I - Survival models and life tables

- Let the random variable X (r.v.) represent the age-at-death of a newborn (nb)
  $\equiv$ Lifetime of nb $\equiv$ time-until-death
- X: continuous, positive r.v.
- Survival function: $s(x) = Pr[X > x]$
  Probability that nb survives to age x

  Properties of $s(x)$
  i) $s(0) = 1$
  ii) $s(\omega) = 0$ $\quad$ ω: terminal age (in practice, around 110; may be ∞ with certain distributions)
  iii) non-increasing function of x - GRAPH.
- Cumulative distribution function (cdf)

$$F_X(x) = Pr[X \leq x] = 1 - s(x) \qquad \text{GRAPH}$$

  Probability that nb dies before age x.
- Probability density function (pdf)

$$f_X(x) = \frac{d}{dx} F_X(x) = -\frac{d}{dx} s(x).$$

- Conditional probability that a nb dies between ages x + $x+t$, given that he survives to age x:

$$\Pr[x < X \le x+t \mid X > x] = \frac{\Pr[x < X \le x+t]}{\Pr[X > x]}$$

$$= \frac{F_X(x+t) - F_X(x)}{1 - F_X(x)} = \frac{s(x) - s(x+t)}{s(x)} = 1 - \frac{s(x+t)}{s(x)}$$

~ Probability of dying between ages x + $x+t$ for a person aged x.

- Force of mortality (hazard rate) at age x

$$\mu_x = \frac{f_X(x)}{1 - F_X(x)} = -\frac{s'(x)}{s(x)} = \frac{-d}{dx} \ln s(x) \quad (*)$$

Properties of μ_x:

i) $\mu_x \ge 0 \quad \forall x$.

ii) from $(*)$, $s(x) \overset{...}{=} \exp\left[-\int_0^x \mu_z \, dz\right]$

iii) as $\lim_{x \to \infty} s(x) = 0$, $\lim_{x \to \infty} \int_0^x \mu_z \, dz = \infty$ (interp.)

iv) from $(*)$, $-s'(x) = s(x)\mu_x$

Integrate between y and $y+n$

$$\Rightarrow s(y) - s(y+n) = \int_y^{y+n} s(x) \mu_x \, dx.$$

v) GRAPH

vi) Interpretation.

• ddf of X $\quad F_X(x) = 1 - s(x) = 1 - \exp\left[-\int_0^x \mu_z\,dz\right]$

X: continuous r.v., so $f_X(x) = \frac{d}{dx} F_X(x)$

$$f_X(x) = -- \exp\left[-\int_0^x \mu_z\,dz\right] \cdot \mu_x$$
$$= \mu_x\, s(x).$$

ACTUARIAL NOTATION

(x): a person aged x.

${}_nq_x$ = Pr[(x) dies within n years] $\qquad$ (ex: ${}_{10}q_{40}$)
$= \Pr[x < X \le x+n \mid X > x]$.

${}_np_x$ = Pr[(x) survives the next n years] $\qquad$ (ex: ${}_{7.5}p_{101}$)
$= 1 - {}_nq_x$

N.B. If $n=1$, it is omitted : q_x and p_x.

${}_{t|u}q_x$ = Pr[(x) dies between ages $x+t$ and $x+t+u$]

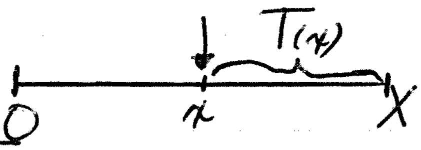
If $u=1$, it is omitted.

Ex: ${}_{10|}q_{50}$ = Pr[(50) dies between ages 60 and 61].

Future lifetime of (x)

let $T(x)$ be the r.v. representing the time-until-death of a person now-aged x.

$T(x)$: continuous positive r.v.



Then $x + T(x) = X. \Rightarrow X - x = T(x)$

let $x = 0 \Rightarrow T(0) = X$

- cdf of $T(x)$

$$\Pr[x < X \le x+t \mid X > x] = \Pr[0 < T(x) \le t]$$

$$\frac{s(x) - s(x+t)}{s(x)} = F_{T(x)}(t)$$

$$_t q_x = 1 - \frac{s(x+t)}{s(x)} = 1 - {_t p_x}.$$

$$= 1 - \exp\left[-\int_x^{x+t} \mu_z \, dz\right] = 1 - \exp\left[-\int_0^t \mu_{x+s} \, ds\right]$$

- pdf of $T(x)$

X continuous r.v., so

$$f_{T(x)}(t) = \frac{d}{dt} F_{T(x)}(t) \stackrel{\dots}{=} {_t p_x} \mu_{x+t}$$

(N.B. $x = 0$ ...)

- Life expectancy at age x $\equiv$ Mean of $T(x)$
  $E[T(x)]$ is denoted $\mathring{e}_x$ ($\mathring{e}_0$: life expectancy at birth)
  Useful to compare 2 populations (ex: Canada vs USA, men vs women, smokers vs non-smokers, white vs black)
  $\mathring{e}_0$, $\mathring{e}_{65}$: socio-economic index.

  By definition, $\mathring{e}_x = E[T(x)] = \int_0^\infty t \; {}_tp_x \, \mu_{x+t} \, dt$
  $= \ldots = \int_0^\infty {}_tp_x \, dt.$
- Variance of $T(x)$
  $$Var(T(x)) = E(T(x)^2) - E^2(T(x))$$
  $$= \ldots = \int_0^\infty 2t \; {}_tp_x \, dt - \mathring{e}_x^2.$$
- Median of $T(x)$
  Find $m(x)$ such that
  $${}_{m(x)}p_x = {}_{m(x)}q_x = 0.5$$
- Mode of $T(x)$: value of t maximizing the pdf $f_{T(x)}(t) = {}_tp_x \, \mu_{x+t}$.

Some parametric laws of mortality used over the years - GRAPHS

- De Moivre (1729)

  $X \sim U[0, \omega]$ - $s(x) = 1 - x/\omega$, $0 \le x \le \omega$

  Linear function of x.

  1 parameter: ω

  Find the distribution of $T(x)$

  Find μ_x.

- Gompertz (1825)

  $\mu_x = Bc^x$ Exponential fct of x.

- Makeham (1860)

  $\mu_x = A + Bc^x$ A: force of accidental death

  Bc^x: force of ageing

- Weibull (1939) - engineering

  $\mu_x = kx^n$ Polynomial fct of x.

- Perks (1932) - British actuary

  $\mu_x = \dfrac{A + Be^{\mu x}}{1 + Ce^{\mu x}}$ - logistic fct of x

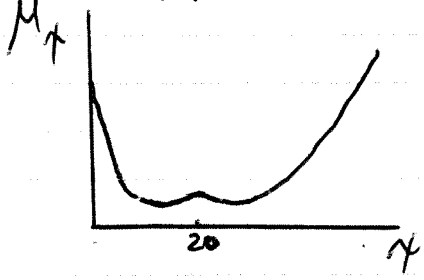
  - asymptote: as $x \to \infty$, $\mu_x \to$

  - Beard (1963): $A = 0$

- Kannisto - demograph (survival to old ages)

  $\mu_x = \dfrac{Be^{\mu x}}{1 + Be^{\mu x}}$

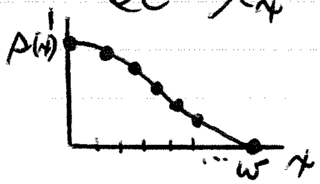
Human Mortality: Graph of μ_x.



No simple function of x (with 3 or 4 parameters) can capture all the characteristics of human mortality over the whole range $[0, 110]$.

Use instead a mortality table which contains, for integer values of $x = 0, 1, 2, \dots, 110$ some basic functions, v.g. a multiple of $s(x)$.

let $l_x = l_0 \cdot s(x)$ where l_0 = radix of table
(ex: $l_0 = 100{,}000$, an arbitrary number)



Illustrative life table (Bowers et al.)

x	l_x	d_x
0	100,000	42
1	97,958	132
2	97,826	70
3	97,756	160
4	97,596	
⋮	⋮	
110	0	

- Construction
- Hypothesis

1- Calculate Pr[(2) survives to age 4]

$$ {}_2p_2 = \frac{s(4)}{s(2)} \cdot \frac{l_0}{l_0} = \frac{l_4}{l_2} = 0.99765 $$

2- Calculate Pr[(1) dies between ages 2 and 3]

$$ {}_{1|}q_1 = \frac{s(2) - s(3)}{s(1)} \cdot \frac{l_0}{l_0} = \frac{l_2 - l_3}{l_1} = 0.0007145 $$

Let $d_x = l_x - l_{x+1}$ be the number of deaths between ages x & $x+1$.

Then ${}_{1|}q_1 = \frac{d_2}{l_1}$ $\qquad q_x = \frac{l_x - l_{x+1}}{l_x} = \frac{d_x}{l_x}$.

$${}_{m|n}q_x = \frac{l_{x+m} - l_{x+m+n}}{l_x}.$$

DISCRETE MODEL

Let the r.v. $K(x)$ be the integral number of future years lived by (x) $\equiv$ Curtate-future-lifetime of (x).

Then $K(x) = \underbrace{\lfloor T(x) \rfloor}_{\text{integer part of } T(x)}$ $\qquad K(x) \in \{0, 1, 2, 3, \ldots\}$

$$Pr[K(x)=0] = q_x = \frac{d_x}{l_x}$$

$$Pr[K(x)=1] = Pr[1 < T(x) \leq 2]$$

$$= Pr[x+1 < X \leq x+2 \mid X > x]$$

$$= \frac{s(x+1) - s(x+2)}{s(x)} = \frac{l_{x+1} - l_{x+2}}{l_x} = \frac{d_{x+1}}{l_x}$$

$$= \frac{d_{x+1}}{l_x}\left(\frac{l_{x+1}}{l_{x+1}}\right) = \frac{d_{x+1}}{l_{x+1}} \cdot \frac{l_{x+1}}{l_x} = p_x\, q_{x+1} = {}_{1|}q_x$$

In general, probability function of r.v. $K(x)$

$$Pr[K(x) = k] = {}_kp_x\, q_{x+k} = {}_{k|}q_x \quad , \; k = 0, 1, 2, \ldots$$

Curtate life expectancy: $E[K(x)] = e_x$

$$e_x = \sum_{k=0}^{\infty} k \cdot {}_{k|}q_x = \ldots = \sum_{k=1}^{\infty} {}_kp_x.$$

Hypothesis for fractionary ages

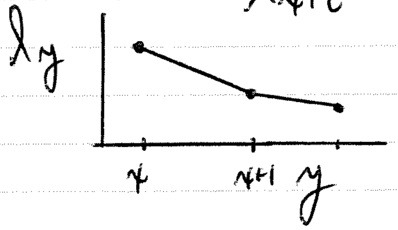
To calculate probabilities like ${}_{1/2}q_x$, ${}_{.4|}q_{x+.2}$ from a life table

(or ${}_{2.7}p_{33.5}$), we need an hypothesis on the behaviour of the curve l_y between 2 integral ages.

H1 : UDD (uniform distribution of death) hypothesis

let x be an integer

l_{x+t} is a linear function for $t \in [0,1]$ (x fixed)



$$l_{x+t} = (1-t)\,l_x + t\,l_{x+1} \quad (*), \quad 0 \le t \le 1$$

x fixed, t varies

Linear interpolation for l_{x+t} between l_x and l_{x+1} . $0 \le t \le 1$.

From $(*)$, $l_x - l_{x+t} = t(l_x - l_{x+1}) = t \cdot d_x$.

Then: 1- $\quad {}_tq_x = \dfrac{l_x - l_{x+t}}{l_x} = t \cdot \dfrac{d_x}{l_x} = t \cdot q_x$

2- $\quad \mu_{x+t} = \dfrac{-\frac{d}{dt}\, s(x+t)}{s(x+t)} = \dfrac{-\frac{d}{dt}\, l_{x+t}}{l_{x+t}} = \dfrac{-\frac{d}{dt}\,(l_x - t\cdot d_x)}{l_{x+t}}$

$$= \frac{d_x}{l_{x+t}} = \frac{d_x}{l_x - t d_x} = \frac{q_x}{1 - t\cdot q_x}$$

3- $\quad {}_yq_{x+t} = \dots = \dfrac{y \cdot q_x}{1 - t\cdot q_x}$, $0 \le t + y \le 1$.

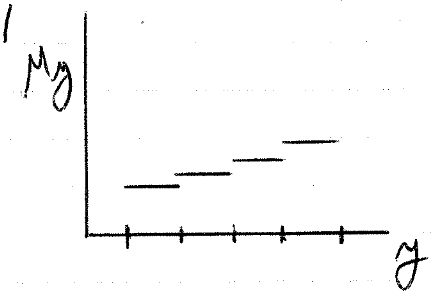
4- $\quad {}_tp_x\, \mu_{x+t} = \dots = q_x$ $\quad$ Constant (independent of t) $\Rightarrow$ UDD

Important result (pdf cst) directly from 1.

Under UDD, $\mathring{e}_x = e_x + 0.5$

H2: CFM (constant force of mortality) hyp.

$\mu_{x+t} = \mu_x$ for $0 \le t \le 1$



If $\mu_{x+t} = \mu_x$,

then ${}_tp_x = \exp\left[-\int_0^t \mu_{x+s}\,ds\right]$

$= e^{-t\cdot\mu_x}$

so ${}_1p_x = e^{-\mu_x}$ and $\mu_x = -\ln p_x = -\ln \frac{l_{x+1}}{l_x}$.

$${}_tp_x = \exp[-t\cdot\mu_x] = (e^{-\mu_x})^t = p_x^{\,t}$$

$$l_{x+t} = l_x \cdot e^{-t\cdot\mu_x} = l_x \cdot p_x^{\,t}$$

$$= l_x\left(\frac{l_{x+1}}{l_x}\right)^t = l_x^{1-t}\cdot l_{x+1}^{\,t}$$ Geometric interpolation.

$${}_tp_x\,\mu_{x+t} = \underline{\mu_x\, e^{-t\mu_x}} \Rightarrow \text{Exponential dist.}$$

Numerical comparisons

If $l_{30} = 96477$ & $l_{31} = 96350$, calculate ${}_{1/2}p_{30}$ under UDD & CFM

UDD $\quad {}_{1/2}p_{30} = \frac{l_{30\frac{1}{2}}}{l_{30}} = \frac{\frac{1}{2}(l_{30}+l_{31})}{l_{30}} = 0.99934181$
arithmetic average

CFM $\quad {}_{1/2}p_{30} = \frac{l_{30\frac{1}{2}}}{l_{30}} = (p_{30})^{1/2} = 0.99934160$
geometric average $l_{30\frac{1}{2}}$

Little numerical differences in practice!

Select and ultimate mortality tables

Insurance companies select their insured lives after a medical exam : underwriting = selection of risks (there are uninsurable lives)

Mortality of insured lives better than that of general population (produced by Statistics Canada).

Mortality does not depend only on attained age, but also on number of years since policy issue (i.e. since medical exam or tests).

Review - Theory of interest (ref. Kellison)

i : interest rate ($\equiv$ effective rate of interest)

v : discount factor

d : discount rate

δ : force of interest

Relations: $v = \frac{1}{1+i}$ $\qquad d = 1 - v = \frac{i}{1+i}$

$\delta = \ln(1+i)$ $\qquad 1+i = e^{\delta}$

$\delta = -\ln v$ $\qquad v = e^{-\delta}$

Accumulation function (in t years)

- simple interest: $a(t) = 1 + it$
- compound interest: $a(t) = (1+i)^t$

Present value of 1 payable in t years (compound int.)

$v^t = (1+i)^{-t} = e^{-\delta t}$.

Interest compounded m times a year: $i^{(m)}$

Accumulated value of \$1 at end of year $\quad \left(1 + \frac{i^{(m)}}{m}\right)^m = 1 + i = e^{\delta}$

Present value of \$1 at end of year $\quad \left(1 - \frac{d^{(m)}}{m}\right)^m = 1 - d = e^{-\delta}$

If $m \to \infty$, $i^{(m)}$ and $d^{(m)} \to \delta$.

P.V. of \$1 per year at BOY for n years

$$\ddot{a}_{\overline{n|}} = 1 + v + \ldots + v^{n-1} = \frac{1-v^n}{d}$$

\$1 payable continuously for n years

$$\bar{a}_{\overline{n|}} = \int_0^n e^{-\delta t}\,dt = \frac{1-v^n}{\delta}$$

II - Life insurance

A- Benefits payable at the moment of death.

a) Whole-life insurance

Let us suppose an insurance policy paying \$ 1 at death is issued to (x). If the interest rate i is constant in the future, the r.v. of the present value, at policy issue (i.e. age x) of the death benefit is

$$Z = 1 \cdot v^{T(x)} = v^T \quad , \quad T \geq 0.$$

The mean of r.v. Z is called the actuarial present value (APV) of the death benefit (or net single premium)

$$E(Z) = \bar{A}_x = \int_0^\infty v^t f_{T(x)}(t)\,dt$$

$$= \int_0^\infty v^t \, {}_tp_x \, \mu_{x+t}\,dt.$$

Variance of r.v. Z = measure of risk to insurer.

$$Var(Z) = E(Z^2) - E^2(Z)$$

$$= E(v^{2T}) - \bar{A}_x^{\,2}$$

$$E(v^{2T}) = \int_0^\infty v^{2t} \, {}_tp_x \, \mu_{x+t}\,dt$$

N.B. $v = e^{-\delta}$ $\qquad v^2 = e^{-2\delta}$

$$E(v^{2T}) = \int_0^\infty e^{-2\delta t} \, {}_tp_x \, \mu_{x+t}\,dt \qquad (\text{let } \delta^* = 2\delta)$$

$$= \int_0^\infty e^{-\delta^* t} \, {}_tp_x \, \mu_{x+t}$$

$$= \bar{A}_x^{*} \quad (\text{i.e. calculated at } \delta^* = 2\delta)$$

denoted ${}^2\bar{A}_x$.

$$Var(Z) = {}^2\bar{A}_x - \bar{A}_x^{\,2}.$$

Ex. If $T(x) \sim$ De Moivre $(\omega = 100)$

and $x = 50$, calculate at δ, for a whole life insurance of 1000 payable at death

a) the APV of the death benefit.
b) the variance of Z.
c) the pdf of Z.
d) the median of Z.

---

b) n-year term insurance

The insurance pays 1 at death of (x) if death occurs within the next n years; 0 if not.

$$Z = \begin{cases} v^T, & 0 < T \leq n. \\ 0, & T > n. \end{cases}$$

APV $\quad E(Z) = \bar{A}^1_{x:\overline{n}|} = \int_0^n v^t \, {}_tp_x \mu_{x+t} \, dt.$

- Principle of notation
- Interpretation.

$$\begin{aligned} Var(Z) &= E(Z^2) - E^2(Z) \\ &= \int_0^n v^{2t} \, {}_tp_x \mu_{x+t} \, dt - \bar{A}^{1\,2}_{x:\overline{n}|} \\ &= \bar{A}^1_{x:\overline{n}|} \text{ calculated at } 2\delta - \bar{A}^{1\,2}_{x:\overline{n}|} \\ &= {}^2\bar{A}^1_{x:\overline{n}|} - \bar{A}^{1\,2}_{x:\overline{n}|} \end{aligned}$$

c) n-year endowment insurance

Policy pays \$1 at death of (x), if (x) dies within n years; if (x) survives n year, \$1 is paid at time n.

$$Z = \begin{cases} v^T, & 0 < T \le n \\ v^n; & T > n \end{cases}$$

A.P.V $\quad E(Z) = \int_0^n v^t \, {}_tp_x \mu_{x+t} \, dt + \int_n^\infty v^n \, {}_tp_x \mu_{x+t} \, dt.$

Benefit composed of 2 parts:
n-year term insurance + survival benefit.

$$E(Z) = \bar{A}_{x:\overline{n|}} = \bar{A}^{1}_{x:\overline{n|}} + v^n \underbrace{\int_n^\infty {}_tp_x \mu_{x+t} \, dt}_{P[T(x) > n]}$$

$$= 1 - P[T(x) \le n]$$

$${}_np_x = 1 - {}_nq_x$$

$$\bar{A}_{x:\overline{n|}} = \bar{A}^{1}_{x:\overline{n|}} + v^n \, {}_np_x.$$

$$Var(Z) = {}^2\bar{A}_{x:\overline{n|}} - \bar{A}_{x:\overline{n|}}^2.$$

Consider $Z = Z_1 + Z_2$

$$Z = \begin{cases} v^T, & 0 < T \le n \\ 0; & T > n \end{cases} + \begin{cases} 0, & 0 < T \le n \\ v^n; & T > n \end{cases}$$

Term insurance + Survival benefit.

$$E(Z) = E(Z_1) + E(Z_2)$$

Z_2 also called pure endowment.

Z_1 and Z_2 not independent (in fact, negatively correlated).

Calculate correlation coefficient ρ between Z_1 and Z_2.

d) m-year deferred insurance

$$Z = \begin{cases} 0, & T \le m \\ v^T, & T > m \end{cases}$$

$$E(Z) = {}_{m|}\bar{A}_x = \int_m^\infty v^t \, {}_tp_x \, \mu_{x+t} \, dt$$

$$= \int_0^\infty \_\_\_ - \int_0^m \_\_\_ = \bar{A}_x - \bar{A}^{\,1}_{x:\overline{m}|}$$

or: $\int_m^\infty v^t \, {}_tp_x \, \mu_{x+t} \, dt$ let $t = m+s$ Change of variable

$$\int_0^\infty v^{m+s} \, {}_{m+s}p_x \, \mu_{x+m+s} \, ds$$

$$= v^m \, {}_mp_x \int_0^\infty v^s \, {}_sp_{x+m} \, \mu_{x+m+s} \, ds$$

$${}_{m|}\bar{A}_x = v^m \, {}_mp_x \, \bar{A}_{x+m}.$$

$Var(Z)$

e) m-year deferred, n-year term insurance

$$Z = \begin{cases} 0, & T \le m \\ v^T, & m < T \le m+n \\ 0, & T > m+n \end{cases}$$

$$E(Z) = {}_{m|n}\bar{A}_x = {}_{m|}\bar{A}^{\,1}_{x:\overline{n}|}$$

$$= \ldots$$

$Var(Z) = \ldots$

B- benefit payable at the end of year of death of (x) $\longrightarrow$ DISCRETE MODEL.

a) Whole life insurance.

$\$1$ is paid at the end of year of death of (x).

Z: r.v. of present value of death benefit

$$Z = v^{K(x)+1} = v^{K+1} \quad , \quad K \in \{0,1,2,\dots\}$$

where $P_r[K=k] = {}_{k|}q_x = {}_kp_x\, q_{x+k} = \dfrac{d_{x+k}}{l_x}$.

APV for whole life ins. $E(Z)$

$$E(Z) = A_x = E(v^{K+1}) = \sum_{k=0}^{\infty} v^{k+1}\, {}_{k|}q_x .$$

$$Var(Z) = E(Z^2) - E^2(Z)$$

$$= \sum_{k=0}^{\infty} v^{2(k+1)}\, {}_{k|}q_x - A_x^2 \qquad \left(\begin{matrix} v = e^{-\delta} \\ v^2 = e^{-2\delta} \end{matrix}\right)$$

$$= {}^2A_x - A_x^2 . \qquad \left({}^2A_x \text{ is calculated at } 2\delta.\right)$$

b) n-year term insurance (n: integer)

$$Z = \begin{cases} v^{K+1} & , \ K = 0,\dots,n-1 \\ 0 & , \ K \geq n \end{cases}$$

$$E(Z) = A^{1}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1}\, {}_{k|}q_x .$$

$$Var(Z) = {}^2A^{1}_{x:\overline{n|}} - \left(A^{1}_{x:\overline{n|}}\right)^2 .$$

c) n-year endowment ins.

$$Z = \begin{cases} v^{K+1}, & K = 0, \ldots, n-1 \\ v^n, & K \geq n. \end{cases}$$

$$E(Z) = A_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_x + \sum_{k=n}^{\infty} v^n {}_{k|}q_x$$

$$= A^{1}_{x:\overline{n|}} + \underbrace{v^n \cdot {}_n p_x}_{A_{x:\overline{n|}}^{\ \ 1}}.$$

$Var(Z) = \ldots$

$\rho(Z_1, Z_2)$... where Z_1 : n-yr term ins.
Z_2 : pure endowment

d) ${}_{m|}A_x$ m-year deferred ins.

e) ${}_{m|n}A_x$. m-yr deferred, n-yr term ins.

m & n are integers.

Ex 1

x	q_x
20	0.010
21	0.015
22	0.020
23	0.025
24	0.030

Consider an endowment insurance policy issued to (20) of \$1000, where \$1000 is paid at the end of yr if death occurs before age 25 or at age 25 if (20) survives 5 years.

a) Calculate the probability fct of Z.
b) Calculate the APV of benefits.
c) Calculate the standard error of Z.

Solution: ...

Ex. 2

b_x: death benefit

x	l_x	d_x	b_x	k
20	1000		1000	0
21	995		1000	1
22	990		1000	2
23	984		1500	3
24	977		2000	4
25	969		0	5
26	959		0	6

a) Calculate the prob. fct of the r.v. Z, the p.v. for a term ins. policy issued to (20) paying b_x at end of yr of death of (20)

b) Calc. the APV of benefits

c) If n similar policies are issued, which risk loading θ will make the portfolio profitable with a 95% probability.

Solution: ...

Approximation for $\bar{A}_x$ in terms of A_x.

1- Approximate method.

$\bar{A}_x > A_x$ since benefit paid sooner.
$\Rightarrow$ higher NSP.

If mortality not taken into account, on average benefit paid 6 months ($= \frac{1}{2}$ year sooner)

With simple interest for a half-year

$$\bar{A}_x \simeq \left(1+\frac{i}{2}\right) A_x .$$

2- Under UDD between ages x & $x+1$.

We know ${}_tp_x \mu_{x+t} = q_x$ for $0 \le t \le 1$.

$$\bar{A}_x = \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \, dt$$

$$= \int_0^1 v^t \, {}_tp_x \mu_{x+t} \, dt + \int_1^2 v^t \, {}_tp_x \mu_{x+t} \, dt + \ldots$$

let $t = 1+s$ $\qquad t = 2+s \ldots$

$$= \int_0^1 v^t \, {}_tp_x \mu_{x+t} dt + v p_x \int_0^1 v^s \, {}_sp_{x+1} \mu_{x+1+s} ds + v^2 \, {}_2p_x \int_0^1 v^s \, {}_sp_{x+2} \mu_{x+2+s} ds + \ldots$$

Under UDD,

$$\bar{A}_x = q_x \int_0^1 v^t dt + v p_x q_{x+1} \int_0^1 v^s ds + \ldots$$

$$\int_0^1 v^s ds = \ldots = \bar{a}_{\overline{1|}} = \frac{1-v}{\delta} = \frac{d}{\delta}$$

$$\Rightarrow \bar{A}_x \overset{UDD}{=} \frac{d}{\delta} q_x + \frac{vd}{\delta}\, {}_{1}p_x\, q_{x+1} + v^2 \frac{d}{\delta}\, {}_{2}p_x\, q_{x+2} + \dots$$

As $d = iv$

$$\bar{A}_x = \frac{i}{\delta}\left[v q_x + v^2\, {}_{1|}q_x + v^3\, {}_{2|}q_x + \dots\right]$$

$$= \frac{i}{\delta} \sum_{k=0}^{\infty} v^{k+1}\, {}_{k|}q_x = \frac{i}{\delta} A_x$$

N.B. $$\frac{i}{\delta} = \frac{i}{\ln(1+i)} = \frac{i}{i - \frac{i^2}{2} + \frac{i^3}{3} - \dots}$$

$$= \left[1 - \frac{i}{2} + \frac{i^2}{3} - \dots\right]^{-1}$$

$$= 1 + \frac{i}{2} + i^2\left(-\frac{1}{3} + \frac{1}{4}\right) + \dots = 1 + \frac{i}{2} - \frac{i^2}{12}$$

$$\simeq 1 + \frac{i}{2} \qquad \left(\text{pi } i = 0.1,\ \frac{i^2}{12} < 0.001\right.$$

Show that under UDD

$$\bar{A}^{\,1}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|}$$

$$\bar{A}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|} + v^n\, {}_{n}p_x$$

$${}_{m|}\bar{A}_x$$

$${}_{m|n}\bar{A}_x$$

Recursive relationships

1- between e_x and e_{x+1}

$$e_x = \sum_{k=1}^{\infty} {}_kp_x = p_x + {}_2p_x + {}_3p_x + \cdots$$

$$= p_x + p_x \left[p_{x+1} + {}_2p_{x+1} + \cdots \right]$$

$$e_x = p_x + p_x\, e_{x+1}$$

With a mortality table and 1 value e_x, all values $e_0, \ldots, e_\omega$ can be calculated recursively.

2- between A_x and A_{x+1}.

$$A_x = A^{1}_{x:\overline{1|}} + {}_{1|}A_x$$

$$A_x = v q_x + v p_x A_{x+1}$$

Same remark + interest rate i.

1- and 2- of the same form

$$u_x = c_x + v p_x\, u_{x+1}$$

(1- with $i=0$ or $v=1$).

3- between $\ddot{a}_x$ and $\ddot{a}_{x+1}$. (of the form above)

III Life annuities

A - Introduction

life insurance: payment made if person dies
life annuity: payment made if person survives.

Life annuity: series of payments made at periodic intervals as long as a person survives.

Types of annuities

- Temporary vs life annuity.
- payments starting immediately vs deferred (ex. to age 65).
- payments made at beginning or at the end of interval (annuity due vs annuity immediate)

Same terminology and notation as used for annuities certain but survival is now a condition for payment.

(Reminder: Pure endowment = survival benefit ; $A_{x:\overline{n}|}^{\;\;1} = 1 \cdot v^n \, {}_np_x$)

discount for interest and mortality.

B- Continuous annuity

- Life annuity

let us consider a life annuity of \$1 per annum, payable continuously, as long as (x) lives.

The r.v. of the present value of payments of this life annuity (until the death of (x))

$$Y = \bar{a}_{\overline{T(x)}|} = \bar{a}_{\overline{T}|} = \frac{1-v^T}{\delta} \qquad (*)$$

The APV of a continuous life annuity issued to (x):

$$\bar{a}_x = E(Y) = E(\bar{a}_{\overline{T}|}) \qquad \text{pdf of } T: {}_tp_x \mu_{x+t}$$

$$= \int_0^\infty \underbrace{\bar{a}_{\overline{t}|}}_{u} \underbrace{{}_tp_x \mu_{x+t} dt}_{dv}$$

Integrate by parts: $v = -{}_tp_x \quad du = v^t$

$$\bar{a}_x = \left[-\bar{a}_{\overline{t}|} \, {}_tp_x \right]_{t=0}^{t=\infty} + \int_0^\infty v^t \, {}_tp_x \, dt$$

$$\bar{a}_x = \int_0^\infty v^t \, {}_tp_x \, dt$$

Interpretation: $v^t \, {}_tp_x \, dt$ is APV (discounted for interest and mortality) of payment dt made at time t; sum these values over all possible values of t.

(analogy to $\bar{a}_{\overline{\infty}|} = \int_0^\infty v^t dt$.

Relation between $\bar{a}_x$ & $\bar{A}_x$

$$Y = \bar{a}_{\overline{T}|} = \frac{1-v^T}{\delta}$$

Take expectation on each side

$$E(Y) = E\left(\frac{1-v^T}{\delta}\right)$$

$$\bar{a}_x = \frac{1-E(v^T)}{\delta} = \frac{1-\bar{A}_x}{\delta}$$

$$\Rightarrow \bar{A}_x = 1-\delta\bar{a}_x \qquad \left(\begin{matrix} 1 = \delta\bar{a}_x + \bar{A}_x \\ \text{Interpretation} \end{matrix}\right)$$

Variance of $\bar{a}_{\overline{T}|}$

$$Var\left(\bar{a}_{\overline{T}|}\right) = Var\left(\frac{1-v^T}{\delta}\right) = \frac{1}{\delta^2} Var(v^T)$$

$$= \frac{1}{\delta^2}\left({}^2\bar{A}_x - \bar{A}_x^2\right)$$

$$= \frac{1}{\delta^2}\left[\left(1-2\delta \cdot {}^2\bar{a}_x\right) - \left(1-\delta\bar{a}_x\right)^2\right]$$

$$= \frac{1}{\delta^2}\left[1-2\delta \cdot {}^2\bar{a}_x - 1 + 2\delta\bar{a}_x - \delta^2\bar{a}_x^2\right]$$

$$= \frac{2}{\delta}\left[\bar{a}_x - {}^2\bar{a}_x\right] - \bar{a}_x^2 .$$

$Y = \bar{a}_{\overline{T|}}$ continuous positive r.v.

To find pdf of Y, let us find first its cdf.

$$F_Y(y) = \Pr[Y \leq y] = \Pr[\bar{a}_{\overline{T|}} \leq y]$$
$$= \Pr[1 - v^T \leq \delta y]$$
$$= \Pr[v^T \geq 1 - \delta y]$$
$$= \Pr\left[T \leq \frac{\ln(1-\delta y)}{-\delta}\right]$$

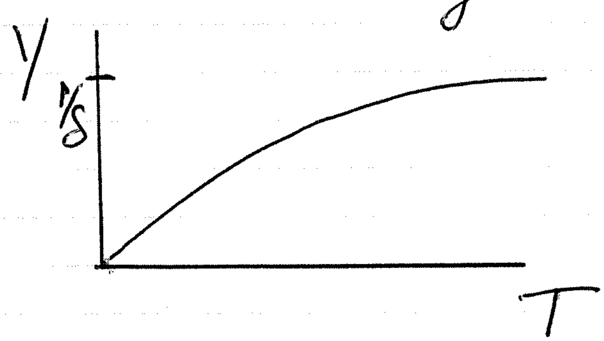
$$F_Y(y) = F_T\left(-\frac{1}{\delta}\ln(1-\delta y)\right).$$

Si $0 < t < \infty$, $0 < \bar{a}_{\overline{t|}} < 1/\delta$

pdf of Y

$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_T\left(-\frac{1}{\delta}\ln(1-\delta y)\right) \cdot \frac{1}{1-\delta y},$$

for $0 < y < 1/\delta$



If $\delta = 0$ $\bar{a}_x = \int_0^\infty t\, p_x\, dt \stackrel{?}{=} \mathring{e}_x.$

Ex. If $\mu_x = \mu \ \forall x$ (i.e. $T(x) \sim Exp(\mu)$), calculate at δ

a) $\bar{a}_x$.

b) $Var(\bar{a}_{\overline{T|}})$

c) the probability that $\bar{a}_{\overline{T|}}$ exceeds $\bar{a}_x$.

$$Pr[\bar{a}_{\overline{T|}} > \bar{a}_x]$$

d) the pdf of $Y = \bar{a}_{\overline{T|}}$.

e) make a graph of $f_Y(y)$ if $\mu = 0.04$, $\delta = 0.06$

- n-year temporary annuity

The APV of an n-year temporary annuity paying \$1 per year continuously while (x) survives during the next n years is denoted $\bar{a}_{x:\overline{n|}}$

The r.v. of the present value of benefits

$$Y = \begin{cases} \bar{a}_{\overline{T|}} & \text{if } 0 \leq T < n \\ \bar{a}_{\overline{n|}} & \text{if } T \geq n. \end{cases}$$

$$\bar{a}_{x:\overline{n|}} = E(Y) = \int_0^n \bar{a}_{\overline{t|}} \, {}_tp_x \, \mu_{x+t} + \int_n^\infty \bar{a}_{\overline{n|}} \, {}_tp_x \mu_{x+t} \, dt$$

integrate by parts

$$= \left[-\bar{a}_{\overline{t|}} \, {}_tp_x \right]_{t=0}^{t=n} + \int_0^n v^t \, {}_tp_x \, dt + \bar{a}_{\overline{n|}} \, {}_np_x$$

$$= -\bar{a}_{\overline{n}|} \cdot {}_n p_x + \int_0^n v^t \, {}_t p_x \, dt + \bar{a}_{\overline{n}|} \cdot {}_n p_x$$

Interpretation: $\bar{a}_{x:\overline{n}|} = \int_0^n v^t \, {}_t p_x \, dt.$

$$Y = \frac{1-Z}{\delta} \quad \text{where} \quad Z = \begin{cases} v^T, & 0 < T < n \\ v^n, & T \geq n. \end{cases}$$

Take $E(\ )$ — r.v. of p.v. for endowment insurance (n-year)

$$\bar{a}_{x:\overline{n}|} = E\left(\frac{1-Z}{\delta}\right) = \frac{1-E(Z)}{\delta} = \frac{1-\bar{A}_{x:\overline{n}|}}{\delta}$$

$$\Rightarrow \bar{A}_{x:\overline{n}|} = 1 - \delta \bar{a}_{x:\overline{n}|}$$

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\frac{1-Z}{\delta}\right) = \frac{1}{\delta^2} \mathrm{Var}(Z)$$

$$= \frac{1}{\delta^2}\left[{}^2\bar{A}_{x:\overline{n}|} - \bar{A}_{x:\overline{n}|}^2\right]$$

Express in terms of annuity symbols! (special case: $n \to \infty$).

- Deferred annuity

$$Y = \begin{cases} 0 & \text{if } T \leq n \\ \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|} & \text{if } T > n. \end{cases}$$

$$E(Y) = {}_{n|}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n}|} = \int_n^\infty v^t \, {}_t p_x \, dt$$

$$= v^n \, {}_n p_x \cdot \bar{a}_{x+n}.$$

Find an expression for $\mathrm{Var}(Y)$.

- n-year certain and life annuity

$$Y = \begin{cases} \bar{a}_{\overline{n}|} & \text{if } T \le n \\ \bar{a}_{\overline{T}|} & \text{if } T > n \end{cases}$$ Payments until $\max[T(x), n]$

VaP $\bar{a}_{\overline{x:\overline{n}|}} = \underset{\text{int. by parts}}{\ldots} = \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x$.

Var(Y) = Var. for n-yr deferred annuity.

$$Y = \begin{pmatrix} \bar{a}_{\overline{n}|} \\ \bar{a}_{\overline{n}|} \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|} \end{pmatrix} \quad \begin{matrix} T \le n \\ T > n. \end{matrix}$$

Y = cst + (r.v. for deferred annuity)

C- Annuity dues with annual payments

- Life annuity: Payment of \$1 at beginning of each year if (x) is then alive

r.v. of P.V. of payments

$$Y = \ddot{a}_{\overline{K+1}|} \qquad K \equiv K(x)$$

$$E(Y) = \ddot{a}_x = E\left(\ddot{a}_{\overline{K+1}|}\right)$$

$$\ddot{a}_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_x$$ Develop series ...

$$\vdots$$

$({}_{k|}q_x = {}_kp_x - {}_{k+1}p_x)$

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x$$

- Analogous to $\bar{a}_x = \int_0^\infty v^t \, {}_tp_x \, dt$.

relation between $\ddot{a}_x$ and A_x.

$$\ddot{a}_x = E\left(\frac{1-v^{K+1}}{d}\right) = \frac{1-A_x}{d}$$

$$A_x = 1 - d\,\ddot{a}_x \quad (\square) \quad \text{Interpretation.}$$

$$\mathrm{Var}\left(\ddot{a}_{\overline{K+1}|}\right) = \mathrm{Var}\left(\frac{1-v^{K+1}}{d}\right) = \frac{1}{d^2}\mathrm{Var}(v^{K+1})$$

$$= \frac{{}^2A_x - A_x^2}{d^2} \quad \text{Use } (\square).$$

- n-year temporary annuity

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|}, & K=0,1,\dots,n-1 \\ \ddot{a}_{\overline{n}|} & K \geq n. \end{cases} \qquad \begin{cases} (1-v^{K+1})/d \\ \dfrac{1-v^n}{d} \end{cases}$$

$$\ddot{a}_{x:\overline{n}|} = E(Y) = \dots = \sum_{k=0}^{n-1} v^k\, {}_kp_x$$

$$Y = \frac{1-Z}{d} \quad \text{where } Z = \begin{cases} v^{K+1}, & \text{if } K=0,\dots,n-1 \\ v^n, & \text{if } K \geq n. \end{cases}$$

$$E(Z) = A_{x:\overline{n}|}$$

$$A_{x:\overline{n}|} = 1 - d\,\ddot{a}_{x:\overline{n}|}$$

$$\mathrm{Var}(Y) = \frac{1}{d^2}\left[{}^2A_{x:\overline{n}|} - A_{x:\overline{n}|}^2\right]$$

- n-year deferred

$${}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k\, {}_kp_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|}$$

$$= v^n\, {}_np_x \cdot \ddot{a}_{x+n}.$$

- n-year certain and life

$$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x$$

D- Annuity-immediate (payment at the end of yr)

Develop series a...

$E(Y) = a_x = \ddot{a}_x - 1$

$Y = a_{\overline{K}|}, \quad K = 0, 1, \ldots$

$\quad = \ddot{a}_{\overline{K+1}|} - 1$

$Var(a_{\overline{K}|}) = Var(\ddot{a}_{\overline{K+1}|})$

$a_{x:\overline{n}|} = v\, p_x\, \ddot{a}_{x+1:\overline{n}|}$

E- Annuities payable m times a year (payments of 1/m)

$$\ddot{a}_x^{(m)} = \sum_{j=0}^{\infty} \frac{1}{m} v^{j/m}\, {}_{j/m}p_x$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{m-1} \frac{1}{m} v^{k+j/m}\, {}_{k+j/m}p_x \qquad (0 \le j/m < 1)$$

Apply UDD ...

$$\ddot{a}_x^{(m)} \overset{\underline{\underline{UDD}}}{=} \alpha(m)\, \ddot{a}_x - \beta(m)$$

where $\alpha(m) = \dfrac{i\,d}{i^{(m)} d^{(m)}}$ and $\beta(m) = \dfrac{i - i^{(m)}}{i^{(m)} d^{(m)}}$

$\alpha(m)$ & $\beta(m)$ $\left\{ \begin{array}{l} \text{depend only on } m \text{ and } i; \\ \text{independent of age.} \end{array} \right.$

$m = 1 \qquad \ddot{a}_x^{(1)} = \ddot{a}_x (1) - 0 \qquad \alpha(1) = 1 \qquad \beta(1) = 0.$

$m \to \infty \qquad \lim_{m \to \infty} \ddot{a}_x^{(m)} = \bar{a}_x = \dfrac{i\,d}{\delta^2}\, \ddot{a}_x - \dfrac{i - \delta}{\delta^2}.$

End of period $\quad a_x^{(m)} = \ddot{a}_x^{(m)} - \dfrac{1}{m}.$

Temporary: $\ddot{a}_{x:\overline{n}|}^{(m)} \overset{UDD}{=} \ldots$

$\qquad = \alpha(m)\, \ddot{a}_{x:\overline{n}|} - \beta(m)\,(1 - {}_nE_x)$

$$a_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m} + \frac{1}{m} v^n\, {}_np_x.$$

Ex. Temporary annuity with varying payments
Benefit b_{k+1} paid at $\boxed{\text{end}}$ of year that (x) survives.

k	b_{k+1}	$_{k+1}p_x$
0	1000	.90
1	2000	.75
2	3000	.55
3	3000	.30
≥ 4	0	0

If $v = 0.9$

find a) the prob. fct of Y
b) the APV of benefit
c) standard error of Y.

Solution:

Recursive relationship between $\ddot{a}_x$ & $\ddot{a}_{x+1}$

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x = 1 + \sum_{k=1}^{\infty} v^k \, {}_kp_x$$

$$= 1 + \ddot{a}_{x+1} v p_x .$$

With one value $\ddot{a}_x$ + mortality table, can generate whole table of $\ddot{a}$ at all other ages.

Add relation to Chapter III, p. 10.

IV Net premiums

Whole life insurance policy usually paid by annual premiums, not a NSP.

Let us define, for a contract, the loss r.v. (L) equal to r.v. of present value of benefits to be paid by insurer minus r.v. of present values of premiums to be paid by insured.

Equivalence principle to determine premium is such that $E(L) = 0$.

Net premiums calculated with equivalence principle.

$$L = (\text{P.V. of benefit}) - (\text{P.V. of net prems})$$

$$E(L) = 0 \Leftrightarrow E(\text{P.V. of benefit}) = E(\text{PV of prems})$$

$$\text{A.P.V of benefit} = \text{APV of prems.}$$

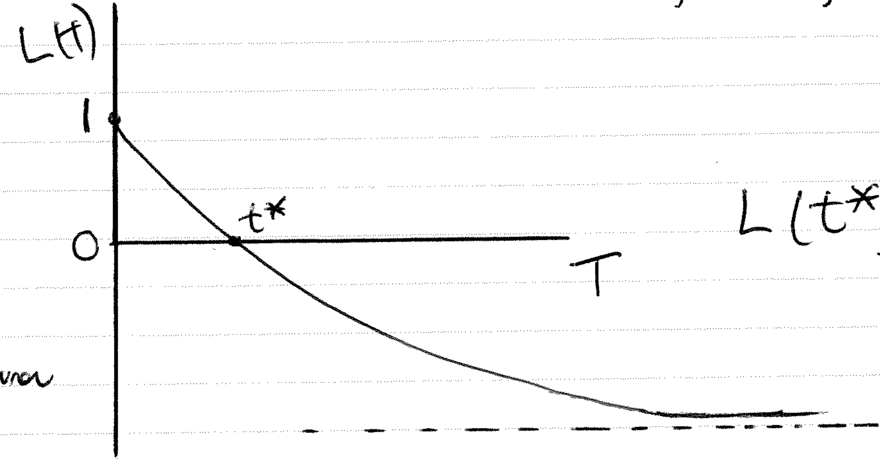
A- Continuous premiums

Ex. Whole life insurance, payable at death of (x), with continuous prem payable for life; r.v. of P.V. of loss to insurer

$$L(T) = v^T - \bar{P}\,\bar{a}_{\overline{T|}} \quad , T > 0.$$

$L(T)$: decreasing fct of T $\quad L(0) = 1$

if $T \to \infty$, $L(T) \to 0 - \bar{P}/\delta$



$L(t^*) = 0$

negative loss = profit to insurer

According to equivalence principle, net premium is such that
$$E[L(T)] \equiv 0$$

Notation for net premium: $\bar{P}(\bar{A}_x)$

$$E[L(T)] = E[v^T - \bar{P}(\bar{A}_x) \times \bar{a}_{\overline{T|}}]$$
$$= \bar{A}_x - \bar{P}(\bar{A}_x)\bar{a}_x = 0$$
$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x} = \frac{1-\delta\bar{a}_x}{\bar{a}_x} = \frac{1}{\bar{a}_x} - \delta$$

To measure variability of losses on a whole life insurance policy (because of the r.v. Time-until-death of (x)), we calculate the variance of $L(T)$.

$$Var(L(T)) = Var[v^T - \bar{P}\bar{a}_{\overline{T|}}]$$
$$= Var\left[v^T - \bar{P}\left(\frac{1-v^T}{\delta}\right)\right]$$
$$= Var\left[v^T\left(1+\frac{\bar{P}}{\delta}\right) - \frac{\bar{P}}{\delta}\right]$$
$$= \left(1+\frac{\bar{P}}{\delta}\right)^2 Var(v^T)$$
$$= \left(1+\frac{\bar{P}}{\delta}\right)^2 \left({}^2\bar{A}_x - \bar{A}_x^2\right).$$

<u>If</u> $\bar{P}$ is net prem (calculated with equivalence principle)

1- $Var(L) = E(L^2)$ since $E(L) = 0$.

2- $\bar{P} = \bar{P}(\bar{A}_x) = \frac{1}{\bar{a}_x} - \delta$ so $Var[L(T)] = \frac{{}^2\bar{A}_x - \bar{A}_x^2}{(\delta\bar{a}_x)^2}$

Ex If $\mu_x = \mu$ $\forall x$, find $\bar{P}(\bar{A}_x)$ and $Var(L(T))$

In general, if b_t is the benefit if death occurs at time t

$\bar{P}$ is a general symbol for a continuous prem,

Y is the r.v. for a continuous annuity

$$L(T) = b_T v^T - \bar{P} Y$$

Equivalence principle $E(L(T)) = 0$

$$\Rightarrow E(b_T v^T) - \bar{P} E(Y) = 0$$

$$\bar{P} = \frac{E(b_T v^T)}{E(Y)}$$

TABLE 6.2.1

Fully Continuous Benefit Premiums

ref: Actuarial Mathematics Bowers + al (1997). p. 183

Plan	Loss Components: $b_T v_T$	Loss Components: $\bar{P}\ Y$ Where Y Is	Premium Formula $\bar{P} = \dfrac{E[b_T v_T]}{E[Y]}$
Whole life insurance	$1\ v^T$	$\bar{a}_{\overline{T}\rceil}$	$\bar{P}(\bar{A}_x) = \dfrac{\bar{A}_x}{\bar{a}_x}$
n-Year term insurance	$1\ v^T$ 0	$\bar{a}_{\overline{T}\rceil},\ T \le n$ $\bar{a}_{\overline{n}\rceil},\ T > n$	$\bar{P}(\bar{A}^{1}_{x:\overline{n}\rceil}) = \dfrac{\bar{A}^{1}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}}$
n-Year endowment insurance	$1\ v^T$ $1\ v^n$	$\bar{a}_{\overline{T}\rceil},\ T \le n$ $\bar{a}_{\overline{n}\rceil},\ T > n$	$\bar{P}(\bar{A}_{x:\overline{n}\rceil}) = \dfrac{\bar{A}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}}$
h-Payment* whole life insurance	$1\ v^T$ $1\ v^T$	$\bar{a}_{\overline{T}\rceil},\ T \le h$ $\bar{a}_{\overline{h}\rceil},\ T > h$	${}_h\bar{P}(\bar{A}_x) = \dfrac{\bar{A}_x}{\bar{a}_{x:\overline{h}\rceil}}$
h-Payment,* n-year endowment insurance	$1\ v^T$ $1\ v^T$ $1\ v^n$	$\bar{a}_{\overline{T}\rceil},\ T \le h$ $\bar{a}_{\overline{h}\rceil},\ h < T \le n$ $\bar{a}_{\overline{h}\rceil},\ T > n$	${}_h\bar{P}(\bar{A}_{x:\overline{n}\rceil}) = \dfrac{\bar{A}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{h}\rceil}}$
n-Year pure endowment	0 $1\ v^n$	$\bar{a}_{\overline{T}\rceil},\ T \le n$ $\bar{a}_{\overline{n}\rceil},\ T > n$	$\bar{P}(A_{x:\overline{n}\rceil}^{\ \ 1}) = \dfrac{A_{x:\overline{n}\rceil}^{\ \ 1}}{\bar{a}_{x:\overline{n}\rceil}}$
n-Year† deferred whole life annuity	0 $\bar{a}_{\overline{T-n}\rceil}\ v^n$	$\bar{a}_{\overline{T}\rceil},\ T \le n$ $\bar{a}_{\overline{n}\rceil},\ T > n$	$\bar{P}({}_{n\mid}\bar{a}_x) = \dfrac{A_{x:\overline{n}\rceil}^{\ \ 1}\ \bar{a}_{x+n}}{\bar{a}_{x:\overline{n}\rceil}}$

*The insurances described in the fourth and fifth rows provide for a premium paying period that is shorter than the period over which death benefits are paid.

†The annuity product described above provides no death benefits and has a level premium with premiums payable for n years. A different, perhaps more realistic, design for an n-year level premium-deferred annuity is given in Example 6.6.2.

Ex. For a whole life insurance paying \$1 at moment of death, with continuous premium $\bar{P}$, find

a) the cdf of L

b) the pdf of L.

c) the probability that a policy is profitable (to insurer).

d) the prem $\bar{P}^*$ such that there is a 90% probability that a policy is profitable.

For n-year endowment insurance

$$L(T) = Z - \bar{P}(\bar{A}_{x:\overline{n|}})\cdot Y, \text{ where } Z = \begin{cases} v^T, & T \le n \\ v^n, & T > n \end{cases} \text{ and } Y = \frac{1-Z}{\delta}$$

$$Var(L(T)) = Var\left[Z\left(1 + \frac{\bar{P}(\bar{A}_{x:\overline{n|}})}{\delta}\right) - \frac{\bar{P}(\bar{A}_{x:\overline{n|}})}{\delta}\right]$$

$$= \left(1 + \frac{\bar{P}(\bar{A}_{x:\overline{n|}})}{\delta}\right)^2 \left({}^2\bar{A}_{x:\overline{n|}} - \bar{A}_{x:\overline{n|}}^2\right)$$

Since $\bar{A}_{x:\overline{n|}} = 1 - \delta\bar{a}_{x:\overline{n|}}$, $\bar{P}(\bar{A}_{x:\overline{n|}}) = \frac{1}{\bar{a}_{x:\overline{n|}}} - \delta$

$$Var(L(T)) = \frac{{}^2\bar{A}_{x:\overline{n|}} - \bar{A}_{x:\overline{n|}}^2}{\delta\bar{a}_{x:\overline{n|}}}$$

B- Fully discrete premiums

- benefit paid at the end of year of death
- premiums paid at the beginning of each year

P_x denotes the net annual prem for a whole life insurance policy of 1, issued at age x, calculated under the equivalence principle.

Loss r.v.

$$L(K) = v^{K+1} - P_x \, \ddot{a}_{\overline{K+1}|} \quad , K=0,1,2,\ldots$$

Equivalence principle : $E(L(K))=0$

$$E\left[v^{K+1} - P_x \, \ddot{a}_{\overline{K+1}|}\right]$$

$$= A_x - P_x \, \ddot{a}_x = 0 \implies P_x = \frac{A_x}{\ddot{a}_x}$$

$$P_x = \frac{1-d\ddot{a}_x}{\ddot{a}_x} = \frac{1}{\ddot{a}_x} - d$$

$$Var(L(K)) = Var\left[v^{K+1}\left(1+\frac{P_x}{d}\right) - \frac{P_x}{d}\right]$$

$$= \left(1+\frac{P_x}{d}\right)^2 Var(v^{K+1}) = \frac{{}^2A_x - A_x^2}{(d\ddot{a}_x)^2}$$

General formula for the loss

$$L(K) = b_{K+1} v^{K+1} - P \cdot Y \quad , \text{where}$$

b_{K+1} : death benefit in year $K+1$ (end of year)
P : general symbol for NAP paid at beginning of yr.
Y : r.v. for discrete annuity

Equivalence principle $E(L(K))=0$

$$\implies P = \frac{E(b_{K+1} v^{K+1})}{E(Y)}$$

Fully Discrete Annual Benefit Premiums

ref: Actuarial Mathematics Bowers et al (1997). p.183

Plan	Loss Components: $b_{K+1}v_{K+1}$	Loss Components: $P\,Y$ Where Y Is	Premium Formula $P = \dfrac{E[b_{K+1}v_{K+1}]}{E[Y]}$
Whole life insurance	$1\ v^{K+1}$	$\ddot{a}_{\overline{K+1}\rvert},\ K = 0, 1, 2, \ldots$	$P_x = \dfrac{A_x}{\ddot{a}_x}$
n-Year term insurance	$1\ v^{K+1}$ 0	$\ddot{a}_{\overline{K+1}\rvert},\ K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rvert},\ K = n, n+1, \ldots$	$P^{1}_{x:\overline{n}\rvert} = \dfrac{A^{1}_{x:\overline{n}\rvert}}{\ddot{a}_{x:\overline{n}\rvert}}$
n-Year endowment insurance	$1\ v^{K+1}$ $1\ v^{n}$	$\ddot{a}_{\overline{K+1}\rvert},\ K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rvert},\ K = n, n+1, \ldots$	$P_{x:\overline{n}\rvert} = \dfrac{A_{x:\overline{n}\rvert}}{\ddot{a}_{x:\overline{n}\rvert}}$
h-Payment whole life insurance	$1\ v^{K+1}$ $1\ v^{K+1}$	$\ddot{a}_{\overline{K+1}\rvert},\ K = 0, 1, \ldots, h-1$ $\ddot{a}_{\overline{h}\rvert},\ K = h, h+1, \ldots$	${}_hP_x = \dfrac{A_x}{\ddot{a}_{x:\overline{h}\rvert}}$
h-Payment, n-year endowment insurance	$1\ v^{K+1}$ $1\ v^{K+1}$ $1\ v^{n}$	$\ddot{a}_{\overline{K+1}\rvert},\ K = 0, 1, \ldots, h-1$ $\ddot{a}_{\overline{h}\rvert},\ K = h, \ldots, n-1$ $\ddot{a}_{\overline{h}\rvert},\ K = n, n+1, \ldots$	${}_hP_{x:\overline{n}\rvert} = \dfrac{A_{x:\overline{n}\rvert}}{\ddot{a}_{x:\overline{h}\rvert}}$
n-Year pure endowment	0 $1\ v^{n}$	$\ddot{a}_{\overline{K+1}\rvert},\ K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rvert},\ K = n, n+1, \ldots$	$P_{x:\overline{n}\rvert}^{\ \ 1} = \dfrac{A_{x:\overline{n}\rvert}^{\ \ 1}}{\ddot{a}_{x:\overline{n}\rvert}}$
n-Year deferred whole life annuity	0 $\ddot{a}_{\overline{K+1-n}\rvert}v^{n}$	$\ddot{a}_{\overline{K+1}\rvert},\ K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rvert},\ K = n, n+1, \ldots$	$P({}_{n\rvert}\ddot{a}_x) = \dfrac{A_{x:\overline{n}\rvert}^{\ \ 1}\,\ddot{a}_{x+n}}{\ddot{a}_{x:\overline{n}\rvert}}$

*** Ex Consider a 3-year endowment insurance policy of \$1000 issued to (30); death benefit is payable at the end of year of death. If $v = 0.9$

$q_{30} = 0.010 \qquad {}_{1|}q_{30} = 0.011 \qquad {}_{2|}q_{30} = 0.012$

1- determine the premium under the equivalence principle

2- determine the prob. fct of $L(K)$ with $P_{30:\overline{3}\rvert}$

3- determine π, the prem such that the probability of a total financial loss be positive for a portfolio of 100 independent policies is 0.05

4- determine π^*, the prem such $Pr[$financial loss on 1 policy$]$ is 0.021.

C- Semi-continuous premiums
- premiums paid annually (BOY)
- death benefit paid at moment of death.

$$P(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_x} \overset{UDD}{=} \frac{\frac{i}{\delta} A_x}{\ddot{a}_x} = \frac{i}{\delta} P_x$$

$$P(\bar{A}^{1}_{x:\overline{n}|}) = \frac{\bar{A}^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \overset{UDD}{=} \frac{i}{\delta} P^{1}_{x:\overline{n}|}$$

$$P(\bar{A}_{x:\overline{n}|}) \overset{UDD}{=} \frac{i}{\delta} P^{1}_{x:\overline{n}|} + P_{x:\overset{1}{\overline{n}|}}$$

D- True m^{thly} payment premium

$P_x^{(m)}$: NAP, payable in m instalments, for a whole life ins. of 1, payable at end of year of death.

True Fractional Benefit Premiums* ref. Bowers & al. Actuarial Mathematics.

Plan	Payment of Proceeds	
	At End of Policy Year	At Moment of Death
Whole life insurance	$P_x^{(m)} = \dfrac{A_x}{\ddot{a}_x^{(m)}}$	$P^{(m)}(\bar{A}_x) = \dfrac{\bar{A}_x}{\ddot{a}_x^{(m)}}$
n-Year term insurance	$P^{1\,(m)}_{x:\overline{n}\vert} = \dfrac{A^{1}_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{n}\vert}}$	$P^{(m)}(\bar{A}^{1}_{x:\overline{n}\vert}) = \dfrac{\bar{A}^{1}_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{n}\vert}}$
n-Year endowment insurance	$P^{(m)}_{x:\overline{n}\vert} = \dfrac{A_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{n}\vert}}$	$P^{(m)}(\bar{A}_{x:\overline{n}\vert}) = \dfrac{\bar{A}_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{n}\vert}}$
h-Payment years, whole life insurance	${}_hP_x^{(m)} = \dfrac{A_x}{\ddot{a}^{(m)}_{x:\overline{h}\vert}}$	${}_hP^{(m)}(\bar{A}_x) = \dfrac{\bar{A}_x}{\ddot{a}^{(m)}_{x:\overline{h}\vert}}$
h-Payment years, n-year endowment insurance	${}_hP^{(m)}_{x:\overline{n}\vert} = \dfrac{A_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{h}\vert}}$	${}_hP^{(m)}(\bar{A}_{x:\overline{n}\vert}) = \dfrac{\bar{A}_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{h}\vert}}$

*The actual amount of each fractional premium, payable m times each policy year, during the premium paying period and the survival of (x), is $P^{(m)}/m$. Note that here h refers to the number of payment years, not to the number of payments.

Under UDD,

$$P_x^{(m)} = \frac{A_x}{\ddot{a}_x^{(m)}} = \frac{A_x}{\alpha(m)\ddot{a}_x - \beta(m)}$$

$$P^{(m)}(\bar{A}_x) = \frac{\frac{i}{\delta}A_x}{\alpha(m)\ddot{a}_x - \beta(m)} = \frac{i}{\delta}P_x^{(m)}$$

** Ex. ① Express $P_x^{(m)}$ in terms of P_x (under UDD).

② Find a recursive relation between P_x + P_{x+1}.

③ Order (in increasing order) the premiums

$P_{x:\overline{n}|}$, $\bar{P}(\bar{A}_{x:\overline{n}|})$, $P^{(m)}(\bar{A}_{x:\overline{n}|})$, $P(\bar{A}_{x:\overline{n}|})$.