Concours Putnam

Atelier de Pratique

Le mardi, 3 octobre 12h30-13h30

Jeus et invariants

1. Let $s(n)$ be the sum of the digits of n written in binary. Find all integer n for which $n + s(n) + s(s(n)) = 1000$.

2. Consider the following two-player game. Ininitally there is a large number of tokens on the table. Each player in their turn can remove between 1 and 10 tokens. The player who must take the last token loses. Give a strategy for the first player to win if we start with 99 tokens; and a strategy for the second player to win if we start with 100 tokens.

3. Borgov places white bishops on a chess board, and Beth Harmon places black bishops in her turn, starting with Borgov, in such a way that a new bishop can only be placed on a square that can be "taken" by a bishop of the other color already on the board. A player loses if they cannot place a bishop during their turn. Give a strategy for Beth Harmon to win.

4. We begin with the set of integers $\{1, 2, \ldots, n\}$. We proceed by replacing any two integers $a \leq b$ in the set with $b - a$, and then perform the same operation on this new set. Note that the new set may have the same integer repeated, but it will have one less element. Keep on doing this until there is just one integer left. Show how this integer will be odd or even depending on the value of $n \pmod 4$.

5. Given 11 red chips, 30 white chips and 19 blue chips, we can replace any two chips of two different colours, by two chips of the third colour. (For example, we may replace a white chip and a blue chip by two red chips.) Can we ever have the same number of chips of two different colours?

6. We play the game of *number solitaire*: Start with a finite set S of distinct integers, with smallest element 0 and largest element n. If $m, m+1 \in S$ but $m+2 \notin S$ then we can remove m and $m+1$ from S and replace them by $m+2$. Show that we can keep on doing this until we obtain a set in which all the integers differ by at least 2, and the largest element is either n or $n+1$.