ON THE HERSCH–PAYNE–SCHIFFER INEQUALITIES FOR STEKLOV EIGENVALUES

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ABSTRACT. We prove that the Hersch–Payne–Schiffer isoperimetric inequality for the n-th nonzero Steklov eigenvalue of a bounded simply–connected planar domain is sharp for all $n \geq 1$. The equality is attained in the limit by a sequence of simply–connected domains degenerating to the disjoint union of n identical disks. Similar results are obtained for the product of two consecutive Steklov eigenvalues. We also give a new proof of the Hersch–Payne–Schiffer inequality for $n = 2$ and show that it is strict in this case.

1. INTRODUCTION AND MAIN RESULTS

1.1. Steklov eigenvalue problem. Let Ω be a simply-connected bounded planar domain with Lipschitz boundary and $\rho \in L^\infty(\partial\Omega)$ be a non-negative nonzero function. The *Steklov eigenvalue problem* [St] is given by

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial \nu} u = \sigma \, \rho \, u & \text{on } \partial\Omega, \end{cases} \tag{1.1.1}$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative. There are several physical interpretations of the Steklov problem [Ba2, Pa]. In particular, it describes the vibration of a free membrane with its whole mass $M(\Omega)$ distributed on the boundary with density ρ:

$$M(\Omega) = \int_{\partial\Omega} \rho(s) ds. \tag{1.1.2}$$

If $\rho \equiv 1$, the mass of Ω is equal to the length of $\partial\Omega$.

The spectrum of the Steklov problem is discrete, and the eigenvalues

$$0 = \sigma_0 < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \sigma_3(\Omega) \leq \cdots \nearrow \infty$$

satisfy the following variational characterization [Ba2, p. 95 and p. 103]:

$$\sigma_n(\Omega) = \inf_{E_n} \sup_{0 \neq u \in E_n} \frac{\int_\Omega |\nabla u|^2 \, dz}{\int_{\partial\Omega} u^2 \rho \, ds}, \quad n = 1, 2, \ldots \tag{1.1.3}$$

Here the infimum is taken over all n-dimensional subspaces E_n of the Sobolev space $H^1(\Omega)$ that are orthogonal to constants on $\partial\Omega$, i.e. $\int_{\partial\Omega} u(s)\rho(s) \, ds = 0$ for all $u \in E_n$. Note that, as in the case of Neumann boundary conditions,

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the Steklov spectrum always starts with the eigenvalue $\sigma_0 = 0$, and the corresponding eigenfunctions are constant.

If the density $\rho \equiv 1$, the Steklov eigenvalues and eigenfunctions coincide with those of the *Dirichlet-to-Neumann operator*

$$\Gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

defined by

$$\Gamma f = \frac{\partial}{\partial \nu}(\mathcal{H} f),$$

where $\mathcal{H}f$ is the unique harmonic extension of the function $f \in H^{1/2}(\partial\Omega)$ to the interior of Ω. If the boundary is smooth, the Dirichlet-to-Neumann operator is a first order elliptic pseudo-differential operator [Ta, pp. 37–38]. It has various important applications, particularly to the study of inverse problems [US].

1.2. **Upper bounds on Steklov eigenvalues.** The present paper is motivated by the following

Question 1.2.1. *How large can the n-th eigenvalue of the Steklov problem be on a bounded simply-connected planar domain of a given mass?*

For $n = 1$, the answer to Question 1.2.1 was given in 1954 by Weinstock [Weinst]. He proved that

$$\sigma_1(\Omega)\, M(\Omega) \le 2\pi \tag{1.2.2}$$

with the equality attained on a disk with $\rho \equiv$ const. Note that the first eigenvalue of the unit disk $\mathbb{D}$ with $\rho \equiv 1$ has multiplicity two and $\sigma_1(\mathbb{D}) = \sigma_2(\mathbb{D}) = 1$. Various extensions of Weinstock's inequality and related results can be found in [Ba1, HPSa, Br, Dit, Ed]; see also [AB, section 8] for a recent survey.

In 1974, Hersch–Payne–Schiffer [HPSc, p. 102] proved the following estimates:

$$\sigma_m(\Omega)\, \sigma_n(\Omega)\, M(\Omega)^2 \le \begin{cases} (m+n-1)^2\, \pi^2 & \text{if } m+n \text{ is odd,} \\ (m+n)^2\, \pi^2 & \text{if } m+n \text{ is even.} \end{cases} \tag{1.2.3}$$

In particular, for $m = n$ and $m = n+1$ we get

$$\sigma_n(\Omega) M(\Omega) \le 2\pi n, \quad n = 1, 2, \dots, \tag{1.2.4}$$

$$\sigma_n(\Omega)\, \sigma_{n+1}(\Omega) M(\Omega)^2 \le 4\pi^2 n^2, \quad n = 1, 2, \dots. \tag{1.2.5}$$

1.3. **Main results.** If $n = 1$, it is easy to see that (1.2.4) and (1.2.5) become equalities on a disk with constant density ρ on the boundary. It was remarked in [HPSc] that estimates (1.2.3) are not expected to be sharp for all m and n. While this is likely to be true, it turns out that if $m = n$ or $m = n + 1$, these inequalities *are* sharp for all $n \ge 1$.

Theorem 1.3.1. *There exists a family of simply-connected bounded Lipschitz domains $\Omega_\varepsilon \subset \mathbb{R}^2$, with $\rho \equiv 1$ on $\partial\Omega_\varepsilon$ for all ε, degenerating to the disjoint union of n identical disks as $\varepsilon \to 0+$, such that*

$$\lim_{\varepsilon\to 0+} \sigma_n(\Omega_\varepsilon)\, M(\Omega_\varepsilon) = 2\pi n, \quad n = 2, 3, \dots \tag{1.3.2}$$

and

$$\lim_{\varepsilon\to 0+} \sigma_n(\Omega_\varepsilon)\, \sigma_{n+1}(\Omega_\varepsilon)\, M(\Omega_\varepsilon)^2 = 4\pi^2 n^2, \quad n = 2, 3, \dots \tag{1.3.3}$$

In particular, the Hersch–Payne–Schiffer inequalities (1.2.4) *and* (1.2.5) *are sharp for all $n \geq 1$.*

Remark 1.3.4. As we show in subsection 2.2, in order to obtain (1.3.2) and (1.3.3), one has to be careful in the choice of a family of domains degenerating to the disjoint union of n identical disks.

It would be interesting to check whether each of the equalities (1.3.2) and (1.3.3) *implies* that the family Ω_ε converges in an appropriate sense to the disjoint union of n identical disks.

Remark 1.3.5. If $\rho \equiv 1$, estimate (1.2.4) and the standard isoperimetric inequality in $\mathbb{R}^2$ imply

$$\sigma_n(\Omega)\, \sqrt{\mathrm{Area}(\Omega)} < n\sqrt{\pi}, \quad n \geq 2.$$

The sharp "isoareal" estimate on σ_n, $n \geq 2$, is unknown (see [Hen, Open Problem 25]).

Theorem 1.3.1 gives an almost complete answer to Question 1.2.1. It remains to establish whether the inequality (1.2.4) is *strict* for all $n \geq 2$. We believe it is true. A modification of the method introduced in [GNP] allows to prove this result for $n = 2$.

Theorem 1.3.6. *Inequality* (1.2.4) *is strict for $n = 2$:*

$$\sigma_2(\Omega)\, M(\Omega) < 4\pi. \tag{1.3.7}$$

The proof of Theorem 1.3.6 uses the Riemann mapping theorem similarly to [Sz, Weinst]. Note that this approach is very different from the techniques of [HPSc].

1.4. **Comparison with the Dirichlet and Neumann case.** To put the inequalities (1.2.2) and (1.3.7) in perspective, let us state similar results for Dirichlet and Neumann eigenvalues. Since Dirichlet and Neumann eigenvalue problems describe vibrations of a membrane of unit density, the mass of the membrane is equal to its area.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots$ and $0 = \mu_0 < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \dots$ be the Dirichlet and Neumann eigenvalues of Ω, respectively. We have:

– *Faber–Krahn inequality*: $\lambda_1(\Omega)\mathrm{Area}(\Omega) \geq \pi\, \lambda_1(\mathbb{D})$ (conjectured in [Ra], proved in [Fa] and [Kra1], a weaker version obtained in [Co]).

– *Krahn's inequality*: $\lambda_2(\Omega)\mathrm{Area}(\Omega) > 2\pi\lambda_1(\mathbb{D})$ ([Kra2]; see [AB, p. 110] for an interesting discussion on the history of this result). The equality is attained in the limit by a sequence of domains degenerating to the disjoint union of two identical disks.
– *Szegö–Weinberger inequality*: $\mu_1(\Omega)\mathrm{Area}(\Omega) \leq \pi\,\mu_1(\mathbb{D})$. This estimate was proved in [Sz] for simply-connected planar domains. In [Weinb], the result was extended to arbitrary domains in all dimensions.
– If Ω is simply-connected, $\mu_2(\Omega)\mathrm{Area}(\Omega) \leq 2\pi\,\mu_1(\mathbb{D})$. This inequality was recently proved in [GNP]. It is an open question whether it holds for multiply connected planar domains. The equality is attained in the limit by a sequence of domains degenerating to the disjoint union of two identical disks.

For higher Dirichlet and Neumann eigenvalues, no sharp estimates of this type are known, and the situation is quite different from the Steklov case. As was mentioned in [GNP, Remark 1.2.8], the disjoint union of n identical disks can not maximize the quantity $\mu_n(\Omega)\,\mathrm{Area}(\Omega)$ for n large enough, because this would contradict Weyl's law. The same argument applies to the minimization problem for $\lambda_n(\Omega)\,\mathrm{Area}(\Omega)$. In fact, it is conjectured that for $n = 3$ the minimizer is a single disk, see [WK, BH].

1.5. **Plan of the paper.** In Section 2 we prove Theorem 1.3.1. We also construct a family of domains whose Steklov spectrum completely "collapses" to zero in the limit as the domains degenerate to the disjoint union of two unit disks. This phenomenon is quite surprising and does not occur for either Dirichlet or Neumann eigenvalues. The rest of the paper is devoted to the proof of Theorem 1.3.6. In Section 3, the "folding and rearrangment" technique, introduced in [Na] and developed in [GNP], is adapted to the Steklov problem. In Section 4 we combine analytic and topological arguments to construct a two-dimensional space of test functions for the variational characterization (1.1.3) of the second Steklov eigenvalue. This space of test functions is then used to prove inequality (1.3.7).

2. Maximization and collapse of Steklov eigenvalues

2.1. **Proof of Theorem 1.3.1.** Let us start with the case $n = 2$. For each $\varepsilon \in (0, \frac{1}{10})$, consider the simply–connected planar domain

$$\Omega_\varepsilon = \{|z - 1 + \varepsilon| < 1\} \cup \{|z + 1 - \varepsilon| < 1\} \subset \mathbb{C}. \tag{2.1.1}$$

As $\varepsilon \to 0+$, Ω_ε degenerates to the disjoint union of two identical unit disks.

Lemma 2.1.2. *Let $\rho \equiv 1$ on $\partial\Omega_\varepsilon$ for any ε. Then*

$$\lim_{\varepsilon\to 0+} \sigma_2(\Omega_\varepsilon) = 1.$$

We recall that if $\rho \equiv 1$ then $\sigma_1(\mathbb{D}) = \sigma_2(\mathbb{D}) = 1$.

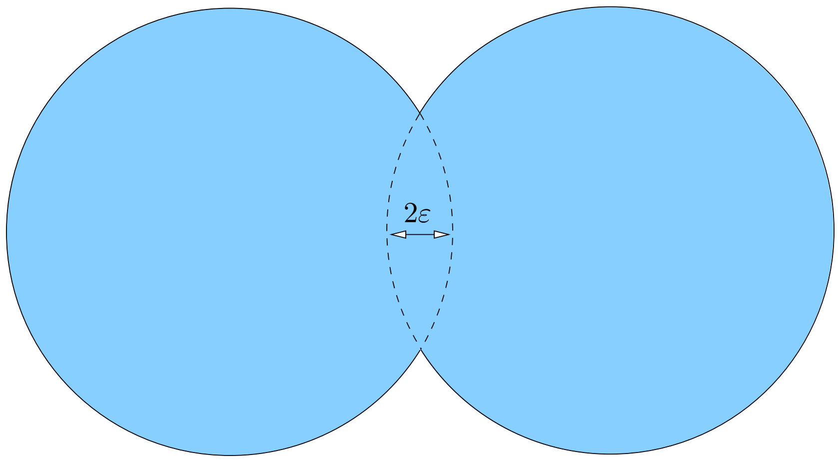

FIGURE 1. The domain Ω_ε for $n = 2$

Remark 2.1.3. While this lemma is not surprising, it does not follow in a straightforward way from general results on convergence of eigenvalues. The difficulty is that the family Ω_ε is not uniformly Lipschitz. Equivalently, the family Ω_ε does not satisfy the uniform cone condition (see [DZ, p. 49] or [HP, p. 53]). This means that one can not choose the Lipschitz constant uniformly in *both* $z \in \partial\Omega_\varepsilon$ and ε. Indeed, it is easy to see that the Lipschitz constant blows up near $z = 0$ as $\varepsilon \to 0$. In this situation, the Steklov eigenvalues may apriori have a rather surprising limiting behavior, see subsection 2.2.

Proof of Lemma 2.1.2. For each $\varepsilon \in (0, \frac{1}{10})$,

$$\sigma_2(\Omega_\varepsilon)M(\Omega_\varepsilon) \leq 4\pi$$

by (1.2.4). Since $\lim_{\varepsilon\to 0+} M(\Omega_\varepsilon) = 4\pi$, we have

$$\limsup_{\varepsilon\to 0+} \sigma_2(\Omega_\varepsilon) \leq 1.$$

It remains to show that

$$\liminf_{\varepsilon\to 0+} \sigma_2(\Omega_\varepsilon) \geq 1. \tag{2.1.4}$$

In view of Remark 2.1.3, in order to apply standard results on convergence of eigenvalues, we need to "desingularize" the family of domains Ω_ε. Let $\Omega'_\varepsilon = \Omega_\varepsilon \cap \{\Re z < 0\}$. Consider the following auxiliary mixed eigenvalue problem on Ω'_ε: impose the Neumann condition on $\Omega_\varepsilon \cap \{\Re z = 0\}$ and keep the Steklov condition on the $\partial\Omega'_\varepsilon \cap \partial\Omega_\varepsilon$. Let $0 = \sigma_0^N(\Omega'_\varepsilon) < \sigma_1^N(\Omega'_\varepsilon) \leq \sigma_2^N(\Omega'_\varepsilon) \ldots$ be the eigenvalues of this mixed problem (it is called a *sloshing* problem, see [FK]). Adding the Neumann condition inside the domain increases the space of test functions, and hence, by the standard monotonicity argument [Ba2, p. 100], it pushes the eigenvalues down. Therefore,

$$\sigma_2(\Omega_\varepsilon) \geq \sigma_1^N(\Omega'_\varepsilon),$$

and hence to prove (2.1.4) it suffices to show that

$$\lim_{\varepsilon\to 0+} \sigma_1^N(\Omega'_\varepsilon) = 1. \tag{2.1.5}$$

The family of domains Ω'_ε converges to $\mathbb{D}$ in the Hausdorff complementary topology (see [BB, p. 101]) as $\varepsilon \to 0+$. Moreover, since the domains Ω'_ε are uniformly Lipschitz in both $z \in \partial\Omega'_\varepsilon$ and $\varepsilon \in (0, \frac{1}{10})$, the extension operators $H^1(\Omega_\varepsilon) \to H^1(\mathbb{R}^2)$ are uniformly bounded [BB, p. 198], and the norms of the trace restriction operators are uniformly bounded as well [Din]. Note also that the Neumann part of $\partial\Omega'_\varepsilon$ given by $\Omega_\varepsilon \cap \{\Re z = 0\}$ tends to the single point $z = 0$ as $\varepsilon \to 0+$. Therefore, using the Rayleigh quotient for the sloshing problem [FK, p. 673]) we get

$$\lim_{\varepsilon\to 0+} \sigma_n^N(\Omega'_\varepsilon) = \sigma_n(\mathbb{D}),\ \ n = 1, 2, \ldots$$

in the same way as [BB, Corollary 7.4.2]. Taking $n = 1$ we get (2.1.5). This completes the proof of the lemma. □

Let us now complete the proof of Theorem 1.3.1. First, it follows from (1.2.5) and the obvious inequality $\sigma_{n+1}(\Omega_\varepsilon) \geq \sigma_n(\Omega_\varepsilon)$, that (1.3.2) implies (1.3.3). Therefore, it suffices to prove (1.3.2). For $n = 2$ it follows from Lemma 2.1.2. For $n > 2$ the proof is analogous. Define Ω_ε as the union of n disks of radius $1 + \varepsilon$ centered at the points $z = 2k$, $k = 0, 1, \ldots, n-1$. We make cuts along the vertical lines $\Re z = 2k - 1$, $k = 1, 2, \ldots, n$, and impose Neumann boundary conditions along these cuts. We get n auxiliary mixed problems. They are of two types: the first and the last disks have just one cut (we denote the corresponding domains by Ω'_ε as before), and the intermediate disks have two cuts — one on the left and the other one on the right (the corresponding domains are denoted by Ω''_ε). The spectra of each of the n auxiliary problems start with the zero eigenvalue. Using the same monotonicity and convergence arguments as above, we get

$$\sigma_n(\Omega_\varepsilon) \geq \min\left(\sigma_1^N(\Omega'_\varepsilon),\, \sigma_1^N(\Omega''_\varepsilon)\right)$$

and

$$\lim_{\varepsilon\to 0+} \sigma_1^N(\Omega'_\varepsilon) = \lim_{\varepsilon\to 0+} \sigma_1^N(\Omega''_\varepsilon) = 1.$$

Therefore, $\liminf_{\varepsilon\to 0+} \sigma_n(\Omega_\varepsilon) \geq 1$. Since $\lim_{\varepsilon\to 0+} M(\Omega_\varepsilon) = 2\pi n$, it follows from (1.2.4) that $\limsup_{\varepsilon\to 0+} \sigma_n(\Omega_\varepsilon) \leq 1$. Hence, $\lim_{\varepsilon\to 0+} \sigma_n(\Omega_\varepsilon) = 1$ and this completes the proof of Theorem 1.3.1.

2.2. **Collapse of the Steklov spectrum: an example.** One could ask why the sequence Ω_ε is constructed by pulling the disks apart, rather than joining them by a tiny passage disappearing as $\varepsilon \to 0$. While this looks geometrically more natural, it turns out that the behavior of the Steklov spectrum under such degeneration can be quite unexpected.

As before, set $\rho \equiv 1$. Let $\Sigma_\varepsilon = \mathbb{D}_1 \cup P_\varepsilon \cup \mathbb{D}_2$, where $\mathbb{D}_1$ and $\mathbb{D}_2$ are two copies of the unit disk joined by a rectangular passage P_ε of length ε and width ε^3 (see Figure 2); the shorter sides of P_ε are chords of the boundary

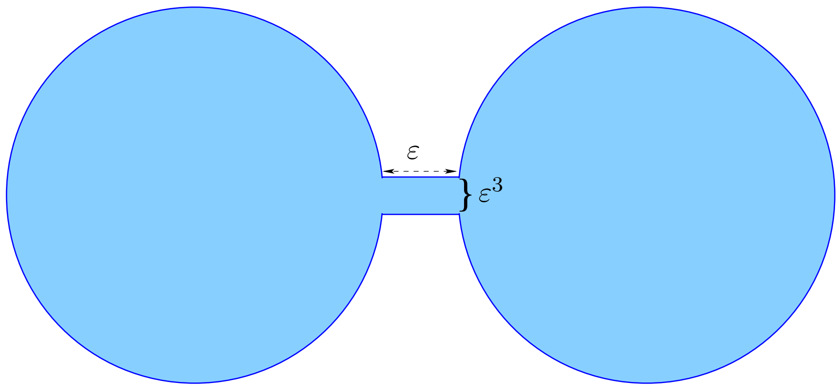

FIGURE 2. The domain Σ_ε

circles $\partial\mathbb{D}_1$ and $\partial\mathbb{D}_2$. What is essential in this construction is that the width of the passage tends to zero much faster than its length. For simplicity we assume that the disks and the passage are chosen in such a way that the domain Σ_ε is symmetric with respect to both coordinate axes. Then, surprisingly enough,

$$\lim_{\varepsilon\to 0+} \sigma_n(\Sigma_\varepsilon) = 0 \text{ for } \textit{all } n = 1, 2, \ldots. \tag{2.2.1}$$

To show this, consider pairwise orthogonal test functions vanishing in the set $(\mathbb{D}_1 \cup \mathbb{D}_2) \setminus P_\varepsilon$, and equal to $\sin \frac{2\pi n x}{\varepsilon}$ in the passage P_ε. For each n, the gradient of the test function is of order n/ε, the area of P_ε is ε^4 and the length of the boundary of P_ε is 2ε. Therefore, for each fixed n, the corresponding Rayleigh quotient is of order $n^2\varepsilon$ and tends to zero as $\varepsilon \to 0+$. This proves (2.2.1).

Similar constructions were studied in the context of Neumann boundary conditions (see [JM, HSS] and references therein). However, the Neumann eigenvalues of Σ_ε converge to the corresponding eigenvalues of the disjoint union of two disks as $\varepsilon \to 0+$. The total "collapse" of the Steklov spectrum in the example above is caused by the fact that the denominator of the Rayleigh quotient is an integral over the *boundary*. Note that the perimeter of the passage P_ε tends to zero much slower than its area, and hence, for every fixed n, the numerator in the Rayleigh quotient vanishes much faster than the denominator.

In the subsequent sections we prove Theorem 1.3.6.

3. FOLDING AND REARRANGEMENT OF MEASURE

3.1. **Conformal mapping to a disk.** Let Ω be a simply connected planar domain with Lipschitz boundary. As before, $\mathbb{D} = \left\{z \in \mathbb{C} \,\middle|\, |z| < 1\right\}$ is the open unit disk. By the Riemann mapping theorem (see [Ta, p. 342]), there exists a conformal equivalence $\phi : \mathbb{D} \to \Omega$ which extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{\Omega}$ (slightly abusing notations, here and further on we denote a conformal map and its extension to the boundary by the same symbol). Let ds be the arc-length measure on $\partial\Omega$, and $d\mu$ be the pull-back by ϕ of the measure $\rho(s)ds$:

$$\int_{\mathcal{O}} d\mu = \int_{\phi(\mathcal{O})} \rho(s)\, ds \tag{3.1.1}$$

for any open set $\mathcal{O} \subset S^1$. Taking (3.1.1) into account and using conformal invariance of the Dirichlet integral, we rewrite the variational characterization (1.1.3) of σ_2 as follows:

$$\sigma_2(\Omega) = \inf_E \sup_{0\neq u\in E} \frac{\int_{\mathbb{D}} |\nabla u|^2\, dz}{\int_{S^1} u^2\, d\mu}. \tag{3.1.2}$$

Here the infimum is taken over all subspaces $E \subset H^1(\mathbb{D})$, such that dim $E = 2$ and $\int_{S^1} u\, d\mu = 0$ for all $u \in E$.

3.2. **Hyperbolic caps.** Let γ be a geodesic in the Poincaré disk model, that is a diameter or the intersection of the disk with a circle which is orthogonal to S^1. Each connected component of $\mathbb{D} \setminus \gamma$ is called a *hyperbolic cap* [GNP].

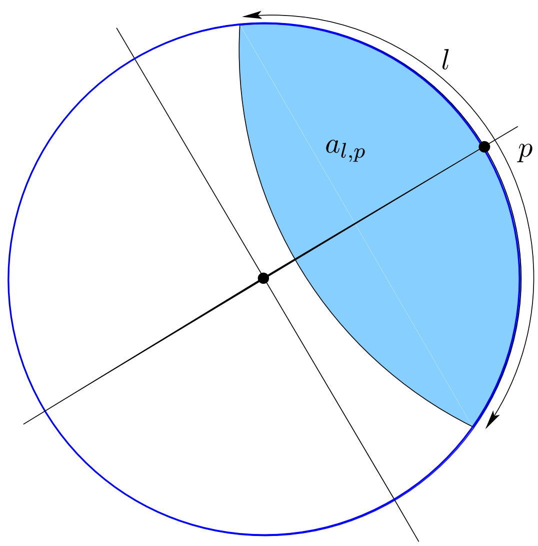

FIGURE 3. The hyperbolic cap $a_{l,p}$

Given $p \in S^1$ and $l \in (0, 2\pi)$, let $a_{l,p}$ be the hyperbolic cap such that the circular segment $\partial a_{l,p} \cap S^1$ has length l and is centered at p (see Figure 3). This gives an identification of the space $\mathcal{HC}$ of all hyperbolic caps with the cylinder $(0, 2\pi) \times S^1$. Given a cap $a \in \mathcal{HC}$, let $\tau_a : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be the reflection across the hyperbolic geodesic bounding a. That is, τ_a is the unique nontrivial conformal involution of $\mathbb{D}$ leaving every point of the geodesic $\partial a \cap \mathbb{D}$

fixed. In particular, $\tau_a(a) = \mathbb{D} \setminus \overline{a}$. The *lift* of a function $u : \overline{a} \to \mathbb{R}$ is the function $\tilde{u} : \overline{\mathbb{D}} \to \mathbb{R}$ defined by

$$\tilde{u}(z) = \begin{cases} u(z) & \text{if } z \in \overline{a}, \\ u(\tau_a z) & \text{if } z \in \overline{\mathbb{D} \setminus a}. \end{cases} \tag{3.2.1}$$

Observe that

$$\begin{aligned} \int_{S^1} \tilde{u}\, d\mu &= \int_{\partial a \cap S^1} u\, d\mu + \int_{\tau_a(\partial a) \cap S^1} u \circ \tau_a\, d\mu \\ &= \int_{\partial a \cap S^1} u\, (d\mu + \tau_a^* d\mu). \end{aligned} \tag{3.2.2}$$

The measure

$$d\mu_a = \begin{cases} d\mu + \tau_a^* d\mu & \text{on } \partial a \cap S^1, \\ 0 & \text{on } S^1 \setminus \partial a \end{cases} \tag{3.2.3}$$

is called the *folded measure*. Equation (3.2.2) can be rewritten as

$$\int_{S^1} \tilde{u}\, d\mu = \int_{S^1} u\, d\mu_a.$$

3.3. **Eigenfunctions on the disk.** Given $t \in \mathbb{R}^2$, define $X_t : \overline{\mathbb{D}} \to \mathbb{R}$ by $X_t(z) = z \cdot t$, the inner product of z and t in $\mathbb{R}^2$. Let (e_1, e_2) be the standard basis of $\mathbb{R}^2$. Then X_{e_1} and X_{e_2} form a basis of the first Steklov eigenspace on the disk with $\rho \equiv 1$. Using Hersch renormalization procedure (see [GNP, subsection 4.1]), we assume that the center of mass of the measure $d\mu$ is at the origin:

$$\int_{S^1} X_t\, d\mu = 0, \ \ \forall t \in \mathbb{R}^2. \tag{3.3.1}$$

Using a rotation if necessary, we may also assume that

$$\int_{S^1} X_{e_1}^2\, d\mu \geq \int_{S^1} X_t^2\, d\mu, \ \ \forall t \in S^1. \tag{3.3.2}$$

3.4. **Rearranged measure.** Let $a \in \mathcal{HC}$ be a hyperbolic cap and let $\psi_a : \mathbb{D} \to a$ be a conformal equivalence. Following the convention adopted in subsection 3.1, we also denote its extension $\overline{\mathbb{D}} \to \overline{a}$ by ψ_a. For each $t \in \mathbb{R}^2$, define $u_a^t : \overline{a} \to \mathbb{R}$ by

$$u_a^t(z) = X_t \circ \psi_a^{-1}(z) = t \cdot \psi_a^{-1}(z).$$

The following auxiliary lemma will be used in the proof of Lemma 4.1.1.

Lemma 3.4.1. *The lift of the function u_a^t is not harmonic in $\mathbb{D}$.*

Proof. Suppose that $\tilde{u}_a^t$ is harmonic. Then it is smooth, and by (3.2.1), the normal derivative of u_a^t vanishes at any point $p \in \partial a \cap \mathbb{D}$. It is well-known that the vanishing of the normal derivative is preserved by conformal transformations. It follows that the normal derivative of the function $X_t = u_a^t \circ \psi_a$

vanishes on $\psi_a^{-1}(\partial a \cap \mathbb{D}) \subset S^1$. However, a straightforward computation shows that for any $s \neq \pm\frac{t}{|t|}$, $\frac{\partial}{\partial n} X_t(s) \neq 0$. □

Let $w_a^t \in C^\infty(\mathbb{D})$ be the unique harmonic extension of $\tilde{u}_a^t\big|_{S^1}$, that is

$$\begin{cases} \Delta w_a^t = 0 & \text{in } \mathbb{D}, \\ w_a^t = \tilde{u}_a^t & \text{on } S^1. \end{cases} \tag{3.4.2}$$

These functions will later be used as test functions in the variational characterization (3.1.2). Observe that

$$\int_{S^1} \tilde{u}_a^t \, d\mu = \int_{S^1} u_a^t \, d\mu_a = \int_{S^1} X_t \, \psi_a^* d\mu_a. \tag{3.4.3}$$

We call the pullback measure

$$d\nu_a = \psi_a^* d\mu_a \tag{3.4.4}$$

the *rearranged measure* on S^1.

A family of conformal transformations $\{\psi_a : \mathbb{D} \to a\}_{a \in \mathcal{HC}}$ is said to be *continuous* if the map $(0, 2\pi) \times S^1 \times \mathbb{D} \to \mathbb{D}$ defined by $(l, p, z) \mapsto \psi_{a_{l,p}}(z)$ is continuous. The next lemma describes the properties of the rearranged measure $d\nu_a$ as the cap a degenerates either to the full disk or to a point $p \in S^1$.

Lemma 3.4.5. *There exists a continuous family of conformal equivalences $\{\psi_a : \mathbb{D} \to a\}_{a \in \mathcal{HC}}$ such that for each cap $a \in \mathcal{HC}$ and each $t \in \mathbb{R}^2$,*

$$\int_{S^1} w_a^t \, d\mu = 0, \tag{3.4.6}$$

$$\lim_{a \to \mathbb{D}} d\nu_a = d\mu, \tag{3.4.7}$$

$$\lim_{a \to p} d\nu_a = R_p^* d\mu, \tag{3.4.8}$$

where w_a^t is defined by (3.4.2), $d\nu_a$ is the rearranged measure given by (3.4.4), and $R_p(x) = x - 2(x \cdot p)$ is the reflection with respect to the diameter orthogonal to the vector p.

A few remarks are in order regarding the last two formulas. As was mentioned in subsection 3.2, the space $\mathcal{HC}$ can be identified with the cylinder $(0, 2\pi) \times S^1$, and convergence in $\mathcal{HC}$ is understood in the sense of of the usual topology on this cylinder. The topology on measures is induced by the norm

$$\|d\nu\| = \sup_{f \in C(S^1), |f| \le 1} \left| \int_{S^1} f \, d\nu \right|. \tag{3.4.9}$$

Proof. Let us give an outline of the proof, for more details, see [GNP, Section 2.5]. Start with any continuous family $\{\phi_a : \mathbb{D} \to a\}_{a \in \mathcal{HC}}$ such that $\lim_{a \to \mathbb{D}} \phi_a = \mathrm{id}$. The maps ψ_a are defined by composing the ϕ_a's on both sides with automorphisms of the disk appearing in the Hersch renormalization procedure. In particular, (3.4.6) is automatically satisfied. As the cap

a converges to the full disk $\mathbb{D}$, the conformal equivalences ψ_a converge to the identity map on $\mathbb{D}$, which implies (3.4.7). Finally, setting $n = 1$ in [GNP, Lemma 4.3.2] one gets (3.4.8). □

From now on, we fix the family of conformal maps ψ_a defined in Lemma 3.4.5. Lemma 3.4.5 implies that the rearranged measure $d\nu_a$ depends continuously on the cap a. This is essential for the topological argument used in the proof of Proposition 4.2.2.

4. Construction of test functions

4.1. **Estimate on the Rayleigh quotient.** It follows from (3.4.6) that the functions w_a^t defined by (3.4.2) are admissible in the variational characterization (3.1.2) for σ_2. For each hyperbolic cap $a \in \mathcal{H}C$ let

$$E_a = \left\{ w_a^t \mid t \in \mathbb{R}^2 \right\}.$$

be a two-dimensional space of test functions.

Lemma 4.1.1. *For any test function $w_a^t \in E_a$,*

$$\int_{\mathbb{D}} |\nabla w_a^t|^2 \, dz < 2\pi.$$

Proof. It is well-known that a harmonic function, such as w_a^t, is the unique minimizer of the Dirichlet energy among all functions in $H^1(\mathbb{D})$ with the same boundary data. By Lemma 3.4.1, the function $\tilde{u}_a^t$ is not harmonic. Since it is continuous, it is not equal to w_a^t in $H^1(\mathbb{D})$. Therefore,

$$\int_{\mathbb{D}} |\nabla w_a^t|^2 \, dz < \int_{\mathbb{D}} |\nabla \tilde{u}_a^t|^2 \, dz = \int_a |\nabla u_a^t|^2 \, dz + \int_{\mathbb{D}\setminus a} |\nabla (u_a^t \circ \tau_a)|^2 \, dz$$

$$= 2 \int_a |\nabla u_a^t|^2 \, dz = 2 \int_{\mathbb{D}} |\nabla X_t|^2 \, dz = 2 \underbrace{\sigma_1(\mathbb{D})}_{1} \overbrace{\int_{S^1} X_t^2 \, d\theta}^{\pi} = 2\pi, \tag{4.1.2}$$

where the second and the third equalities follow from the conformal invariance of the Dirichlet energy. □

Let $t_1, t_2 \in S^1$ be such that $t_1 \cdot t_2 = 0$. Given a hyperbolic cap $a \in \mathcal{HC}$, we have

$$\int_{S^1} (w_a^{t_1})^2 \, d\mu = \int_{S^1} (X_{t_1})^2 d\nu_a$$

$$\geq \frac{1}{2} \int_{S^1} \overbrace{(X_{t_1})^2 + (X_{t_2})^2}^{1} \, d\nu_a = \frac{1}{2} \int_{\partial\Omega} \rho(s) \, ds. \tag{4.1.3}$$

Here the first equality follows from (3.4.2) and (3.4.3), the last equality follows from (3.2.3) and (3.1.1), and we may assume without loss of generality that the inequality in the middle is true (if not, we interchange t_1 and t_2).

Remark 4.1.4. Since $X_{t_1}^2 + X_{t_2}^2 = 1$ on S^1, the estimate (4.1.3) is proved as in [Her], and it is much easier than the analogous result [GNP, Lemma 2.7.5] for the Neumann problem.

Consider the one-dimensional space of test functions

$$V_{t_1} = \left\{\alpha w_a^{t_1} \;\mid\; \alpha \in \mathbb{R}\right\}.$$

It follows from Lemma 4.1.1 and formula (4.1.3) that any function $u \in V_{t_1}$ satisfies

$$\frac{\int_{\mathbb{D}} |\nabla u|^2 \, dz}{\int_{S^1} u^2 \, d\mu} \leq \frac{4\pi}{M(\Omega)}. \tag{4.1.5}$$

Our next goal is to show that there exists a hyperbolic cap $a \subset \mathbb{D}$ such that (4.1.5) holds not only for $u \in V_{t_1}$ but for *each* $u \in E_a$. Since E_a is two-dimensional, the estimate (1.3.7) will follow from (4.1.5) and (3.1.2).

4.2. **Simple and multiple measures.** Given a finite measure $d\nu$ on S^1, consider the quadratic form $V_{d\nu} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$V_{d\nu}(t) = \int_{S^1} X_t^2 \, d\nu.$$

Let $\mathbb{R}P^1 = S^1/\mathbb{Z}_2$ be the projective line. We denote by $[t] \in \mathbb{R}P^1$ the element of the projective line corresponding to the pair of points $\pm t \in S^1$. We say that $[t] \in \mathbb{R}P^1$ is a *maximizing direction* for the measure $d\nu$ if $V_{d\nu}([t]) \geq V_{d\nu}([s])$ for any $[s] \in \mathbb{R}P^1$. The measure $d\nu$ is called *simple* if there is a unique maximizing direction. Otherwise, the measure $d\nu$ is said to be *multiple*.

Lemma 4.2.1. *A measure $d\nu$ is multiple if and only if $V_{d\nu}(t)$ does not depend on $t \in S^1$.*

Proof. The lemma follows from the fact that $V_{d\nu}(t)$ is a quadratic form, and is proved analogously to [GNP, Lemma 2.6.1]. □

Note that by (3.3.2), $[e_1]$ is a maximizing direction for the measure $d\mu$.

Proposition 4.2.2. *If the measure $d\mu$ is simple, then there exists a cap $a \in \mathcal{HC}$ such that the rearranged measure $d\nu_a$ is multiple.*

Proposition 4.2.2 is proved by contradiction. Assume that the measure $d\mu$, as well as the measures $d\nu_a$ for all $a \in \mathcal{HC}$, are simple. Given a hyperbolic cap a, let $[m(a)] \in \mathbb{R}P^1$ be the unique maximizing direction for $d\nu_a$.

By construction, the folded measures $d\mu_a$ depend continuously on the cap a. The family ψ_a is continuous by Lemma 3.4.5, and hence the rearranged measures $d\nu_a$ depend continuously on a. Therefore, the functions $V_{d\nu_a}$ and the unique maximizing direction $[m(a)]$ also depend continuously on a.

Let us understand the behavior of the maximizing direction as the cap a degenerates either to the full disk or to a point.

Lemma 4.2.3. *Let the measure $d\mu$ as well as the measures $d\nu_a$ for all $a \in \mathcal{HC}$ be simple. Then*

$$\lim_{a\to\mathbb{D}}[m(a)] = [e_1] \tag{4.2.4}$$

$$\lim_{a\to e^{i\theta}}[m(a)] = [e^{2i\theta}]. \tag{4.2.5}$$

Proof. First, note that formula (4.2.4) immediately follows from (3.4.7) and (3.3.2). Let us prove (4.2.5). Set $p = e^{i\theta}$. Formula (3.4.8) implies

$$\lim_{a\to p}\int_{S^1} X_t^2\, d\nu_a = \int_{S^1} X_t^2\, R_p^* d\mu = \int_{S^1} X_t^2 \circ R_p\, d\mu = \int_{S^1} X_{R_p t}^2\, d\mu. \tag{4.2.6}$$

Since $d\mu$ is simple, $[e_1]$ is the unique maximizing direction for $d\mu$ and the right hand side of (4.2.6) is maximal for $R_p t = \pm e_1$. Applying R_p on both sides we get $t = \pm e^{2i\theta}$ and hence $[m(a)] = [e^{2i\theta}]$. ☐

Proof of Proposition 4.2.2. Suppose that for each hyperbolic cap $a \in \mathcal{HC}$ the measure $d\nu_a$ is simple. Recall that the space $\mathcal{HC}$ is identified with the open cylinder $(0, 2\pi) \times S^1$. Define $h : (0, 2\pi) \times S^1 \to \mathbb{R}P^1$ by $h(l, p) = [m(a_{l,p})]$. As was mentioned above, the maximizing direction depends continuously on the cap a. Therefore, it follows from Lemma 4.2.3 that h extends to a continuous map on the closed cylinder $[0, 2\pi] \times S^1$ such that

$$h(0, e^{i\theta}) = [e_1],\ \ h(2\pi, e^{i\theta}) = [e^{2i\theta}].$$

This means that h is a homotopy between a trivial loop and a non-contractible loop on $\mathbb{R}P^1$. This is a contradiction. ☐

4.3. **Proof of Theorem 1.3.6.** Assume that the measure $d\mu$ is simple. By Proposition 4.2.2, there exists a cap $a \in \mathcal{HC}$ such that the measure $d\nu_a$ is multiple so that inequality (4.1.5) holds for any $u \in E_a$. Theorem 1.3.6 then immediately follows from the variational characterization (3.1.2) of σ_2.

Suppose now that the measure $d\mu$ is multiple. In this case the proof is easier. Indeed, it follows from Lemma 4.2.1, that any direction $[s] \in \mathbb{R}P^1$ is maximizing for $d\mu$ so that we can use the space

$$E = \left\{X_t \,\middle|\, t \in \mathbb{R}^2\right\}$$

of test functions in the variational characterization (3.1.2) of σ_2. Replacing w_a^t by X_t and inspecting (4.1.2) we notice that the factor 2 disappears. Therefore, (3.1.2) implies

$$\sigma_2(\Omega)\, M(\Omega) \le 2\pi, \tag{4.3.1}$$

which is an even better bound than (1.3.7). This completes the proof of Theorem 1.3.6.

Remark 4.3.2. When $d\mu$ is multiple, Lemma 3.4.1 is not applicable, since we are not using formula (3.2.1), and therefore the inequality (4.3.1) is not strict. Indeed, the equality is attained on a disk with $\rho \equiv \text{const}$.

It is easy to show that if the domain Ω is symmetric of order $q \geq 3$ in the sense of [Ba1] and [Ba2, pp. 136-140] (for instance, if Ω is a regular q-gon), then the measure $d\mu$ is multiple, provided the density ρ satisfies the same symmetry condition. Under these assumptions (4.3.1) is a special case of [Ba2, Theorem 3.15]. In fact, one can show using Courant's nodal domain theorem for Steklov eigenfunctions [KS, section 3] that if the domain Ω and the density ρ are symmetric of order q, then $\sigma_1 = \sigma_2$ so that (4.3.1) is just a consequence of (1.2.2). Indeed, in this case Ω has at least two axes of symmetry, and each of them is a nodal line of an eigenfunction corresponding to σ_1. Therefore, $\mathrm{mult}(\sigma_1) \geq 2$. We are not aware of any examples for which the measure $d\mu$ is multiple but the eigenvalue σ_1 is simple.

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