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An upper bound on the mean value of the Erdős–Hooley Delta function

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Abstract

The Erdős–Hooley Delta function is defined for $n \in \mathbb{N}$ as $\Delta(n) = \sup_{u\in\mathbb{R}} \#\{d|n : e^u < d \leqslant e^{u+1}\}$. We prove that $\sum_{n\leqslant x} \Delta(n) \ll x(\log\log x)^{11/4}$ for all $x \geqslant 100$. This improves on earlier work of Hooley, Hall–Tenenbaum, and La Bretèche–Tenenbaum.

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1 | INTRODUCTION

The *Erdős–Hooley Delta function* (oeis.org/A226898) is defined for a natural number n as

$$\Delta(n) := \sup_{u\in\mathbb{R}} \#\{d|n : e^u < d \leqslant e^{u+1}\}.$$

Erdős introduced this function in the 1970s [2, 3] and studied certain aspects of its distribution in joint work with Nicolas [4, 5]. However, it was not until the work of Hooley in 1979 that Δ was studied in more detail [13]. Specifically, Hooley proved that

$$\sum_{n\leqslant x}\Delta(n)\ll x(\operatorname{Log}x)^{\frac{4}{\pi}-1} \tag{1.1}$$

for any $x\geqslant 1$. Here and in the sequel, we use the notation

$$\operatorname{Log}x := \max\{1,\log x\}\quad \text{for } x>0,$$

and also define

$$\operatorname{Log}_2 x := \operatorname{Log}(\operatorname{Log}x)\quad\text{and}\quad \operatorname{Log}_3 x := \operatorname{Log}(\operatorname{Log}_2 x);$$

see also Section 2 below for our asymptotic notation conventions.

To put Hooley's estimate (1.1) into context, let us note that $1\leqslant\Delta\leqslant\tau$ with $\tau(n)=\#\{d|n\}$ the divisor function. Thus, we have the trivial bounds

$$x\ll\sum_{n\leqslant x}\Delta(n)\ll x\operatorname{Log}x \tag{1.2}$$

for $x\geqslant 1$. Comparing (1.1) with (1.2), we see that Δ is on average of genuinely smaller order than τ. This savings is crucial: as Hooley demonstrated (see [13, 19], and Remarks 2 and 4 below), it can be exploited to count solutions to certain Diophantine equations that are not amenable to more "standard" techniques, as well as to improve bounds on certain Diophantine approximation results.

In a series of papers, Hall and Tenenbaum improved significantly Hooley's estimate for Δ and for various generalizations of it; see [9–11], and also [12]. Their work culminated in the following estimates [12, Theorems 60 and 70]: for every fixed $\varepsilon>0$ and for every $x\geqslant 1$, we have

$$x\operatorname{Log}_2 x\ll\sum_{n\leqslant x}\Delta(n)\ll_\varepsilon x\exp\left(\left(\sqrt{2}+\varepsilon\right)\sqrt{\operatorname{Log}_2 x\operatorname{Log}_3 x}\right). \tag{1.3}$$

The upper bound was improved recently by La Bretèche and Tenenbaum [1] to

$$\sum_{n\leqslant x}\Delta(n)\ll_\varepsilon x\exp\left(\left(\sqrt{2}\log 2+\varepsilon\right)\sqrt{\operatorname{Log}_2 x}\right)$$

for every fixed $\varepsilon>0$ and for every $x\geqslant 1$.

The main result of this note is the following further sharpening of the upper bound.

Theorem 1 (Mean value bound). *For $x\geqslant 1$, we have*

$$\sum_{n\leqslant x}\Delta(n)\ll x(\operatorname{Log}_2 x)^{11/4}.$$

Remark 1. The average value of Δ is dominated by "atypical" integers. Indeed, we know from results in [1] and in [7] that, for every fixed $\varepsilon>0$, we have

$$(\operatorname{Log}_2 x)^{\eta-\varepsilon}\leqslant\Delta(n)\leqslant(\operatorname{Log}_2 x)^{\theta+\varepsilon}$$

for all but $o(x)$ integers $n \in [1, x]$, where $\theta := \frac{\log 2}{\log 2+1/\log 2-1} = 0.6102\ldots$ and $\eta = 0.3533\ldots$ is another constant.† However, the leftmost inequality in (1.3) implies that the mean value of $\Delta(n)$ over $n \in [1, x]$ is of larger order. As a matter of fact, it appears that the average value has significant contributions from integers for which $\Delta(n)$ is as large as $(\log x)^{\log 4-1}$. Indeed, in a recent preprint of Kevin Ford and the two authors of the present paper [8], it was shown that

$$\sum_{n\leqslant x} \Delta(n) \gg x(\mathrm{Log}_2\, x)^{1+\eta},$$

with η as above. Ignoring factors of $(\mathrm{Log}_2\, x)^{O(1)}$, this paper shows roughly that for any choice of $\mathrm{Log}_2\, y \in [\varepsilon\,\mathrm{Log}_2\, x, (1-\varepsilon)\,\mathrm{Log}_2\, x]$, we have $\Delta(n) \gtrapprox (\mathrm{Log}\, y)^{\log 4-1}$ for $\gtrapprox x/(\mathrm{Log}\, y)^{\log 4-1}$ integers $n \leqslant x$ with $\omega(n) = \mathrm{Log}_2\, y + \mathrm{Log}_2\, x + O(1)$ (those that have about $2\,\mathrm{Log}_2\, y$ prime factors $\leqslant y$, and about $\mathrm{Log}_2\, x - \mathrm{Log}_2\, y$ prime factors in $(y, x]$).

Remark 2. As indicated before, estimates on the partial sums of the Δ-function have applications to counting solutions to certain Diophantine equations. In [18], Olivier Robert studied the following question: given integers $k \geqslant 2$, $\ell_k \geqslant \cdots \geqslant \ell_1 \geqslant 2$ and $c_0, c_1, \ldots, c_k \geqslant 1$ such that $\sum_{j=1}^{k} 1/\ell_j = 1/2$, let $S^{\neq}(x)$ denote the number of tuples $(m_0, m_1, \ldots, m_k, n_0, n_1, \ldots, n_k) \in \mathbb{N}^{2k+2}$ such that

$$c_0 m_0^2 + \sum_{j=1}^{k} c_j m_j^{\ell_j} = c_0 n_0^2 + \sum_{j=1}^{k} c_j n_j^{\ell_j} \leqslant x. \tag{1.4}$$

A straightforward adaptation of [18] leads to the estimate

$$S^{\neq}(x) \ll x(\mathrm{Log}_2\, x)^{2^{4L}+15/4} \tag{1.5}$$

with $L = \max\{\ell_1, \ldots, \ell_k\}$ and the implied constant depending at most on parameters $k, c_1, \ldots, c_k$ and $\ell_1, \ldots, \ell_k$, which improves Theorem 1.1 of [18]. In turn, this leads to a similar improvement of Theorem 1.2 of [18]. We will outline the proof of (1.5) in Section 8.

Remark 3. Theorem 1 has applications to a problem of Erdős on sets whose subset sums are not squares. Specifically, assume that c is a constant such that

$$\sum_{n\leqslant x} \Delta(n) \ll x(\mathrm{Log}_2\, x)^c \quad \text{for all } x \geqslant 1. \tag{1.6}$$

In an upcoming paper, David Conlon, Jacob Fox, and Huy Pham developed a new combinatorial argument that deduces from (1.6) that any subset A of $\{1, 2, \ldots, N\}$ with $|A| \geqslant N^{1/3}(\mathrm{Log}_2\, N)^{c'}$ for some appropriate $c' = c'(c)$ has the property that its set of subset sums $\{\sum_{b\in B} b : B \subseteq A\}$ contains a square. This improves the earlier bound of $N^{1/3}(\mathrm{Log}\, N)^C$ with $C > 0$ of Nguyen and Vu [16].

Remark 4. In [13], Hooley used the bound (1.1) to show that for any irrational θ and real γ, and any $\varepsilon > 0$, the inequality $\|n^2\theta - \gamma\| \leqslant n^{-1/2}(\log n)^{\frac{2}{\pi}-\frac{1}{2}+\varepsilon}$ holds for infinitely many n where $\|x\|$

† The precise definition is $\eta = (\log 2)/\log(2/\varrho)$, where ϱ is the unique number in $[0, 1/3]$ satisfying the equation $1 - \varrho/2 = \lim_{j\to\infty} 2^{j-2}/\log a_j$ with $a_1 = 2$, $a_2 = 2 + 2^{\varrho}$ and $a_j = a_{j-1}^2 + a_{j-1}^{\varrho} - a_{j-2}^{2\varrho}$ for $j \in \mathbb{Z}_{\geqslant 3}$.

denotes the distance of the real number x from the nearest integer. Tenenbaum [19] improved the logarithmic factor in this bound using (1.3). Similarly, it should be possible to use Theorem 1 to improve further the logarithmic factor, but we will not pursue this matter here. In the homogeneous case $\gamma = 0$, the more significant improvement $\|n^2\theta\| \leqslant n^{-2/3+\varepsilon}$ was achieved (for arbitrary real θ) by Zaharescu [20].

2 | NOTATION

We use $X \ll Y$, $Y \gg X$, or $X = O(Y)$ to denote a bound of the form $|X| \leqslant CY$ for a constant C. If we need this constant to depend on parameters, we indicate this by subscripts, for instance, $X \ll_k Y$ denotes a bound of the form $|X| \leqslant C_k Y$ where C_k can depend on k. We also write $X \asymp Y$ for $X \ll Y \ll X$. All sums will be over natural numbers unless the variable is p, in which case the sum will be over primes. We use 1_E to denote the indicator of a statement E; thus, 1_E equals 1 when E is true and 0 otherwise.

Given an integer n, we write $\tau(n) := \sum_{d|n} 1$ for its divisor-function and $\omega(n) := \sum_{p|n} 1$ for the number of its distinct prime factors.

It will be convenient, for each $x \geqslant 1$, to work with the set $\mathcal{S}_{<x}$ which denotes the set of square-free numbers, all of whose prime factors p are such that $p < x$. Observe that if $1 \leqslant y \leqslant x$, then every $n \in \mathcal{S}_{<x}$ has a unique factorization $n = n_{<y} n_{\geqslant y}$, where $n_{<y} \in \mathcal{S}_{<y}$ and $n_{\geqslant y}$ lies in the set $\mathcal{S}_{[y,x)}$ of square-free numbers, all of whose prime factors p are in the interval $[y, x)$.

3 | METHODS OF PROOF

Similarly to other authors, we shall work with logarithmic weights. Specifically, for all $x \geqslant 1$, we have [12, Theorem 61]

$$\sum_{n \leqslant x} \Delta(n) \ll \frac{x}{\operatorname{Log} x} \sum_{n \in \mathcal{S}_{<x}} \frac{\Delta(n)}{n}. \tag{3.1}$$

Now, for each $u \in \mathbb{R}$, let us define

$$\Delta(n; u) := \#\{d|n : e^u < d \leqslant e^{u+1}\}, \tag{3.2}$$

so that

$$\Delta(n) = \sup_{u \in \mathbb{R}} \Delta(n; u).$$

As with previous work, we introduce the moments

$$M_q(n) := \int_{\mathbb{R}} \Delta(n; u)^q \,\mathrm{d}u \tag{3.3}$$

for $q \geqslant 1$. Thus, for instance,

$$M_1(n) = \tau(n)$$

and

$$\Delta(n) = \lim_{q\to\infty} M_q(n)^{1/q}. \tag{3.4}$$

In view of (3.4), it is then natural to try to control $M_q(n)$ for large q, keeping track of the dependence of constants on q. In order to exploit the multiplicative nature of Δ, we employ the identity

$$\Delta(np;u) = \Delta(n;u) + \Delta(n;u-\log p)$$

whenever n is a natural number, p is a prime not dividing n, and u is a real. Taking the qth moments of both sides of this identity, we obtain

$$M_q(pn) = \sum_{\substack{a+b=q\\ 0\leqslant b\leqslant q}} \binom{q}{a} \int_{\mathbb{R}} \Delta(n;u)^a \Delta(n;u-\log p)^b \,\mathrm{d}u.$$

Extracting out the extreme terms with $b \in \{0,q\}$, we can write this as

$$M_q(pn) = 2M_q(n) + \sum_{\substack{a+b=q\\ 1\leqslant b\leqslant q-1}} \binom{q}{a} \int_{\mathbb{R}} \Delta(n;u)^a \Delta(n;u-\log p)^b \,\mathrm{d}u. \tag{3.5}$$

By the use of Hölder's inequality and other tools, one can use this identity to recursively control expressions such as

$$\sum_{\substack{n\geqslant 1\\ \omega(n)\geqslant k}} \frac{M_q(n)^{1/q}}{n^\sigma}$$

for various $\sigma > 1$ and $k \geqslant 1$, where $\omega(n)$ denotes the number of distinct prime factors of n. See, for instance, [1] for an example of this approach.

In our work, we use a variation of the above ideas. Our main guiding heuristic is that $\Delta(n)$ behaves roughly as

$$\max_{y\in[1,x]} \frac{\tau(n_{<y})}{\operatorname{Log} y} \tag{3.6}$$

for integers $n \in [1,x]$. To give some support to this heuristic, let us note that

$$\tau(a) = M_1(a) = \int_{-1}^{\log a} \Delta(a;u)\,\mathrm{d}u \leqslant (1+\log a)\Delta(a)$$

for any $a \in \mathbb{N}$. Applying this with $a = n_{<y}$ and noticing that $\Delta(n_{<y}) \leqslant \Delta(n)$ and that $\log n_{<y}$ is typically of size $\operatorname{Log} y$, we find that the expression in (3.6) is morally a lower bound (up to constants) for $\Delta(n)$.

Motivated by the discussion of the above paragraph, we introduce certain sets that are meant to act roughly as level sets of the Δ-function. Precisely, given a parameter $A \geqslant 1$, we define $\tilde{\mathcal{S}}_{<x}^{A}$ to be the set of integers $n \in \mathcal{S}_{<x}$ such that

$$\tau(n_{<y}) \leqslant A \operatorname{Log} y \qquad \text{for all } y \in [1, x].$$

Using a simple Markov inequality, we may show that a proportion of $1 - O(1/A)$ integers in $\mathcal{S}_{<x}$ lies also in $\tilde{\mathcal{S}}_{<x}^{A}$. As a matter of fact, using a more careful analysis, the same statement holds if we replace $\tilde{\mathcal{S}}_{<x}^{A}$ by the set $\mathcal{S}_{<x}^{A}$ of integers $n \in \mathcal{S}_{<x}$ such that

$$\tau(n_{<y}) \leqslant A e^{-f_A(y)} \operatorname{Log} y \qquad \text{for all } y \in [1, x], \tag{3.7}$$

where $e^{-f_A(y)}$ is a Gaussian-type weight concentrated around the region

$$\operatorname{Log}_2 y = \frac{\operatorname{Log} A + O\left(\sqrt{\log A}\right)}{\log 4 - 1}$$

(cf. Proposition 5.1).

Our goal would then be to also show that $\Delta(n) \lessapprox A$ for most $n \in \mathcal{S}_{<x}^{A}$. (In fact, we will only be able to show a weaker version of this, which is why the exponent in Theorem 1 is larger than in the lower bound of (1.3).) In order to achieve this goal, we use (3.5) and a recursive argument that allows us to control averages of $M_q(n)$ when n ranges over $\mathcal{S}_{<x}^{q-1,A}$, defined to be the set of $n \in \mathcal{S}_{<x}^{A}$ such that

$$M_j(n) \leqslant \tau(n) \cdot m_{j,A} \quad \text{for } j = 2, 3, \ldots, q-1, \tag{3.8}$$

where the $m_{j,A}$'s are certain suitable quantities growing roughly like $(jA)^j (\log A)^{3j/4}$.

It is important to note that our recursive argument makes use of the following simple but crucial observation: the integral

$$\binom{q}{a} \int_{\mathbb{R}} \Delta(n; u)^a \Delta(n; u - \log p)^b \, \mathrm{d}u \tag{3.9}$$

is *symmetric* in a, b. Indeed, we have $\Delta(n; v) = \Delta(n; \log n - v - 1)$ for all but finitely many values of $v \in \mathbb{R}$, because $d \in (e^v, e^{v+1}]$ if and only if $n/d \in [e^{\log n - v - 1}, e^{\log n - v})$. Thus,

$$\begin{aligned} \int_{\mathbb{R}} \Delta(n; u)^a \Delta(n; u - \log p)^b \, \mathrm{d}u &= \int_{\mathbb{R}} \Delta(n; \log n - u - 1)^a \Delta(n; \log n - u - 1 + \log p)^b \, \mathrm{d}u \\ &= \int_{\mathbb{R}} \Delta(n; v - \log p)^a \Delta(n; v)^b \, \mathrm{d}v. \end{aligned}$$

This proves our claim that the integral in (3.9) is symmetric in a, b.

Now, combining (3.5) with the symmetry of (3.9), we have the inequality

$$M_q(pn) \leqslant 2M_q(n) + 2 \sum_{\substack{a+b=q \\ 1 \leqslant b \leqslant q/2}} \binom{q}{a} \int_{\mathbb{R}} \Delta(n; u)^a \Delta(n; u - \log p)^b \, du. \tag{3.10}$$

To eliminate the factors of 2, we observe that $\tau(pn) = 2\tau(n)$ (recall that $p \nmid n$ here), and hence,

$$\frac{M_q(pn)}{\tau(pn)} \leqslant \frac{M_q(n)}{\tau(n)} + \frac{1}{\tau(n)} \sum_{\substack{a+b=q \\ 1\leqslant b\leqslant q/2}} \binom{q}{a} \int_{\mathbb{R}} \Delta(n;u)^a \Delta(n;u-\log p)^b \, du. \tag{3.11}$$

We then can apply Hölder's inequality (treating the $\Delta(n;u)^a$ and $\Delta(n;u-\log p)^b$ terms differently) to (3.11), and use our pointwise bounds (3.7) and (3.8), which will allow us to inductively obtain efficient estimates for the sum

$$\sum_{n\in\mathcal{S}^{q-1,A}_{<x}} \frac{M_q(n)/\tau(n)}{n},$$

where $q \geqslant 1$, $A \geqslant 1$, $x \geqslant 1$ are parameters.

4 | BASIC ESTIMATES

We record here a couple of simple lemmas for easy reference, starting with the following standard consequence of Mertens' theorem.

Lemma 4.1 (Mertens' theorem estimate). *Fix $k \geqslant 0$. For $x \geqslant y \geqslant 1$, we have*

$$\sum_{n\in\mathcal{S}_{[y,x)}} \frac{\tau^k(n)}{n} = \prod_{y\leqslant p<x} \left(1 + \frac{2^k}{p}\right) \asymp_k \left(\frac{\operatorname{Log} x}{\operatorname{Log} y}\right)^{2^k}.$$

Proof. We have

$$\log \prod_{y\leqslant p<x} \left(1 + \frac{2^k}{p}\right) = \sum_{y\leqslant p<x} \frac{2^k}{p} + O_k(1),$$

so the lemma follows by a classical estimate of Mertens [14, Theorem 3.4(b)]. □

We also note the following estimate.

Lemma 4.2 (Brun–Titchmarsh inequality). *For $z \geqslant y \geqslant z/100 \geqslant 1$, we have*

$$\sum_{y\leqslant p\leqslant z} \frac{1}{p} \ll \frac{\log(z/y)}{\log y} + \frac{1}{y^{1/2}}.$$

Proof. Note that $\log(z/y) \asymp (z-y)/y$ and that $1/p \asymp 1/y$ for all primes $p \in [y,z] \subseteq [y,100y]$. Hence, it suffices to show that

$$\#\{y \leqslant p \leqslant z\} \ll \frac{z-y}{\log y} + y^{1/2}. \tag{4.1}$$

If $z \leqslant y + y^{1/2}$, there are at most $y^{1/2}$ primes in $[y, z]$. On the other hand, if $100y \geqslant z > y + y^{1/2}$, then (4.1) follows from the Brun–Titchmarsh inequality (see, e.g., [14, Theorem 20.1]) □

5 | CONTROL ON THE DIVISOR FUNCTION

Let $x > 1$. Let us recall our heuristic argument that $\Delta(n)$ behaves like $\max_{y \in [1,x]}(\tau(n_{<y})/\mathrm{Log}\, y)$ for integers $n \in \mathcal{S}_{<x}$. Our ultimate goal is to understand the probability that $\Delta(n) > A$. Motivated by our heuristic, we first study the probability of the event that $\max_{y \in [1,x]}(\tau(n_{<y})/\mathrm{Log}\, y) > A$. Equivalently, this is the event that there exists some $y \in [1, x]$ such that $\tau(n_{<y}) > A \log y$. From Mertens' theorem, we have

$$\sum_{n \in \mathcal{S}_{<x}} \frac{\tau(n_{<y})}{n} = \prod_{p<y} \left(1 + \frac{2}{p}\right) \prod_{y \leqslant p < x} \left(1 + \frac{1}{p}\right) \asymp (\mathrm{Log}\, x)(\mathrm{Log}\, y),$$

and hence, by Markov's inequality, we see that $\tau(n_{<y}) \leqslant A\, \mathrm{Log}\, y$ for all $n \in \mathcal{S}_{<x}$ outside of an exceptional set $\mathcal{E}_{A,y}$ with

$$\sum_{n \in \mathcal{E}_{A,y}} \frac{1}{n} \ll \frac{\mathrm{Log}\, x}{A}. \tag{5.1}$$

We now give a refinement of this simple analysis, in which we have a single exceptional set that covers all $y \in [1, x]$, and furthermore, there is an additional Gaussian-type decay outside of the critical regime $\mathrm{Log}_2\, y = \frac{\mathrm{Log}\, A + O(\sqrt{\mathrm{Log}\, A})}{\log 4 - 1}$.

Proposition 5.1. *Let $A \geqslant 1$. For any $x > 1$, let $\mathcal{S}^A_{<x}$ denote the collection of all $n \in \mathcal{S}_{<x}$ such that*

$$\tau(n_{<y}) \leqslant A e^{-f_A(y)}\, \mathrm{Log}\, y \qquad \textit{for all } y \in [1, x], \tag{5.2}$$

where

$$f_A(y) := \delta \min \left\{ \frac{\left(\mathrm{Log}_2\, y - \frac{\mathrm{Log}\, A}{\log 4 - 1}\right)^2}{\mathrm{Log}\, A}, \mathrm{Log}\, A + \mathrm{Log}_2\, y \right\}, \tag{5.3}$$

and $\delta > 0$ is a sufficiently small absolute constant. Then

$$\sum_{n \in \mathcal{S}_{<x} \setminus \mathcal{S}^A_{<x}} \frac{1}{n} \ll \frac{\mathrm{Log}\, x}{A}. \tag{5.4}$$

Remark. The upper bound (5.4) is sharp. When $\mathrm{Log}_2\, y = \frac{\mathrm{Log}\, A}{\log 4 - 1}$, relation (5.2) becomes $\tau(n_{<y}) \leqslant (\log y)^{\log 4}$ or, equivalently, $\omega(n_{<y}) \leqslant 2\, \mathrm{Log}_2\, y$. This event occurs with probability roughly equal to $1 - (\log y)^{-(\log 4 - 1)} = 1 - 1/A$. A more refined analysis that uses appropriately adapted results of Ford [6] can show that the left-hand side of (5.4) is $\asymp \frac{\mathrm{Log}\, x}{A}$. Hence, the naive Markov bound (5.1) is actually close to the truth in the critical range of y.

Proof. We may assume that A is large, as the claim is immediate from Mertens' inequality otherwise.

Suppose $n \in \mathcal{S}_{<x} \setminus \mathcal{S}_{<x}^A$. Then there exists $y_0 \in [1, x]$ such that

$$\tau(n_{<y_0}) > Ae^{-f_A(y_0)} \operatorname{Log} y_0.$$

We claim that this implies the existence of an absolute constant $c > 0$ such that

$$\tau(n_{<y}) \geqslant cAe^{-f_A(y)} \operatorname{Log} y \qquad \text{for all } y \in [y_0, y_0^2]. \tag{5.5}$$

Indeed, if $\operatorname{Log}_2 y_0 \geqslant 10 \operatorname{Log} A$, then $f_A(y) = \delta(\operatorname{Log} A + \operatorname{Log}_2 y)$ for all $y \in [y_0, y_0^2]$, so (5.5) holds for some appropriate choice of $c > 0$; on the other hand, if $\operatorname{Log}_2 y_0 \leqslant 10 \operatorname{Log} A$, then both functions in the right-hand side of (5.3) change by at most $O(1)$ when y ranges in $[y_0, y_0^2]$, so (5.5) holds again provided we choose $c > 0$ to be small enough.

Now, using (5.5), we find that

$$\int_1^{x^2} 1_{\tau(n_{<y}) \geqslant cAe^{-f_A(y)} \operatorname{Log} y} \frac{\mathrm{d}y}{y \operatorname{Log} y} \gg 1.$$

We conclude that

$$\sum_{n \in \mathcal{S}_{<x} \setminus \mathcal{S}_{<x}^A} \frac{1}{n} \ll \int_1^{x^2} \sum_{n \in \mathcal{S}_{<x}} \frac{1_{\tau(n_{<y}) \geqslant cAe^{-f_A(y)} \operatorname{Log} y}}{n} \cdot \frac{\mathrm{d}y}{y \operatorname{Log} y}.$$

Factoring $n = n_{<y} n_{\geqslant y}$ and using Mertens' theorem, we have

$$\sum_{n \in \mathcal{S}_{<x}} \frac{1_{\tau(n_{<y}) \geqslant cAe^{-f_A(y)} \operatorname{Log} y}}{n} \asymp \frac{\operatorname{Log} x}{\operatorname{Log} y} \sum_{n \in \mathcal{S}_{<y}} \frac{1_{\tau(n) \geqslant cAe^{-f_A(y)} \operatorname{Log} y}}{n},$$

so, it suffices to show that

$$\int_1^{x^2} \sum_{n \in \mathcal{S}_{<y}} \frac{1_{\tau(n) \geqslant cAe^{-f_A(y)} \operatorname{Log} y}}{n} \cdot \frac{\mathrm{d}y}{y \operatorname{Log}^2 y} \ll \frac{1}{A}. \tag{5.6}$$

First, we dispose of some easy contributions. If $\operatorname{Log} y \leqslant A^{0.01}$, then we bound

$$\sum_{n \in \mathcal{S}_{<y}} \frac{1_{\tau(n) \geqslant cAe^{-f_A(y)} \operatorname{Log} y}}{n} \leqslant \frac{1}{(cAe^{-f_A(y)} \operatorname{Log} y)^2} \sum_{n \in \mathcal{S}_{<y}} \frac{\tau(n)^2}{n} \ll \frac{\operatorname{Log}^2 y}{A^2} e^{2 f_A(y)}$$

by Lemma 4.1, and the contribution of this case to the left-hand side of (5.6) is easily seen to be acceptable for $\delta \leqslant 1/3$, which we may assume.

In the other extreme, if $\operatorname{Log} y \geqslant A^{100}$, then we bound

$$\sum_{n \in \mathcal{S}_{<y}} \frac{1_{\tau(n) \geqslant cAe^{-f_A(y)} \operatorname{Log} y}}{n} \leqslant \frac{1}{(cAe^{-f_A(y)} \operatorname{Log} y)^{1/2}} \sum_{n \in \mathcal{S}_{<y}} \frac{\tau(n)^{1/2}}{n} \ll \frac{(\operatorname{Log} y)^{\sqrt{2}-1/2}}{A^{1/2}} e^{f_A(y)/2}$$

using Lemma 4.1 again, and one can check here too that this contribution to the left-hand side of (5.6) is acceptable if $\delta \leqslant 1/20$, which we may assume.

In conclusion, in order to prove (5.6), it will suffice to establish a bound of the form

$$\sum_{n\in\mathcal{S}_{<y}} \frac{1_{\tau(n)\geqslant cAe^{-f_A(y)}\operatorname{Log} y}}{n} \ll \frac{e^{-f_A(y)}}{A(\operatorname{Log} A)^{1/2}}\operatorname{Log} y \tag{5.7}$$

whenever $A^{0.01} \leqslant \operatorname{Log} y \leqslant A^{100}$. This essentially follows by work of Norton [17] (see also [12, Theorems 08 and 09]). We give the details below.

We have $\tau(n) = 2^{\omega(n)}$, and thus, $\tau(n) \geqslant cAe^{-f_A(y)}\operatorname{Log} y$ if, and only if,

$$\omega(n) \geqslant k_y := \left\lceil \frac{\log c + \log A - f_A(y) + \log(\operatorname{Log} y)}{\log 2} \right\rceil .$$

In addition, for each $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$\sum_{\substack{n\in\mathcal{S}_{<y}\\ \omega(n)=k}} \frac{1}{n} \leqslant \frac{1}{k!}\Bigg(\sum_{p<y}\frac{1}{p}\Bigg)^k \leqslant \frac{(\operatorname{Log}_2 y + C)^k}{k!}$$

for some constant $C > 0$, by Mertens' theorem [14, Theorem 3.4(b)]. Notice that $k_y \geqslant 1.1(\operatorname{Log}_2 y + C)$, which implies that the quantities $\frac{1}{k!}(\operatorname{Log}_2 y + C)^k$ decay at least exponentially fast for $k \geqslant k_y$. We thus conclude that

$$\sum_{n\in\mathcal{S}_{<y}} \frac{1_{\tau(n)\geqslant cAe^{-f_A(y)}\operatorname{Log} y}}{n} \leqslant \sum_{k\geqslant k_y}\frac{(\operatorname{Log}_2 y + C)^k}{k!} \ll \frac{(\operatorname{Log}_2 y + C)^{k_y}}{k_y!}.$$

By Stirling's formula and the bounds $k_y \asymp \operatorname{Log}_2 y \asymp \operatorname{Log} A$, we then have

$$\sum_{n\in\mathcal{S}_{<y}} \frac{1_{\tau(n)\geqslant cAe^{-f_A(y)}\operatorname{Log} y}}{n} \ll \frac{(\operatorname{Log} y)^{1-Q(t_y)}}{(\operatorname{Log} A)^{1/2}}, \tag{5.8}$$

where

$$Q(t) = t\log t - t + 1 \quad \text{and} \quad t_y = \frac{k_y}{\operatorname{Log}_2 y + C} = \frac{\operatorname{Log} A - f_A(y) + \operatorname{Log}_2 y}{(\log 2)\operatorname{Log}_2 y} + O\bigg(\frac{1}{\operatorname{Log}_2 y}\bigg).$$

Observe that $t_y \in [1.1, 150]$ when $A^{0.01} \leqslant \operatorname{Log} y \leqslant A^{100}$, $\delta \leqslant 1/5$ and A is large enough.

Now, note that

$$t_y - 2 = \frac{\operatorname{Log} A - (\log 4 - 1)\operatorname{Log}_2 y - f_A(y)}{(\log 2)\operatorname{Log}_2 y} + O\bigg(\frac{1}{\operatorname{Log}_2 y}\bigg). \tag{5.9}$$

In addition, we have $0 \leqslant f_A(y) \leqslant 100\delta|\operatorname{Log}_2 y - \frac{\operatorname{Log} A}{\log 4 - 1}|$, and thus,

$$\frac{|\operatorname{Log}_2 y - \frac{\operatorname{Log} A}{\log 4-1}|}{2\operatorname{Log}_2 y} \leqslant |t_y - 2| \leqslant \frac{|\operatorname{Log}_2 y - \frac{\operatorname{Log} A}{\log 4-1}|}{\operatorname{Log}_2 y}, \tag{5.10}$$

if δ is small enough and A is large enough. We shall now use Taylor's theorem to approximate $Q(t_y)$ by $Q(2)$. Since $t_y \in [1.1, 150]$, there must exist some $\xi \in [1.1, 150]$ such that

$$Q(t_y) = Q(2) + Q'(2)(t_y - 2) + Q''(\xi)\frac{(t_y - 2)^2}{2}.$$

We have $Q(2) = \log 4 - 1$, $Q'(2) = \log 2$ and $Q''(\xi) = 1/\xi \geqslant 1/150$. We then use (5.10) to obtain a lower bound on $(t_y - 2)^2$, and subsequently (5.9) to estimate $t_y - 2$. In conclusion, we have

$$\begin{aligned} Q(t_y)\operatorname{Log}_2 y &\geqslant (\log 4 - 1)\operatorname{Log}_2 y + (t_y - 2)(\log 2)\operatorname{Log}_2 y + 2f_A(y) \\ &= \operatorname{Log} A + f_A(y) + O(1), \end{aligned}$$

as long as δ is small enough. Inserting this estimate into (5.8) completes the proof of (5.7), and thus of the proposition. □

6 | THE KEY MOMENT ESTIMATE

For inductive purposes, we will need to introduce a quantity $m_{q,A}$ depending on several parameters C_0, A, q. According to these quantities, we shall then define $\mathcal{S}_{<x}^{q,A}$ to be the set of all integers $n \in \mathcal{S}_{<x}^{A}$ such that

$$M_a(n)/\tau(n) \leqslant m_{a,A} \qquad \text{for all } a = 1, 2, \dots, q. \tag{6.1}$$

Observe that $M_1(n) = \tau(n)$, and thus, the above inequality is trivially satisfied when $a = 1$ as long as we ensure that

$$m_{1,A} \geqslant 1.$$

In particular,

$$\mathcal{S}_{<x}^{1,A} = \mathcal{S}_{<x}^{A}. \tag{6.2}$$

Clearly, we have the inclusions

$$\mathcal{S}_{<x} \supset \mathcal{S}_{<x}^{1,A} \supset \mathcal{S}_{<x}^{2,A} \supset \dots.$$

In addition, from (3.5), we have

$$M_a(pn)/\tau(pn) \geqslant M_a(n)/\tau(n)$$

whenever p is a prime, n is coprime to p, and $a \geqslant 1$. In particular, $M_a(n_{<y})/\tau(n_{<y})$ is a nondecreasing function of y, and thus,

$$M_a(n_{<y})/\tau(n_{<y}) \leqslant m_{a,A} \qquad \text{for } a = 1, 2, \dots, q \text{ and } y \in [1, x].$$

In other words, we have that

$$n_{<y} \in \mathcal{S}^{q,A}_{<y} \quad \text{whenever } n \in \mathcal{S}^{q,A}_{<x} \text{ and } y \in [1,x]. \tag{6.3}$$

We shall choose

$$m_{q,A} := \frac{q!}{q^2}(C_0A)^{q-1}(\operatorname{Log} A)^{\frac{1}{2}(q-1+\lfloor q/2 \rfloor)}, \tag{6.4}$$

where C_0 is a large enough constant to be determined. We now show that our choice satisfies certain properties.

Lemma 6.1 (The recursive upper bound). *The following properties hold, with all implied constants independent of q, A, and C_0:*

(i) *One has $m_{1,A} \geqslant 1$, $m_{2,A} \gg A \operatorname{Log} A$, and $m_{q,A} \gg (C_0A/3)^{q-1}q^q$.*
(ii) *For any $q \geqslant 3$, one has*

$$\sum_{\substack{a+b=q \\ 1 \leqslant b \leqslant q/2}} \binom{q}{a} m_{b,A} m_{a,A} \ll \frac{1}{C_0 A (\operatorname{Log} A)^{1/2}} \cdot m_{q,A}.$$

(iii) *For any $q \geqslant 1$, one has*

$$(Am_{q,A})^{1/q} \ll qC_0A(\operatorname{Log} A)^{3/4}.$$

Proof. The claims (i) and (iii) are clear from (6.4) (bounding $q! \leqslant q^q$ and $q-1+\lfloor q/2 \rfloor \leqslant 3q/2$). For (ii), we calculate

$$\binom{q}{a} m_{b,A} m_{a,A} = \frac{q!}{a^2b^2}(C_0A)^{a+b-2}(\operatorname{Log} A)^{\frac{1}{2}(a+b-2+\lfloor a/2 \rfloor + \lfloor b/2 \rfloor)}.$$

Noticing that $a+b=q$, $\lfloor a/2 \rfloor + \lfloor b/2 \rfloor \leqslant \lfloor q/2 \rfloor$, and $a^2 \asymp q^2$, the claim follows from the summability of $\sum_{b=1}^{\infty} \frac{1}{b^2}$. □

We now prove the following key moment estimate. In its proof, we shall only use the three properties of the parameters $m_{q,A}$ given in Lemma 6.1. We may thus think of these properties as the only axioms our parameters need to satisfy.

Proposition 6.2 (Key moment estimate). *Suppose that $C_0 \geqslant 1$ is a sufficiently large constant, and $A \geqslant 1$. Then, for any $q \geqslant 2$ and $x > 1$, we have the bound*

$$\sum_{n \in \mathcal{S}^{q-1,A}_{<x}} \frac{M_q(n)/\tau(n)}{n} \leqslant \frac{C_0}{q^2 A} m_{q,A} \operatorname{Log} x. \tag{6.5}$$

Proof. We induct on q, assuming that the claim has already been proven for all smaller values of q (this assumption is vacuous for $q = 2$). We fix A and introduce the notation

$$T_q(x) := \sum_{n \in \mathcal{S}_{<x}^{q-1,A}} \frac{M_q(n)/\tau(n)}{n}.$$

Every natural number $n \in \mathcal{S}_{<x}^{q-1,A}$ other than 1 is expressible in the form $n = pm$ with $p < x$ a prime and $m \in \mathcal{S}_{<p}^{q-1,A}$ (here we use (6.3)). Thus,

$$T_q(x) \leqslant 1 + \sum_{p<x} \sum_{n \in \mathcal{S}_{<p}^{q-1,A}} \frac{M_q(pn)/\tau(pn)}{pn}.$$

Applying (3.11), we conclude that

$$T_q(x) \leqslant \sum_{p<x} \frac{T_q(p)}{p} + Q_q(x),$$

where

$$Q_q(x) := 1 + \sum_{p<x} \sum_{n \in \mathcal{S}_{<p}^{q-1,A}} \frac{1}{\tau(n)pn} \sum_{\substack{a+b=q \\ 1 \leqslant b \leqslant q/2}} \binom{q}{a} \int_{\mathbb{R}} \Delta(n;u)^a \Delta(n;u-\log p)^b \,\mathrm{d}u. \tag{6.6}$$

We can iterate this inequality in the obvious fashion to arrive at

$$T_q(x) \leqslant Q_q(x) + \sum_{\substack{n \in \mathcal{S}_{<x} \\ n>1}} \frac{Q_q(P^-(n))}{n},$$

where $P^-(n)$ is the least prime factor of n with the convention that $P^-(1) = +\infty$. Note that

$$\sum_{\substack{n \in \mathcal{S}_{<x} \\ P^-(n)=p_0}} \frac{1}{n} = \frac{1}{p_0} \prod_{p_0<p<x} \left(1 + \frac{1}{p}\right) \asymp \frac{1}{p_0} \cdot \frac{\operatorname{Log} x}{\operatorname{Log} p_0}$$

for any prime $p_0 < x$, and thus,

$$T_q(x) \ll Q_q(x) + \sum_{p<x} \frac{Q_q(p)}{p} \cdot \frac{\operatorname{Log} x}{\operatorname{Log} p}. \tag{6.7}$$

We now turn to the estimation of $Q_q(x)$. Recall its definition in (6.6). Note that if $n \in \mathcal{S}_{<p}^{q-1,A}$, then $n \in \mathcal{S}_{<y}^{q-1,A}$ for all $y \in [p, p^2]$ because $n_{<y} = n_{<p}$ for all such values of y and the function

$w \to e^{-f_A(w)} \operatorname{Log} w$ is increasing. Since $\int_p^{p^2} dy/(y \operatorname{Log} y) \asymp 1$, we conclude that

$$Q_q(x) \ll 1 + \sum_{p<x} \int_p^{p^2} \sum_{n \in S_{<y}^{q-1,A}} \frac{1}{\tau(n)pn} \sum_{\substack{a+b=q \\ 1 \leqslant b \leqslant q/2}} \binom{q}{a} \int_{\mathbb{R}} \Delta(n;u)^a \Delta(n;u-\log p)^b \, du \frac{dy}{y \operatorname{Log} y}$$

$$\leqslant 1 + \int_1^{x^2} \int_{\mathbb{R}} \sum_{\substack{a+b=q \\ 1 \leqslant b \leqslant q/2}} \binom{q}{a} \sum_{n \in S_{<y}^{q-1,A}} \sum_{p \geqslant y^{1/2}} \frac{1}{\tau(n)pn} \Delta(n;u)^a \Delta(n;u-\log p)^b \, du \frac{dy}{y \operatorname{Log} y}.$$

From (3.2) followed by Lemma 4.2, we have

$$\begin{aligned} \sum_{p \geqslant y^{1/2}} \frac{1}{p} \Delta(n;u-\log p)^b &= \sum_{p \geqslant y^{1/2}} \frac{1}{p} \sum_{\substack{d_1,\dots,d_b | n \\ u-\log p < \log d_1,\dots,\log d_b \leqslant u - \log p + 1}} 1 \\ &= \sum_{\substack{d_1,\dots,d_b | n \\ \log d_{\max} < \log d_{\min}+1}} \sum_{\substack{p \geqslant y^{1/2} \\ u - \log d_{\min} < \log p \leqslant u - \log d_{\max} + 1}} \frac{1}{p} \\ &\ll \sum_{\substack{d_1,\dots,d_b | n \\ \log d_{\max} < \log d_{\min}+1}} \left(\frac{\log d_{\min} + 1 - \log d_{\max}}{\operatorname{Log} y} + \frac{1}{y^{1/4}} \right), \end{aligned}$$

where we adopt the shorthand $d_{\min} := \min(d_1, \dots, d_b)$ and $d_{\max} := \max(d_1, \dots, d_b)$. A similar computation also gives

$$M_b(n) = \sum_{\substack{d_1,\dots,d_b | n \\ \log d_{\max} < \log d_{\min}+1}} \int_{u < \log d_1,\dots,\log d_b \leqslant u+1} du = \sum_{\substack{d_1,\dots,d_b | n \\ \log d_{\max} < \log d_{\min}+1}} (\log d_{\min} + 1 - \log d_{\max}),$$

while

$$\begin{aligned} \sum_{\substack{d_1,\dots,d_b | n \\ \log d_{\max} < \log d_{\min}+1}} 1 &\leqslant \sum_{\substack{d_1,\dots,d_b | n \\ \log d_{\max} < \log d_{\min}+2}} (\log d_{\min} + 2 - \log d_{\max}) \\ &= \int_{\mathbb{R}} (\Delta(n;u) + \Delta(n;u+1))^b \, du \\ &\leqslant 2^b M_b(n) \end{aligned} \tag{6.8}$$

thanks to the triangle inequality in L^b (the proof of inequality (6.8) goes back to Maier and Tenenbaum [15]). Combining all these estimates, we obtain the bound

$$Q_q(x) \ll 1 + \int_1^{x^2} \sum_{\substack{a+b=q \\ 1 \leqslant b \leqslant q/2}} \binom{q}{a} \sum_{n \in S_{<y}^{q-1,A}} \left(\frac{1}{\operatorname{Log} y} + \frac{2^b}{y^{1/4}} \right) \frac{M_a(n) M_b(n)}{\tau(n)n} \cdot \frac{dy}{y \operatorname{Log} y}. \tag{6.9}$$

At this point, we split our analysis into the base case $q = 2$ and the inductive case $q > 2$.

Base case $q = 2$. We must then have $a = b = 1$. Since $M_1(n) = \tau(n)$ and $\mathcal{S}^{1,A}_{<x} = \mathcal{S}^{A}_{<x}$ (cf. (6.2)), the bound (6.9) simplifies to

$$Q_2(x) \ll 1 + \int_1^{x^2} \sum_{n\in\mathcal{S}^A_{<y}} \frac{\tau(n)}{n} \cdot \frac{\mathrm{d}y}{y \operatorname{Log}^2 y}.$$

On the one hand, we have from Mertens' theorem that

$$\sum_{n\in\mathcal{S}^A_{<y}} \frac{\tau(n)}{n} \leqslant \prod_{p<y}\left(1+\frac{2}{p}\right) \ll \operatorname{Log}^2 y.$$

On the other hand, from (5.2) and Lemma 4.1, one has

$$\sum_{n\in\mathcal{S}^A_{<y}} \frac{\tau(n)}{n} \leqslant \left(Ae^{-f_A(y)}\operatorname{Log} y\right)^{1/2} \sum_{n\in\mathcal{S}_{<y}} \frac{\tau(n)^{1/2}}{n} \ll A^{1/2}(\operatorname{Log} y)^{1/2+\sqrt{2}}.$$

Consequently,

$$Q_2(x) \ll 1 + \int_1^{x^2} \min\left\{A^{1/2}(\operatorname{Log} y)^{-0.01}, 1\right\}\frac{\mathrm{d}y}{y} \ll \min\left\{A^{1/2}(\operatorname{Log} x)^{0.99}, \operatorname{Log} x\right\},$$

and thus, by (6.7)

$$T_2(x) \ll \min\left\{A^{1/2}(\operatorname{Log} x)^{0.99}, \operatorname{Log} x\right\} + \sum_{p<x} \frac{\min\left\{A^{1/2}(\operatorname{Log} p)^{0.99}, \operatorname{Log} p\right\}}{p} \cdot \frac{\operatorname{Log} x}{\operatorname{Log} p}.$$

Dividing the summation into the ranges $\operatorname{Log} p \leqslant A^{50}$ and $\operatorname{Log} p > A^{50}$, and using Mertens' theorem, we conclude that

$$T_2(x) \ll (\operatorname{Log} A)(\operatorname{Log} x) \ll \frac{1}{A} \cdot m_{2,A} \operatorname{Log} x$$

thanks to Lemma 6.1(ii). Thus, the claim (6.5) follows for C_0 large enough. This concludes the treatment of the base case $q = 2$.

Inductive case $q > 2$. We first handle the lower order term

$$R_q(x) := \int_1^{x^2} \sum_{\substack{a+b=q \\ 1\leqslant b\leqslant q/2}} \binom{q}{a} \sum_{n\in\mathcal{S}^{q-1,A}_{<y}} \frac{2^b}{y^{1/4}} \cdot \frac{M_a(n)M_b(n)}{\tau(n)n} \cdot \frac{\mathrm{d}y}{y \operatorname{Log} y}$$

appearing in (6.9). We crudely use Hölder's inequality to bound

$$M_a(n)M_b(n) \leqslant M_1(n)M_{q-1}(n) \leqslant \tau(n)^q(1 + \log n).$$

Since we also have $\sum_{a+b=q}\binom{q}{a}2^b = 3^q$, we conclude that

$$R_q(x) \leqslant 3^q \int_1^{x^2} \sum_{n\in S_{<y}^{q-1,A}} \frac{\tau(n)^{q-1}(1+\log n)}{n} \cdot \frac{\mathrm{d}y}{y^{5/4}\,\mathrm{Log}\,y}.$$

From (5.2), we have

$$\tau(n)^{q-1} \leqslant (A\,\mathrm{Log}\,y)^{q-2}\tau(n),$$

while

$$\sum_{n\in S_{<y}} \frac{\tau(n)(1+\log n)}{n} \leqslant \left(1 + 2\sum_{p<y}\frac{\log p}{p}\right)\prod_{p<y}\left(1+\frac{2}{p}\right) \ll (\mathrm{Log}\,y)^3.$$

Thus,

$$R_q(x) \ll 3^q A^{q-2}\int_1^{\infty}\frac{(\mathrm{Log}\,y)^q\,\mathrm{d}y}{y^{5/4}} = 3^q A^{q-2}\cdot 4^{q+1}q! \leqslant 12^{q+1}q^qA^{q-2},$$

as can be seen by the change of variables $y = e^{4u}$. Inserting this into (6.9), we conclude that

$$Q_q(x) \ll 12^q q^q A^{q-2} + Q_q'(x), \tag{6.10}$$

where

$$Q_q'(x) := \int_1^{x^2} \sum_{\substack{a+b=q\\ 1\leqslant b\leqslant q/2}} \binom{q}{a} \sum_{n\in S_{<y}^{q-1,A}} \frac{M_a(n)M_b(n)}{\tau(n)n}\cdot\frac{\mathrm{d}y}{y\,\mathrm{Log}^2\,y}.$$

Applying successively (6.1) and (5.2), we find that

$$M_b(n) \leqslant m_{b,A}Ae^{-f_A(y)}\,\mathrm{Log}\,y,$$

and thus,

$$Q_q'(x) \leqslant \int_1^{x^2} \sum_{\substack{a+b=q\\ 1\leqslant b\leqslant q/2}} \binom{q}{a} m_{b,A}Ae^{-f_A(y)}T_a(y)\frac{\mathrm{d}y}{y\,\mathrm{Log}\,y}.$$

Since $q > 2$, $a+b=q$, and $1\leqslant b\leqslant q/2$, we have $2\leqslant a<q$, and hence by induction hypothesis

$$T_a(y) \leqslant \frac{C_0}{a^2A}m_{a,A}\,\mathrm{Log}\,y.$$

Since $a \geqslant q/2$, we have $a^2 \geqslant q^2/4$. As a consequence,

$$Q_q'(x) \leqslant \frac{4C_0}{q^2}\int_1^{x^2} \sum_{\substack{a+b=q\\ 1\leqslant b\leqslant q/2}} \binom{q}{a} m_{a,A}m_{b,A}e^{-f_A(y)}\frac{\mathrm{d}y}{y},$$

and hence, by Lemma 6.1(ii),

$$Q_q'(x) \ll \frac{m_{q,A}}{q^2 A(\operatorname{Log} A)^{1/2}} \int_1^{x^2} e^{-f_A(y)} \frac{\mathrm{d}y}{y}.$$

We make the change of variables $y = e^{e^t}$ to find that

$$\int_1^{x^2} e^{-f_A(y)} \frac{\mathrm{d}y}{y} \leqslant \int_{-\infty}^{\operatorname{Log}_2 x+1} e^{t-f_A(\exp\exp(t))}\,\mathrm{d}t \ll e^{-f_A(x)} \operatorname{Log} x,$$

where we used (5.3) with δ small enough to show that the function $t - f_A(\exp\exp(t))$ is piece-wise differentiable with derivative bounded from below by an absolute positive constant. In conclusion,

$$Q_q'(x) \ll \frac{e^{-f_A(x)} m_{q,A} \operatorname{Log} x}{q^2 A(\operatorname{Log} A)^{1/2}}.$$

Together with (6.10), this implies that

$$Q_q(x) \ll 12^q q^q A^{q-2} + \frac{e^{-f_A(x)} m_{q,A} \operatorname{Log} x}{q^2 A(\operatorname{Log} A)^{1/2}}.$$

Inserting the above bound into (6.7), and using Mertens' theorem, we conclude that

$$T_q(x) \ll 12^q q^q A^{q-2} \operatorname{Log} x + \frac{m_{q,A} \operatorname{Log} x}{q^2 A(\operatorname{Log} A)^{1/2}} \left(1 + \sum_p \frac{e^{-f_A(p)}}{p}\right), \tag{6.11}$$

where we used that the sum $\sum_p \frac{1}{p\log p}$ converges. Finally, we break up the sum $\sum_p \frac{e^{-f_A(p)}}{p}$ over p on the right-hand side of (6.11) into intervals such that $j \leqslant \operatorname{Log}_2 p < j+1$ for some $j \in \mathbb{Z}_{\geqslant 0}$. For each fixed j, we have $f_A(p) = f_A(\exp\exp(j)) + O(1)$ as well as $\sum_{j \leqslant \operatorname{Log}_2 p < j+1} \frac{1}{p} \ll 1$ by Mertens' theorem. Consequently,

$$\sum_p \frac{e^{-f_A(p)}}{p} \ll \sum_{j \geqslant 1} e^{-f_A(\exp\exp(j))} \ll (\operatorname{Log} A)^{1/2},$$

by the definition of f_A (cf. (5.3)). Hence, using Lemma 6.1(i), we conclude (for C_0 large enough) that

$$T_q(x) \leqslant \frac{C_0}{q^2 A} m_{q,A} \operatorname{Log} x.$$

This completes the proof of the proposition. □

7 | CLOSING THE ARGUMENT

Henceforth, we fix C_0 so that Proposition 6.2 applies, and allow implied constants to depend on C_0.

Corollary 7.1 (Weak type estimate). *Uniformly for $\lambda \geqslant 1$, we have*

$$\sum_{\substack{n \in \mathcal{S}_{<x} \\ \Delta(n) \geqslant \lambda \operatorname{Log}_2 x}} \frac{1}{n} \ll \frac{(\operatorname{Log} \lambda)^{3/4}}{\lambda} \cdot \operatorname{Log} x.$$

Proof. Let C_1 be a large constant and define $A > 0$ implicitly via the equation

$$\lambda = C_1 A (\operatorname{Log} A)^{3/4}.$$

We may assume that $A \geqslant 1$, as the estimate is trivial otherwise. Our task is now to show that

$$\sum_{\substack{n \in \mathcal{S}_{<x} \\ \Delta(n) \geqslant \lambda \operatorname{Log}_2 x}} \frac{1}{n} \ll \frac{\operatorname{Log} x}{A}.$$

From Proposition 5.1 and relation (6.2), we have

$$\sum_{n \in \mathcal{S}_{<x} \setminus \mathcal{S}_{<x}^{1,A}} \frac{1}{n} \ll \frac{\operatorname{Log} x}{A}. \tag{7.1}$$

Also, from (6.1), Proposition 6.2, and Markov's inequality, we have for all $j \geqslant 2$ that

$$\sum_{n \in \mathcal{S}_{<x}^{j-1,A} \setminus \mathcal{S}_{<x}^{j,A}} \frac{1}{n} \leqslant \frac{1}{m_{j,A}} \sum_{n \in \mathcal{S}_{<x}^{j-1,A}} \frac{M_j(n)/\tau(n)}{n} \ll \frac{\operatorname{Log} x}{j^2 A}. \tag{7.2}$$

Summing (7.1) and (7.2) for $j = 2, \dots, q$, we conclude that

$$\sum_{n \in \mathcal{S}_{<x} \setminus \mathcal{S}_{<x}^{q,A}} \frac{1}{n} \ll \frac{\operatorname{Log} x}{A} \qquad \text{for all } q \in \mathbb{N}.$$

The corollary will then follow if we can show that there exists $q \in \mathbb{N}$ such that

$$\Delta(n) < \lambda \operatorname{Log}_2 x \quad \text{for all } n \in \mathcal{S}_{<x}^{q,A}. \tag{7.3}$$

Indeed, let us fix $q \in \mathbb{N}$ to be chosen later and let $n \in \mathcal{S}_{<x}^{q,A}$. From Theorem 72 in [12], we know that[†]

$$\Delta(n)^q \leqslant 2^q M_q(n).$$

Hence, by (6.1) and (5.2), we have

$$\Delta(n)^q \ll 2^q A m_{q,A} \operatorname{Log} x.$$

[†] For completeness, we give the short proof of this inequality. We have $\Delta(n) = \Delta(n; u_0)$ for some real u_0, hence $\Delta(n)^q \leqslant (\Delta(n;u) + \Delta(n;u+1))^q \leqslant 2^{q-1}(\Delta(n;u)^q + \Delta(n;u+1)^q)$ for all $u \in [u_0 - 1, u_0]$. Integrating both sides over $u \in [u_0 - 1, u_0]$ yields the inequality $\Delta(n)^q \leqslant 2^q M_q(n)$.

Taking qth roots and using Lemma 6.1(iii), we find that

$$\Delta(n) \ll qA(\operatorname{Log} A)^{3/4}(\operatorname{Log} x)^{1/q}.$$

We take $q := \lfloor \operatorname{Log}_2 x \rfloor$ to optimize constants. Recalling the definition of A in terms of λ, and assuming the constant C_1, there is chosen to be large enough, and we conclude that (7.3) does hold for all $n \in \mathcal{S}^{q,A}_{<x}$. This completes the proof of the corollary. □

Corollary 7.2 (Strong type estimate). *For any $x \geqslant 1$, we have*

$$\sum_{n \in \mathcal{S}_{<x}} \frac{\Delta(n)}{n} \ll (\operatorname{Log}_2 x)^{11/4} \operatorname{Log} x.$$

Proof. For those n with $\Delta(n) \geqslant (\operatorname{Log} x)^{10}$, we use the trivial bound $\Delta(n) \leqslant \tau(n)^2/(\operatorname{Log} x)^{10}$, and this contribution is acceptable by Lemma 4.1.

On the other hand, those n with $\Delta(n) \leqslant \operatorname{Log}_2 x$ also have an acceptable contribution because $11/4 > 1$.

We then subdivide the remaining range $\operatorname{Log}_2 x \leqslant \Delta(n) < (\operatorname{Log} x)^{10}$ into $O(\operatorname{Log}_2 x)$ dyadic ranges $2^j \operatorname{Log}_2 x \leqslant \Delta(n) < 2^{j+1} \operatorname{Log}_2 x$ with $j \in \mathbb{Z}_{\geqslant 0}$. In each range, we use Corollary 7.2. Thus,

$$\begin{aligned}
\sum_{\substack{n \in \mathcal{S}_{<x} \\ \operatorname{Log}_2 x \leqslant \Delta(n) < (\operatorname{Log} x)^{10}}} \frac{\Delta(n)}{n} &\leqslant \sum_{0 \leqslant j \ll \operatorname{Log}_2 x} \sum_{\substack{n \in \mathcal{S}_{<x} \\ 2^j \leqslant \Delta(n)/\operatorname{Log}_2 x < 2^{j+1}}} \frac{\Delta(n)}{n} \\
&\leqslant \sum_{0 \leqslant j \ll \operatorname{Log}_2 x} (2^{j+1} \operatorname{Log}_2 x) \sum_{\substack{n \in \mathcal{S}_{<x} \\ \Delta(n) \geqslant 2^j \operatorname{Log}_2 x}} \frac{1}{n} \\
&\ll \sum_{0 \leqslant j \ll \operatorname{Log}_2 x} (2^{j+1} \operatorname{Log}_2 x) \cdot \frac{j^{3/4}}{2^j} \operatorname{Log} x \ll (\operatorname{Log}_2 x)^{11/4} \operatorname{Log} x.
\end{aligned}$$

This completes the proof. □

Lastly, Theorem 1 follows immediately by Corollary 7.2 and inequality (3.1).

8 | PROOF OF (1.5)

Fix $k, c_1, \dots, c_k, \ell_1, \dots, \ell_k$ as in Remark 2. All implied constants might depend on these parameters without further notice.

Following the proof of Theorem 1.1 in Section 5 of [18], we have

$$S^{\neq}(x) \ll x + \frac{x}{\log x}(\operatorname{Log}_2 x)^{2+2^{4L}} \sum_{p | m \Rightarrow p < y} \frac{\Delta(m) f(m)}{m} \tag{8.1}$$

with $y = \exp(c\frac{\operatorname{Log} x}{\operatorname{Log}_2 x})$ for some constant $c > 0$ and $f(m) = N(\underline{\ell}; \underline{c}; m)/(m^{2k-2}\varphi(m))$, where $\varphi(m) = \#(\mathbb{Z}/m\mathbb{Z})^*$ is Euler's totient function and $N(\underline{\ell}; \underline{c}; m)$ is defined to be the number of tuples $(m_1, \dots, m_k, n_1, \dots, n_k) \in (\mathbb{Z}/m\mathbb{Z})^{2k}$ such that $\sum_{j=1}^k c_j m_j^{\ell_j} \equiv \sum_{j=1}^k c_j n_j^{\ell_j} \pmod m$.

Now, in view of [18, Lemma 3.4] and our assumption that[†] $k \geqslant 2$, we have $f(p) = 1 + O(1/p)$ and $f(p^\nu) \leqslant \nu^{O(1)}$ for $\nu \geqslant 2$. Therefore,

$$\sum_{p|m \Rightarrow p<y} \frac{\Delta(m)f(m)}{m} \ll \sum_{m \in S_{<y}} \frac{\Delta(m)f(m)}{m} \tag{8.2}$$

$$\ll \sum_{m \in S_{<y}} \frac{\Delta(m)}{m} \tag{8.3}$$

$$\ll (\operatorname{Log} y)(\operatorname{Log}_2 y)^{11/4} \asymp (\operatorname{Log} x)(\operatorname{Log}_2 x)^{7/4}, \tag{8.4}$$

where (8.2) is proven by writing $m = m_1 m_2$ with m_1 square-free, m_2 square-full and $(m_1, m_2) = 1$, so that $\Delta(m) \leqslant \Delta(m_1)\tau(m_2)$, (8.3) is proven by writing $f = 1 * g$ so that $f(m)\Delta(m) \leqslant \sum_{ab=m} \Delta(a)|g(b)|\tau(b)$ for m square-free (because we must then have $(a, b) = 1$ whenever $m = ab$, and thus, $\Delta(m) \leqslant \Delta(a)\tau(b)$), and (8.4) follows by Corollary 7.2 and the definition of y.

Combining (8.1) and (8.4) completes the proof of (1.5).

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[†] When $k = 1$, we have $f(p) = 2 + O(1/p)$, and the behavior of $\sum_{p|m \Rightarrow p<y} f(m)\Delta(m)/m$ changes. Indeed, the case $k = 1$ of (1.4) corresponds to the classical problem of which integers n can be written in the form $c_0 m_0^2 + c_1 m_1^2$. In particular, a correction is needed in [18, Theorem 1.1] to indicate that k must be at least 2.

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