Rational approximations of irrational numbers

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Rational approximations

Fundamental Question

Given an irrational number x, find fractions a/q that approximate it “well”.

- the error $|x - a/q|$ must be small
- q must be small (fractions of “low complexity”)

Remark

Often, q must lie in a restricted set of denominators (e.g. primes, squares etc.)

Dirichlet's theorem

Dirichlet (c.1840):

$\forall x$ irrational, we have i.o.

$$\left|x - \frac{a}{q}\right| < \frac{1}{q^2}.$$

Improving Dirichlet's theorem:

1. Can we replace $1/q^2$ by something smaller?
2. Can we restrict q to lie in some special set of denominators?

Improving the precision of rational approximations

Irrationality measure:

$$\mu(x) := \sup \left\{ E \geqslant 0 \, : \, 0 < \left| x - \frac{a}{q} \right| \leqslant \frac{1}{q^E} \quad \text{i.o.} \right\}$$

Results:

- Roth (1955): $\mu(x) = 2$ for every algebraic irrational x.
- Zeilberger–Zudilin (2020): $\mu(\pi) \leqslant 7.10320533\ldots$

Restricting the denominators

Zaharescu (1995):

Fix $\varepsilon > 0$ and $x \in \mathbb{R} \setminus \mathbb{Q}$. Then

$$\left|x - \frac{a}{q^2}\right| \leqslant \frac{1}{q^{8/3-2\varepsilon}} = \frac{1}{(q^2)^{4/3-\varepsilon}} \quad \text{i.o.}$$

Matomäki (2009):

Fix $\varepsilon > 0$ and $x \in \mathbb{R} \setminus \mathbb{Q}$. Then

$$\left|x - \frac{a}{p}\right| \leqslant \frac{1}{p^{4/3-\varepsilon}} \quad \text{i.o. with } p \text{ prime.}$$

Hard open problems:

Improve “4/3” to “3/2” in Zaharescu’s theorem and “4/3” to “2” in Matomäki’s theorem.

Metric Diophantine approximation

Diophantine approximation:

Approximate a fixed irrational number x $\rightsquigarrow$ hard open problems

Metric Diophantine approximation:

- Prove results about almost all numbers.
- Exclusion of small pathological sets $\rightsquigarrow$ simple-to-state, general results

The basic set-up:

Given "permissible errors" $\Delta_1, \Delta_2, \ldots \geqslant 0$, let

$$\mathcal{A} := \left\{ x \in [0,1] : \left| x - \frac{a}{q} \right| < \Delta_q \quad \text{i.o.} \right\}$$

Khinchin's theorem

Khinchin (1924):

Let $\mathcal{A} = \{x \in [0,1] : |x - a/q| < \Delta_q \quad \text{i.o.}\}$.

1. If $\sum q\Delta_q < \infty$, then $\mathrm{m}(\mathcal{A}) = 0$.
2. If $\sum q\Delta_q = \infty$ and $q^2\Delta_q \searrow$, then $\mathrm{m}(\mathcal{A}) = 1$.

The Borel–Cantelli lemmas:

$E_1, E_2, \ldots$ events; E event that ∞-many E_j occur.

1. If $\sum \mathbb{P}(E_j) < \infty$, then $\mathbb{P}(E) = 0$.
2. If $\sum \mathbb{P}(E_j) = \infty$ and the E_j's are independent, then $\mathbb{P}(E) = 1$.

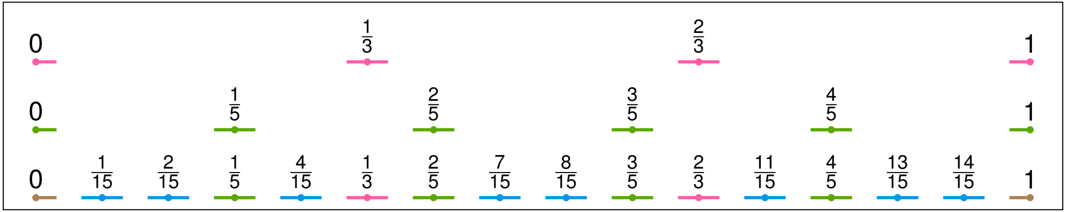
Proof of 1: consider the events $\mathcal{A}_q := \{x \in [0,1] : |x - a/q| < \Delta_q\}$.

A cautionary tale

Duffin–Schaeffer (1941):

Khinchin's theorem fails in full generality: $\exists \Delta_1, \Delta_2, \ldots \geqslant 0$ such that:

1. $\sum q\Delta_q = \infty$;
2. $m(\mathcal{A}) = 0$.



Example with $\mathcal{A}_3, \mathcal{A}_5 \subset \mathcal{A}_{15}$

The Duffin–Schaeffer conjecture

Removing repetitions:

$$\mathcal{A}^* := \left\{ x \in [0,1] : \left| x - \frac{a}{q} \right| < \Delta_q \quad \text{i.o. with } \gcd(a,q) = 1 \right\}$$

The Duffin–Schaeffer conjecture (1941):

1. If $\sum \phi(q)\Delta_q < \infty$, then $\mathrm{m}(\mathcal{A}^*) = 0$.
2. If $\sum \phi(q)\Delta_q = \infty$, then $\mathrm{m}(\mathcal{A}^*) = 1$.

DSC proven by K.–Maynard in 2019.

Earlier results: *Duffin–Schaeffer, Gallagher, Erdős, Vaaler, Pollington – Vaughan, Beresnevich – Velani, Aistleitner, Harman, Haynes, Lachman, Munsch, Technau, Zafeiropoulos, . . .*

Consequences of the DSC

Application 1: Catlin's conjecture (1976):

Let $\Delta'_q := \sup\{\Delta_q, \Delta_{2q}, \ldots\}$. Then

$$\mathrm{m}(\mathcal{A}) = 1 \quad \iff \quad \sum \phi(q)\Delta'_q = \infty.$$

Application 2: Hausdorff dimensions

Assume $\sum \phi(q)\Delta_q < \infty$ so that $\mathrm{m}(\mathcal{A}^*) = 0$. Using a mass-transference principle of Beresnevich–Velani (2006), we have

$$\dim(\mathcal{A}^*) = \inf\left\{s > 0 : \sum \phi(q)\Delta_q^s < \infty\right\}.$$

- Same result for $\mathcal{A}$ by replacing Δ_q with Δ'_q.

New developments since 2019

Aistleitner–Borda–Hauke (2023): quantitative DSC

Assume that $\sum \phi(q)\Delta_q = \infty$. Given $x \in \mathbb{R}$ and $Q \geqslant 1$, let

$$N_Q(x) = \#\{\tfrac{a}{q} \text{ reduced} : q \leqslant Q,\ \left|x - \tfrac{a}{q}\right| < \Delta_q\},$$

and note that

$$\int_0^1 N_Q(x)\,\mathrm{d}x = \sum_{q \leqslant Q} 2\phi(q)\Delta_q =: S_Q.$$

Let C be arbitrarily large but fixed. Then, for a.a. $x \in \mathbb{R}$, we have

$$N_Q(x) = S_Q + O_C\left(S_Q/(\log S_Q)^C\right) \qquad \text{as } Q \to \infty.$$

Conjecture

Presumably the error term can be improved to $S_Q^{1/2+\varepsilon}$ for any fixed $\varepsilon > 0$.

Borel–Cantelli without independence

$$\mathcal{A}_q^* := \left\{ x \in [0,1] : \exists a \in \mathbb{Z} \text{ co-prime to } q \text{ s.t. } \left| x - \frac{a}{q} \right| < \Delta_q \right\}$$

Goal: prove $\mathbb{P}(\mathcal{A}_q^* \cap \mathcal{A}_r^*) \leqslant (1+\varepsilon) \cdot \mathbb{P}(\mathcal{A}_q^*) \cdot \mathbb{P}(\mathcal{A}_r^*)$ on average.

Gallagher's ergodic theorem (1961): $\mathrm{m}(\mathcal{A}^*) \in \{0, 1\}$.

Revised goal: prove $\mathbb{P}(\mathcal{A}_q^* \cap \mathcal{A}_r^*) \leqslant 10^{10^{10}} \cdot \mathbb{P}(\mathcal{A}_q^*) \cdot \mathbb{P}(\mathcal{A}_r^*)$ on average.

Erdős (1970), Vaaler (1978), Pollington–Vaughan (1990):

$$\frac{\mathbb{P}(\mathcal{A}_q^* \cap \mathcal{A}_r^*)}{\mathbb{P}(\mathcal{A}_q^*) \cdot \mathbb{P}(\mathcal{A}_r^*)} > 10^{10^{10}} \quad \Longrightarrow$$

(1) $qr/\gcd(q,r)^2$ has "too many" small prime factors
(2) $\gcd(q,r)$ is "large"

An important special case

- $\mathcal{S} \subset [x, 2x] \cap \mathbb{Z}$ s.t. $\sum_{q \in \mathcal{S}} \frac{\phi(q)}{q} \asymp x^c$ "$\iff$" $|\mathcal{S}| \approx x^c$
- $\Delta_q = \frac{1}{q^{1+c}} \; \forall q \in \mathcal{S} \implies \sum_{q \in \mathcal{S}} \phi(q) \Delta_q \asymp 1.$

Goal: prove that $\mathrm{m}(\bigcup_{q \in \mathcal{S}} \mathcal{A}_q^*) \gg 1$

Pollington–Vaughan: OK unless positive proportion of pairs (q, r) s.t.

(1) $qr/\gcd(q,r)^2$ has "too many" small prime factors
(2) $\gcd(q, r) \geqslant x^{1-c}$

- $c = 1$ (Erdős–Vaaler): (2) trivial; use that (1) can't hold for too many pairs (q, r)
- $c < 1$: savings from (1) insufficient; must exploit structural condition (2)

The model problem

A guiding example

Let $d = \lceil x^{1-c} \rceil$ and $\mathcal{S} = \{dm : \frac{x}{d} \leqslant m \leqslant \frac{2x}{d}\}$.

- $\#\mathcal{S} \sim \frac{x}{d} \sim x^c$
- $\gcd(q, r) \geqslant x^{1-c} \ \forall q, r \in \mathcal{S}$

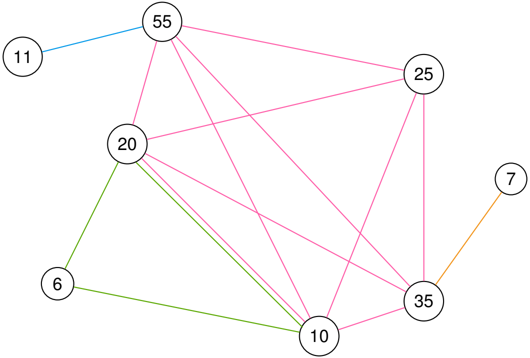
The model problem

- Let $\mathcal{S} \subseteq [x, 2x]$ with $\#\mathcal{S} = x/D$
- Assume $\gcd(q, r) > D$ for $\geqslant 1\%$ of pairs $(q, r) \in \mathcal{S} \times \mathcal{S}$.
- Must there exist a single $d > D$ diving 0.01% of elements of $\mathcal{S}$?

- If YES, then re-calibrate and pass to $\mathcal{S}' = \{m : dm \in \mathcal{S}\}$:
  - $\mathcal{S}' \subseteq [1, x/D]$ and $|\mathcal{S}'| \geqslant 0.0001x/D$
  - $\mathcal{S}'$ dense, so condition (1) can now be exploited.

The graph of dependencies

$$G = \{(q, r) \in \mathcal{S} \times \mathcal{S} : \gcd(q, r) > D\}$$



The graph of dependencies when $D = 1$

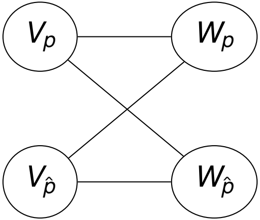
An iterative compression algorithm

- $\mathcal{S} \subseteq \{\text{square-frees}\}$, $G_0 := (\mathcal{S}, \mathcal{S}, \mathcal{E})$ (view as bipartite graph)
- $G_0 \geqslant G_1 \geqslant \cdots \geqslant G_J = (\mathcal{S}', \mathcal{T}', \mathcal{E}')$ s.t.
  - at each step gain info about divisibility of vertices/edges w.r.t. a new prime p_j
  - while maintaining control of edge/vertex counts
- In the end, we have full divisibility info:
  - $\exists a$ dividing all of $\mathcal{S}'$
  - $\exists b$ dividing all of $\mathcal{T}'$
  - $\gcd(q, r) = \gcd(a, b) > D$ for all $(q, r) \in \mathcal{E}'$
- In G_J, condition (1) can be exploited to control $|\mathcal{E}'|$
- Pigeonhole to find G_J. This uses a quality increment argument (inspired by Roth):

$$q(G_0) \leqslant q(G_1) \leqslant \cdots \leqslant q(G_J)$$

The inductive step

- Assume we have constructed $G_{j-1} = (V, W, E)$
- Consider a new prime $p = p_j$ and let $V_p = \{v \in V : p|v\}$ and $V_{\hat{p}} = V \setminus V_p$



Subgraph	Gain in V	Gain in W	Loss in E	Total gain
(V_p, W_p)	p	p	p	1
$(V_{\hat{p}}, W_{\hat{p}})$	1	1	1	1
$(V_p, W_{\hat{p}})$	p	1	1	p
$(V_{\hat{p}}, W_{\hat{p}})$	1	p	1	p

Defining the quality of a graph

$$\frac{q(G_j)}{q(G_{j-1})} = \frac{|\mathcal{E}_j|}{|\mathcal{E}_{j-1}|} \cdot \left(\frac{\delta_j}{\delta_{j-1}}\right)^{10} \cdot \begin{cases} 1 & \text{in symmetric cases} \\ p & \text{in asymmetric cases} \end{cases}$$

Thank you for your attention