The distribution of the maximum of character sums

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Background & motivation

Let χ be a Dirichlet character modulo q and define

$$M(\chi) = \max_{1\le z\le q} \left| \sum_{n\le z} \chi(n) \right| .$$

If χ is non-principal, then Pólya and Vinogradov showed in 1918 that

$$M(\chi) \ll \sqrt{q} \log q.$$

Assuming GRH, Montgomery and Vaughan improved this in 1977 to

$$M(\chi) \ll \sqrt{q} \log\log q.$$

This is best possible: Paley had already shown in 1932 that

there is a sequence $q_n \to \infty$ such that $M\left(\left(\frac{q_n}{\cdot}\right)\right) \gg \sqrt{q_n} \log\log q_n$.

However, such extremal examples should be rather rare. Our goal is to study *how rare* they are.

The distribution of $M(\chi)$: random models

We shall study

$$P_q(\tau) := \frac{\#\left\{\chi \,(\mathrm{mod}\, q) : M(\chi) > \frac{e^\gamma}{\pi}\tau\sqrt{q}\right\}}{\phi(q)} = \textbf{Prob}\left(M(\chi) > \frac{e^\gamma}{\pi}\tau\sqrt{q}\right).$$

We always assume for simplicity that q is prime.

Two questions:

1. Is there a random model that describes $P_q(\tau)$ accurately?
2. How big is $P_q(\tau)$?

Let $(X_p)_{p\nmid q}$ be a sequence of independent random variables, uniformly distributed on $\{z \in \mathbb{C} : |z| = 1\}$, and $X_p = 0$ if $p|q$.
(They should model $\chi(p)$ as χ runs through characters modulo q.)

Then we define $X_n = \prod_{p^r \| n} X_p^r$, which serves a model for $\chi(n)$.

First attempt: model $\sum_{n \le z} \chi(n)$ by $\sum_{n \le z} X_n$.

For z large compared to q, this will fail: periodicity is not taken into account.

The distribution of $M(\chi)$: random models, continued

$$P_q(\tau) = \textbf{Prob}\left(M(\chi) > \frac{e^\gamma}{\pi}\tau\sqrt{q}\right).$$

$(X_p)_{p\nmid q}$ sequence of independent random variables, uniformly distributed on $\{z \in \mathbb{C} : |z| = 1\}$, $X_p = 0$ if $p|q$, $X_n = \prod_{p^r\|n} X_p^r$.

Second attempt: use Pólya's expansion (χ primitive, $e(x) = e^{2\pi i x}$):

$$\sum_{n\le z}\chi(n) = \frac{\tau(\chi)}{2\pi i}\sum_{1\le|n|\le w}\frac{\overline{\chi}(n)(1-e(-nz/q))}{n} + O\left(\frac{q\log q}{w}\right) \quad (1 \le w \le q).$$

Our model for $\sum_{n\le z}\chi(n)$ then becomes

$$S(z) := \frac{\tau(\chi)}{2\pi i}\sum_{1\le|n|\le q}\frac{\overline{X_n}\cdot(1-e(-nz/q))}{n}.$$

This model captures the periodicity of χ.

Remark. The standard deviation of $S(z)/\sqrt{q}$ is $\ll 1$. (Compare this to 1st model: the SD of $T(z) = \sum_{n\le z} X_n$ is $\sqrt{z}$, and one might expect $T(z)/\sqrt{z}$ to get large relatively often.) As a result, $P_q(\tau)$ will be rather small.

Known results on $P_q(\tau)$

In 1979, Montgomery and Vaughan showed that

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} M(\chi)^{2k} \ll_k q^k.$$

An immediate corollary is that

$$P_q(\tau) = \textbf{Prob}\left(M(\chi) > \frac{e^\gamma}{\pi} \tau \sqrt{q} \right) \ll_A \frac{1}{\tau^A}.$$

In 2011, Bober-Goldmakher proved that, for fixed τ and $q \to \infty$ over primes,

$$\exp\left\{ -\frac{Ce^\tau}{\tau}(1 + o_{\tau\to\infty}(1)) \right\} \le P_q(\tau) \le \exp\left\{ -e^{B\sqrt{\tau}/(\log \tau)^{1/4}} \right\},$$

where $C = 1.09258\ldots$ This supports the claim that $P_q(\tau)$ is very small.

Question: why do the tails of the distribution of $M(\chi)$ have this double exponential decay?

The distribution of $M(\chi)$ vs the distribution of $L(1,\chi)$

$$\sum_{n\le \alpha q}\chi(n)=\frac{\tau(\chi)}{2\pi i}\sum_{1\le |n|\le q}\frac{\overline{\chi}(n)(1-e(-n\alpha))}{n}+O(\log q).$$

In view of Pólya's expansion, one might conjecture that

$$P_q(\tau):=\textbf{Prob}\left(M(\chi)>\frac{e^{\gamma}}{\pi}\tau\sqrt{q}\right)\approx \textbf{Prob}\left(|L(1,\chi)|>c\tau\right).$$

Granville-Soundararjan: for q prime and $e^{\tau}=o(\log q)$,

$$\textbf{Prob}\left(|L(1,\chi)|>e^{\gamma}\tau\right)=\exp\left\{-\frac{Ce^{\tau}}{\tau}(1+o_{\tau\to\infty}(1))\right\}.$$

Compare this to

$$\exp\left\{-\frac{Ce^{\tau}}{\tau}(1+o_{\tau\to\infty}(1))\right\}\le P_q(\tau)\le \exp\left\{-e^{B\sqrt{\tau}/(\log\tau)^{1/4}}\right\}.$$

The distribution of $L(1,\chi)$: main ideas

1. We shall take moments of $L(1,\chi)$, so we need to 'shorten' it. We have that $\log L(1,\chi) = \sum_p \chi(p)/p + C_\chi$, where C_χ is a constant.

   PNT $\Rightarrow$ $\log L(1,\chi) \sim \sum_{p \le e^{q^\epsilon}} \chi(p)/p + C_\chi$,
   GRH $\Rightarrow$ $\log L(1,\chi) \sim \sum_{p \le (\log q)^{2+\epsilon}} \chi(p)/p + C_\chi$.

   But we study $L(1,\chi)$ *statistically*: for most $\chi \pmod q$,
   Zero-density estimates $\Rightarrow$ $\log L(1,\chi) \sim \sum_{p \le (\log q)^{100}} \chi(p)/p + C_\chi$.
2. Take moments of

$$L(1,\chi;y) := \prod_{p \le y} \left(1 - \frac{\chi(p)}{p}\right)^{-1} = \sum_{p|n \Rightarrow p \le y} \frac{\chi(n)}{n} \quad (y = (\log q)^{100}) :$$

$$\frac{1}{\phi(q)} \sum_{\chi \pmod q} |L(1,\chi;y)|^{2k} = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \left| \sum_{p|n \Rightarrow p \le y} \frac{\tau_k(n)\chi(n)}{n} \right|^2$$

$$= \sum_{\substack{m \equiv n \pmod q \\ p|mn \Rightarrow p \le y,\, p \nmid q}} \frac{\tau_k(m)\tau_k(n)}{mn}.$$

The distribution of $L(1,\chi)$, continued

Ignoring the off-diagonal terms (assumption that X_n is a good model for $\chi(n)$), and assuming that q is prime,

$$M_{2k} := \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} |L(1,\chi;y)|^{2k} = \sum_{\substack{m \equiv n \,(\mathrm{mod}\, q) \\ p|mn \,\Rightarrow\, p \le y,\, p\nmid q}} \frac{\tau_k(m)\tau_k(n)}{mn}$$

$$\approx \sum_{p|n \,\Rightarrow\, p \le y} \frac{\tau_k(n)^2}{n^2} = \prod_{p \le y} \left(1 + \frac{\tau_k(p)^2}{p^2} + \frac{\tau_k(p^2)}{p^4} + \cdots\right).$$

Then Granville and Soundararajan proceed to show that $\log M_{2k} = 2e^{\gamma}k + C'k/\log k + O(k/\log^2 k)$, which allows them to estimate $\mathbf{Prob}\left(|L(1,\chi)| > e^{\gamma}\tau\right)$ quite accurately.

Remark. In fact, they observe that

$$\log\left(1 + \frac{\tau_k(p)^2}{p^2} + \frac{\tau_k(p^2)}{p^4} + \cdots\right) = \log I_0(2k/p) + O(k/p^2),$$

where $I_0(t) = \sum_{n \ge 0} \left(\frac{t/2}{n!}\right)^2$ is the modified Bessel function of the 1st kind. In particular, most of the contribution to M_{2k} comes from primes $p \approx k$.

New results on $P_q(\tau) = \textbf{Prob}\left(M(\chi) > \frac{e^\gamma}{\pi}\tau\sqrt{q}\right)$

Recall Bober-Goldmakher's result: for τ fixed and $q \to \infty$ over primes,

$$\exp\left\{-\frac{Ce^\tau}{\tau}(1 + o_{\tau\to\infty}(1))\right\} \le P_q(\tau) \le \exp\left\{-e^{B\sqrt{\tau}/(\log\tau)^{1/4}}\right\}.$$

There are two issues to be addressed:

- There is a discrepancy between upper and lower bounds.
- The result is not uniform in τ and q.

Theorem (Bober, Goldmakher, Granville, K. (2013))

Let $\theta > 14/15$, q be prime and $2 \le \tau \le \log\log q - \log\log\log q - 5$. Then

$$\exp\left\{-\frac{Ce^\tau}{\tau}(1 + o_{\tau\to\infty}(1))\right\} \le P_q(\tau) \le \exp\left\{-e^{\tau + O_\theta(\tau^\theta)}\right\}.$$

Remark. On GRH, the theorem holds when $\tau \le \log_2 q - \log_4 q + O(1)$. It seems likely that it can be shown unconditie$^\tau = o(\log q)$ can be obtained unconditionally.

A reduction to the distribution of $L(1,\chi)$: lower bounds

For lower bounds on $P_q(\tau)$, we follow Bober-Goldmakher and note that

$$\sum_{n\le q/2}\chi(n)\sim\frac{\tau(\chi)}{2\pi i}\sum_{1\le|n|\le q}\frac{\overline{\chi}(n)(1-e(-n/2))}{n}$$

$$=\frac{\tau(\chi)}{\pi i}\sum_{\substack{1\le|n|\le q\\ n\text{ odd}}}\frac{\overline{\chi}(n)}{n}=\frac{2\tau(\chi)}{\pi i}\begin{cases}\sum_{\substack{1\le n\le q\\ n\text{ odd}}}\frac{\overline{\chi}(n)}{n} & \text{if } \chi(-1)=-1,\\ 0 & \text{if } \chi(-1)=1.\end{cases}$$

When χ is odd, the right hand side is essentially $L(1,\overline{\chi})$, divided by the Euler factor at $p=2$.

One can then obtain the claimed lower bound on $P_q(\tau)$ using the methods of Granville-Soundararajan.

The upper bound is significantly harder. The main issue is to understand where $\sum_{n\le z}\chi(n)$ is maximized (ideas about pretentious characters).

A detour: pretentious characters

Granville-Soundararajan (2006) and Goldmakher (2010) improved the previously known bounds for $M(\chi)$ when χ has odd order g to

$$M(\chi) \ll \begin{cases} \sqrt{q}(\log q)^{1-\delta_g+o(1)} & \text{unconditionally,} \\ \sqrt{q}(\log\log q)^{1-\delta_g+o(1)} & \text{on GRH.} \end{cases} \qquad \left(\delta_g = 1 - \frac{g}{\pi}\sin\frac{\pi}{g}\right)$$

Idea of the proof: g odd $\Rightarrow$ $\chi(-1) = 1$. So Pólya's expansion becomes

$$\sum_{n\le\alpha q} \chi(n) \sim \frac{\tau(\chi)}{2\pi i} \sum_{1\le|n|\le q} \frac{\overline{\chi}(n)(1-e(-n\alpha))}{n} = -\frac{\tau(\chi)}{2\pi i} \sum_{1\le|n|\le q} \frac{\overline{\chi}(n)e(-n\alpha)}{n}.$$

Let $|\alpha - a/b| < 1/(bB)$, $b \le B := e^{\sqrt{\log q}}$. Montgomery-Vaughan showed

$$\sum_{n\le x} \frac{\chi(n)e(n\alpha)}{n} \ll \log\log x + \log b + \frac{(\log b)^{3/2}}{\sqrt{b}}\log x \quad (x \ge 2).$$

So, we may assume that $b \le (\log q)^{1/3}$. Also, let $\alpha = a/b$ for simplicity.

Pretentious characters, continued

$\mathrm{ord}(\chi) = g = odd$, $\chi(-1) = 1$, $\alpha = a/b$, $b \le (\log q)^{1/3}$. We need to estimate

$$\sum_{n \le \alpha q} \chi(n) \sim -\frac{\tau(\chi)}{2\pi i} \sum_{1 \le |n| \le q} \frac{\overline{\chi}(n) e(-na/b)}{n}.$$

Expand $e(-na/b)$ in terms of characters $\psi \pmod d$, $d|b$, to replace $\sum_{n \le \alpha q} \chi(n)$ by sums of the form

$$S = \sum_{1 \le |n| \le z} \frac{\overline{\chi}(n)\psi(n)}{n} = (1 - \chi(-1)\psi(-1)) \sum_{n \le z} \frac{\overline{\chi}(n)\psi(n)}{n}.$$

For S to be big, χ must be 'close' to ψ ($\chi(p) \approx \psi(p)$). Indeed,

$$\sum_{n \le z} \frac{\chi(n)\overline{\psi}(n)}{n} \ll \frac{\log z}{\exp\{\mathbb{D}(\chi, \psi; z)/2\}}, \quad \mathbb{D}^2(\chi, \psi; z) = \sum_{p \le z} \frac{1 - \Re(\chi(p)\overline{\psi}(p))}{p}.$$

If $\mathbb{D}(\chi, \psi; z)$ is small, we say that *χ pretends to be ψ*.

Also, $\psi(-1) = -\chi(-1) = -1 \implies \mathrm{ord}(\psi) = even \ne g$.

But then $\chi \approx \psi \implies 1 = \chi^g \approx \psi^g \ne 1$, a contradiction.

A reduction to the distribution of $L(1,\chi)$: upper bounds

In bounding $P_q(\tau)$ from above, the key step is the following:

Theorem (Bober, Goldmakher, Granville, K. (2013))

Let $\theta > 14/15$, q be prime and $2 \le \tau \le \log\log q - \log\log\log q - 5$. With the exception of $\ll q\exp\{-20\tau e^{\tau}\}$ characters mod q, if $M(\chi) > \frac{e^{\gamma}}{\pi}\tau\sqrt{q}$, then χ is odd, and there is a $b \le \tau^{10}$ such that

$$\left|\sum_{\substack{n\in\mathbb{N},\,(n,b)=1\\ p|n \,\Rightarrow\, p\le e^{\tau}}} \frac{\chi(n)}{n}\right| \ge e^{\gamma}\tau + O_{\theta}(\tau^{\theta}).$$

Then $P_q(\tau) \le \exp\left\{-e^{\tau+O_{\theta}(\tau^{\theta})}\right\}$, by Granville-Soundararajan.

Main ideas involved in proving the above theorem:

1. A high moment bound to truncate Pólya's expansion.
2. Use "pretentious characters" to locate the max of $|\sum_{n\le x}\chi(n)|$.
3. Slow variance of $\sum_{n\le x}\chi(n)$ (Lipschitz bounds).

Truncating Pólya's expansion

When χ is primitive, we have that

$$M(\chi) = \max_{\alpha\in[0,1]} \left| \sum_{n\le \alpha q} \chi(n) \right| = \frac{\sqrt{q}}{2\pi} \max_{\alpha\in[0,1]} \left| \sum_{1\le|n|\le q} \frac{\chi(n)(1-e(n\alpha))}{n} \right|.$$

Using a moments argument, we show that, for most χ,

$$\sum_{1\le|n|\le q} \frac{\chi(n)(1-e(n\alpha))}{n} \sim \sum_{1\le|n|\le q,\, P^+(n)\le y} \frac{\chi(n)(1-e(n\alpha))}{n},$$

with $y \approx e^{\tau}$ (here $P^+(n) = \max\{p|n\}$ and $P^-(n) = \min\{p|n\}$). This is done by observing that their difference equals

$$\sum_{\substack{1\le|n|\le q \\ P^+(n)>y}} \frac{\chi(n)(1-e(n\alpha))}{n} = \sum_{\substack{1\le|g|\le q \\ P^+(g)\le y}} \frac{\chi(g)}{g} \sum_{\substack{y<h\le q/g \\ P^-(h)>y}} \frac{\chi(h)(1-e(gh\alpha))}{h}$$

$$\ll \sum_{P^+(g)\le y} \frac{1}{g} \max_{\alpha\in[0,1]} \left| \sum_{y<h\le q/g,\ P^-(h)>y} \frac{\chi(h)e(h\alpha)}{h} \right|.$$

Truncating Pólya's expansion, continued

$$\sum_{\substack{1\leq|n|\leq q\\ P^+(n)>y}} \frac{\chi(n)(1-e(n\alpha))}{n} \ll \sum_{P^+(g)\leq y} \frac{1}{g} \max_{\alpha\in[0,1]} \left| \sum_{y<h\leq q/g,\, P^-(h)>y} \frac{\chi(h)e(h\alpha)}{h} \right|.$$

We raise both sides to $2k$. Then $max_{\alpha\in[0,1]}$ is removed by noticing that $|\alpha - r/R|$ for some $r \in \{1,\dots,R\}$. It remains to estimate

$$\sum_{\chi \,(\mathrm{mod}\, q)} \left| \sum_{y<h\leq q/g,\, P^-(h)>y} \frac{\chi(h)e(hr/R)}{h} \right|^{2k}.$$

Then we find that this is $\lesssim \sum_{P^-(n)>y,\, n>y^k} \tau_k(n)^2/n^2 = o(1)$ if $k \leq y/(\log y)^{100}$. (If $y > k$, the primes $p \approx k$ that give most of the contribution to the sum $\sum_{n\geq 1} \tau_k(n)^2/n^2$ are not present.)

$$y \approx e^\tau \;\Rightarrow\; P_q(\tau) \sim \mathbf{Prob}\left(\max_{\alpha\in[0,1]} \left| \sum_{P^+(|n|)\leq e^\tau} \frac{\chi(n)(1-e(n\alpha))}{n} \right| > 2e^\gamma\tau \right).$$

Locating the maximum

$$P_q(\tau) \sim \textbf{Prob}\left(\max_{\alpha\in[0,1]}\left|\sum_{P^+(|n|)\le e^\tau}\frac{\chi(n)(1-e(n\alpha))}{n}\right| > 2e^\gamma\tau\right).$$

Write $N(\chi)$ for the above maximum, and let α_χ be its location.

Let $|\alpha_\chi - a/b| < 1/(bB)$, $b \le B := e^{\sqrt{\tau}}$. Also, let ξ be the primitive character of conductor $\le \tau$ that lies the 'closest' to χ, i.e.

$$\mathbb{D}(\chi,\xi;e^\tau) = \min_{\substack{\psi \bmod d\le\tau \\ \psi \text{ prim.}}} \mathbb{D}(\chi,\psi;e^\tau), \quad \mathbb{D}^2(f,g;y) = \sum_{p\le y}\frac{1-\Re(f(p)\overline{g}(p))}{p}.$$

Claim: If $N(\chi) > 2e^\gamma\tau$, then $\xi = 1$ and χ is odd.

Assume not. Then

$$\sum_{P^+(|n|)\le e^\tau}\frac{\chi(n)}{n} = (1-\chi(-1))\sum_{P^+(n)\le e^\tau}\frac{\chi(n)}{n} = o(\tau).$$

Also, $b \le \tau^{1/10}$; else, $N(\chi) \sim \sum_{P^+(|n|)\le e^\tau}\chi(n)e(n\alpha)/n = o(\tau)$, by Montgomery-Vaughan, a contradiction to "$N(\chi) > 2e^\gamma\tau$".

Locating the maximum, continued

If $\xi \pmod{D}$ is the 'closest' character to χ, and either $\xi \neq 1$ or χ is even:

$$2e^{\gamma}\tau < N(\chi) \sim \left| \sum_{P^+(|n|)\le e^{\tau}} \frac{\chi(n)e(n\alpha)}{n} \right|, \quad |\alpha_\chi - a/b| \le \frac{1}{be^{\sqrt{\tau}}}, \quad b \le \tau^{1/10}.$$

Assume that $\alpha_\chi = a/b$, and expand $e(na/b)$ using characters, to get sums

$$\sum_{P^+(|n|)\le e^{\tau}} \frac{\chi(n)\overline{\psi}(n)}{n} = (1-\chi(-1)\psi(-1)) \sum_{P^+(n)\le e^{\tau}} \frac{\chi(n)\overline{\psi}(n)}{n}.$$

Small unless $\chi\overline{\psi}$ odd and $\chi \approx \psi$. So ψ induced by ξ and $\chi\overline{\xi}$ odd. Then

$$\left| \sum_{P^+(|n|)\le e^{\tau}} \frac{\chi(n)e(\frac{na}{b})}{n} \right| \sim \frac{2D^{\frac{1}{2}}}{b} \left| \sum_{D|d|b} \frac{\chi(\frac{b}{d})\mu(\frac{d}{D})\overline{\xi}(\frac{d}{D})}{\phi(d)/d} \sum_{P^+(n)\le e^{\tau}}^{(n,d)=1} \frac{\chi(n)\overline{\xi}(n)}{n} \right|.$$

$$\implies 2e^{\gamma}\tau \lesssim \frac{2\sqrt{D}}{b} \sum_{D|d|b} \frac{d}{\phi(d)} \cdot \frac{\phi(d)}{d} e^{\gamma}\tau = \frac{2e^{\gamma}\tau(b/D)}{\sqrt{D}b/D} \le \frac{2e^{\gamma}\tau}{\sqrt{D}}.$$

If $D > 1$, this is a contradiction. So $\xi = 1$ and χ is odd.

Locating the maximum, continued

To summarize,

$$N(\chi) := \max_{\alpha\in[0,1]} \left| \sum_{P^+(|n|)\le e^\tau} \frac{\chi(n)(1-e(n\alpha))}{n} \right| > 2e^\gamma\tau \;\Rightarrow\quad \chi \approx 1 \text{ and } \chi \text{ odd.}$$

Also, recall that α_χ location of max, $|\alpha_\chi - a/b| < 1/(be^{\sqrt{\tau}})$.

Claim. $\exists c \le \tau$ with $|\sum_{P^+(n)\le e^\tau,\,(n,c)=1} \chi(n)/n| \gtrsim e^\gamma \tau \phi(b)/b$.

If $b > \tau$, we take $c = 1$ (by Mont-Vaughan: $N(\chi) \sim 2|\sum_{P^+(n)\le e^\tau} \frac{\chi(n)}{n}|$).

If $b \le \tau$, then we use a result of Fouvry-Tenenbaum on smooth numbers in APs to get asymptotics for the sum $\sum_{n\le N,\,P^+(n)\le e^\tau} \chi(n)/n$.

We then find that $\exists N \in (e^{\sqrt{\tau}}, e^{\tau\log\tau}]$ (related to $|\alpha_\chi - a/b|$) such that

$$\left| \sum_{\substack{n\le N,\,(n,b)=1\\ P^+(n)\le e^\tau}} \frac{\chi(n)}{n} \right| + \left| \sum_{\substack{n> N,\,(n,b)=1\\ P^+(n)\le e^\tau}} \frac{\chi(n)}{n} \right| \gtrsim \frac{\phi(b)}{b} e^\gamma \tau.$$

Lipschitz bounds for averages of χ

$$S_1 = \sum_{\substack{n\leq N,\,(n,b)=1 \\ P^+(n)\leq e^\tau}} \frac{\chi(n)}{n}, \quad S_2 = \sum_{\substack{n> N,\,(n,b)=1 \\ P^+(n)\leq e^\tau}} \frac{\chi(n)}{n}.$$

We have $|S_1| + |S_2| \gtrsim e^\gamma \tau \frac{\phi(b)}{b}$, we want to show that $|S_1 + S_2| \gtrsim e^\gamma \tau \frac{\phi(b)}{b}$.

Note that $|S_1| + |S_2| \lesssim e^\gamma \tau \phi(b)/b$.

So, if $S_j = \lambda_j |S_j|$ with $|\lambda_j| = 1$, $j \in \{1, 2\}$, then

$$0 \leq \sum_{\substack{n\leq N,\,(n,b)=1 \\ P^+(n)\leq e^\tau}} \frac{\chi(n)\overline{\lambda_1}}{n} + \sum_{\substack{n> N,\,(n,b)=1 \\ P^+(n)\leq e^\tau}} \frac{\chi(n)\overline{\lambda_2}}{n} = o(\tau\phi(b)/b).$$

So $\chi(n) \sim \lambda_1$ for most $n \leq N$ and $\chi(n) \sim \lambda_2$ for most $n > N$.

Averages of mult. fncs vary slowly. Ideas from Halász's theorem + $\chi \approx 1$:

$$\frac{\sum_{n\leq x^{1+\delta}} \chi(n)}{x^{1+\delta}} - \frac{\sum_{n\leq x} \chi(n)}{x} \lesssim \delta \log(1/\delta) \qquad (\delta \geq 1/\log x).$$

So $\lambda_1 \sim \lambda_2$, which implies that $|S_1| + |S_2| \sim |S_1 + S_2|$.

Thank you!