is the Morse differential and ∂_1 is an operator acting as $\partial_1 : CM_*(f) \to CM_{*-1+n}(f)$. Put $E^1_* = (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$. By the results of [15], $\partial_1 t$ descends to a differential d_1 defined on the homology $E^1_* = (H(C(f), \partial_0) \otimes \Lambda)_* \cong (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$, which has the form $d_1 = \delta_1 t$ where $\delta_1 : H_*(L;\mathbb{Z}_2) \to H_{*-1+n}(L;\mathbb{Z}_2)$. Moreover the homology $H_*(d_1)$ is isomorphic to $QH_*(L)$. By grading reasons δ_1 is zero on $H_k(L;\mathbb{Z}_2)$ for every $k \geq 2$.

We claim that $\delta_1 = 0$ (also on $H_0(L;\mathbb{Z}_2)$ and $H_1(L;\mathbb{Z}_2)$). To prove this we use the work of Buhovsky [16], by which δ_1 satisfies Leibniz rule with respect to the cap product on $H_*(L;\mathbb{Z}_2)$, i.e. $\delta_1(\alpha \cap \beta) = \delta_1(\alpha) \cap \beta + \alpha \cap \delta_1(\beta)$ for every $\alpha, \beta \in H_*(L;\mathbb{Z}_2)$. By assumption there exists a non-trivial element $\alpha_j \in H_j(L;\mathbb{Z}_2)$ for some $2 \leq j \leq n-2$. By Poincaré duality there exists $\alpha_{n-j} \in H_{n-j}(L;\mathbb{Z}_2)$ such that $\alpha_j \cap \alpha_{n-j} = \gamma$ where $\gamma \in H_0(L;\mathbb{Z}_2)$ is the class of a point. As $\delta_1(\alpha_j)$, $\delta_1(\alpha_{n-j}) = 0$ it follows from Leibniz rule that $\delta_1(\gamma) = 0$. For degree reasons γ cannot be a d_1-boundary. Therefore $QH_0(L) \neq 0$. Next, note that δ_1 maps $H_1(L;\mathbb{Z}_2)$ to $H_n(L;\mathbb{Z}_2) = \mathbb{Z}_2 \alpha_n$, where α_n is the fundamental class. If δ_1 were not 0 on $H_1(L;\mathbb{Z}_2)$ we would get that $\alpha_n = \delta_1(\alpha_1)$ for some $\alpha_1 \in H_1(L;\mathbb{Z}_2)$, hence $[\alpha_n]$ would be 0 in $QH(L)$. Since $[\alpha_n]$ is the unity in $QH_*(L)$ it follows that $QH_*(L) = 0$, a contradiction. This proves that δ_1 vanishes also on $H_1(L;\mathbb{Z}_2)$. Summarizing, we have $\delta_1 = 0$ and therefore $QH_*(L) \cong (H(L) \otimes \Lambda)_*$ as claimed. Note that $QH_0(L) \cong H_0(L;\mathbb{Z}_2) \oplus H_n(L;\mathbb{Z}_2)t$. The statement on the canonicity of these isomorphisms for various values of $*$ follows easily from degree reasons.

We turn to the proof of the other statements of the theorem. Write

$$p * \alpha_0 = s_0 \alpha_0 t^2 + s_n \alpha_n t^3, \quad p * \alpha_n = t_0 \alpha_0 t + t_n \alpha_n t^2, \tag{122}$$

for some $s_0, s_n, t_0, t_n \in \mathbb{Z}_2$. By Proposition 6.3.1 we have $p * p = ut^4$. Multiplying both sides of equations (122) by p and comparing coefficients on both sides we obtain three possibilities for the values of s_0, s_n, t_0, t_n:

(1) $s_0 = 0,\ s_n = 1,\ t_0 = 1,\ t_n = 0$.
(2) $s_0 = 1,\ t_0 = 0,\ t_n = 1$.
(3) $s_0 = 1,\ t_0 = 1,\ s_n = 0,\ t_n = 1$.

The first case leads to possibility 1 of the theorem. The two other two cases lead to possibility 2.

We now prove identities 1', 2' assuming $n =$odd. We can write $i_L(\alpha_n) = gut$ for some $g \in \mathbb{Z}_2$. Since $\{\alpha_0, \alpha_n t\}$ are linearly independent it is easy to see that $\epsilon_L(\alpha_0) = 1$. It follows that $i_L(\alpha_0)$ has the form $i_L(\alpha_0) = p + dut^2$ for some $d \in \mathbb{Z}_2$. Using the fact that

$i_L(p * \alpha_j) = p * i_L(\alpha_j)$ and the identities 1, 2 just proved, gives the desired result on d, g. $\square$

Remark 6.3.7. (1) Using similar methods, when $n =$even one can obtain information on $a * \alpha_0$, $b * \alpha_0$, $a * \alpha_n$, $a * \alpha_n$ and also on $i_L(\alpha_0)$, $i_L(\alpha_n)$. We leave these computations to the reader.

(2) The identities for $p * \alpha_j$ in Theorem 6.3.5 suggest that for some Lagrangians in Q a stronger version of Corollary 6.3.6 should hold. An interesting case seems to be for example when identity 1 of Theorem 6.3.5 holds.

(3) As was shown above, when $n =$even, $H_1(L;\mathbb{Z}) = 0$, $2H_1(L;\mathbb{Z}) \neq 0$ we have $N_L = n$. A priori this need not be the case when $n =$odd (i.e. it may happen that $N_L = 2n$). However we are not aware of any examples of this sort.

6.4. **Complete intersections.** Some of the result in this section and in §6.5 are somewhat non-rigorous. The reason for this is that we need to use here Floer homology with coefficients in $\mathbb{Q}$ rather than $\mathbb{Z}_2$. For this end one has to orient the moduli spaces of pearls arising in our constructions. Orientations of moduli spaces of pseudo-holomorphic disks in the context of Floer homology have been worked out by Fukaya, Oh, Ohta and Ono [34]. Still, it remains to check whether some ingredients of the theory of the present paper, e.g. the quantum module structure and the quantum inclusion maps, are indeed compatible with the coherent orientations arising in Floer homology over $\mathbb{Q}$. While it seems very likely that the two theories are compatible we have not checked all the details. On the other hand we found it worth presenting here possible applications of our theory over $\mathbb{Q}$. Below we state such applications with the convention that theorems and corollaries are marked with $*$ whenever they depend on the validity of our theory over $\mathbb{Q}$.

Let $X \subset \mathbb{C}P^{n+r}$ be a smooth complete intersection of degree $(d_1, \ldots, d_r)$, i.e. a transverse intersection of r complex hypersurfaces of degrees $d_1, \ldots, d_r$ in $\mathbb{C}P^{n+r}$. Note that $\dim_{\mathbb{C}} X = n$. We will assume that $n \geq 3$ hence by the Lefschetz hyperplane section theorem we have $\dim H^2(X;\mathbb{R}) \cong \mathbb{R}$. It follows by Moser argument that all Kähler forms on X are symplectically equivalent up to a constant factor. We endow X with the symplectic structure induced from $\mathbb{C}P^{n+r}$. We will also assume that X is non-linear, i.e. that at least one of the d_i's is > 1. Put $N = n + r + 1 - \sum_{i=1}^r d_i$. When $N > 0$, X is monotone with minimal Chern number N.

Note that such a complete intersection X must have Lagrangian submanifolds L with $H_1(L;\mathbb{Z}) = 0$. Indeed, X can be degenerated to a variety with isolated singularities, hence by symplectic Picard-Lefschetz theory (see e.g. [6, 26, 63, 60, 10]) X must have Lagrangian spheres.

Let $\mathbb{K}$ be one of the fields $\mathbb{Z}_2$ or $\mathbb{Q}$. Let $\Lambda^{\mathbb{K}}_* = \mathbb{K}[t, t^{-1}]$ where $\deg t = -2N$. Let $QH_*(X; \Lambda^{\mathbb{K}}) = (H(X; \mathbb{K}) \otimes \Lambda^{\mathbb{K}})_*$ be the quantum homology of X with coefficients in $\Lambda^{\mathbb{K}}$. Let $L \subset X$ be a Lagrangian submanifold with $H_1(L; \mathbb{Z}) = 0$. Clearly L is monotone with minimal Maslov number $N_L = 2N$. Denote by $QH_*(L; \Lambda^{\mathbb{K}})$ the Floer homology of L. When $\mathbb{K} = \mathbb{Q}$ we assume that L is orientable and relatively spin (see [34]). Note that when $n \geq 2\sum_{i=1}^r d_i - 2r$ we have $2N \geq n+2$ hence there exists a canonical isomorphism $QH_*(L; \Lambda^{\mathbb{K}}) \cong (H(L; \mathbb{K}) \otimes \Lambda^{\mathbb{K}})_*$. In this case denote by $\alpha_0 \in H_0(L; \mathbb{K})$ the class of a point in L, by $p \in H_0(X; \mathbb{K})$ the class of a point in X and by $h \in H_{2n-2}(X; \mathbb{K})$ the class of a hyperplane section of X.

Proposition* 6.4.1. *Let $L \subset X$ be a Lagrangian submanifold as above. Assume that $n \geq 3$ and $n \geq 2\sum_{i=1}^r d_i - 2r + 1$. Then*

$$i_L(\alpha_0) = p - \big(\prod_{i=1}^r (d_i - 1)!\big) h^{\cap (n-N)} t.$$

In particular for every $x \in L$ and every almost complex structure J compatible with the symplectic structure of X there exists a J-holomorphic disk $u : (D, \partial D) \to (X, L)$ with $\mu([u]) = 2N$ and $x \in u(\partial D)$. In fact, the number of such disks intersecting a complex N-dimensional hyperplane in X is $\prod_{i=1}^r (d_i - 1)!$, when counted appropriately.

The identity on $i_L(\alpha_0)$ is completely rigorous when $\mathbb{K} = \mathbb{Z}_2$ but the coefficient of $h^{\cap(n-N)} t$ is $0 \in \mathbb{Z}_2$ unless $d_i \leq 2$ for every i.

Proof. Put $\mathbb{K} = \mathbb{Q}$. Since $4N > 2n$ we have $QH_0(X; \Lambda^{\mathbb{K}}) = H_0(X; \mathbb{K}) \oplus H_{2N}(X; \mathbb{K})t$. Also note that since $2N \geq n+3$ we have $H_{2N}(X; \mathbb{K}) \cong \mathbb{K} h^{\cap (n-N)}$. Therefore we can write

$$i_L(\alpha_0) = p + \tau h^{\cap (n-N)} t \tag{123}$$

for some $\tau \in \mathbb{K}$.

We first claim that $h * \alpha_0 = 0$. Indeed $h * \alpha_0 \in QH_{-2}(L; \Lambda^{\mathbb{K}}) \cong H_{2N-2}(L; \mathbb{K})t$. But $2N - 2 \geq n+1$ hence $H_{2N-2}(L; \mathbb{K}) = 0$.

A straightforward computation based on the results of Beauville [9] gives:

$$h * p = \big(\prod_{i=1}^r (d_i - 1)!\big) h^{\cap (n-N+1)} t, \quad h^{*j} = h^{\cap j} \ \forall\ 0 \leq j \leq N.$$

We now obtain from (123) that

$$0 = i_L(h * \alpha_0) = h * p + \tau (h * h^{(n-N)})T = \prod_{i=1}^r (d_i - 1)! h^{\cap (n-N+1)} t + \tau h^{\cap (n-N+1)} t.$$

Here we have used the fact that $n - N + 1 \leq N$ hence $h * h^{(n-N)} = h^{\cap (n-N+1)} \neq 0$. It immediately follows that $\tau = -\prod_{i=1}^r (d_i - 1)!$. □

6.5. **Algebraic properties of quantum homology and Lagrangian submanifolds.** Here we present relations between algebraic properties of the quantum homology of an (ambient) symplectic manifold and the kind of Lagrangians it contains. Interestingly enough the existence of certain types of Lagrangian submanifolds (e.g. spheres) imposes strong restrictions on the quantum homology of the ambient manifold.

Let (M^{2n}, ω) be a spherically monotone symplectic manifold with minimal Chern number N. Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{Z}_2$. Let $\Lambda_*^{\mathbb{K}}$ be either $\mathbb{K}[t, t^{-1}]$ or the field $\mathbb{K}[t]]$ of formal Laurent series with finitely many terms having negative powers of t. Define $\deg t = -2N$. Let $QH_*(M; \Lambda^{\mathbb{K}}) = (H(M; \mathbb{K}) \otimes \Lambda^{\mathbb{K}})_*$ be the quantum homology over $\mathbb{K}$ endowed with the quantum cap product. Let

$$\mathrm{pr}_l : QH_*(M; \Lambda^{\mathbb{K}}) \longrightarrow QH_l(M; \Lambda^{\mathbb{K}}) = \bigoplus_{k \in \mathbb{Z}} H_{l+2Nk}(M; \mathbb{K}) t^k$$

be the projection on the degree-l component of QH.

Proposition* 6.5.1. *Suppose that (M^{2n}, ω) has a Lagrangian sphere. Assume that $n = \dim_{\mathbb{C}} M \geq 2$ and $N \nmid (n+1)$. Let $a \in QH_*(M; \Lambda^{\mathbb{K}})$ be an invertible element. Then either $\mathrm{pr}_l(a) \neq 0$ for some l with $l \equiv 2n (\mathrm{mod} 2N)$ or there exist two indices l_1, l_2 with $l_1 \equiv n (\mathrm{mod} 2N)$, $l_2 \equiv 3n (\mathrm{mod} 2N)$ such that $\mathrm{pr}_{l_1}(a)$, $\mathrm{pr}_{l_2}(a) \neq 0$. If we assume in addition that $N \nmid n$ then the indices l_1, l_2 must be distinct hence the only invertible elements $a \in QH(M; \Lambda^{\mathbb{K}})$ of pure degree satisfy $\deg a \equiv 2n (\mathrm{mod} 2N)$.*

As before, the statement of the Theorem is completely rigorous when $\mathbb{K} = \mathbb{Z}_2$.

Proof. Let $L \subset M$ be a Lagrangian sphere. Clearly L is monotone with minimal Maslov number $2N$. Since $N \nmid n+1$, a standard argument as in [15] shows that Oh's spectral sequence collapses at stage $r = 1$, hence $QH_*(L; \Lambda^{\mathbb{K}}) \cong (H(S^n; \mathbb{K}) \otimes \Lambda^{\mathbb{K}})_*$. Choose two non-zero elements $\alpha_0 \in QH_0(L)$, $\alpha_n \in QH_n(L)$ (e.g. under the previous isomorphism we can take α_0, α_n to correspond to the class of a point in $H_0(S^n; \mathbb{K})$ and to the fundamental class in $H_n(S^n; \mathbb{K})$). Note that when $n \geq 3$, $L \approx S^n$ is automatically relatively spin hence $QH(L)$ is well defined in case $\mathbb{K} = \mathbb{Q}$. The same holds for $n = 2$ since the 2'nd Stiefel-Whitney class of S^2 vanishes.

Write $a = \sum_{j=k_1}^{k_2} a_j$, for some $k_1 \leq k_2$, where $a_j = \mathrm{pr}_j(a)$. Since a is invertible there exists l_1 such that $a_{l_1} * \alpha_n \neq 0$. As $a_{l_1} * \alpha_n \in QH_{l_1 - n}(L)$ it follows that either $l_1 \equiv 2n (\mathrm{mod} 2N)$ or $l_1 \equiv n (\mathrm{mod} 2N)$. Similarly there exists l_2 such that $\alpha_{l_2} * \alpha_0 \neq 0$. This leads to the following two possibilities: either $l_2 \equiv 2n (\mathrm{mod} 2N)$ or $l_2 \equiv 3n (\mathrm{mod} 2N)$. $\square$

Proposition* 6.5.2. *Let (M, ω) be a spherically monotone (resp. monotone) symplectic manifold. Suppose that (M, ω) contains a Lagrangian submanifold L which is simply connected (resp. has $H_1(L; \mathbb{Z}) = 0$) and is relatively spin. Assume that the minimal*

Chern number N of (M,ω) satisfies $N \geq \frac{n}{2}+1$ where $n = \dim_{\mathbb{C}} M$. Then for every $3n+1-2N \leq l \leq 2n-1$ all elements of $QH_l(M;\Lambda^{\mathbb{K}})$ are divisors of 0.

As before, the statement of the Theorem is completely rigorous when $\mathbb{K} = \mathbb{Z}_2$.

Proof. Clearly L is monotone with minimal Maslov number $N_L = 2N \geq n+2$. Therefore there exists a canonical isomorphism $QH_*(L;\Lambda^{\mathbb{K}}) \cong (H(L;\mathbb{K}) \otimes \Lambda^{\mathbb{K}})_*$. In particular $QH_j(L;\Lambda^{\mathbb{K}}) = 0 \ \forall n+1-2N \leq j \leq -1$.

Let $i_L : QH_*(L;\Lambda^{\mathbb{K}}) \longrightarrow QH_*(M;\Lambda^{\mathbb{K}})$ be the quantum inclusion homomorphism (see §5.4). Let $x \in H_0(L;\mathbb{K})$ be the class of a point. Put $a = i_L(x) \in QH_0(M;\Lambda^{\mathbb{K}})$. Note that $a \neq 0$ since $i_L(x)$ has the form $p + \sum_{i\geq 1} a_i t^i$, where $p \in H_0(M;\mathbb{K})$ is the class of a point and $a_i \in H_{2iN}(M;\mathbb{K})$.

We claim that for every $b \in QH_l(M;\Lambda^{\mathbb{K}})$ with $3n+1-2N \leq l \leq 2n-1$ we have $b*a = 0$. Indeed let b be such an element. Then $b*a = i_L(b*x)$. But $b*x \in QH_{l-2n}(L;\Lambda^{\mathbb{K}}) = 0$ since $n+1-2N \leq l-2n \leq -1$. □

Semi-simplicity of quantum homology. Recall that a commutative algebra A over a field $\mathbb{F}$ is called semi-simple if it splits into a direct sum of finite dimensional vector spaces over $\mathbb{F}$, $A = A_1 \oplus \cdots \oplus A_r$ such that the splitting is compatible with the multiplication of A (i.e. $(a_1,\ldots,a_r)\cdot(a_1',\ldots,a_r') = (a_1a_1',\ldots,a_ra_r')$) and such that each A_i is a field with respect to the ring structure induced from A.

Remark 6.5.3. There exist several different notions of semi-simplicity in the context of quantum homology, or more generally in the context of Frobenius algebras and Frobenius manifolds (see e.g. [27, 39, 66]). The semi-simplicity we use here was first considered in the context of quantum homology by Abrams [1]. It is in general not equivalent to semi-simplicity in the sense of Dubrovin [27] since we work with a different coefficient ring.

Let (M^{2n},ω) be a spherically monotone symplectic manifold with minimal Chern number N. For simplicity we will work here with the even quantum homology

$$QH_*^{\mathrm{ev}}(M;\mathbb{F}) = \bigoplus_{i\in\mathbb{Z}} QH_{2i}(M;\mathbb{F}) = \bigoplus_{j=0}^{n} H_{2j}(M;\mathbb{Q}) \otimes \mathbb{F},$$

where $\mathbb{F}$ is the field $\mathbb{F} = \mathbb{Q}[t]]$ with $\deg t = -2N$. We could work here also with the full quantum homology but then semi-simplicity should be considered in the framework of skew-commutative algebras.

Proposition* 6.5.4. *Let (M^{2n},ω) be a monotone (resp. spherically monotone) symplectic manifold with minimal Chern number $N \geq \frac{n}{2}+1$. Assume that $QH_*^{ev}(M;\mathbb{F})$ is semi-simple. Let $L \subset (M,\omega)$ be a Lagrangian submanifold which is relatively spin and has*

$H_1(L;\mathbb{Z}) = 0$ *(resp.* $\pi_1(L) = 1$*). Put*

$$I = \{0 \leq i \leq n \mid i \equiv ln(\text{mod} 2N) \text{ for some } l \in \mathbb{Z}\},$$
$$J = \{0 \leq j \leq n \mid j \equiv q + 2nk(\text{mod} 2N) \text{ for some } n+1 \leq q \leq 2N-1, k \in \mathbb{Z}\},$$
$$J' = \{0 \leq j \leq n \mid j \equiv 1 + nk'(\text{mod} 2N) \text{ for some } k' \in \mathbb{Z}\},$$
$$J'' = \{0 \leq j \leq n \mid j \equiv -1 + nk''(\text{mod} 2N) \text{ for some } k'' \in \mathbb{Z}\}.$$

Then $H_i(L;\mathbb{Q}) \cong \mathbb{Q}$ *for every* $i \in I$ *and* $H_i(L;\mathbb{Q}) = 0$ *for every* $j \in J \cup J' \cup J''$*.*

We will give the proof of Proposition 6.5.4 later in this section. In the meanwhile here is an immediate corollary.

Corollary* 6.5.5. *Let* (M^{2n}, ω) *be a monotone (resp. spherically monotone) symplectic manifold with minimal Chern number* N*. Assume that* $\frac{3n+1}{4} \leq N \leq n-1$ *and that* (M,ω) *has a Lagrangian submanifold* L *which is relatively spin and has* $H_1(L;\mathbb{Z}) = 0$ *(resp.* $\pi_1(L) = 1$*). Then* $QH_*^{ev}(M;\mathbb{F})$ *is not semi-simple.*

Proof of Corollary 6.5.5. Suppose that $QH_*^{ev}(M;\mathbb{F})$ is semi-simple. Let J be the set of indices defined in Theorem 6.5.4. As $\frac{3n+1}{4} \leq N \leq n-1$, we have $n+2 \leq 3n-2N \leq 2N-1$ hence $n \in J$ (take $q = 3n-2N$, $k=-1$). By Theorem 6.5.4, $H_n(L;\mathbb{Q}) = 0$ which is impossible since L is orientable. A contradiction. □

Here is another restriction on semi-simplicity.

Proposition* 6.5.6. *Let* (M^{2n}, ω) *be a spherically monotone symplectic manifold of* $\dim_{\mathbb{C}} M \geq 2$*. Suppose that* (M,ω) *contains a Lagrangian sphere and that its minimal Chern number* N *satisfies* $N \nmid n$ *and* $N \nmid (n+1)$*. Then* $QH_*^{ev}(M;\mathbb{F})$ *is not semi-simple.*

In order to prove Propositions 6.5.6, 6.5.4 we will need some preparations regarding semi-simplicity. Let

$$\eta : QH_*(M;\mathbb{F}) \longrightarrow H_0(M;\mathbb{Q}) \otimes \mathbb{F} \cong \mathbb{F}$$

be the projection. The identification of the last isomorphism here is made via $H_0(M;\mathbb{Q}) = \mathbb{Q}p$, where p is the class of a point. The projection η assigns to an element $a \in QH_*^{\mathrm{ev}}(M;\mathbb{F})$ a power series in t which is the coefficient of a at p. Define a pairing

$$\Delta : QH_*^{\mathrm{ev}}(M;\mathbb{F}) \otimes QH_*^{\mathrm{ev}}(M;\mathbb{F}) \longrightarrow \mathbb{F}, \quad \Delta(a,b) = \eta(a * b).$$

It is well known that Δ is a non-degenerate pairing (see [44]). Let $e_1, \ldots, e_\nu$ be a basis over $\mathbb{F}$ of $QH_*^{\mathrm{ev}}(M;\mathbb{F})$ and denote by $e_1^\#, \ldots, e_\nu^\#$ the dual basis with respect to Δ. Define an element $\mathcal{E}_Q^{ev}(M) \in QH_*^{\mathrm{ev}}(M;\mathbb{F})$ by

$$\mathcal{E}_Q^{ev}(M) = \sum_{i=1}^{\nu} e_i * e_i^\#. \tag{124}$$

This element is called the even quantum Euler class. It does not depend on the choice of the basis $e_1, \ldots, e_\nu$. A Theorem due to Abrams [1] asserts that $QH_*^{\mathrm{ev}}(M;\mathbb{F})$ *is semi-simple iff* $\mathcal{E}_Q^{ev}(M)$ *is invertible*. Of course, it is possible to define a more complete quantum Euler class by taking an alternate sum as in (124) over a basis of the whole of $QH_*(M;\mathbb{F})$ (not just the even part). In this setting the quantum Euler is indeed a deformation of the classical Euler class, hence its name.

We will now need the following proposition. Denote by $\cdot : H_*(M;\mathbb{Q}) \otimes H_*(M;\mathbb{Q}) \to \mathbb{Q}$ the classical intersection pairing, with the convention that $a \cdot b = 0$ whenever a, b are elements of pure degree with $\deg(a) + \deg(b) \neq 2n$.

Proposition 6.5.7. *Let $e', e'' \in H_*(M;\mathbb{Q})$ and view e', e'' as elements of $QH_*(M;\mathbb{F})$. Then $\eta(e' * e'') = e' \cdot e''$.*

Proof. Without loss of generality we may assume that e', e'' have pure degrees. Then $\eta(e' * e'') = st^j$ for some $j \geq 0$ and $s \in \mathbb{Q}$. We may also assume that $s \neq 0$ since otherwise the statement is obvious.

We claim that $j = 0$. To prove this, choose generic $\mathbb{Q}$-cycles C', C'' representing e', e''. Suppose that $j > 0$ and $s \neq 0$. Then for a generic ω-compatible almost complex structure J there exists a simple J-holomorphic rational curve u passing through C' and C'' with $c_1([u]) = jN$. Denote by $\mathcal{M}(J)$ the space of simple J-holomorphic curves in the class $[u]$. Consider the evaluation map:

$$ev : \big(\mathcal{M}(J) \times \mathbb{C}P^1 \times \mathbb{C}P^1\big)/G \longrightarrow M \times M, \quad ev(w, p_1, p_2) = (w(p_1), w(p_2)),$$

where $G = \mathrm{Aut}(\mathbb{C}P^1)$. As $ev^{-1}(C' \times C'') \neq \emptyset$ we obtain that $\dim(ev^{-1}(C' \times C'') = \deg(e') + \deg(e'') - 2n + 2Nj - 2 \geq 0$. On the other hand since $\eta(e' * e'') = st^j$, we have $\deg(e') + \deg(e'') - 2n = -2Nj$. A contradiction. This proves that $j = 0$, hence $\eta(e' * e'') = s = e' \cdot e''$. $\square$

Lemma 6.5.8. *The even quantum Euler class is an element of pure degree* 0.

Proof. Let $e_1, \ldots, e_\nu$ be elements of pure degree which form a basis of $H_*^{ev}(M;\mathbb{Q})$ over $\mathbb{Q}$. Let $e_1^\#, \ldots, e_\nu^\# \in H_*^{ev}(M;\mathbb{Q})$ be a dual basis to $e_1, \ldots, e_\nu$ with respect to the classical intersection pairing $(-)\cdot(-)$. Thus $e_i \cdot e_j^\# = \delta_{i,j}$. Note that both $\{e_i\}$ and $\{e_i^\#\}$ are bases of $QH_*^{ev}(M, \mathbb{F})$ over $\mathbb{F}$. We claim that the basis $\{e_i^\#\}$ is dual to $\{e_i\}$ also with respect to the pairing Δ. (This is, by the way, contrary to what is written in [1].) Indeed, by Proposition 6.5.7 we have $\Delta(e_i, e_j^\#) = \eta(e_i * e_j^\#) = e_i \cdot e_j^\# = \delta_{i,j}$.

Finally, recall that the even quantum Euler class does not depend on the choice of the basis, hence $\mathcal{E}_Q^{ev}(M) = \sum_{i=1}^{\nu} e_i * e_i^\#$. Since $\deg(e_i^\#) = 2n - \deg(e_i)$ it follows that $\mathcal{E}_Q^{ev}(M)$ has degree 0. $\square$

We are now in position to prove Propositions 6.5.4, 6.5.6.

Proof of Proposition 6.5.6. By Proposition 6.5.1 $QH_*(M;\mathbb{F})$ has no invertible elements of pure degree $\not\equiv 2n(\mathrm{mod} 2N)$. As $0 \not\equiv 2n(\mathrm{mod} 2N)$, the even quantum Euler class $\mathcal{E}_Q^{ev}(M)$ is not invertible, hence $QH_*^{ev}(M;\mathbb{F})$ is not semi-simple. □

Proof of Proposition 6.5.4. The Lagrangian $L^n \subset M^{2n}$ is monotone and has minimal Maslov number $2N$. By assumption $2N \geq n+2$, hence there exists a canonical isomorphism $QH_*(L;\mathbb{F}) \cong (H(L;\mathbb{Q}) \otimes \mathbb{F})_*$, or more specifically:

$$
\begin{aligned}
&QH_{i+2kN}(L;\mathbb{F}) \cong H_i(L;\mathbb{Q}), \quad \forall\, 0 \leq i \leq n, k \in \mathbb{Z}, \\
&QH_{j+2kN}(L;\mathbb{F}) = 0, \quad \forall\, n+1 \leq j \leq 2N-1, k \in \mathbb{Z}.
\end{aligned} \tag{125}
$$

Since $QH_*^{ev}(M,\mathbb{F})$ is semi-simple the even quantum Euler class $\mathcal{E}_Q^{ev}(M) \in QH_0(M;\mathbb{F})$ is invertible. Therefore exterior multiplication by $\mathcal{E}_Q^{ev}(M)$ gives isomorphisms $QH_l(L;\mathbb{F}) \cong QH_{l-2n}(L;\mathbb{F})$ for every $l \in \mathbb{Z}$. The rest of the proof follows from these isomorphisms together with (125) and the fact that for every $k \in \mathbb{Z}$ we have:

$$
\begin{aligned}
&QH_{2kN}(L;\mathbb{F}) \cong H_0(L;\mathbb{Q}) \cong \mathbb{Q}, \quad QH_{n+2kN}(L;\mathbb{F}) \cong H_n(L;\mathbb{Q}) \cong \mathbb{Q}, \\
&QH_{1+2kN}(L;\mathbb{F}) \cong H_1(L;\mathbb{Q}) = 0, \quad QH_{n-1+2kN}(L;\mathbb{F}) \cong H_{n-1}(L;\mathbb{Z}) = 0.
\end{aligned}
$$

□

Examples. Here are a few examples of symplectic manifolds with semi-simple quantum homology.

(1) $\mathbb{C}P^n$.
(2) The smooth complex quadric Q (defined in §6.3 above). See [1] for the computation of the quantum Euler class.
(3) Complex Grassmannians. See [1] for the proof.
(4) Semi-simplicity is preserved when taking products. More precisely, let (M_1,ω_1), (M_2,ω_2) be two spherically monotone symplectic manifolds with the same proportionality factor between the $\omega_i|_{\pi_2}$'s and the $c_1(M_i)|_{\pi_2}$'s. Then $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ is also spherically monotone. If (M_1,ω_1) and (M_2,ω_2) both have semi-simple quantum homology then so does $(M_1 \times M_2, \omega_1 \oplus \omega_2)$. This follows from the quantum Künneth formula (see e.g. [44]). Of course, here one should consider semi-simplicity in the skew-commutative framework (see [1]).

Let $X \subset \mathbb{C}P^{n+r}$ be a smooth complete intersection of degree $(d_1,\ldots,d_r)$ as in §6.4. Let us now examine the semi-simplicity of complete intersections in relation to Lagrangian submanifolds. Assume that $2(\sum_{i=1}^r d_i - 1) \leq n$ and that $n \geq 3$. Put $d = \prod_{i=1}^r d_i$. Consider the following cases:

1. $d = 1$. In this case $X \cong \mathbb{C}P^n$. The quantum homology $QH_*(X;\mathbb{F})$ in this case is a field (see §6.2), hence semi-simple.
2. $d = 2$. In this case X is isomorphic to a smooth quadric. A direct computation of the quantum Euler class shows that it is an invertible element (see [1]) hence $QH_*(X;\mathbb{F})$ is semi-simple.
3. $d = 3$. In this case either $d_i \geq 3$ for some i or there exist at least two d_i's that are ≥ 2. Therefore the minimal Chern number N of X satisfies $N = n+1-\sum_{i=1}^r (d_i - 1) \leq n-1$. The assumption that $2(\sum_{i=1}^r d_i - 1) \leq n$ implies that we also have $\frac{n}{2}+1 \leq N$. It follows that $N \nmid n$ and $N \nmid n+1$. Next, note that X has a Lagrangian sphere. This follows from symplectic Picard-Lefschetz theory since for $d \geq 2$, X can be degenerated to a variety with isolated singularities. By Theorem 6.5.6 we conclude that $QH_*(X;\mathbb{F})$ is not semi-simple. Of course, this can be also verified by computing the quantum Euler class (see [1]).

It is not clear to us how large is the class of symplectic manifolds with semi-simple quantum homology. It seems that it is in fact a rather restricted class of manifolds.

Finally, let us mention that Entov and Polterovich [28] have also found restrictions on semi-simplicity of quantum homology related to Lagrangian submanifolds. The methods they use are based on the theory of spectral numbers and are quite different than ours.

6.6. **Gromov radius and Relative symplectic packing.** Let (M^{2n},ω) be a $2n$-dimensional symplectic manifold and $L \subset M$ a Lagrangian submanifold. Denote by $B(r) \subset \mathbb{R}^{2n}$ the closed $2n$-dimensional Euclidean ball of radius r endowed with the standard symplectic structure ω_{std} of $\mathbb{R}^{2n}$. Denote by $B_{\mathbb{R}}(r) \subset B(r)$ the "real" part of $B(r)$, i.e. $B_{\mathbb{R}}(r) = B(r) \cap (\mathbb{R}^n \times 0)$. Note that $B_{\mathbb{R}}(r)$ is Lagrangian in $B(r)$.

By a *relative symplectic embedding* $\varphi : (B(r), B_{\mathbb{R}}(r)) \to (M, L)$ of a ball in (M, L) we mean a symplectic embedding $\varphi : B(r) \to (M,\omega)$ which satisfies the following conditions:

(1) $\varphi(B_{\mathbb{R}}(r)) \subset L$.
(2) $\varphi(x) \notin L$ for every $x \in B(r) \setminus B_{\mathbb{R}}(r)$.

These two conditions can be rewritten as "$\varphi^{-1}(L) = B_{\mathbb{R}}(r)$". Condition (2) may look strange at first sight. We will explain its role soon (see Remark 6.6.2).

In analogy with the (absolute) Gromov radius, we define here the Gromov radius of $L \subset M$ to be

$$Gr(L) = \sup\{r \mid \exists \text{ a relative symplectic embedding } (B(r), B_{\mathbb{R}}(r)) \to (M, L)\}.$$

We will consider also symplectic embeddings of balls in the complement of L, i.e. symplectic embeddings $\psi : B(r) \to (M \setminus L, \omega)$. We denote the Gromov radius of $(M \setminus L, \omega)$ by $Gr(M \setminus L)$ i.e.

$$Gr(M \setminus L) = \sup\{r \mid \exists \text{ a symplectic embedding } B(r) \to (M \setminus L)\}.$$

Denote by $\mathcal{J}(M, \omega)$ the space of almost complex structure on M which are compatible with ω.

Proposition 6.6.1. *Let $L \subset (M, \omega)$ be a Lagrangian submanifold. Let $E', E'' > 0$.*

(1) Suppose that there exists a dense subset $\mathcal{J}' \subset \mathcal{J}(M, \omega)$ and a dense subset $\mathcal{U} \subset M$ such that for every $J \in \mathcal{J}'$ and every $p \in \mathcal{U}'$ there exists a non-constant J-holomorphic disk $u : (D, \partial D) \to (M, L)$ with $u(\mathrm{Int}\, D) \ni p$ and $\mathrm{Area}_\omega(u) \leq E'$. Then

$$\pi Gr(M \setminus L)^2 \leq E'.$$

(2) Suppose that there exists a dense subset $\mathcal{J}'' \subset \mathcal{J}(M, \omega)$ and a dense subset $\mathcal{U}'' \subset L$ such that for every $J \in \mathcal{J}''$ and every $q \in \mathcal{U}''$ there exists a non-constant J-holomorphic disk $u : (D, \partial D) \to (M, L)$ with $u(\partial D) \ni q$ and $\mathrm{Area}_\omega(u) \leq E''$. Then

$$\frac{\pi Gr(L)^2}{2} \leq E''.$$

Proof of Proposition 6.6.1. The proof is based on an argument of Gromov from [36]. Variants of the proof below can be found in [7, 23].

We begin with the proof of statement 2. Let $\varphi : (B(r), B_{\mathbb{R}}(r)) \to (M, L)$ be a relative symplectic embedding. Let J_0 be the standard complex structure of $B(r)$. Let $J \in \mathcal{J}(M, \omega)$ be an almost complex structure which extends the complex structure $\varphi_*(J_0)$ defined on the image of the ball $\varphi(B(r))$.

Put $q = \varphi(0) \in L$. By Gromov compactness theorem there exists a non-constant J-holomorphic disk $u : (D, \partial D) \to (M, L)$ with $u(\partial D) \ni q$ and $\mathrm{Area}_\omega(u) \leq E''$. Put $S' = \varphi^{-1}(u(D))$ and let $S'' = \overline{S'}$ be the complex conjugate copy of S'. Put $S = S' \cup S''$, $S^\circ = S \cap \mathrm{Int}\, B(r)$. Clearly S° is an analytic subvariety of $(\mathrm{Int}\, B(r), J_0)$. Note also that due to condition (2) (in the definition of relative symplectic embedding) the subvariety $S^\circ \subset \mathrm{Int}\, B(r)$ is properly embedded (with respect to the induced topology from $\mathrm{Int}\, B(r)$). By the Lelong inequality [35] we have

$$1 \leq \mathrm{mult}_0 S^\circ \leq \frac{\mathrm{Area}_{\omega_{\mathrm{std}}}(S^\circ)}{\pi r^2} = \frac{2\mathrm{Area}_{\omega_{\mathrm{std}}}(S')}{\pi r^2} \leq \frac{2E'}{\pi r^2}.$$

This proves statement 2.

The proof of Statement 1 is very similar (but now no reflection argument is needed). □

Remark 6.6.2. Let us explain the role of the condition (2) in the definition of relative symplectic embeddings. If this condition is dropped and we only require that $\varphi^{-1}(L) \supset B_{\mathbb{R}}(r)$ then the problem becomes equivalent to the problem of absolute symplectic embeddings.

The point is that given any symplectic embedding $\varphi : B(r) \to M$, there exists another symplectic embedding $\varphi' : B(r) \to M$ with $\varphi'(B_{\mathbb{R}}(r)) \subset L$. Indeed by a straightforward argument there exists a Hamiltonian diffeomorphism $h : (M, \omega) \to (M, \omega)$ such that $h \circ \varphi(B_{\mathbb{R}}(r)) \subset L$. Put $\varphi' = h \circ \varphi$.

Thus if we drop condition (2) in the definition of relative symplectic embedding we lose the effect of the presence of the Lagrangian submanifold L. This can be easily illustrated already in dimension $2n = 2$. Let $L \subset \mathbb{R}^2$ be a circle of radius r. Clearly $Gr(L) = \pi r^2$. However, it is easy to see that for every $R > 0$ there exists a symplectic embedding $\varphi : B^2(R) \to \mathbb{R}^2$ with $\varphi(B_{\mathbb{R}}(r)) \subset L$.

Before we continue would like to make a general remark concerning the Gromov radius of Lagrangians. Given a monotone Lagrangian $L \subset (M, \omega)$ we denote by τ the monotonicity constant $\tau = \frac{\omega}{\mu}\big|_{\pi_2(M,L)}$ (see formula (1)).

Remark 6.6.3. Assume that the monotone Lagrangian $L^n \subset (M^{2n}, \omega)$ has the property that $QH_*(L) = 0$. We claim that this implies that for any almost complex structure J, through each point in L passes a non-trivial J-holomorphic of area at most $(n+1)\tau$. This argument appears in [23] (where it is described mainly in the more delicate cluster setup) and goes as follows. Let f be a Morse function on L with a single maximum, x_n. We know from Remark 5.1.4 that $dx_n = 0$ in $\mathcal{C}(L; f, J)$. Thus x_n is a boundary in this complex. But for this to happen some moduli space $\mathcal{P}(y, x_n; \mathbf{A}, J)$ has to be non-trivial. To have $x_n t^{\mu(\mathbf{A})/N_L} = dx + \ldots$ we need that $|x| - 1 = n - \mu(\mathbf{A})$ which means $\mu(\mathbf{A}) \leq n+1$. Thus, there is a J-holomorphic disk passing through x_n of Maslov class at most $n+1$ and of area at most $(n+1)\tau$. As x_n can be chosen as generic point of L this shows the claim.

In case L is displaceable a variant of this argument that also takes into account the action filtration on the Floer complex shows that the area of these disks can be bounded by the displacement energy, $E(L)$.

The existence of these disks implies by Proposition 6.6.1 that, in general we have

$$\pi Gr(L)^2/2 \leq (n+1)\tau$$

and, in the displaceable case, we also have

$$\pi Gr(L)^2/2 \leq E(L) \ .$$

The last inequality (in the displaceable case) can be proven using the cluster machinery even if L is not monotone but is relative spin, orientable and $H_*(L; \mathbb{Q}) = 0$ for all even $*$ different from 0 and $dim(L)$ - see again [23] for this and more details on this argument.

The following Corollary bounds the Gromov radius for Lagrangian tori. Recall from Proposition 6.1.4 that for a monotone Lagrangian torus $T \subset (M, \omega)$ we either have

$QH_*(T) = (H(T) \otimes \Lambda)_*$ or $QH_*(T) = 0$. In the latter case Proposition 6.1.4 implies that for generic J and a generic point $x \in T$ there exists a J-holomorphic disk $u : (D, \partial D) \to (M, T)$ with $u(\partial D) \ni x$ and $\mu([u]) = 2$. Combining this with Proposition 6.6.1 we the following.

Corollary 6.6.4. *If $T \subset (M, \omega)$ is a monotone Lagrangian torus with monotonicity constant τ and $QH_*(T) \neq H_*(T) \otimes \Lambda$, then*

$$\frac{\pi Gr(T)^2}{2} \leq 2\tau.$$

Endow $\mathbb{C}P^n$ with the standard Kähler symplectic structure ω_{FS}, normalized so that $\int_{\mathbb{C}P^1} \omega_{\mathrm{FS}} = \pi$. Note that with this normalization $(\mathbb{C}P^n \setminus \mathbb{C}P^{n-1}, \omega_{\mathrm{FS}}) \cong (\mathrm{Int}\, B(1), \omega_{\mathrm{std}})$, hence the (absolute) Gromov radius of $\mathbb{C}P^n$ is $Gr(\mathbb{C}P^n) = 1$.

Corollary 6.6.5. *Let $L \subset \mathbb{C}P^n$ be a monotone Lagrangian with minimal Maslov number N_L.*

(1) *If $QH_*(L) \neq 0$, then we have:*

$$Gr(\mathbb{C}P^n \setminus L)^2 \leq \frac{[\frac{2n}{N_L}]N_L}{2(n+1)}.$$

In particular,

$$Gr(\mathbb{C}P^n \setminus L)^2 \leq \frac{n}{n+1}.$$

(2) *Suppose that $QH_*(L) \cong H_*(L) \otimes \Lambda$, then:*

$$\frac{Gr(L)^2}{2} + Gr(\mathbb{C}P^n \setminus L)^2 \leq 1.$$

Remark 6.6.6. (1) In most of the examples below we will be in a special situation in which the pearl and Morse differentials agree, $d = \partial_0$, which greatly simplifies the proof of the Corollary (∂_0 is the Morse differential). More precisely, suppose that $QH_*(L) \cong (H(L; \mathbb{Z}_2) \otimes \Lambda)_*$ and that L admits a perfect Morse function $f : L \to \mathbb{R}$. In this case the pearl complex differential d satisfies $d = \partial_0 = 0$. Indeed, since $\partial_0 = 0$ we have $\mathcal{C}_*(L; f, J) \cong (H(L; \mathbb{Z}_2) \otimes \Lambda)_*$. Therefore if $d \neq 0$ we would have $\dim QH_i(L) < \dim (H(L; \mathbb{Z}_2) \otimes \Lambda)_i$ for some i. A contradiction.

(2) Note that the term $[\frac{2n}{N_L}]$ in the first inequality of Corollary 6.6.5 cannot be 0. This is because for any monotone Lagrangian submanifold $L \subset \mathbb{C}P^n$ we have $N_L \leq n+1$. This can be easily proved by the techniques from [15].

Proof of Corollary 6.6.5. We will use here the module structure described in §5.3. As in that section we fix a Morse function $f : L \to \mathbb{R}$ and a Morse function $g : \mathbb{C}P^n \to \mathbb{R}$. We will fix the following notation for the coefficients of the module operation. For $a \in \mathrm{Crit(g), x \in Crit(f)}$ we write

$$a * x = \sum_y n(a, x, y)y + \sum_{y,\mathbf{A}} n(a, x, y; \mathbf{A}, J)yt^{\bar{\mu}(\mathbf{A})} \tag{126}$$

as in formula (72).

We assume that f has a single maximum denoted by x_n. We also assume that g is a perfect Morse function so that we may identify its critical points with the generators of $H_*(\mathbb{C}P^n)$. In particular, the minimum of g, will be denoted by p and is identified with $h^{\cap n}$ where $h = [\mathbb{C}P^{n-1}] \in H_{2n-2}(\mathbb{C}P^n; \mathbb{Z}_2)$ is the homology class of the hyperplane. We recall from the quantum homology of $\mathbb{C}P^n$ that $h * p = h^{*(n+1)} = ut^{2(n+1)/N_L}$ where $u \in H_{2n}(\mathbb{C}P^n; \mathbb{Z}_2)$ is the fundamental class of $\mathbb{C}P^n$. By Remark 5.2.5 we know that as $QH_*(L) \neq 0$, x_n can not be a boundary and hence its homology class $w = [x_n]$ does not vanish in $QH_*(L)$. Put $\alpha = p * w \in QH_{-n}(L)$. Notice that

$$h * \alpha = h * (p * w) = h^{*(n+1)} * w = (u * w)t^{2(n+1)/N_L} = wt^{2(n+1)/N_L}. \tag{127}$$

We deduce that $\alpha \neq 0$. For dimension reasons the classical terms $n(p, x_n, -)$ in formula (126) for $p * x_n$ vanish. Therefore we can write

$$\alpha = p * w = \Big[\sum_{z,\mathbf{A}} n(p, x_n, z; \mathbf{A}, J)zt^{\bar{\mu}(\mathbf{A})}\Big]$$

where $z \in \mathrm{Crit}(f)$ and $[-]$ indicates taking homology classes. We have $-n = |zt^{\bar{\mu}(\mathbf{A})}| = |z| - \mu(\mathbf{A})$, $0 \leq |z| \leq n$ so that if $n(p, x_n, z; \mathbf{A}, J) \neq 0$, as p is the minimum of g, we deduce that there exists a J-holomorphic disk of Maslov index at most $2n$ which passes through p and with boundary on L. As the Maslov index comes in multiples of N_L the Maslov index of that disk is in fact $\leq [\frac{2n}{N_L}]N_L$. As p may be chosen generically and the monotonicity constant τ is here $\pi/(2n+2)$ we deduce the statement at (1).

For statement (2) we first pursue with the analysis of equation (127) under the additional asumption that the function f is so that the differential of the pearl complex coincides with the Morse differential, $d = \partial_0$. As mentioned in Remark 6.6.6 this condition is often satisfied and the proof in this case is relatively straightforward.

We write:

$$\begin{aligned} w\ t^{2(n+1)/N_L} = h * \alpha = & \Big[\sum_{z,\mathbf{A}} n(p,x_n,z;\mathbf{A},J)(h*z)t^{\bar{\mu}(\mathbf{A})}\Big] \\ = & \Big[\sum_{z,\mathbf{A},y} n(p,x_n,z;\mathbf{A},J)n(h,z,y)yt^{\bar{\mu}(\mathbf{A})} \\ & + \sum_{z,\mathbf{A},z',\mathbf{A}'} n(p,x_n,z;\mathbf{A},J)n(h,z,z';\mathbf{A}',J)z't^{\bar{\mu}(\mathbf{A})+\bar{\mu}(\mathbf{A}')}\Big], \end{aligned}$$

This means that in the chain complex $\mathcal{C}(L;f,J)$ we have:

$$\begin{aligned} & x_n - \sum_{z,\mathbf{A},y} n(p,x_n,z;\mathbf{A},J)n(h,z,y)yt^{(\mu(\mathbf{A})-2(n+1))/N_L} \\ & - \sum_{z,\mathbf{A},z',\mathbf{A}'} n(p,x_n,z;\mathbf{A},J)n(h,z,z';\mathbf{A},J)z't^{(\mu(\mathbf{A})+\mu(\mathbf{A}')-2(n+1))/N_L} \in \text{image}\,(d). \end{aligned} \tag{128}$$

Since $d = \partial_0$ and since x_n is not homologous to any element in

$$\bigoplus_{j\neq 0} \mathbb{Z}_2 x_n t^j \bigoplus (\mathbb{Z}_2\langle \text{Crit}(f) \setminus \{x_n\}\rangle \otimes \Lambda)$$

it follows that x_n must appear in one of the two sums in formula (128). But the y's in the first sum of (128) satisfy $|y| = |z| - 2 \leq n-2$, hence $y \neq x_n$. Therefore x_n must appear as one of the summands in the second sum of (128). In other words there exist z, and vectors $\mathbf{A}, \mathbf{A}'$ of non-zero classes in $H_2(M,L)$ such that:

(1) $n(p,x_n,z;\mathbf{A},J) \neq 0$, $n(h,z,x_n;\mathbf{A}',J) \neq 0$.
(2) $\mu(\mathbf{A}) + \mu(\mathbf{A}') = 2(n+1)$.

This means that through p (the minimum of h) passes a non-constant J-holomorphic disk of Maslov index at most $\mu(\mathbf{A})$ and through x_n passes the boundary of non-constant J-holomorphic disk of Maslov index at most $\mu(\mathbf{A}') = 2(n+1) - \mu(\mathbf{A})$. By Proposition 6.6.1 we have:

$$\frac{\pi Gr(L)^2}{2} + \pi Gr(\mathbb{C}P^n, L)^2 \leq \tau\mu(\mathbf{A}) + \tau\mu(\mathbf{A}') = \tau(2n+2) = \pi.$$

To conclude the proof of the point (2) of the proposition we now need to describe an argument in the absence of the condition $d = \partial_0$. To do so we will make use of the minimal model technology as described in §2.2.3. Thus, recall that there is a complex $(\mathcal{C}_{min}, \delta)$, $\mathcal{C}_{min} = H(L;\mathbb{Z}_2) \otimes \Lambda$, and chain morphisms $\phi : \mathcal{C}(L;f,J) \to \mathcal{C}_{min}$, $\psi : \mathcal{C}_{min} \to \mathcal{C}(L;f,J)$ so that $\delta_0 = 0$ and $\phi \circ \psi = id$, ϕ, ψ (respectively, ϕ_0, ψ_0) induce isomorphisms in pearl (respectively, Morse) homology. As in Remark 2.2.6 we use the applications ϕ and ψ to transport the module structure on $\mathcal{C}_{min}$. For $u \in \mathrm{Crit(h)}$ and $x \in H_*(L;\mathbb{Z}_2)$ this module structure has the form:

$$u * x = \sum_{k, y \in H_*(L;\mathbb{Z}_2)} n'(u, x, y; k) y t^k \tag{129}$$

so that $u * x = \phi(u * \psi(x))$. It is important to note that, as the minimal pearl model is constructed (in §5.9) with coefficients in Λ^+ we have that, for degree reasons, $\psi([L]) = x_n$, $\phi(x_n) = [L]$ (where $[L] \in H_n(L;\mathbb{Z}_2)$ is the fundamental class). It follows that all the argument above can now be applied to $\mathcal{C}_{min}$ instead of $\mathcal{C}(L; f, J)$. It leads to the fact that there is $z \in H_*(L;\mathbb{Z}_2)$ for which there are coefficients $n'(-,-,-)$ which verify:

(1') $n'(p, [L], z; k) \neq 0$, $n'(h, z, [L]; k') \neq 0$.
(2') $(k + k')N_L = 2(n + 1)$.

The condition $QH(L) \cong H(L;\mathbb{Z}_2) \otimes \Lambda$ means that $\delta = 0$ in $\mathcal{C}_{min}$ (see Remark 2.2.6). With this in mind we take another look at equation (129) and taking into account that both ϕ and ψ are defined over Λ^+ we deduce from $n'(p, [L], z; k) \neq 0$ that there exists $u \in \mathrm{Crit(f)}$ and $\mathbf{A}$ so that $\mu(\mathbf{A}) \leq kN_L$ and $n(p, x_n, u; \mathbf{A}, J) \neq 0$ and so through the minimum of h passes a disk of Maslov class at most kN_L. We now want to interpret the equation $n'(h, z, [L]; k') \neq 0$. This obviously means that in $H_*(\mathcal{C}_{min})$ we have $h*z = \ldots + [L]t^{k'} + \ldots$. In other words, if we write $H_*(\mathcal{C}_{min}) = H(L;\mathbb{Z}_2) \otimes \Lambda = E \oplus \mathbb{Z}_2 < [L]t^{k'} >$ where E is some complement of the $\mathbb{Z}_2$ vector space $\mathbb{Z}_2 < [L]t^{k'} >$ we have $h * z = \xi + [L]t^{k'}$ with $\xi \in E$. This implies, in homology, $h * \psi(z) = \psi(\xi) + [x_n]t^{k'}$ so that at the chain level we have $h * \psi(z) + dw = x_n t^{k'} + \psi(\xi)$. We also split $\mathcal{C}(L; f, J) = E' \oplus \mathbb{Z}_2 < x_n t^{k'} >$ and we consider two cases. First suppose that there is some cycle $\xi' \in E'$ so that $[\xi'] = [x_n t^{k'}]$. The only possibility for that to happen is that there is some $u'' \in \mathrm{Crit(f)}$, $\mathbf{A}''$ with $\mu(\mathbf{A}'') \leq k'N_L$ so that $n(u'', x_n; \mathbf{A}'', J) \neq 0$ where $n(-,-)$ are the coefficients of the differential in the pearl complex $\mathcal{C}(L; f, J)$. In this case, it follows that through the maximum of f passes a disk of Maslov class at most $k'N_L$ and the proof ends as in the special case treated before. Assume therefore from now on that for all the cycles ξ' in E' we have $[\xi'] \neq [x_n t^{k'}]$. In particular, this means that $x_n t^s$ does not appear in the boundary expression of any critical point of f for $s \leq k'$. At this point we need to recall the explicit construction of the map ϕ: it is defined on a basis of $\mathbb{Z}_2 < \mathrm{Crit(f)} >$ which is chosen so that it is formed by three types of elements x,'s with $\partial_0 x = 0$, y's with $\partial_0 y = 0$ and y''s with $\partial_0 y' = y$ and is given by $\phi(x) = x$, $\phi(y') = 0$, $\phi y = \phi^{n-s+1}(dy' - y)$ (the last equality appears in the inductive step - see again §5.9). The critical point x_n is itself a generator of type x and it is the only one in dimension n. Under our assumption, this means that we have $\phi(E') = E$. Indeed, the only thing to check is that each generator of type y is sent to E but this follows inductively because dy' does not contain $x_n t^s$ for $s \leq k'$.

We now write $\psi(\xi) = \xi'' + \epsilon x_n t^{k'}$ where ϵ is 0 or 1 and $\xi'' \in E'$. In case $\epsilon = 1$ we have, in homology, $h * \psi(z) = [\xi'']$ and, so $\phi(h * \psi(z)) = h * z = \phi[\xi''] = \xi + [L]t^k$ which contradicts $\phi(\xi'') \in E$ (we use here again $\delta = 0$).

Thus we are left to discuss the case when $\epsilon = 0$. Therefore we have $h * \psi(z) + dw = x_n t^{k'} + \xi''$ and given our assumption this means that there exists $u' \in \mathrm{Crit(f)}$, $\mathbf{A}'$ with $\mu(\mathbf{A}') \leq k' N_L$ so that $n(h, u', x_n; \mathbf{A}', J) \neq 0$. It again follows that through the maximum of f passes a disk of Maslov class at most $k' N_L$ and concludes the proof. □

6.7. **Examples.** Endow $\mathbb{C}P^n$ with the standard Kähler symplectic structure ω_{FS}, normalized so that $\int_{\mathbb{C}P^1} \omega_{\mathrm{FS}} = \pi$.

Corollary 6.7.1. *Let $L \subset \mathbb{C}P^n$ be a Lagrangian submanifold with $2H_1(L; \mathbb{Z}) = 0$. Then $Gr(\mathbb{C}P^n \setminus L)^2 \leq \frac{1}{2}$.*

Proof. This follows immediately from Proposition 6.2.12 in conjunction with Proposition 6.6.1.

Alternatively, one can use Proposition 6.2.5 by which $N_L = n + 1$ and $QH_*(L) \neq 0$. Then Corollary 6.6.5 implies that:

$$Gr(\mathbb{C}P^n \setminus L)^2 \leq \frac{[\frac{2n}{n+1}](n+1)}{2(n+1)} = \frac{1}{2}.$$

□

When L is the standard real projective space $\mathbb{R}P^n \subset \mathbb{C}P^n$ the inequality in Corollary 6.7.1 has been proved before by Biran [14] by different methods. In this case the inequality turns out to be sharp. Note also that (by an explicit construction) we have $Gr(\mathbb{R}P^n) = 1$, thus the inequality in Corollary 6.6.5-(2) is sharp in this case.

Consider the n-dimensional Clifford torus $\mathbb{T}^n_{\mathrm{clif}} \subset \mathbb{C}P^n$. (See Example 6.1.6.)

Corollary 6.7.2. $Gr(\mathbb{T}^n_{\mathrm{clif}})^2 \leq \frac{2}{n+1}$, $\quad Gr(\mathbb{C}P^n \setminus \mathbb{T}^n_{\mathrm{clif}})^2 = \frac{n}{n+1}$.

Remark. By an explicit packing construction, communicated to us by Buhovsky, it seems that the first inequality in Corollary 6.7.2 is in fact sharp, i.e. $Gr(\mathbb{T}^n_{\mathrm{clif}})^2 = \frac{2}{n+1}$.

Proof of Corollary 6.7.2. A straightforward construction using moment maps (in the spirit of [37, 67]) shows that $Gr(\mathbb{C}P^n \setminus \mathbb{T}_{\mathrm{clif}}) \geq \frac{n}{n+1}$. On the other hand, by Example 6.1.6 we have $QH_*(\mathbb{T}^n_{\mathrm{clif}}) \cong (H(\mathbb{T}^n_{\mathrm{clif}}; \mathbb{Z}_2) \otimes \Lambda)_*$ hence by Corollary 6.6.5-(1) we have $Gr(\mathbb{C}P^n \setminus \mathbb{T}^n_{\mathrm{clif}}) \leq \frac{n}{n+1}$. This proves that $Gr(\mathbb{C}P^n \setminus \mathbb{T}^n_{\mathrm{clif}}) = \frac{n}{n+1}$.

The first inequality follows now from the fact that $QH_*(\mathbb{T}^n_{\mathrm{clif}}) \cong (H(\mathbb{T}^n_{\mathrm{clif}}; \mathbb{Z}_2) \otimes \Lambda)_*$ in conjunction with Corollary 6.6.5-(2) and Remark 6.6.6. An alternative, more direct, proof goes as follows. By Example 6.1.6 it follows that for generic $J \in \mathcal{J}(\mathbb{C}P^n)$ and $x \in \mathbb{T}^n_{\mathrm{clif}}$

there exists a J-holomorphic disk $u : (D, \partial D) \to (\mathbb{C}P^n, L)$ with $\mu([u]) = 2$. This disk has area $\frac{\pi}{n+1}$. By Proposition 6.6.1-(2) we obtain $Gr(\mathbb{T}^n_{\text{clif}}) \leq \frac{1}{2(n+1)}$. $\square$

6.8. **Mixed symplectic packing.** Let $l, m \geq 0$ and $r_1, \ldots, r_l > 0$, $\rho_1, \ldots, \rho_m > 0$. A mixed symplectic packing of (M, L) by balls of radii $(r_1, \ldots, r_l; \rho_1, \ldots, \rho_m)$ is given by l relative symplectic embeddings $\varphi_i : (B(r_i), B_{\mathbb{R}}) \to (M, L)$, $i = 1, \ldots, l$, and m symplectic embeddings $\psi_j : B(r_j) \to (M \setminus L, \omega)$, $j = 1, \ldots, m$, such that the images of all the φ_i and ψ_j are mutually disjoint, i.e.

(1) $\varphi_{i'}(B(r_{i'})) \cap \varphi_{i''}(B(r_{i''})) = \emptyset$ for every $i' \neq i''$.
(2) $\psi_{j'}(B(r_{j'})) \cap \psi_{j''}(B(r_{j''})) = \emptyset$ for every $j' \neq j''$.
(3) $\varphi_i(B(r_i)) \cap \psi_j(B(r_j)) = \emptyset$ for every i, j.

The following can be proved in a similar way to Proposition 6.6.1.

Proposition 6.8.1. *Let $L \subset (M, \omega)$ be a Lagrangian submanifold and $E > 0$. Suppose that there exists a dense subset $\mathcal{J}_* \subset \mathcal{J}(M, \omega)$, a dense subset of m-tuples $\mathcal{U}' \subset (M \setminus L)^{\times m}$, and a dense subset of l-tuples $\mathcal{U}'' \subset L^{\times l}$ such that for every $J \in \mathcal{J}_*$, $(q_1, \ldots, q_l) \in \mathcal{U}'$, $(p_1, \ldots, p_m) \in \mathcal{U}''$ there exists a non-constant J-holomorphic disk $u : (D, \partial D) \to (M, L)$ with $u(\text{Int}\, D) \ni p_1, \ldots, p_m$, $u(\partial D) \ni q_1, \ldots, q_l$ and $\text{Area}_\omega(u) \leq E$. Then for every mixed symplectic packing of (M, L) by balls of radii $(r_1, \ldots, r_l; \rho_1, \ldots, \rho_m)$ we have:*

$$\sum_{i=1}^{l} \frac{\pi r_i^2}{2} + \sum_{j=1}^{m} \pi \rho_j^2 \leq E.$$

6.9. **Examples.**

Corollary 6.9.1. *For every mixed symplectic packing of $(\mathbb{C}P^2, \mathbb{T}^2_{\text{clif}})$ by two balls of radii $(r; \rho)$ we have $\frac{1}{2}r^2 + \rho^2 \leq \frac{2}{3}$. In particular if the two balls are assumed to have the same radius $r = \rho$ then $r^2 \leq \frac{4}{9}$.*

Note that the inequality in Corollary 6.9.1 is stricter than the inequalities for (absolute) packing of $\mathbb{C}P^2$ by 2 balls. Indeed, for every symplectic packing of $\mathbb{C}P^2$, $B(r) \coprod B(\rho) \hookrightarrow \mathbb{C}P^2$ the (optimal) packing inequality reads $r^2 + \rho^2 \leq 1$ (See [36, 43]). In particular, if the balls are of equal radii $r = \rho$ the latter inequality reads $r^2 \leq \frac{1}{2}$ (and this is sharp), whereas the mixed packing inequality gives $r^2 \leq \frac{4}{9}$.

We do not know whether the inequality in Corollary 6.9.1 is sharp.

Proof of Corollary 6.9.1. We will use here the notation of §6.2.1 and Proposition 6.2.3. Fix two Morse functions $g : \mathbb{C}P^2 \to \mathbb{R}$, $f : \mathbb{T}^2_{\text{clif}} \to \mathbb{R}$ and two generic Riemannian metrics ν, $\nu_{\mathbb{T}}$ on $\mathbb{C}P^2$ and $\mathbb{T}^2_{\text{clif}}$. We assume that both g and f are perfect Morse functions so that we can identify their critical points with homology classes in $H_*(\mathbb{C}P^2; \mathbb{Z}_2)$ and $H_*(\mathbb{T}^2_{\text{clif}}; \mathbb{Z}_2)$.

Denote by $h \in H_2(\mathbb{C}P^2;\mathbb{Z}_2)$ the class of the hyperplane and by $p \in H_0(\mathbb{C}P^2;\mathbb{Z}_2)$ the class of a point. Denote by m the minimum of f. For every generic $J \in \mathcal{J}(\mathbb{C}P^2)$, m defines an element, still denoted m, in $QH_0(\mathbb{T}^2_{\text{clif}}; f, \nu_{\mathbb{T}}, J)$. (See the discussion in the beginning of §6.11.) According to Proposition 6.2.3 we have

$$p * m = h * h * m = h * mt = mt^2.$$

The pearly trajectories that potentially contribute to this computation appear in figure 20.

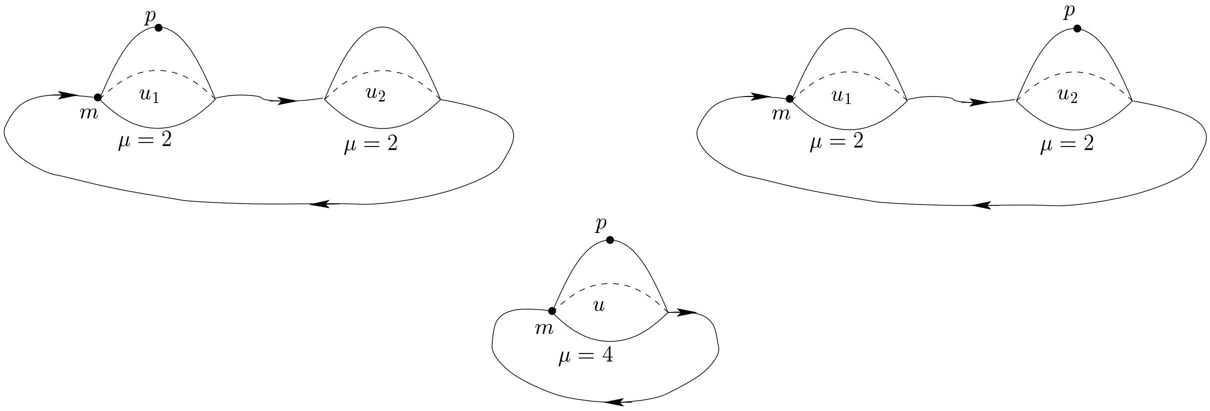

FIGURE 20.

It follows that for generic J one of the following three possibilities occur:

(1) There exists a J-holomorphic disk $u_1 : (D, \partial D) \to (\mathbb{C}P^2, \mathbb{T}^2_{\text{clif}})$ with $\mu([u_1]) = 2$ and $u_1(-1) = m$, $u_1(0) = p$.
(2) There exist two J-holomorphic disks $u_1, u_2 : (D, \partial D) \to (\mathbb{C}P^2, \mathbb{T}^2_{\text{clif}})$ with $\mu([u_1]) = \mu([u_2]) = 2$ and $u_1(-1) = m$, $u_2(0) = p$.
(3) There exists a J-holomorphic disk $u : (D, \partial D) \to (\mathbb{C}P^2, \mathbb{T}^2_{\text{clif}})$ with $\mu([u]) = 4$ and $u(-1) = m$, $u(0) = p$.

Note that the in cases (1), (2) we have $\text{Area}_{\omega_{\text{FS}}}(u_1) = \text{Area}_{\omega_{\text{FS}}}(u_2) = 2\tau = 1/3$ and in case (3) $\text{Area}_{\omega_{\text{FS}}}(u) = 4\tau = 2/3$.

Let $\varphi : B(r) \to (\mathbb{C}P^2, \mathbb{T}^2_{\text{clif}})$, $\psi : B(\rho) \to \mathbb{C}P^2 \setminus \mathbb{T}^2_{\text{clif}}$ be a mixed symplectic packing of $(\mathbb{C}P^2, \mathbb{T}^2_{\text{clif}})$ by two balls of radii $(r; \rho)$. Take m to be $\varphi(0)$ and $p = \psi(0)$. Arguing as in the proof of Propositions 6.6.1 and 6.8.1 we obtain the following inequality:

$$\frac{1}{2}\pi r^2 + \pi \rho^2 \leq \max\{2\tau, 2\tau + 2\tau, 4\tau\} = 4\tau = \frac{2\pi}{3}.$$

□

The proof of Corollary 6.9.1 suggests that for mixed packing of $(\mathbb{C}P^n, \mathbb{T}^n_{\text{clif}})$ by two balls of radii $(r;\rho)$ the following packing inequality should hold: $\frac{1}{2}r^2 + \rho^2 \leq \frac{n}{n+1}$.

Consider the smooth complex quadric $Q \subset \mathbb{C}P^{n+1}$ endowed with the symplectic structure ω induced from $\mathbb{C}P^{n+1}$ (See §6.3). With our normalization of the symplectic structure on $\mathbb{C}P^{n+1}$ the symplectic structure on Q and the first Chern class c_1 (of the tangent bundle of Q) have the following relation $c_1 = n[\omega]/\pi$.

Corollary 6.9.2. *Let $L \subset Q$ be a Lagrangian submanifold with $H_1(L;\mathbb{Z}) = 0$ (e.g. a Lagrangian sphere). Then for every mixed symplectic packing of (Q, L) with 2 balls of radii $(r;\rho)$ we have: $\frac{1}{2}r^2 + \rho^2 \leq 1$.*

Note that inequality in Corollary 6.9.2 is stricter than the absolute packing inequalities, at least in dimension $2n = 4$. Indeed in that case $(Q, \omega) \cong (\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{\text{FS}} \oplus \omega_{\text{FS}})$ and the (optimal) absolute packing inequalities for two balls $B(r) \coprod B(\rho) \hookrightarrow (\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{\text{FS}} \oplus \omega_{\text{FS}})$ read $r^2 \leq 1$, $\rho^2 \leq 1$.

We do not know whether the inequality in Corollary 6.9.2 is sharp.

Proof of Corollary 6.9.2. This follows from Corollary 6.3.4 and Proposition 6.8.1. □

6.10. **Further questions.** The results in §6.6- 6.9 give rise to several questions. The first one is whether the packing inequalities above are sharp. This is especially relevant for the results in Corollaries 6.6.5, 6.7.1, 6.7.2, 6.9.1, 6.9.2. This would require to obtain lower bounds on the radii of balls in relative/mixed symplectic packing. Such bounds can sometimes be obtained by an explicit packing construction, but one would like a more systematic method. In the theory of absolute symplectic packing an essential ingredient is the symplectic blowing up and down constructions (see [43]). It seems relevant to establish a relative (with respect to a Lagrangian L) version of the symplectic blowing up and down constructions. Another important ingredient in the theory of absolute symplectic packing consists of criteria for realizing 2-dimensional cohomology classes by symplectic/Kähler forms such as the Nakai-Moishezon criterion (see [11, 13, 12] for symplectic analogues of this). In the relative version one would expect to have analogous criteria with the additional requirement that the resulting symplectic form makes a given submanifold Lagrangian.

Another interesting question is what happens for larger number of balls. This would require to establish existence of holomorphic disks (or surfaces with boundary) passing through many points on L. It seems likely that our methods combined with the A_∞ approach of Fukaya-Oh-Ohta-Ono [34] or the cluster homology approach of Cornea-Lalonde [23] would be relevant for this purpose. In the same direction, it would be interesting to find out whether the packing obstructions disappear in the relative case once the number of balls becomes large enough, as happens in the absolute case in dimension 4 (see [11, 12]).

Some of the results on existence of holomorphic disks in §6.2.4 and 6.3.4 give rise to redundant packing inequalities in the sense that they coincide (or can be derived from) the absolute packing inequalities. For example, Proposition 6.2.12 implies that for every relative symplectic packing

$$(B(r_1), B_{\mathbb{R}}(r_1)) \coprod (B(r_2), B_{\mathbb{R}}(r_2)) \hookrightarrow (\mathbb{C}P^n, \mathbb{R}P^n)$$

we have $\frac{1}{2}r_1^2 + \frac{1}{2}r_2^2 \leq \frac{1}{2}$. However this is precisely the (optimal) absolute packing inequality for two balls in $\mathbb{C}P^n$ (see [36]). In fact, an explicit construction as in [37] shows that for $l \leq 3$ balls there is no difference between relative and absolute symplectic packing (i.e. the same packing inequalities hold for both cases). We do not know whether this continues to hold for general l (even in dimension $2n = 4$). It would also be interesting to find a geometric explanation to why the relative symplectic packing problem coincides with the absolute one for some Lagrangians while for others it gives stricter restrictions.

6.11. **Quantum product on tori and enumerative geometry.** The goal of this section is to give a geometric interpretation of the quantum cap product for Lagrangian tori in terms of enumeration of holomorphic disks. Related results in this direction have recently been obtained for torus fibres of Fano toric manifolds by Cho [19] by a different approach. For simplicity we consider here only the case of 2-dimensional tori.

Let $L^2 \subset (M^4, \omega)$ be a monotone Lagrangian torus with minimal Maslov number $N_L = 2$. Assume that $QH_*(L) \neq 0$. By Proposition 6.1.4 we have $QH_*(L) \cong (H(L; \mathbb{Z}_2) \otimes \Lambda)_*$. It follows that

$$QH_0(L) \cong H_0(L; \mathbb{Z}_2) \oplus H_2(L; \mathbb{Z}_2)t, \tag{130}$$

$$QH_1(L) \cong H_1(L; \mathbb{Z}_2). \tag{131}$$

We will see in a moment that the isomorphism (131) is canonical. However, the splitting in (130) is *not canonical* in the sense that it is not compatible with the canonical identifications in Morse homology. As we will see below, the "$H_2(L; \mathbb{Z}_2)t$" part is canonical but the inclusion in $QH_0(L)$ of the summand $H_0(L; \mathbb{Z}_2)$ actually depends on the Morse function $f : L \to \mathbb{R}$ and the almost complex structure J used to compute $QH(L)$. Let us explain this point in more detail.

Let $f, g : L \to \mathbb{R}$ be two *perfect* Morse functions. Fix two generic Riemannian metrics ρ, τ on L and a generic almost complex structure $J \in \mathcal{J}(M, \omega)$. Denote:

$$\mathrm{Crit}_0(f) = \{x_0\}, \quad \mathrm{Crit}_1(f) = \{x_1', x_1''\}, \quad \mathrm{Crit}_2(f) = x_2,$$
$$\mathrm{Crit}_0(g) = \{y_0\}, \quad \mathrm{Crit}_1(g) = \{y_1', y_1''\}, \quad \mathrm{Crit}_2(g) = y_2.$$

Since $QH_*(L) \cong (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$ the Floer differential vanishes and we have:

$$QH_0(f, \rho, J) = \mathbb{Z}_2 x_0 \oplus \mathbb{Z}_2 x_2 t, \quad QH_1(f, \rho, J) = \mathbb{Z}_2 x_1' \oplus \mathbb{Z}_2 x_1'',$$
$$QH_0(g, \tau, J) = \mathbb{Z}_2 y_0 \oplus \mathbb{Z}_2 y_2 t, \quad QH_1(g, \tau, J) = \mathbb{Z}_2 y_1' \oplus \mathbb{Z}_2 y_1''.$$

Denote by $\phi^M : H_*(g, \tau) \to H_*(f, \rho)$ the canonical isomorphisms of Morse homologies. Denote by $\phi^F : QH_*(g, \tau, J) \to QH_*(f, \rho, J)$ the canonical isomorphisms of Floer homologies as described in §5.1.2. Then for degree reasons ϕ^F coincides with ϕ^M on $QH_1(g, \tau, J)$. Clearly we also have $\phi^M(y_0) = x_0$ and $\phi^M(y_2) = x_2$. Moreover $\phi^F(y_2) = x_2$ since y_2, x_2 are the unities of the respective Floer homologies. However it might happen that $\phi^F(y_0) \neq \phi^M(y_0)$. More precisely, following the description in §5.1.2 we have $\phi^F(y_0) = x_0 + \epsilon x_2 t$ where the coefficient $\epsilon \in \mathbb{Z}_2$ is determined by counting the number of pearly trajectories appearing in figure 21.

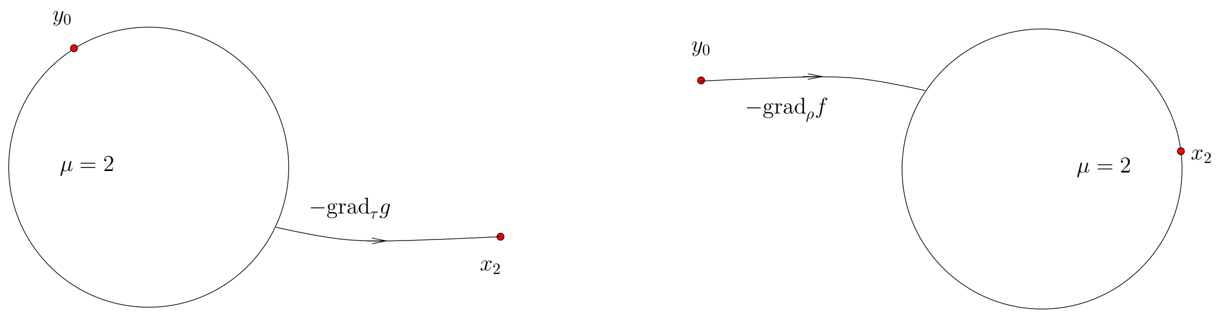

FIGURE 21. Pearly trajectories contributing to the coefficient ϵ.

As f, g, ρ, τ, J are taken to be generic we may assume that $y_0 \in W^s_{x_0}(-\mathrm{grad}_\rho f)$, $x_2 \in W^u_{y_2}(-\mathrm{grad}_\tau g)$ and that no two points from x_0, y_0, x_2, y_2 lie on the boundary of the same J-holomorphic disk with Maslov number 2. Denote by $\gamma_{y_2,x_2}(g)$ the $-\mathrm{grad}_\tau(g)$ trajectory connecting y_2 to x_2 and by $\gamma_{y_0,x_0}(f)$ the $-\mathrm{grad}_\rho(f)$ trajectory connecting y_0 to x_0. Using this and the notation from (102) we have (see figure 22):

$$\epsilon = \#_{\mathbb{Z}_2}\big(\delta_{y_0}(J) \cap \gamma_{y_2,x_2}(g)\big) + \#_{\mathbb{Z}_2}\big(\delta_{x_2}(J) \cap \gamma_{y_0,x_0}(f)\big). \tag{132}$$

By Proposition 6.1.4 the cycles $\delta_{y_0}(J)$ and $\delta_{x_2}(J)$ are 0 in the homology $H_1(L;\mathbb{Z}_2)$. Therefore, the path $\gamma_{y_2,x_2}(g)$ in formula (132) can be replaced by any path $\ell(y_2, x_2) \subset L$

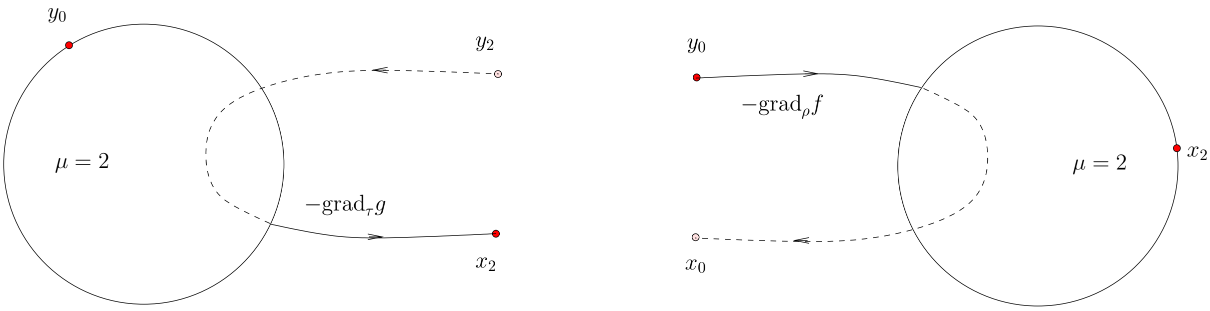

FIGURE 22.

connecting y_2 to x_2 (as long as the intersection of that path with $\delta_{y_0}(J)$ is transverse). Similarly $\gamma_{y_0,x_0}(f)$ can be replaced by any path $\ell(y_0, x_0)$.

Summarizing the above we have:

Proposition 6.11.1. *The isomorphism $\phi^F : QH_*(g, \tau, J) \longrightarrow QH_*(f, \rho, J)$ is given by:*

$$\phi^F(y) = \phi^M(y)\ \forall y \in QH_1(g, \tau, J), \quad \phi^F(y_2) = x_2, \quad \phi^F(y_0) = x_0 + \epsilon x_2 t,$$

where

$$\epsilon = \#_{\mathbb{Z}_2}\big(\delta_{y_0}(J) \cap \ell(y_2, x_2)\big) + \#_{\mathbb{Z}_2}\big(\delta_{x_2}(J) \cap \ell(y_0, x_0)\big) \tag{133}$$

and $\ell(y_2, x_2), \ell(y_0, x_0) \subset L$ are any two paths in L joining y_2 to x_2, resp. y_0 to x_0, and meeting the cycles $\delta_{y_0}(J)$, resp. $\delta_{x_2}(J)$, transversely. In particular, the isomorphism $QH_1(L) \cong H_1(L; \mathbb{Z}_2)$ as well as the inclusion $H_2(L; \mathbb{Z}_2) \to QH_2(L; \mathbb{Z}_2)$ are canonical.

Remark 6.11.2.
1. From formula (133) it follows that once we fix a generic J and take f, g with the critical points x_2 close enough to y_2 and x_0 close enough to y_0 then the coefficient ϵ is 0. Thus in this case $\phi^F = \phi^M$.
2. One can derive a similar formula to (133) when we have two different almost complex structures J_0, J_1. In this case one takes a generic path $\{J_t\}$ connecting J_0 to J_1. Then there is an additional contribution to the coefficient ϵ coming from J_t-holomorphic disks (for some t) with Maslov index 2 whose boundaries pass through the points y_0, x_2.

Let L be as above. Denote by $w \in QH_2(L) = H_2(L; \mathbb{Z}_2)$ the fundamental class. Choose $m \in QH_0(L; \mathbb{Z}_2)$ so that $\{m, wt\}$ consists of a basis for $QH_0(L)$. As $m * m \in QH_{-2}(L) = QH_0(L) \otimes t$ we can write

$$m * m = s_1 m t + s_2 w t^2, \tag{134}$$

for some $s_1, s_2 \in \mathbb{Z}_2$. We first claim that the coefficient s_1 does not depend on the choice of the element m. Indeed, if we replace m by $m' = m + wt$ then

$$m' * m' = m * m + wt^2 = s_1 mt + (s_2 + 1)wt^2 = s_1 m't + (s_1 + 1 + s_2)wt^2.$$

This also shows that when $s_1 = 1$ the coefficient s_2 is also independent of the choice of m.

Proposition 6.11.3. *The coefficient s_1 can be computed as follows. Let $J \in \mathcal{J}$ be a generic almost complex structure. Let $p_1, p_2, p_3 \in L$ be a generic triple of points. Let $\ell(p_1, p_2)$, $\ell(p_2, p_3)$, $\ell(p_3, p_1) \subset L$ be three paths such that $\ell(p_i, p_j)$ connects p_i to p_j and intersects $\delta_{p_k}(J)$ transversely, where $p_k \in \{p_1, p_2, p_3\}$ is the third point (i.e. $k \neq i, j$). Then:*

$$s_1 = \#_{\mathbb{Z}_2}\big(\delta_{p_1}(J) \cap \ell(p_2, p_3)\big) + \#_{\mathbb{Z}_2}\big(\delta_{p_2}(J) \cap \ell(p_3, p_1)\big) + \#_{\mathbb{Z}_2}\big(\delta_{p_3}(J) \cap \ell(p_1, p_2)\big). \tag{135}$$

In case $s_1 = 1$ the coefficient s_2 can be computed as follows. Denote by $n_4(p_1, p_2, p_3)$ the number modulo 2 of simple J-holomorphic disks $u : (D, \partial D) \to (M, L)$ with $\mu([u]) = 4$ and such that $u(e^{-2\pi i l/3}) = p_l$ for every $1 \leq l \leq 3$. Then:

$$s_2 = \#_{\mathbb{Z}_2}\big(\delta_{p_3}(J) \cap \ell(p_1, p_2)\big) \cdot \#_{\mathbb{Z}_2}\big(\delta_{p_1}(J) \cap \ell(p_2, p_3)\big) + n_4(p_1, p_2, p_3). \tag{136}$$

Moreover, when $s_1 = 1$ we have the following identities:

$$\begin{aligned} &\#_{\mathbb{Z}_2}\big(\delta_{p_2}(J) \cap \ell(p_3, p_1)\big) \cdot \#_{\mathbb{Z}_2}\big(\delta_{p_3}(J) \cap \ell(p_1, p_2)\big) \\ =&\#_{\mathbb{Z}_2}\big(\delta_{p_3}(J) \cap \ell(p_1, p_2)\big) \cdot \#_{\mathbb{Z}_2}\big(\delta_{p_1}(J) \cap \ell(p_2, p_3)\big) \\ =&\#_{\mathbb{Z}_2}\big(\delta_{p_1}(J) \cap \ell(p_2, p_3)\big) \cdot \#_{\mathbb{Z}_2}\big(\delta_{p_2}(J) \cap \ell(p_3, p_1)\big). \end{aligned} \tag{137}$$

$$n_4(p_1, p_2, p_3) = n_4(p_1, p_3, p_2). \tag{138}$$

Remark 6.11.4.
1. It is interesting to note that the numbers $n_4(p_1, p_2, p_3)$ depend on the position of the three points p_1, p_2, p_3 (and of course on J). Thus the number of $\mu = 4$ simple J-holomorphic disks with boundary passing through 3 points on L is *not a symplectic invariant.*
2. The J-holomorphic disks counted by $n_4(p_1, p_2, p_3)$ are different than those counted by $n_4(p_1, p_3, p_2)$. Nevertheless these two numbers (mod 2) are equal. Thus the number of J-holomorphic disks of Maslov index 4 whose boundary passes through p_1, p_2, p_3 in any possible order is even.
3. Essentially the same result as Proposition 6.11.3 holds for any orientable monotone Lagrangian L with $N_L = 2$ and $\text{genus}(L) > 0$. However, other than tori we are not aware of any examples of such Lagrangians. Note that if such a Lagrangian $L^2 \subset M^4$ exists then M cannot have $b_2^+ = 1$. By classification results this means that M cannot be a monotone symplectic manifold (i.e. $c_1(M) = \lambda[\omega] \in H^2(M; \mathbb{R})$

for some $\lambda > 0$) or even birationally equivalent to it. Similarly M cannot be a (blow up of a) ruled surface.

Proof of Proposition 6.11.3. Fix a generic $J \in \mathcal{J}(M, \omega)$. Let $f, g : L \to \mathbb{R}$ be two perfect Morse functions. Fix a generic Riemannian metric on L. Denote by x_0 (resp. y_0) the minimum of f (resp. g) and by x_2 (resp. y_2) the maximum of f (resp. g). We can choose the functions f, g so that $x_0 = p_1$, $y_0 = p_2$, $x_2 = p_3$ and so that y_2 is very close to $x_2 = p_3$.

With this data fixed we have two versions of the quantum cap product:

$$*_{f,g} : QH(f, J) \otimes QH(g, J) \to QH(f, J),$$
$$* : QH(f, J) \otimes QH(f, J) \to QH(f, J).$$

The relation between these products is that for $x \in QH(f, J)$, $y \in QH(g, J)$, we have $x *_{f,g} y = x * \phi^F(y)$, where ϕ^F is the isomorphism from Proposition 6.11.1.

Viewing x_0, x_2, y_0, y_2 as elements of $QH(f, J)$ and $QH(g, J)$ we can write:

(139) $$x_0 *_{f,g} y_0 = r_1 x_0 t + r_2 x_2 t, \quad \text{for some } r_1, r_2 \in \mathbb{Z}_2.$$

From the definition of the quantum cap product it follows that the coefficient r_1 is given by counting trajectories as in figure 23. Denote by $\gamma_{x_2,y_0}(f)$ the $-\mathrm{grad} f$ trajectory connecting

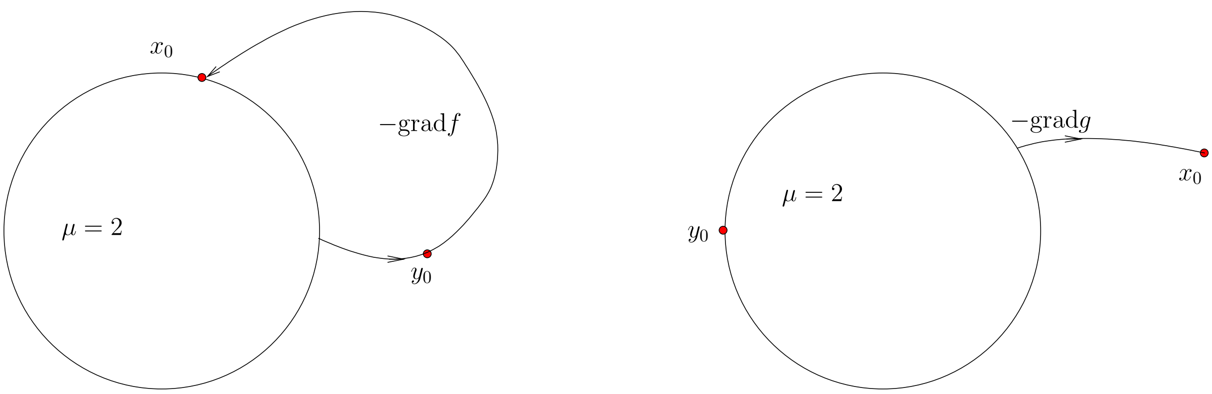

FIGURE 23. Pearly trajectories contributing to the coefficient r_1.

x_2 to y_0 and by $\gamma_{y_2,x_0}(g)$ the $-\mathrm{grad} g$ trajectory connecting y_2 to x_0. (by taking the Riemannian metric generic we may assume that y_0 and x_0 lie on trajectories as above). It follows that

(140) $$r_1 = \#_{\mathbb{Z}_2}\big(\delta_{x_0}(J) \cap \gamma_{x_2,y_0}(f)\big) + \#_{\mathbb{Z}_2}\big(\delta_{y_0}(J) \cap \gamma_{y_2,x_0}(g)\big).$$

See figure 24. Since $\delta_{x_0}(J)$, $\delta_{y_0}(J)$ are null-homologous, $\gamma_{x_2,y_0}(f)$, $\gamma_{y_2,x_0}(g)$ from formula (140) can be replaced by any paths $\ell(x_2, y_0)$, $\ell(y_2, x_0)$ in L connecting x_2 to y_0 and

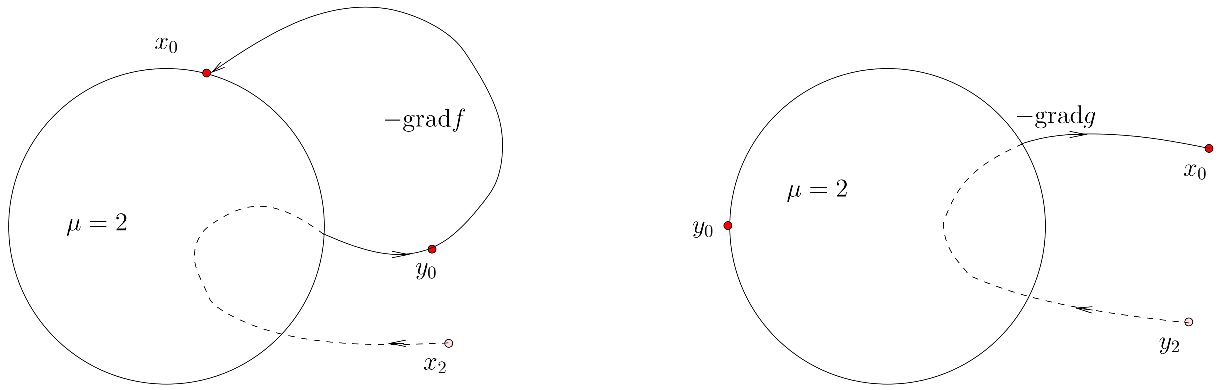

FIGURE 24.

y_2 to x_0. Moreover, since y_2 can be chosen arbitrary close to x_2 we can replace $\ell(y_2, x_0)$ by any path $\ell(x_2, x_0)$ connecting x_2 to x_0. Thus

$$r_1 = \#_{\mathbb{Z}_2}\big(\delta_{x_0}(J) \cap \ell(x_2, y_0)\big) + \#_{\mathbb{Z}_2}\big(\delta_{y_0}(J) \cap \ell(x_2, x_0)\big). \tag{141}$$

By Proposition 6.11.1

$$\phi^F(y_0) = x_0 + \#_{\mathbb{Z}_2}\big(\delta_{x_2}(J) \cap \ell(y_0, x_0)\big) x_2 t.$$

Here we have used the fact that y_2 is very close to x_2. Therefore:

$$\begin{aligned} x_0 * x_0 = x_0 *_{f,g} (\phi^F)^{-1}(x_0) &= x_0 *_{f,g} \Big(y_0 + \#_{\mathbb{Z}_2}\big(\delta_{x_2}(J) \cap \ell(y_0, x_0)\big) y_2 t\Big) \\ &= \Big(r_1 + \#_{\mathbb{Z}_2}\big(\delta_{x_2}(J) \cap \ell(y_0, x_0)\big)\Big) x_0 t + r_2 x_2 t^2 \\ &= \Big(\#_{\mathbb{Z}_2}\big(\delta_{x_0}(J) \cap \ell(x_2, y_0)\big) + \#_{\mathbb{Z}_2}\big(\delta_{y_0}(J) \cap \ell(x_2, x_0)\big) + \#_{\mathbb{Z}_2}\big(\delta_{x_2}(J) \cap \ell(y_0, x_0)\big)\Big) x_0 t \\ &\quad + r_2 x_2 t^2. \end{aligned}$$

Finally, take the element m to be x_0 and recall that $x_0 = p_1$, $y_0 = p_2$, $x_2 = p_3$. Formula (135) follows.

We turn to the proof of formula (136). From the computations above we see that $s_2 = r_2$, i.e. the coefficient of $x_2 t^2$ in $x_0 *_{f,g} y_0$. By the definition of the quantum cap product this coefficient counts the number of trajectories that appear in figure 25. The trajectories in the left-hand side of figure 25 contribute $n_4(x_0, y_0, x_2) = n_4(p_1, p_2, p_3)$ to the coefficient r_2. As for the trajectories in the right-hand side, a similar argument to

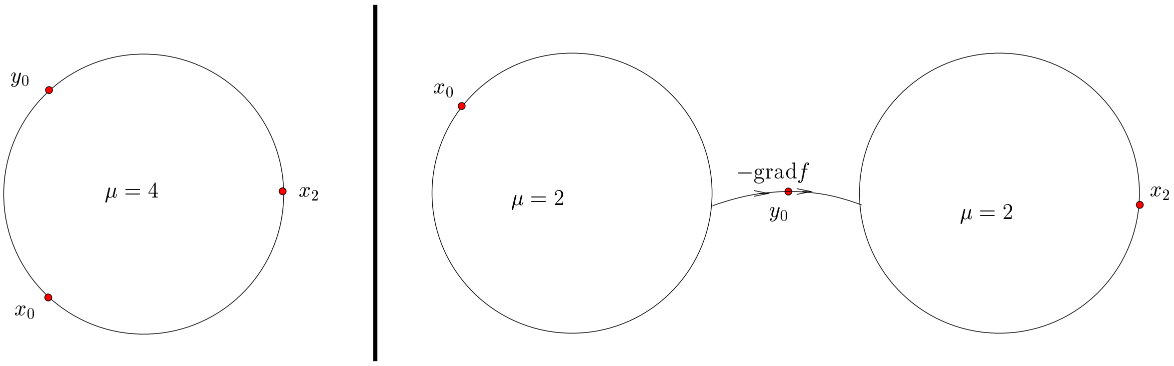

FIGURE 25. Pearly trajectories contributing to the coefficient $s_2 = r_2$.

what we had before shows that their number is:

$$\begin{aligned}
&\#_{\mathbb{Z}_2}\big(\delta_{x_0}(J) \cap \gamma_{x_2,y_0}(f)\big) \cdot \#_{\mathbb{Z}_2}\big(\delta_{x_2}(J) \cap \gamma_{y_0,x_0}(f)\big) \\
&= \#_{\mathbb{Z}_2}\big(\delta_{x_0}(J) \cap \ell(x_2, y_0)\big) \cdot \#_{\mathbb{Z}_2}\big(\delta_{x_2}(J) \cap \ell(y_0, x_0)\big) \\
&= \#_{\mathbb{Z}_2}\big(\delta_{p_1}(J) \cap \ell(p_2, p_3)\big) \cdot \#_{\mathbb{Z}_2}\big(\delta_{p_3}(J) \cap \ell(p_1, p_2)\big).
\end{aligned}$$

See figure 26. This completes the proof of formula (136).

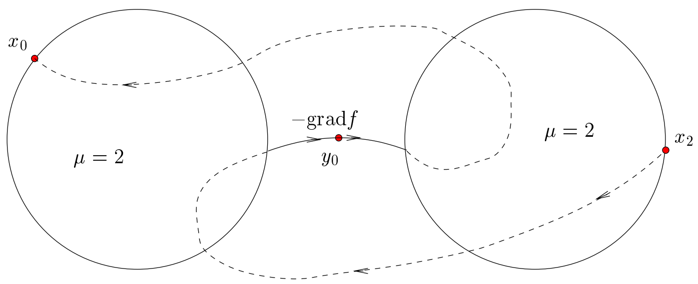

FIGURE 26.

Formulae (137) and (138) follow from the fact that when $s_1 = 1$ the coefficient s_2 does not depend on the choice of the element m. Thus we can take the points x_0, x_2, y_2 to be any permutation of the points p_1, p_2, p_3 and formula (136) will still give us the same number s_2. Note also that when $s_1 = 1$ it follows from formula (135) that either all three numbers

$$\#_{\mathbb{Z}_2}\big(\delta_{p_1}(J) \cap \ell(p_2, p_3)\big), \quad \#_{\mathbb{Z}_2}\big(\delta_{p_2}(J) \cap \ell(p_3, p_1)\big), \quad \#_{\mathbb{Z}_2}\big(\delta_{p_3}(J) \cap \ell(p_1, p_2)\big)$$

are 1, or precisely two of them are 0 and one of them is 1. □

Let us examine in view of Proposition 6.11.3 the case of the 2-dimensional Clifford torus $L = \mathbb{T}^2_{\text{clif}} \subset \mathbb{C}P^2$. By Proposition 6.2.3 there exist generators a, b of $H_1(L; \mathbb{Z}_2)$ and a basis for $QH_0(L)$ of the form $\{m, wt\}$ such that

$$a * b = m + wt, \quad b * a = m, \quad a * a = b * b = wt.$$

Using the associativity of the quantum product we obtain: $m * a = b * a * a = b * wt = bt$. Therefore:

$$m * m = b * a * b * a = b * (m + wt) * a = b * (bt + at) = mt + wt^2.$$

Thus $s_1 = s_2 = 1$. Since $s_1 = 1$ the coefficient $s_2 = 1$ is independent of the choice of m. It follows from Proposition 6.11.3 that the following three numbers

$$\#_{\mathbb{Z}_2}\big(\delta_{p_1}(J) \cap \ell(p_2, p_3)\big), \quad \#_{\mathbb{Z}_2}\big(\delta_{p_2}(J) \cap \ell(p_3, p_1)\big), \quad \#_{\mathbb{Z}_2}\big(\delta_{p_3}(J) \cap \ell(p_1, p_2)\big)$$

can be either all 1 or exactly one of them is 1 and the other two are 0. Moreover, in the second case there exists a simple J-holomorphic disk u with $\mu([u]) = 4$ and such that $u(\partial D) \ni p_1, p_2, p_3$.

It is instructive to consider the case when $J = J_0$ is the standard complex structure of $\mathbb{C}P^2$ (or a small perturbation of it). In this case, for every $p \in L$ the cycle $\delta_p(J_0)$ consist of three embedded circles passing through p (see §6.1.6). It is easy to see that $L \setminus \delta_p(J_0)$ has *two* connected components. Thus, if p_1, p_2 lie in the same connected component of $L \setminus \delta_{p_3}(J_0)$ then for a small enough perturbation J of J_0 we will have $n_4(p_1, p_2, p_3) = 1$. Of course, it is possible to find a different configurations of points p'_1, p'_2, p'_3 for which $n_4(p'_1, p'_2, p'_3) = 0$. Figure 27 shows these two possibilities. In this figure the torus $\mathbb{T}^2_{\text{clif}}$ is represented as a square with opposite sides identified. The three lines through each of the points p_i represent the boundaries of the three J_0-holomorphic disks passing through p_i. Using formula (136) and the fact that $s_2 = 1$ it is easy to see that $n_4(p_1, p_2, p_3) = 1$ while $n_4(p'_1, p'_2, p'_3) = 0$.

We continue with analyzing the coefficient s_2 from formula (134). As discussed above the coefficient s_2 may depend on the choice of m (this happens only when $s_1 = 0$). A natural class of choices for m can be made as follows. Choose a generic almost complex structure J and a generic pair of points $p, q \in L$. Choose a Riemannian metric ρ on L and a perfect Morse function $f : L \to \mathbb{R}$ so that p is its single (local) minimum and q is its single (local) maximum. As noted before the differential of the pearl complex vanishes hence we can view p as an element of $QH_0(f, \rho, J)$. It follows from formula (133) of Proposition 6.11.1 that this element is independent of the function f and metric ρ as long as f has p and q as its minimum and maximum. (Of course, once J is fixed, a slight perturbation of p and q will define the same element.) Thus a choice of a generic pair of

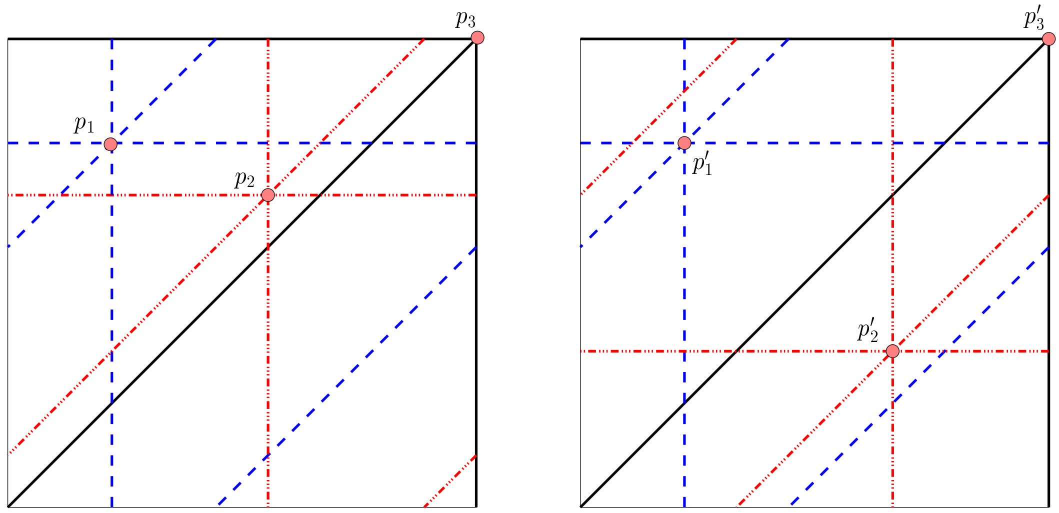

FIGURE 27. On the left $n_4(p_1, p_2, p_3) = 1$. On the right $n_4(p'_1, p'_2, p'_3) = 0$.

points p, q and J determines in a canonical way an element $m(p, q, J) = [p] \in QH_0(L)$. Put $w = [q]$. Clearly $\{m(p, q, J), wt\}$ form a basis for $QH_0(L)$. Note also that different functions f as above may give rise to different generators for $QH_1(L) \cong H_1(L; \mathbb{Z}_2)$ however, the element $m(p, q, J)$ is not affected by this.

Fix a generic J and a generic pair of points p, q. Denote by $s_2(p, q, J)$ the coefficient s_2 corresponding to the choice of $m = m(p, q, J)$. The following Proposition shows how the coefficient s_2 changes when we change p, q to a another pair p', q', hence $m(p, q, J)$ to $m(p', q', J)$.

Proposition 6.11.5. $s_2(p', q', J) = s_2(p, q, J) + \eta(s_1 + 1)$, *where* η *is given by:*

$$\eta = \#_{\mathbb{Z}_2}\big(\delta_{p'}(J) \cap \ell(q', q)\big) + \#_{\mathbb{Z}_2}\big(\delta_q(J) \cap \ell(p', p)\big).$$

In particular, when $q = q'$ *we have* $\eta = \#_{\mathbb{Z}_2}\big(\delta_q(J) \cap \ell(p', p)\big)$.

Proof. Put $m = m(p, q, J)$, $m' = m(p', q', J)$. By Proposition 6.11.1, we have $m' = m + \eta wt$. A direct computation (over $\mathbb{Z}_2$!) gives:

$$\begin{aligned} m' * m' &= m * m + \eta wt^2 = s_1 mt + s_2(p, q, J)wt^2 + \eta wt^2 \\ &= s_1 m't + s_1\eta wt^2 + s_2(p, q, J)wt^2 + \eta wt^2 \\ &= s_1 m't + \big(s_2(p, q, J) + \eta(s_1 + 1)\big)wt^2. \end{aligned}$$

□

Proposition 6.11.6. *With the notation of Proposition 6.11.3 we have:*

$$s_2(p_1, p_3, J) = \#_{\mathbb{Z}_2}\big(\delta_{p_3}(J) \cap \ell(p_1, p_2)\big) \cdot \#_{\mathbb{Z}_2}\big(\delta_{p_1}(J) \cap \ell(p_2, p_3)\big) + n_4(p_1, p_2, p_3). \tag{142}$$

The proof is essentially the same as the proof of Proposition 6.11.3.

6.11.1. *Explicit formulae for the quantum product of* 2*-dimensional Lagrangian tori.* Here we develop formulae which allow us to reproduce the quantum cap product for every (monotone) 2-dimensional Lagrangian torus L from minimal information on holomorphic disks with boundaries on L. As it turns out, it is enough to know the number of Maslov index 2 - holomorphic disks passing through a generic point on L in every homology class.

Let $L^2 \subset (M^4, \omega)$ be a 2-dimensional monotone Lagrangian torus with minimal Maslov number $N_L = 2$. Assume that $QH_*(L) \neq 0$ (which is equivalent to $QH_*(L) \cong (H(L;\mathbb{Z}_2) \otimes \Lambda)_*$. See Proposition 6.1.4).

Fix generators a, b of the *integral* homology $H_1(L;\mathbb{Z})$ so that $H_1(L;\mathbb{Z}) = \mathbb{Z}a \oplus \mathbb{Z}b$. Define a function $\nu : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2$ as follows:

$$\nu(k,l) = \sum_{\substack{A \in \mathcal{E}_2, \\ \partial A = ka + lb}} \deg_{\mathbb{Z}_2} ev_{A,J}. \tag{143}$$

where $J \in \mathcal{J}_{\text{reg}}$ is a generic almost complex structure. In other words, $\nu(k,l)$ counts, mod 2, the number of J-holomorphic disks of Maslov index $\mu = 2$ passing through a generic point in L and whose boundaries realize the homology class $ka + lb \in H_1(L;\mathbb{Z})$. By the discussion at the beginning of §6.1.1, $\nu(k,l)$ does not depend on the choice of $J \in \mathcal{J}_{\text{reg}}$. Moreover, $\nu(k,l) = 0$ for all but a finite number of pairs (k,l).

Denote by $w \in H_2(L;\mathbb{Z}_2)$ the fundamental class. Let $m \in QH_0(L;\mathbb{Z}_2)$ be an element so that $\{m, wt\}$ forms a basis for $QH_0(L)$. Then we can write

$$\begin{aligned} & a * a = \alpha wt, \quad b * b = \beta wt, \\ & a * b = m + \gamma' wt, \quad b * a = m + \gamma'' wt, \\ & a * b + b * a = (\gamma' + \gamma'')wt, \end{aligned} \tag{144}$$

for some $\alpha, \beta, \gamma', \gamma'' \in \mathbb{Z}_2$.

Proposition 6.11.7. *The coefficients α, β are given by:*

$$\alpha = \sum_{k,l} \nu(k,l) \frac{l(l+1)}{2} \,(\text{mod} 2), \quad \beta = \sum_{k,l} \nu(k,l) \frac{k(k+1)}{2} \,(\text{mod} 2).$$

The sum $\gamma' + \gamma''$ is independent of the choice of the element m and we have:

$$\gamma' + \gamma'' = \sum_{k,l} \nu(k,l) kl \,(\text{mod} 2).$$

The proof is given in §6.11.2 below. Note that using the coefficients $\alpha, \beta, \gamma', \gamma''$ we can recover the quantum product. Indeed, a simple computation based on (144) gives:

$$\begin{aligned} m * a &= \alpha bt + \gamma'' at, \quad a * m = \alpha bt + \gamma' at \\ m * b &= \beta at + \gamma' bt, \quad b * m = \beta at + \gamma'' bt \\ m * m &= (\gamma' + \gamma'')mt + (\alpha\beta + \gamma'\gamma'')wt^2. \end{aligned} \tag{145}$$

Notice that the quantum product is commutative iff $\gamma' + \gamma'' = 0$ (recall that we work here over $\mathbb{Z}_2$). Thus the function $\nu(k, l)$ determines whether or not the quantum product is commutative. It is also worth noting that when $\gamma' + \gamma'' = 1$ we must have $\gamma'\gamma'' = 0$, hence in this case $m * m = mt + \alpha\beta wt^2$.

Let us apply the above to two examples. We start with our favorite example, the 2-dimensional Clifford torus $L = \mathbb{T}^2_{\text{clif}} \subset \mathbb{C}P^2$. We will use here the notation of §6.1.6. We take the basis a, b for $H_1(L; \mathbb{Z})$ to be $a = \partial A_1$, $b = \partial A_2$. With this choice the function $\nu : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2$ is:

$$\begin{aligned} &\nu(1, 0) = 1, \quad \nu(0, 1) = 1, \quad \nu(-1, -1) = 1, \\ &\nu(k, l) = 0 \text{ for all other } k, l. \end{aligned}$$

It easily follows from Proposition 6.11.7 that $\alpha = \beta = 1$ and that $\gamma' + \gamma'' = 1$. It follows that precisely one of γ', γ'' equals 0 and the other one equals 1. We thus recover again the quantum product for $\mathbb{T}^2_{\text{clif}}$.

Our second example is the split Lagrangian torus in $S^2 \times S^2$. Endow $S^2 \times S^2$ with the split symplectic form $\omega = \omega_{S^2} \oplus \omega_{S^2}$, where ω_{S^2} is the standard symplectic form of S^2. Let $Eq \subset S^2$ be the equator. Then $L = Eq \times Eq \subset S^2 \times S^2$ is a monotone Lagrangian with $N_L = 2$. Denote by D_0, D_1 the two oriented disk obtained from $S^2 \setminus Eq$. Put

$$A_0 = [D_0 \times pt], A_1 = [D_1 \times pt], B_0 = [pt \times D_0], B_1 = [pt \times D_1] \in H_2(S^2 \times S^2, L; \mathbb{Z}).$$

Clearly $H_2^D = H_2(S^2 \times S^2, L; \mathbb{Z}) \cong \mathbb{Z}A_0 \oplus \mathbb{Z}A_1 \oplus \mathbb{Z}B_0 \oplus \mathbb{Z}B_1$.

Let $J_0 = j_0 \oplus j_0$ be the standard split complex structure where j_0 is the complex structure of $S^2 \cong \mathbb{C}P^1$. Let $u_0, u_1 : (D, \partial) \to (S^2, Eq)$ be the obvious j_0-holomorphic disks of Maslov index 2 parametrizing D_0 and D_1. It is easy to see that the only J_0-holomorphic disks with Maslov index 2 are in one of the classes A_0, A_1, B_0, B_1 and in fact they are all given (up to reparametrization by an element of $\text{Aut}(D)$) by

$$u_0 \times pt, \quad u_1 \times pt, \quad pt \times u_0, \quad pt \times u_1.$$

Moreover J_0 is regular for all the classes $A \in H_2^D$ with Maslov index 2. It follows that for every $p = (x, y) \in L$ we have $\delta_p(J_0) = Eq \times x - Eq \times x + y \times Eq - y \times Eq$, hence $D_1 = 0$

(See (102) and (103)). By Proposition 6.1.4 we have $QH_*(L) \cong (H(L) \otimes \Lambda)_*$. (This can be easily verified also by the Floer Künneth formula).

We now fix the following generators $a = [Eq \times pt], b = [pt \times Eq] \in H_1(L; \mathbb{Z})$. Let $m \in QH_0(L)$ be an element such that $\{m, wt\}$ forms a basis for $QH_0(L)$. The function $\nu : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2$ is:

$$\begin{aligned} &\nu(1,0) = 1, \quad \nu(-1,0) = 1, \quad , \nu(0,1) = 1, \quad \nu(0,-1) = 1, \\ &\nu(k,l) = 0 \text{ for all other } k, l. \end{aligned}$$

By Proposition 6.11.7 we have $\alpha = 1$, $\beta = 1$ and $\gamma' + \gamma'' = 0$ (i.e. $\gamma' = \gamma''$). It follows that the coefficient s_1 in $m * m$ is 0. This means that the coefficient s_2 may depend on the choice of m. To choose m let $p, q \in L$ be two generic points. Let J_ϵ be a small enough perturbation of J_0. This defines the element $m = m(p, q; J_\epsilon)$. A straightforward computation shows that $a * b = m$ and $b * a = m$, hence $\gamma' = \gamma'' = 0$. Using (144) and (145) we obtain (for the above choice of m):

$$\begin{aligned} &a * a = b * b = wt, \quad a * b = b * a = m, \\ &a * m = m * a = bt, \quad b * m = m * b = at, \\ &m * m = wt^2. \end{aligned}$$

Thus $s_2(p, q, J_\epsilon) = 1$.

Let us interpret the value of s_2 using Proposition 6.11.6. Put $p_1 = p$, $p_3 = q$ and let $p_2 \in L \setminus \delta_{p_3}(J_0)$ be a generic point. Note that $L \setminus \delta_{p_3}(J_0)$ is connected (it is an open rectangle). If the perturbation J_ϵ of J_0 is small enough then the cycle $\delta_{p_3}(J_\epsilon)$ will be close to $\delta_{p_3}(J_0)$ in the C^0-topology. Therefore, for small enough perturbation the point p_1 and p_2 will still remain in the same connected component of $L \setminus \delta_{p_3}(J_\epsilon)$. In particular $\#_{\mathbb{Z}_2}\big(\delta_{p_3}(J) \cap \ell(p_1, p_2)\big) = 0$. It follows from Proposition 6.11.6 that $n_4(p_1, p_2, p_3) = 1$. In other words, we obtain the following.

Corollary 6.11.8. *Let $p_1, p_2, p_3 \in L$ be generic points. Then there exists a neighbourhood $\mathcal{U} \subset \mathcal{J}$ of J_0 (which depends on p_1, p_2, p_3) such that for generic $J \in \mathcal{U}$ there exists a simple J-holomorphic disk $u : (D, \partial D) \to (S^2 \times S^2, L)$ with $\mu(L) = 4$ and such that $u(e^{-2\pi i l/3}) = p_l$ for every $1 \leq l \leq 3$.*

6.11.2. *Proof of Proposition 6.11.7.* Let $f : L \to \mathbb{R}$ be a perfect Morse function and ρ a Riemannian metric on L. Denote by x_2 the unique maximum of f, by x_0 the unique minimum and by x_1', x_1'' the critical points of index 1. Since a, b generate $H_1(L; \mathbb{Z})$ the pair (f, ρ) can be chosen so that the closures of the unstable submanifolds $\overline{W}^u_{x_1'}$, $\overline{W}^u_{x_1''}$ represent the homology classes $\pm a, \pm b$. Fix an orientation on L so that $a \cdot b = 1$ and pick orientations $\mathbf{o}'$, $\mathbf{o}''$ on $\overline{W}^u_{x_1'}$, $\overline{W}^u_{x_1''}$ so that $[\overline{W}^u_{x_1'}] = a$, $[\overline{W}^u_{x_1''}] = b$.

Having fixed the above let $g : L \to \mathbb{R}$ be a small perturbation of f chosen in the following way. Denote by y_0, y_1', y_1'', y_2 the critical points of g (with the convention that subscripts denote the Morse index). We choose g so that $W^u_{y_1'}$ is a "parallel" translate of $W^u_{x_1'}$ pushed in a direction $\vec{\mathbf{n}}'$ normal to $W^u_{x_1'}$ such that the pair $(\mathbf{o}', \vec{\mathbf{n}}')$ give the positive orientation on L. Similarly we require that $W^u_{y_1''}$ is obtained by pushing $W^u_{x_1''}$ in the direction $\vec{\mathbf{n}}''$ so that the pair $(\mathbf{o}'', \vec{\mathbf{n}}'')$ give the negative orientation on L.

By choosing g to be close enough to f we may assume that the canonical quasi-isomorphism between $\mathcal{C}_*(f, J)$ and $\mathcal{C}_*(g, J)$ is in fact a base preserving isomorphism (i.e. x_i is mapped to y_i, x_1' to y_1' and x_1'' to y_1''). Thus in order to compute the coefficient α in $a * a = \alpha w t$ we will compute $[x_1'] * [y_1']$.

Choose a generic $J \in \mathcal{J}(M, \omega)$. Let $u : (D, \partial) \to (M, L)$ be a J-holomorphic disk with Maslov index 2 that contributes to $\nu(k, l)$, i.e. $u(\partial D) \ni x_2$ and $[u(\partial D)] = ka + lb$. After reparametrization we may assume that $u(1) = x_2$. The contribution of this disk to the coefficient α in $[x_1'] * [y_1']$ is given by the number, mod 2, of pairs (θ_g, θ_f) such that:

$$0 < \theta_g < \theta_f < 2\pi, \quad u(e^{i\theta_g}) \in W^u_{y_1'}, \quad u(e^{i\theta_f}) \in W^u_{x_1'}.$$

In other words we have to mark on the boundary $u(\partial D)$ of the disk u two types of points: the points hit by a $-\text{grad} f$ trajectory coming from x_1' and the points hit by a $-\text{grad} g$ trajectory coming from y_1'. Let us call the first set "points of type f" and the second "points of type g". Then we have to count how many pairs of points (q, p), p of type f and q of type g, are there such that q appears "before" p along $u(\partial D)$. We denote this number α_u. The wording "before" means that we take $x_2 \in u(\partial D)$ as the origin and use the standard orientation of ∂D for ordering.

To compute α_u first note that points of types f and g appear in pairs (provided that the function g is close enough to f). In fact, each time $W^u_{x_1'}$ intersects positively the boundary of the disk $u(\partial D)$ then $W^u_{y_1'}$ intersects it positively too (in a nearby point) and we we obtain two points p_+, q_+, of types f and g, where p_+ comes before q_+. When $W^u_{x_1'}$ intersects $u(\partial D)$ negatively we obtain two points q_-, p_-, of types g and f, where q_- comes before p_-. This is described in figure 28. Note that the arrows along $W^u_{x_1'}, W^u_{y_1'}$ in this figure might be a bit confusing. These arrows represent the orientation $\mathbf{o}'$ discussed above, *not* the direction of the gradient (or minus gradient) flow.

Denote by i_+, i_- the number of positive, resp. negative, intersection points of $W^u_{x_1'}$ with $u(\partial D)$. As for the contribution to α_u, each point $q_\pm$ of type g (positive or negative) can be paired to any point $p_\pm$ coming "after" it. In addition, each point q_- appearing in a

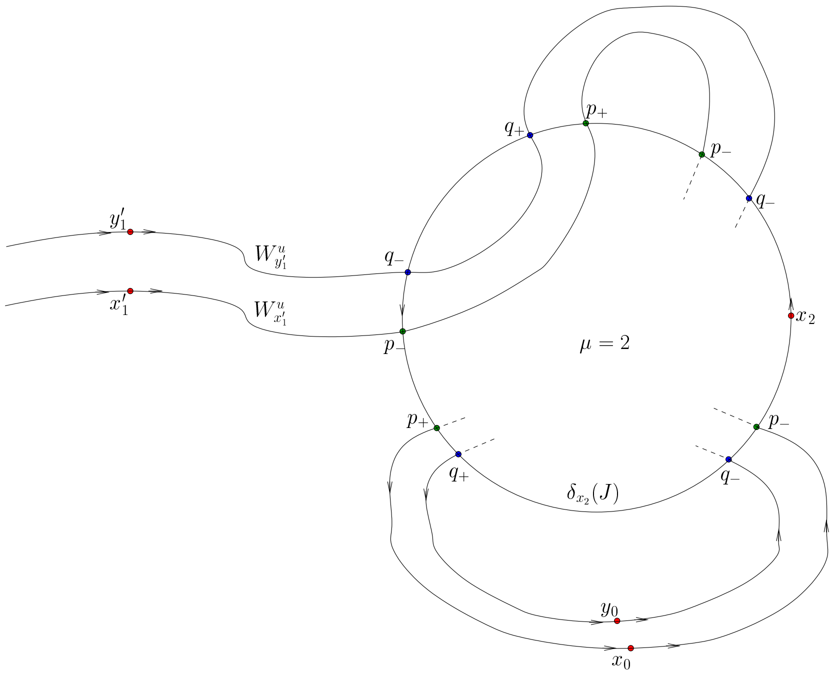

FIGURE 28.

negative intersection as above can also be paired with its adjacent point p_-. Therefore

$$
\begin{aligned}
\alpha_u &= \Big(\sum_{j=i_+ + i_-} j\Big) + i_- \\
&= \frac{(i_+ + i_-)(i_+ + i_- + 1)}{2} + i_- = \frac{i_+^2 + 2i_+i_- + i_-^2 + i_+ + i_-}{2} + i_-.
\end{aligned} \tag{146}
$$

On the other hand since $[u(\partial D)] = ka + lb$ we have $i_+ - i_- = [W_{x'_1}] \cdot [u(\partial D)] = l$. Therefore we obtain from (146):

$$
\begin{aligned}
\alpha_u &\equiv \frac{i_+^2 - 2i_+i_- + i_-^2 + i_+ - i_-}{2} (\mathrm{mod} 2) \equiv \frac{(i_+ - i_-)(i_+ - i_- + 1)}{2} (\mathrm{mod} 2) \\
&\equiv \frac{l(l+1)}{2} (\mathrm{mod} 2).
\end{aligned}
$$

It follows that the contribution to coefficient α of the disks u with $[u(\partial D)] = ka + lb$ is $\nu(k,l)\frac{l(l+1)}{2}(\mathrm{mod} 2)$, hence

$$
\alpha = \sum_{k,l} \nu(k,l) \frac{l(l+1)}{2} (\mathrm{mod} 2).
$$

The formula for the coefficient β is proved in a similar way.

The formula for $\gamma' + \gamma''$ is more straightforward. Let $u : (D, \partial D) \to (M, L)$ be a J-holomorphic disk with Maslov index 2, with $u(1) = x_2$ and such that $[u(\partial D)] = ka + lb$.

Denote by i', i'' the number of intersection points (counted without signs) of $W^u_{x'_1}$, resp. $W^u_{x''_1}$ with $u(\partial D)$. Note that if g is chosen to be close enough to f we have:

$$i' = \#(W^u_{y'_1} \cap u(\partial D)), \quad i'' = \#(W^u_{y''_1} \cap u(\partial D)).$$

Since $a * b + b * a = [x'_1] * [y''_1] + [x''_1] * [y'_1] = (\gamma' + \gamma'')wt$ the contribution of the disk u to the coefficient $\gamma' + \gamma''$ is $i'i''$. (This is because we are computing the symmetric expression $a*b+b*a$, hence the order of the 3 points on $u(\partial D)$ involved in the quantum product does not matter.) Since $[\overline{W}^u_{x'_1}] \cdot [u] = l$, $[\overline{W}^u_{x''_1}] \cdot [u] = -k$ we have $i' \equiv l(\text{mod}2)$, $i'' \equiv k(\text{mod}2)$. It follows that

$$\gamma' + \gamma'' \equiv \sum_{k,l} \nu(k,l)lk \,(\text{mod}2).$$

□

References

[1] L. Abrams. The quantum euler class and the quantum cohomology of the grassmannians. *Israel J. Math.*, 117:335–352, 2000.

[2] M. Akaho. Gluing constructions of pseudo-holomorphic discs and desingularization. Preprint, 2006.

[3] M. Akveld and D. Salamon. Loops of lagrangian submanifolds and pseudoholomorphic discs. *Geom. Funct. Anal.*, 11(4):609–650, 2001.

[4] P. Albers. A Lagrangian Piunikhin-Salamon-Schwarz morphism and two comparison homomorphisms in Floer homology. Preprint (2005), can be found at `http://lanl.arxiv.org/pdf/math/0512037`.

[5] P. Albers. On the extrinsic topology of Lagrangian submanifolds. *Int. Math. Res. Not.*, 2005(38):2341–2371, 2005.

[6] V. Arnold. Some remarks on symplectic monodromy of Milnor fibrations. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 99–103. Birkhäuser Verlag, Basel, 1995.

[7] J.-F. Barraud and O. Cornea. Lagrangian intersections and the Serre spectral sequence. Preprint, can be found at math.DG/0401094.

[8] J.-F. Barraud and O. Cornea. Homotopic dynamics in symplectic topology. In P. Biran, O. Cornea, and F. Lalonde, editors, *Morse theoretic methods in nonlinear analysis and in symplectic topology*, volume 217 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 109–148, Dordrecht, 2006. Springer.

[9] A. Beauville. Quantum cohomology of complete intersections. *Mat. Fiz. Anal. Geom.*, 2(3-4):384–398, 1995.

[10] P. Biran. Algebraic families and Lagrangian cycles. In preparation.

[11] P. Biran. Symplectic packing in dimension 4. *Geom. Funct. Anal.*, 7(3):420–437, 1997.

[12] P. Biran. A stability property of symplectic packing. *Invent. Math.*, 136(1):123–155, 1999.

[13] P. Biran. From symplectic packing to algebraic geometry and back. In *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, volume 202 of *Progr. Math.*, pages 507–524, Basel, 2001. Birkhäuser Verlag.

[14] P. Biran. Lagrangian barriers and symplectic embeddings. *Geom. Funct. Anal.*, 11(3):407–464, 2001.

[15] P. Biran. Lagrangian non-intersections. *Geom. Funct. Anal.*, 16(2):279–326, 2006.

[16] L. Buhovsky. Multiplicative structures in lagrangian floer homology. Preprint, can be found at math.SG/0608063.

[17] L. Buhovsky. Homology of Lagrangian submanifolds in cotangent bundles. *Israel J. Math.*, 143:181–187, 2004.

[18] R. Chiang. New Lagrangian submanifolds of $\mathbb{C}P^n$. *Int. Math. Res. Not.*, 2004(45):2437–2441, 2004.

[19] C.-H Cho. On the counting of holomorphic discs in toric Fano manifolds. Preprint (2006). Can be found at `http://xxx.lanl.gov/pdf/math/0604502`.

[20] C.-H Cho. Holomorphic discs, spin structures, and floer cohomology of the Clifford torus. *Int. Math. Res. Not.*, 2004(35):1803–1843, 2004.

[21] C.-H Cho. Products of Floer cohomology of torus fibers in toric Fano manifolds. *Comm. Math. Phys.*, 260(3):613–640, 2005.

[22] C.-H Cho and Y.-G. Oh. Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds. *Asian J. Math.*, 10(4, pages = 773–814), 2006.

[23] O. Cornea and F. Lalonde. Cluster homology. Preprint (2005), can be found at `http://xxx.lanl.gov/pdf/math/0508345`.

[24] O. Cornea and F. Lalonde. Cluster homology: an overview of the construction and results. *Electron. Res. Announc. Amer. Math. Soc.*, 12:1–12, 2006.

[25] O. Cornea and A. Ranicki. Rigidity and gluing for Morse and Novikov complexes. *J. Eur. Math. Soc.*, 5(4):343–394, 2003.

[26] S. Donaldson. Polynomials, vanishing cycles and Floer homology. In *Mathematics: frontiers and perspectives*, pages 55–64. Amer. Math. Soc., Providence, RI, 2000.

[27] B. Dubrovin. Geometry of 2d topological field theories. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, volume 1620 of *Lecture Notes in Math.*, pages 120–348. Springer, Berlin, 1996.

[28] M. Entov and L. Polterovich. Rigid subsets of symplectic manifolds. Preprint (2007).

[29] A. Floer. A relative morse index for the symplectic action. *Comm. Pure Appl. Math.*, 41(4):393–407, 1988.

[30] A. Floer. The unregularized gradient flow of the symplectic action. *Comm. Pure Appl. Math.*, 41(6):775–813, 1988.

[31] A. Floer. Symplectic fixed points and holomorphic spheres. *Comm. Math. Phys.*, 120(4):575–611, 1989.

[32] U. Frauenfelder. Gromov convergence of pseudo-holomorphic disks. Diploma thesis, ETH Zürich, 2000.

[33] K. Fukaya. Morse homotopy and its quantization. In *Geometric topology (Athens, GA, 1993)*, volume 2.1 of *AMS/IP Stud. Adv. Math.*, pages 409–440, Providence, RI, 1997. Amer. Math. Soc., Amer. Math. Soc.

[34] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. Lagrangian intersection Floer theory - anomaly and obstruction. Preprint.

[35] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Pure and Applied Mathematics. John Wiley & Sons, Inc., New York, 1978.

[36] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[37] Y. Karshon. Appendix to "symplectic packings and algebraic geometry", by D. McDuff and L. Polterovich. *Invent. Math.*, 115(3):405–434, 1994.

[38] J. Katić and D. Milinković. Piunikhin-Salamon-Schwarz isomorphisms for Lagrangian intersections. *Differential Geom. Appl.*, 22(2):215–227, 2005.

[39] M. Kontsevich and Yu. Manin. Gromov-witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.*, 164(3):525–562, 1994.

[40] D. Kwon and Y.-G. Oh. Structure of the image of (pseudo)-holomorphic discs with totally real boundary condition. With an appendix by Jean-Pierre Rosay. *Comm. Anal. Geom.*, 8(1):31–82, 2000.

[41] L. Lazzarini. Decompostion of a J-holomorphic curve. Preprint. Can be downloaded at `http://www.math.jussieu.fr/~lazzarin/articles.html`.

[42] L. Lazzarini. Existence of a somewhere injective pseudo-holomorphic disc. *Geom. Funct. Anal.*, 10(4):829–862, 2000.

[43] D. McDuff and L. Polterovich. Symplectic packings and algebraic geometry. *Invent. Math.*, 115(3):405–434, 1994.

[44] D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[45] Y.-G. Oh. Floer mini-max theory, the Cerf diagram, and the spectral invariants. Preprint (2004), can be found at `http://xxx.lanl.gov/pdf/math/0406449`.

[46] Y.-G. Oh. Mini-max theory, spectral invariants and geometry of the Hamiltonian diffeomorphism group. Preprint (2002), can be found at `http://xxx.lanl.gov/pdf/math/0206092`.

[47] Y.-G. Oh. Removal of boundary singularities of pseudo-holomorphic curves with lagrangian boundary conditions. *Comm. Pure Appl. Math.*, 45(1):121–139, 1992.

[48] Y.-G. Oh. Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I. *Comm. Pure Appl. Math.*, 46(7):949–993, 1993.

[49] Y.-G. Oh. Floer cohomology, spectral sequences, and the Maslov class of Lagrangian embeddings. *Internat. Math. Res. Notices*, 1996(7):305–346, 1996.

[50] Y.-G. Oh. Relative floer and quantum cohomology and the symplectic topology of lagrangian submanifolds. In C. B. Thomas, editor, *Contact and symplectic geometry*, volume 8 of *Publications of the Newton Institute*, pages 201–267. Cambridge Univ. Press, Cambridge, 1996.

[51] Y.-G. Oh. Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds. In J. Marsden and T. Ratiu, editors, *The breadth of symplectic and Poisson geometry. Festschrift in honor of Alan Weinstein*, volume 232 of *Progr. Math.*, pages 525–570, Boston, MA, 2005. Birkhäuser Boston.

[52] Y.-G. Oh. Spectral invariants, analysis of the Floer moduli space, and geometry of the Hamiltonian diffeomorphism group. *Duke Math. J.*, 130(2):199–295, 2005.

[53] Y.-G. Oh. Lectures on Floer theory and spectral invariants of Hamiltonian flows. In P. Biran, O. Cornea, and F. Lalonde, editors, *Morse theoretic methods in nonlinear analysis and in symplectic topology*, volume 217 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 321–416, Dordrecht, 2006. Springer.

[54] S. Piunikhin, D. Salamon, and M. Schwarz. Symplectic Floer-Donaldson theory and quantum cohomology. In C. B. Thomas, editor, *Contact and symplectic geometry*, volume 8 of *Publications of the Newton Institute*, pages 171–200. Cambridge Univ. Press, Cambridge, 1996.

[55] J. Robbin and D. Salamon. The spectral flow and the maslov index. *CBull. London Math. Soc.*, 27(1):1–33, 1995.

[56] D. Salamon. Lectures on floer homology. In *Symplectic geometry and topology (Park City, UT, 1997)*, volume 7 of *IAS/Park City Math. Ser.*, pages 143–229. Amer. Math. Soc., Providence, RI, 1999.

[57] M. Schwarz. *Cohomology operations from S^1-cobordisms in Floer homology.* PhD thesis, ETH Zürich, 1995.

[58] M. Schwarz. Equivalences for morse homology. In *Geometry and topology in dynamics*, volume 246 of *Contemp. Math.*, pages 197–216. Amer. Math. Soc., Providence, RI, 1999.

[59] M. Schwarz. On the action spectrum for closed symplectically aspherical manifolds. *Pacific J. Math.*, 193(2):419–461, 2000.

[60] P. Seidel. *Floer homology and the symplectic isotopy problem.* PhD thesis, Oxford University, 1997.

[61] P. Seidel. π_1 of symplectic automorphism groups and invertibles in quantum homology rings. *Geom. Funct. Anal.*, 7(6):1046–1095, 1997.

[62] P. Seidel. Graded Lagrangian submanifolds. *Bull. Soc. Math. France*, 128(1):103–149, 2000.

[63] P. Seidel. Vanishing cycles and mutation. In *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, volume 202 of *Progr. Math.*, pages 65–85, Basel, 2001. Birkhäuser Verlag.

[64] P. Seidel. Exact Lagrangian submanifolds of T^*S^n and the graded Kronecker quiver. In Y. Eliashberg S. Donaldson and M. Gromov, editors, *Different faces of geometry*, Int. Math. Ser. (N. Y.), pages 349–364. Kluwer/Plenum, New York, 2004.

[65] J. Solomon. Intersection theory on the moduli space of holomorphic curves with lagrangian boundary conditions. Preprint, can be found at math.SG/0606429.

[66] G. Tian and G. Xu. On the semi-simplicity of the quantum cohomology algebras of complete intersections. *Math. Res. Lett.*, 4(4):481–488, 1997.

[67] L. Traynor. Symplectic packing constructions. *J. Differential Geom.*, 42(2):411–429, 1995.

[68] J.-Y. Welschinger. Enumerative invariants of strongly semipositive real symplectic manifolds. Preprint, can be found at math.AG/0509121.

[69] J.-Y. Welschinger. Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry. *C. R. Math. Acad. Sci. Paris*, 336(4):341–344, 2003.

[70] J.-Y. Welschinger. Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry. *Invent. Math.*, 162(1):195–234, 2005.

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