Sieve weights and their smoothings

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Joint work with Andrew Granville[1,2] and James Maynard[3]

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Selberg's sieve

$$\sum_{d|(a,m)} \mu(d) \leq \left(\sum_{d|(a,m)} \lambda_d \right)^2,$$

for any $\lambda_d \in \mathbb{R}$ with $\lambda_1 = 1$.

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$$\sum_{a \in \mathcal{A}} \left(\sum_{d|(a,m)} \lambda_d \right)^2 = \sum_{\substack{d_1, d_2 | m \\ D = [d_1, d_2]}} \lambda_{d_1} \lambda_{d_2} \cdot \#\{a \in \mathcal{A} : \ D|a\}.$$

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Optimizing (making assumptions on $\mathcal{A}$) yields

$$\lambda_d \approx c \cdot \mu(d) \cdot \left(\frac{\log(R/d)}{\log R} \right)^{\kappa} \cdot \mathbf{1}_{d \le R}\,, \quad \kappa = \textit{sieve dimension}$$

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More generally, consider

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$\mathrm{supp}(f) \subset (-\infty, 1]$, $f(0) = 1$.

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$M_f(n;R)$ should behave like a sieve weight for f sufficiently smooth, i.e.

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Maynard and Tao used k-dimensional generalization of $M_f(n; R)$ to detect small gaps between primes.

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Ford: $\#\{n \le x : \exists! d \in (R/2, R],\ d|n\} \asymp x(\log R)^{-\delta}(\log\log R)^{-3/2}$,

where $\delta \approx 0.086 < 1$, i.e. $\sum_{d|m,\ d \le R} \mu(d) \neq 0$ too often.

How much to smooth?

For $f \in C^1(\mathbb{R})$, $n = p^v m$ with $p \nmid m$,

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For $f \in C^A(\mathbb{R})$, $n = p_1^{v_1} \cdots p_A^{v_A} m$ with $p_1, \ldots, p_A \nmid m$,

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$$\Rightarrow \quad M_f(n;R) \lesssim M_{f^{(A)}}(m;R) \prod_{a=1}^{A} \frac{\log p_a}{\log R}.$$

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Guess : $$\sum_{n \le x} M_f(n; R)^{2k} \lesssim \max \left\{ \frac{x}{\log R}, \frac{\sum_{n \le x} (\sum_{d|n,\, d \le R} \mu(d))^{2k}}{(\log R)^{2kA}} \right\}$$

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If $\sum_{n \le x} (\sum_{d|n,\, d \le R} \mu(d))^{2k} \sim c_k x (\log R)^{E_k}$, then we would need $A > E_k/2k$ for $M_f(n; R)^{2k}$ to act as a sieve weight.

Theorem (Granville, K., Maynard (201?))

Let $k, A \in \mathbb{N}$. Assume that:

- $\sum_{n\le x}(\sum_{d|n,\ d\le R}\mu(d))^{2k} \sim c_k x(\log R)^{E_k}$ *when* $x/R^{2k} \to \infty$;
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(a) *If* $A > E_k/2k+1$, *then*

- $$\sum_{\substack{n\le x\\ \exists p|n,\ p\le R^\eta}} M_f(n;R)^{2k} \ll_{f,k,A} \frac{\eta^{3/2}x}{\log R} \quad (x\ge R\ge 2)$$

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(b) If $A\le E_k/2k+1$ *and* $x\ge R\ge 2$, *then*

$$\sum_{n\le x} M_f(n;R)^{2k} \ll x(\log R)^{E_k-2k(A-1)}.$$

Dominant contribution when $\#\{p|n : p\le R\}\sim(E_k+2k)\log\log R$.

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Finite field analogy: $$\frac{1}{q^n}\sum_{\substack{F\in\mathbb{F}_q[t]\\ \deg(F)=n}}\Big(\sum_{G|F,\ \deg(G)=m}\mu(G)\Big)^{2k}.$$

The analogy for permutations

$$\mathrm{Perm}(n; m, k) := \frac{1}{n!} \sum_{\sigma \in S_n} \Big(\sum_{\substack{T \subset [n],\ |T|=m \\ \sigma(T)=T}} \mu(\sigma\big|_T) \Big)^{2k}$$

with $\mu(\tau) := (-1)^{\#\{\text{cycles of } \tau\}}$.

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$$= \#\{(r_I)_{\emptyset\neq I\subset[2k]} : r_I\in\{0,1\}\ (|I|\text{ odd}), \sum_{I:\ i\in I} r_I = m,\ \forall i\}.$$

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We have $2^{2k-1} - 1$ free variables of size $\leq m$ and $2k$ constraints :

$$\mathrm{Perm}(n; m, k) \asymp m^{2^{2k-1} - 2k - 1} \quad (k \geq 2).$$

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Guess: $\quad E_k = 2^{2k-1} - 2k - 1 \quad$ in $\quad \sum_{n \leq x} \Big(\sum_{d | n,\, d \leq R} \mu(d) \Big)^{2k} \sim c_k x (\log R)^{E_k}.$

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$$\mathrm{Perm}(n; m, k) \asymp m^{2^{2k-1}-2k-1} \quad (k \geq 2).$$

Remark: The analogous results holds for $\mathbb{F}_q[t]$.

Guess: $E_k = 2^{2k-1} - 2k - 1$ in $\sum_{n \leq x} (\sum_{d|n,\, d \leq R} \mu(d))^{2k} \sim c_k x (\log R)^{E_k}$.

Motohashi proved $E_2 = 2$, but $2^{4-1} - 4 - 1 = 3$ (the analogy between $\mathbb{Z}$ and $\mathbb{F}_q[t]$ analogy breaks down).

The source of the extra cancellation

Write $d \approx R$ if $R/2 < d \le R$, so that

$$\frac{1}{x}\sum_{n\le x}\Big(\sum_{d|n,\ d\approx R}\mu(d)\Big)^{2k} \sim \sum_{d_1,\dots,d_{2k}\approx R}\frac{\mu(d_1)\dots\mu(d_{2k})}{[d_1,\dots,d_{2k}]}.$$

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$$\implies \quad E_k < 2^{2k-1} - 2k - 1.$$

The exponent E_k: encore

$$\sum_{n\le x}\Big(\sum_{d|n,\ d\le R}\mu(d)\Big)^{2k} \sim \frac{x}{(2\pi i)^{2k}} \underset{\substack{\Re(s_j)=\frac{1}{\log R}\\ 1\le j\le 2k}}{\int\cdots\int} F(\boldsymbol{s}) \frac{\prod_I^{\text{even}}\zeta(1+s_I)}{\prod_I^{\text{odd}}\zeta(1+s_I)} \prod_{j=1}^{2k}\frac{R^{s_j}}{s_j}\mathrm{d}\boldsymbol{s},$$

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Theorem (Granville, K., Maynard (201?))

$$E_k=\binom{2k}{k}-2k,\quad \textit{i.e.}\quad \sum_{n\le x}\Big(\sum_{d|n,\ d\le R}\mu(d)\Big)^{2k}\sim c_k x(\log R)^{\binom{2k}{k}-2k}.$$

Interpolating between integers and polynomials

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(b) If $L(\beta,\chi) = 0$ and $(\log q)^{C_k} \leq \log R \leq \frac{1}{1-\beta}$, then

$$\frac{1}{x}\sum_{n\leq x}\Big(\sum_{d|n,\ d\approx R}\chi(d)\Big)^{2k} = (\log q)^{O_k(1)} \cdot (\log R)^{\binom{2k}{k}-2k}.$$

Thank you!

$$\mathrm{Poly}_q(n,m;k) := \frac{1}{q^n} \sum_{\substack{N\in\mathbb{F}_q[t] \\ \deg N=n}} \Big| \sum_{\substack{M|N \\ \deg M=m}} \mu(M) \Big|^{2k}$$

$$= \int_{[0,1]^{2k}} \tilde{F}(r,\boldsymbol{\theta}) \frac{\prod_I^{\mathrm{even}} \mathcal{Z}_q(r^{|I|} e(\theta_I))}{\prod_I^{\mathrm{odd}} \mathcal{Z}_q(r^{|I|} e(\theta_I))} \cdot \frac{\mathrm{d}\boldsymbol{\theta}}{\prod_{j=1}^{2k} (r \cdot e(\theta_j))^m},$$

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The torsion of $\mathbb{R}/\mathbb{Z}$, i.e. the discrete structure of $\mathbb{F}_q[t]$, yields a fundamentally different pole structure.