Pretentious multiplicative functions

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How many primes up to x?

$$\pi(x) := \#\{p \le x : p \text{ prime}\} \sim ?$$

$$\text{Gauss's guess}: \quad \pi(x) \sim \mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}$$

x	$\pi(x)$	$\mathrm{Li}(x) - \pi(x)$
10^{13}	346065536839	108970
10^{14}	3204941750802	314889
10^{15}	29844570422669	1052618
10^{16}	279238341033925	3214631
10^{17}	2623557157654233	7956588
10^{18}	24739954287740860	21949554
10^{19}	234057667276344607	99877774
10^{20}	2220819602560918840	222744643
10^{21}	21127269486018731928	597394253
10^{22}	201467286689315906290	1932355207
10^{23}	1925320391606803968923	7250186214

Riemann's plan

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (\Re(s) > 1).$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{\substack{p \text{ prime} \\ k \geq 1}} \frac{\log p}{p^{ks}}$$

Mellin's inversion : $$\sum_{p^k \leq x} \log p = \frac{1}{2\pi i} \int_{\Re(s)=2} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s} ds.$$

$$|x^s| = x^{\Re(s)} \quad \rightsquigarrow \quad \text{want to have } \Re(s) \text{ small}$$

Riemann's remarkable discoveries:

- ζ has meromorphic continuation to $\mathbb{C}$ (simple pole at 1 of residue 1)
- Functional equation: $\pi^{-\frac{1-s}{2}} \zeta(1-s)\Gamma(\frac{1-s}{2}) = \pi^{-\frac{s}{2}} \zeta(s)\Gamma(\frac{s}{2})$

$\rightsquigarrow$ Explicit Formula : $$\sum_{p^k \leq x} \log p = x - \sum_{\rho:\, \zeta(\rho)=0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}$$

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \pi^{-\frac{1-s}{2}}\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right) = \pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right)$$

$$\sum_{p^k \le x} \log p = x - \sum_{\rho\colon \zeta(\rho)=0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)}$$

$$\zeta(\rho) = 0 \overset{\text{Product}}{\Longrightarrow} \Re(\rho) \le 1 \overset{\text{F.E.+Prod.}}{\Longrightarrow} 0 \le \Re(\rho) \le 1 \text{ or } \rho \in \{-2, -4, \dots\}.$$

- $\pi(x) \sim \mathrm{Li}(x)$ (Prime Number Theorem) $\quad \Leftrightarrow \; \Re(\rho) < 1, \forall\, \rho$
- $\pi(x) = \mathrm{Li}(x) + O(x^{\frac{1}{2}+\epsilon})$ (Riemann Hypothesis) $\quad \Leftrightarrow \; \Re(\rho) \le 1/2, \forall\, \rho$

Results on $\pi(x)$ and proof ideas

$$\text{Korobov-Vinogradov}: \quad \pi(x) = \mathrm{Li}(x) + O\left(\frac{x}{e^{c(\log x)^{3/5}(\log\log x)^{1/5}}}\right).$$

Follows by: $\zeta(\sigma + it) \neq 0$ for $\sigma \geq 1 - \frac{c}{(\log|t|)^{2/3}(\log\log|t|)^{1/3}}$.

Idea: If $\zeta(1 + it) = 0$ with $t \neq 0$, then $\zeta(\sigma + it) \sim c(\sigma - 1)$ as $\sigma \to 1^+$.

$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0$. So, as $\sigma \to 1^+$:

$$1 \leq |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \sim \frac{1}{(\sigma - 1)^3} c^4(\sigma - 1)^4 |\zeta(\sigma + 2it)|$$

$\implies \quad \zeta(1 + 2it) = \infty. \quad$ Contradiction! (only pole of ζ at 1)

Better upper bounds on $\zeta(1 + 2it)$ lead to improvements of this argument

$\rightsquigarrow \quad$ Need estimates for the exponential sums $\sum_{N<n\leq 2N} n^{2it}$

Claim: this argument uses little input specific to ζ. Rather, it uses general facts about multiplicative functions.

Multiplicative Functions

Definition

An arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is called multiplicative if

$$f(mn) = f(m)f(n) \quad \text{whenever} \quad \gcd(m,n) = 1.$$

- the Euler function $\phi(n) = \#\{1 \leq a \leq n : \gcd(a,n) = 1\}$
  e.g. $\phi(12) = 4 = \phi(4)\phi(3)$
- the divisor function $\tau(n) = \#\{d \in \mathbb{N} : d|n\}$
  e.g. $\tau(6) = 4 = \tau(2)\tau(3))$
- the sum-of-divisors function $\sigma(n) = \sum_{d|n} d$
  e.g. $\sigma(28) = 56 = \sigma(4)\sigma(7)$
- $2^{\omega(n)}$, where $\omega(n) = \#\{p|n\}$
  e.g. $\omega(18) = 2 = \omega(2) + \omega(9)$
- the Dirichlet characters χ (periodic extensions of characters of the group $(\mathbb{Z}/q\mathbb{Z})^* = \{a \pmod q : (a,q) = 1\})$
  e.g. $\chi(a) = \left(\frac{a}{p}\right) = 1$ or -1, according to whether $a \equiv \square \pmod p$ or not.

Zeroes and Möbius

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \mu(n) = \begin{cases} (-1)^r & \text{if } n = p_1 \cdots p_r, \\ & p_1 < \cdots < p_r, \\ 0 & \text{otherwise.} \end{cases}$$

$\rightsquigarrow$ The Möbius function μ is **multiplicative**.

$$\sum_{n \le x} \mu(n) \ll x^{\theta + o(1)} \quad \Leftrightarrow \quad \zeta(s) \neq 0 \text{ for } \Re(s) > \theta$$

$$\sum_{n \le x} \mu(n) = o(x) \quad \Leftrightarrow \quad \zeta(s) \neq 0 \text{ for } \Re(s) = 1 \quad \Leftrightarrow \quad \text{PNT}$$

Proof of the PNT, recast

- $\zeta(1 + it) = 0, \ t \neq 0 \quad \Leftrightarrow \quad \prod_{p \le x}(1 + \frac{1+p^{-it}}{p}) \approx c \quad$ as $x \to \infty$.
- But $\prod_p (1 + \frac{\epsilon}{p}) = \infty$. Thus $p^{it} \approx -1$ often (i.e. $\mu(n) \approx n^{it}$).
- But then $p^{2it} \approx 1$ often (i.e. $n^{2it} \approx 1$).
- *Impossible*: $\sum_{n \le x} 1/n \sim \log x$ but $\sum_{n \le x} n^{2it}/n \ll_t 1$.

Statistical questions about multiplicative functions

1. If $A \subset \mathbb{C}$, then $\#\{n \leq x : f(n) \in A\} \sim ?$ (Distribution of values of f)
2. How big is $S(x;f) := \sum_{n \leq x} f(n)$? (Average value of f)

- PNT $\Leftrightarrow S(x;\mu) = o(x)$. RH $\Leftrightarrow S(x;\mu) = O_\epsilon(x^{1/2+\epsilon})$.
- $S(x;\mu\chi) = o(x), \quad \forall \chi \pmod q \quad \Leftrightarrow$ primes are equidistributed among arithmetic progressions $a \pmod q$ with $(a,q) = 1$.
- $2^{\omega(n)} \in A \;\Leftrightarrow\; \omega(n) \in \frac{\log A}{\log 2}$ (integers with a given number of prime factors)
  Erdős-Kac: $\omega(n),\; n \leq x$, is Gaussian with $\mu \sim \log\log x$, $\sigma \sim \log\log x$
- $\#\{n \leq x : \sigma(n)/n = 2\} = \#\{n \leq x : n$ perfect number$\}$.
  The distribution of $\sigma(n)/n$ was shown to be continuous by Erdős.

Remark: Knowing $S(x,f^k)$ for $k \in \mathbb{N}$, or $S(x,f^{it})$ for $t \in \mathbb{R}$, means knowing the distribution of values of f. So, we only need to study Question 2.

Methods for studying the average value of f

1. Complex-analytic methods (à la Riemann): analytic continuation, zeroes, functional equations of $L(s,f) := \sum_{n=1}^{\infty} f(n)/n^s$

Under this category, we also find the Selberg-Delange method:
If $f(p) \sim v$ on average, then

$$L(s,f) = G(s)\zeta(s)^v,$$

where G is analytic in a "large" region (say, $\Re(s) > 1 - \epsilon$).

So the most important analytic properties of $L(s,f)$ (poles, rate of growth, etc.) are captured by $\zeta(s)^v$.

What if we know *nothing* about $f(p)$ on average?

2. 'Elementary' methods: theory of general multiplicative functions, harmonic analysis
   $\rightsquigarrow$ pretentious multiplicative functions

Particularity implies structure

Idea/goal of the 'pretentious' approach: assume that the multiplicative function f has some special behaviour on average (e.g. some *extremality*).

Then we wish to show that f has some nice structure: f *pretends to be* some simpler multiplicative function g.

$f(n) = f(m)f(p)$ if $n = pm$, $p \nmid m$. So if we know $f(m)$ and $f(p)$ (**past of f**), we know $f(n)$ (**present of f**):

$$S(x;f) := \sum_{n\leq x} f(n) \approx \sum_{p\leq x} f(p)\frac{\log p}{\log x} S(x/p;f)$$

$S(x;f)$ is an average of its 'history' (Integral-delay equations)

$\rightsquigarrow$ if $f(p)$ is close to $g(p)$ *on average*, then $S(f;x)$ and $S(g;x)$ can be related.

As a measure of the *distance* of f and g, we use

$$\mathbb{D}(f,g;x)^2 := \sum_{p\leq x} \frac{1-\Re(f(p)\overline{g(p)})}{p}.$$

Halász's theorem

Goal: If $|f(n)| \le 1$, when is $S(x;f) = \sum_{n \le x} f(n) = o(x)$?

Counterexamples:

- If $f(n) = 1$, then $S(x;f) \sim x$.
- More generally, if $f(n) = n^{it}$, then $S(x;f) \sim x^{1+it}/(1+it)$.
- Also, if $f(n) \approx n^{it}$, then we should still have that $S(x;f) \sim cx^{1+it}/(1+it)$.

Halász showed that these are the only counterexamples:

Theorem (Halász)

Let f be multiplicative with $|f(n)| \le 1$. Then

$$\begin{aligned} S(x;f) = o(x) \quad &\Leftrightarrow \quad f(n) \not\approx n^{it}, \quad \forall t \in \mathbb{R} \\ &\Leftrightarrow \quad \mathbb{D}^2(f(n), n^{it}; \infty) = \sum_p \frac{1 - \Re(f(p)/p^{it})}{p} = \infty, \ \forall\, t \in \mathbb{R}. \end{aligned}$$

Prime Number Theorem via Halász and limitations

Theorem (Halász for μ)

$$\begin{aligned} S(x;\mu) = o(x) \quad &\Leftrightarrow \quad \mu(n) \not\approx n^{it}, \quad \forall t \in \mathbb{R} \\ &\Leftrightarrow \quad \sum_p \frac{1+\Re(p^{it})}{p} = \infty, \ \forall\, t \in \mathbb{R}. \end{aligned}$$

Recall: Prime Number Theorem $\quad \Leftrightarrow \quad S(x;\mu) = o(x)$.

Granville-Soundararajan: Need to show that $p^{it} \not\approx -1$. Use sieve methods to show that $|1+p^{it}| \geq \epsilon$ most of the time $\quad \Longrightarrow \quad$ PNT

(Alternatively, use that $p^{2it} \not\approx 1$, like de la Vallée-Poussin - Hadamard.)

Problem: best result one can get is $S(x;\mu) \lesssim x/\log x$ but we expect that $S(x;\mu) \ll x^{1/2+\epsilon}$.

A converse problem

Question

For which multiplicative functions $f : \mathbb{N} \to \{z \in \mathbb{C} : |z| \le 1\}$ is it true that

$$S(x;f) \ll \frac{x}{(\log x)^{100}}, \quad \text{for all } x \ge 2? \tag{*}$$

Assume (*) and that $f(p) \sim v \quad \Longrightarrow \quad S(x;f) \sim \frac{c_f}{\Gamma(v)} x(\log x)^{v-1}, \quad c_f \neq 0$

$$\overset{(*)}{\Longrightarrow} \quad \Gamma(v) = \infty \quad \overset{|v| \leqslant 1}{\Longrightarrow} \quad v = 0 \text{ or } v = -1.$$

- If $v = -1$, then f looks like μ and $S(x;f)$ is small by the PNT.
- If $v = 0$, then $f(n)$ is small on average by an elementary argument.

Theorem (K. (2013))

Fix $A > 2$, f mult. with $|f(n)| \le 1$ and $S(x;f) \ll x/(\log x)^A$ for $x \ge 2$. Then

- *either $f(n) \approx \mu(n) n^{it}$ for some $t \in \mathbb{R}$ (i.e. $\sum_p \frac{1+\Re(f(p)p^{-it})}{p} < \infty$)*
- *or $\sum_{p \le x} f(p) = o(\pi(x))$.*

A converse theorem (K. (2013))

Fix $A > 2$, f mult. with $|f(n)| \leq 1$ and $S(x;f) \ll x/(\log x)^A$ for $x \geq 2$. Then

- either $f(n) \approx \mu(n)n^{it}$ for some $t \in \mathbb{R}$ (i.e. $\sum_p \frac{1+\Re(f(p)p^{-it})}{p} < \infty$)
- or $\sum_{p \leq x} f(p) = o(\pi(x))$.

More precisely, in the second case, if

$$\sum_{p \leq x} \frac{1 + \Re(f(p)p^{it})}{p} \geq \epsilon \log\log x \quad (|t| \leq (\log x)^{A-2}),$$

then

$$\sum_{p \leq x} f(p) \ll_A \pi(x)/(\log x)^{\epsilon(A-2)/4}.$$

An application to the distribution of primes in APs

Is it true that $\#\{p \le x : p \equiv a \pmod q\} \sim \pi(x)/\phi(q)$, when $\gcd(a,q) = 1$?

How to detect the condition $p \equiv a \pmod q$?

1st idea: use additive characters ($n \to e^{2\pi i b n/q}$). Problem: they do not mix well with primes, which are multiplicative objects.

2nd idea: use multiplicative characters (characters of the group $(\mathbb{Z}/q\mathbb{Z})^*$), the so-called *Dirichlet characters*:

$$\begin{aligned}
\sum_{\substack{p\le x\\ p\equiv a \pmod q}} 1 &= \sum_{\substack{p\le x\\ p\nmid q}} \frac{1}{\phi(q)} \sum_{\chi \pmod q} \chi(a^{-1}p) \\
&= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \chi(a)^{-1} \sum_{\substack{p\le x\\ p\nmid q}} \chi(p) \\
&= \frac{\pi(x)}{\phi(q)} + \frac{1}{\phi(q)} \sum_{\chi\ne\chi_0} \bar{\chi}(a) \sum_{p\le x} \chi(p) + O(\log q).
\end{aligned}$$

So if $\chi \ne \chi_0$, is $\chi(p)$ small on average?

Question

If χ is a non-trivial Dirichlet character mod q, is it true that

$$\sum_{p\leq x}\chi(p) = o(\pi(x))?$$

For such a χ, we know that $\sum_{n\leq x}\chi(n) = O(\sqrt{x}\log x)$ when $x \geq q$. So…

- either $\chi(p)$ is small on average $\implies$ PNT for APs
- or $\chi(n) \approx \mu(n)n^{it} \quad \Rightarrow \quad \chi^2(n)n^{-2it} \approx \mu^2(n) \quad \Rightarrow \quad t = 0,\ \chi$ real.

The case when $\chi \approx \mu$, χ real, means we have a *Siegel zero* (different techniques).

In the end, we have a new proof (K., 2013) of…

The prime number theorem for arithmetic progressions

Fix $A > 0$. If $1 \leq q \leq (\log x)^A$ and $\gcd(a, q) = 1$, then

$$\#\{p \leq x : p \equiv a \pmod q\} = \frac{\mathrm{Li}(x)}{\phi(q)}\left\{1 + O\left(\frac{1}{e^{c_A(\log x)^{3/5}(\log\log x)^{1/5}}}\right)\right\}.$$

The proof of the converse theorem

$$\sum_{p\le x} f(p) \approx \frac{(-1)^m}{2\pi i(\log x)^m}\int_{\Re(s)=1}\left(\frac{L'}{L}\right)^{(m-1)}(s,f)\frac{x^s}{s}ds,$$

where $L(s,f)=\sum_{n\ge 1} f(n)/n^s$. So...

$$\sum_{p\le x} f(p) = \text{small} \quad\Leftrightarrow\quad \left(\frac{L'}{L}\right)^{(m-1)}(1+it,f) = \text{small}$$

$$\Leftrightarrow\quad \frac{L^{(j)}(1+it,f)}{L(1+it,f)} = \text{small} \quad (1\le j\le m)$$

$$\sum_{n\le x} f(n) \ll \frac{x}{(\log x)^A} \quad\Rightarrow\quad L^{(j)}(1+it,f) = \sum_{n=1}^{\infty}\frac{f(n)(-\log n)^j}{n^{1+it}} \ll 1 \quad (j<A-1)$$

$$|L(1+it,f)| = \text{large} \quad\Leftrightarrow\quad \mathbb{D}^2(f(n),\mu(n)n^{it};\infty) = \text{large}.$$

Pretentiousness' status

We now have "pretentious" proofs of all major results in classical analytic number theory:

- Prime number theorem for arithmetic progressions
- Prime number theorem for short intervals (Hoheisel's result)
- Linnik's Theorem about primes in short APs
- Asymptotic behaviour of $S(x;f)$ for 'regular' f (Selberg-Delange method)

Moral: the underlying ideas of many proofs of classical analytic number theory are, in reality, ideas about multiplicative functions
(use of multiplicativity, i.e. Euler product)

The use of the functional equation (symmetry of ζ) is absent. Could this be the missing ingredient?

Applications of pretentious ideas to **previously unsolved problems**:

- Character sum bounds (Granville-Soundararajan, '07; Goldmakher, '12)
- Quantum Unique Ergodicity (Holowinsky-Soundararajan, '10)
- Distribution of the max of char. sums (Bober-Goldmakher-Graville-K.)

Bounds for character sums

Why care? Multiplicative Fourier Inversion $\implies$ applications to problems about equidistribution in arithmetic progressions

for a Dirichlet character χ mod q, $\quad M(\chi) := \max_{1\le t\le q} |\sum_{n\le t}\chi(n)|$.

- Pólya-Vinogradov: if χ is non trivial, then $M(\chi) \ll \sqrt{q}\log q$.
- Montgomery-Vaughan: if GRH holds, then $M(\chi) \ll \sqrt{q}\log\log q$.

Remark: Improvement seems very modest even on GRH because cancellation is much milder due to the presence of a logarithmic weight $1/n$:

$$\text{Pólya}: \quad M(\chi) = \frac{\sqrt{q}}{2\pi}\max_{0\le\alpha\le1}\left|\sum_{1\le|n|\le q}\frac{\chi(n)(1-e^{2\pi i n\alpha})}{n}\right| + O(\log q)$$

Granville-Soundararajan and Goldmakher: if χ has odd order g, then

$$M(\chi) \ll_g \begin{cases} \sqrt{q}(\log q)^{1-\delta_g+o(1)} & \text{unconditionally,} \\ \sqrt{q}(\log\log q)^{1-\delta_g+o(1)} & \text{on GRH,} \end{cases}$$

where $\delta_g = 1 - \frac{g}{\pi}\sin\frac{\pi}{g}$.

Improving Polya-Vinogradov

How to show that $M(\chi) = o(\sqrt{q}\log q)$?

$$M(\chi) \sim \frac{\sqrt{q}}{2\pi} \max_{\alpha\in[0,1]} \left| \sum_{1\le|n|\le q} \frac{\chi(n)(1-e^{2\pi i\alpha n})}{n} \right|.$$

Montgomery-Vaughan: if $|\alpha - a/b| < 1/b^2$ and $b \to \infty$ slowly, then $\sum_{n\le q} \chi(n)e^{2\pi i n\alpha}/n = o(\log q)$

Assume now that $b \ll 1$. We have that

$$\sum_{1\le|n|\le q} \frac{\chi(n)e^{2\pi i\alpha n}}{n} \sim \sum_{1\le|n|\le q} \frac{\chi(n)e^{\frac{2\pi i a n}{b}}}{n}.$$

So if $M(\chi)/\sqrt{q} \gg \log q$, then Granville-Soundararajan observe that $\chi(n)$ must resonate with some harmonic $e^{2\pi i a n/b}$, i.e. χ exhibits b-pseudoperiodicity.

Multiplicative functions correlate with multiplicative functions

$\Rightarrow$ χ pretends to be a character $\psi \pmod b$ (and, also, $\psi(-1) = -\chi(-1)$).

Granville-Soundararajan show this is impossible if χ has odd order.

The distribution of $M(\chi)$

$$P_q(\tau) := \frac{1}{\phi(q)} \# \left\{ \chi \pmod q : M(\chi) > (e^{\gamma}/\pi)\tau\sqrt{q} \right\}.$$

Recall: $M(\chi) \ll \sqrt{q} \log\log q$ on GRH.

Paley: conversely, there are ∞-ly many $\chi \pmod q$ with $M(\chi) \gg \sqrt{q}\log\log q$.

Such extremal examples should be rather rare.

Improving on results of Montgomery-Vaughan and of Bober-Goldmakher:

Theorem (Bober, Goldmakher, Granville, K.)

Fix $\theta > 1/2$. If q is prime and $1 \le \tau \le \log\log q - K$, for some $K \ge 1$, then

$$\exp\left\{ -\frac{Ce^{\tau}}{\tau}(1 + o_{\tau,K\to\infty}(1)) \right\} \le P_q(\tau) \le \exp\left\{ -e^{\tau + O(\tau^{\theta})} \right\}.$$

Remark: extremely fast decay of tails, due to the weight $1/n$:

$$M(\chi) \sim \frac{\sqrt{q}}{2\pi} \max_{\alpha\in[0,1]} \left| \sum_{1\le|n|\le q} \frac{\chi(n)(1-e^{2\pi i\alpha n})}{n} \right|.$$

$$P_q(\tau) := \frac{1}{\phi(q)} \# \left\{ \chi \pmod q : M(\chi) > (e^\gamma/\pi)\tau\sqrt{q} \right\}.$$

$$M(\chi) \sim \frac{\sqrt{q}}{2\pi} \max_{\alpha\in[0,1]} \left| \sum_{1\le|n|\le q} \frac{\chi(n)(1-e^{2\pi i\alpha n})}{n} \right|.$$

Theorem (Bober, Goldmakher, Granville, K.)

Fix $\theta > 1/2$. If q is prime and $1 \le \tau \le \log\log q - K$, for some $K \ge 1$, then

$$\exp\left\{ -\frac{Ce^\tau}{\tau}(1 + o_{\tau,K\to\infty}(1)) \right\} \le P_q(\tau) \le \exp\left\{ -e^{\tau+O(\tau^\theta)} \right\}.$$

Three key steps:

1. Bound high moments to truncate Pólya's expansion for most χ
2. The remaining characters are 1-pretentious
3. Relate $M(\chi)$ to $\sum_{n\ge1} \frac{\chi(n)}{n}$; use distribution results about the latter due to Granville-Soundararjan.

Thank you!