Concours Putnam

Atelier de Pratique
Le mardi, 10 octobre 12h30-13h30
5448 Pav. André Aisenstadt
Fonctions

1. The functions $f(x) = 4x - 4x^2$ and $\sin \pi x$ agree at $x = 0, 1/2$, and 1. Show that $f(x) \geq \sin \pi x$ for $0 \leq x \leq 1$.

Solution: Since $f(x)$ and $\sin \pi x$ are symmetric about $x = 1/2$, it suffices to prove the inequality for $0 \leq x \leq 1/2$. Let $g(x) = f(x) - \sin \pi x$. We have that $g'(x) = 4 - 8x - \pi \cos \pi x$ and $g''(x) = -8 + \pi^2 \sin \pi x$. Thus $g''(x)$ increases monotonically from -8 to $\pi^2 - 8 > 0$ as x ranges over the interval $[0, 1/2]$ and therefore has a unique zero x_0 in this interval. It follows that $g'(x)$ is increasing for $0 \leq x \leq x_0$ and decreasing for $x_0 < x \leq 1$. Since $g(0) = 0$ and $g'(0) = 4 - \pi > 0$, this implies that $g(x) \geq 0$ in the interval $[0, x_0]$ and $g(x_0) > 0$. In the interval $[x_0, 1]$, $g(x)$ is convave downwards, and since $g(x_0) > 0$ and $g(1) = 0$ it follows that $g(x) \geq 0$ in that interval.

2. Determine, with proof, all functions f defined on the set of integers and satisfying

$$f(n + m) + f(n - m) = 2(f(m) + f(n))$$

for all n and m.

Solution: Setting $m = n = 0$ gives $2f(0) = 4f(0)$ which implies $f(0) = 0$. Setting $n = 0$, $f(m)+f(-m) = 2(f(m)+f(0)) = 2f(m)$, which implies that $f(-m) = f(m)$ for all m. Let $\alpha = f(1)$ and apply the equation for $m = 1$. $f(n + 1) + f(n - 1) = 2(\alpha + f(n))$, or $f(n+1) = 2f(n) - f(n-1) + 2\alpha$ for all n. By induction one can prove that $f(n) = \alpha n^2$ for all positive n which also holds for the negatives. Conversely, any function of the form $f(n) = \alpha n^2$ satisfies the equation.

3. Let $f(x) = \frac{x^3 e^{x^2}}{(1-x^2)^2}$. Find $f^{(2012)}(0)$. (Here $f^{(n)}$ denotes the nth derivative of f.)

Solution: The answer is $f^{(2012)}(0) = 0$. If $f(x)$ has Taylor series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $a_n = \frac{f^{(n)}(0)}{n!}$. The Taylor series of $f(x)$ is the product of the Taylor series of the three functions $x^3, e^{-x^2}, \frac{1}{(1-x^2)^2}$. For the latter two functions, the Taylor series involve only even powers of x, and when we multiply by x^3, the Taylor series has only odd powers of x. Thus, all the even-indexed coefficients are zero and $f^{(n)}(0) = 0$ for n even.

4. Let

$$f(x) = \frac{1}{1-x}.$$

Let $f_1(x) = f(x)$ and for each $n = 2, 3, \ldots$, let $f_n(x) = f(f_{n-1}(x))$. What is the value of $f_{2012}(2012)$?

Solution: $f_{2012}(2012) = 1 - \frac{1}{2012}$.

Observe that

$$f_1(x) = \frac{1}{1-x},$$

$$f_2(x) = \frac{1}{1 - \frac{1}{1-x}} = \frac{1-x}{-x} = 1 - \frac{1}{x},$$

$$f_3(x) = \frac{1}{1 - \left(1 - \frac{1}{x}\right)} = x.$$

Thus $f_4(x) = f_1(x) = \frac{1}{1-x}$, $f_5(x) = f_2(x) = 1 - \frac{1}{x}$, etc. Since 2010 is a multiple of 3, we have $f_{2010}(x) = x$, and $f_{2012}(x) = 1 - \frac{1}{x}$.

5. Evaluate $\int_0^{\frac{\pi}{2}} \ln(\sin x)dx$.

Solution: Let $I = \int_0^{\frac{\pi}{2}} \ln(\sin x)dx$. By making $y = \frac{\pi}{2} - x$, and using $\sin\left(\frac{\pi}{2} - y\right) = \cos y$, we see that $I = \int_0^{\frac{\pi}{2}} \ln(\cos y)dy$. Thus,

$$2I = \int_0^{\frac{\pi}{2}} (\ln \sin x + \ln \cos x)dx = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x)dx$$

$$= \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin(2x)\right)dx = \left(\ln \frac{1}{2}\right)\frac{\pi}{2} + \int_0^{\frac{\pi}{2}} \ln \sin(2x)dx.$$

The change of variables $y = 2x$ shows that the last integral equals I. Solving the resulting equation yields $I = -\frac{\pi}{2}\ln 2$.

6. Let

$$I_\alpha = \int_0^\infty \frac{dx}{x^\alpha(1+x)}, \quad 0 < \alpha < 1.$$

Find the choice of α that minimizes I_α. Explain.

Solution: We will show that the minimum occurs when $\alpha = \frac{1}{2}$. Split I_α into $\int_0^1$ and $\int_1^\infty$. Setting $u = \frac{1}{x}$, $du = -\frac{dx}{x^2}$ in the first integral leads to

$$\int_0^1 = \int_1^\infty \frac{du}{u^2 u^{-\alpha}\left(1+\frac{1}{u}\right)} = \int_1^\infty \frac{du}{u^{1-\alpha}(u+1)}.$$

Hence,

$$I_\alpha = \int_0^1 + \int_1^\infty = \int_1^\infty (x^{-\alpha} + x^{\alpha-1})\frac{dx}{x+1}.$$

To show that I_α is minimal at $\alpha = 1/2$ one could take the arithmetic-geometric mean inequality which gives

$$\frac{x^{-\alpha} + x^{\alpha-1}}{2} \geq \frac{1}{\sqrt{x}}$$

The equality is attained when $x^{-\alpha} = x^{\alpha-1}$ which corresponds to $\alpha = 1/2$.

7. Let f be a continuous, decreasing function on $[0,1]$. Show that

$$\int_0^1 f(x)(1-2x)dx \geq 0.$$

Solution: Splitting the range of integration in two parts $0 \leq x \leq 1/2$ and $1/2 \leq x \leq 1$ and making the change of variables $y = 1 - x$ in the integral over the latter range, the given integral can be written as

$$\int_0^{1/2} f(x)(1-2x)dx + \int_0^{1/2} f(1-y)(2y-1)dy = \int_0^{1/2} (f(x) - f(1-x))(1-2x)dx.$$

Since f is decreasing, we have $f(x) - f(1-x) \geq 0$ for $0 \leq x \leq 1/2$. Hence the integrand in the last integral is nonnegative in the range of integration, and the integral is therefore nonnegative as well.

8. Evaluate

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x}dx.$$

Solution: This is (Putnam 1982, A3). Answer $\frac{\pi}{2}\ln\pi$.

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx$$

$$\int_0^\infty \int_1^\pi \frac{dy}{1+(xy)^2} dx = \int_1^\pi \int_0^\infty \frac{dx}{1+(xy)^2} dy$$

$$\int_1^\pi \frac{\arctan(xy)}{y}\bigg|_{x=0}^{x=\infty} dy = \frac{\pi}{2}\int_1^\pi \frac{dy}{y} = \frac{\pi}{2}\ln\pi$$

9. Let T be the triangle with vertices $(0,0), (a,0)$, and $(0,a)$. Find

$$\lim_{a\to\infty} a^4 e^{-a^3} \int_T e^{x^3+y^3} dxdy.$$

Solution: This is (Putnam 1983, A6). Answer: $\frac{2}{9}$.

The corresponding indefinite integral is intractable and the definite integral diverges. But $\frac{e^{a^3}}{a^4}$ also diverges. So we have the ratio of two divergent quantities. We apply l'Hôpital's rule.

The first step is to rearrange the integral so that a only occurs as an integration limit for one of the variables (thus making it easier to differentiate with respect to it).

After a little experimentation we take $s = x+y$, $t = x-y$. The Jacobian is $\frac{1}{2}$ and so we get

$$\frac{1}{2}\int_0^a \int_{-s}^s e^{\frac{s^3}{4}+\frac{3st^2}{4}} dtds.$$

Differentiating with respect to a, we get

$$\frac{1}{2}\int_{-a}^a e^{\frac{a^3}{4}+\frac{3at^2}{4}} dt = \frac{1}{2}e^{\frac{a^3}{4}} \int_{-a}^a e^{\frac{3at^2}{4}} dt.$$

Similarly, differentiating the denominator gives $\left(\frac{3}{a^2} - \frac{4}{a^5}\right) e^{a^3}$. We can cancel out the $e^{\frac{a^3}{4}}$ to get

$$\frac{\int_{-a}^a e^{\frac{3at^2}{4}} dt}{2\left(\frac{3}{a^2} - \frac{4}{a^5}\right) e^{\frac{3a^3}{4}}}$$

but both these still diverge. Accordingly, we must apply the rule again.

We would like to eliminate the a in the integrand to make differentiation simpler. This can be achieved by setting $s = a^{1/2}t$. Notice that the integrand has the same value for t and $-t$ (or s and $-s$) so we can further simplify by taking the integration from 0 to a and doubling. Thus we get

$$\frac{a^{-1/2}\int_0^{a^{3/2}} e^{\frac{3s^2}{4}}ds}{\left(\frac{3}{a^2}-\frac{4}{a^5}\right)e^{\frac{3a^3}{4}}} = \frac{\int_0^{a^{3/2}} e^{\frac{3s^2}{4}}ds}{\left(\frac{3}{a^{3/2}}-\frac{4}{a^{9/2}}\right)e^{\frac{3a^3}{4}}}.$$

Now differentiating the numerator and the denominator give

$$\frac{\frac{3a^{1/2}}{2}e^{\frac{3a^3}{4}}}{\left(-\frac{9}{2a^{5/2}}+\frac{18}{a^{11/2}}\right)e^{\frac{3a^3}{4}}+\left(\frac{3}{a^{3/2}}-\frac{4}{a^{9/2}}\right)e^{\frac{3a^3}{4}}\frac{9a^2}{4}} = \frac{\frac{3a^{1/2}}{2}}{\left(-\frac{9}{2a^{5/2}}+\frac{18}{a^{11/2}}\right)+\left(\frac{27a^{1/2}}{4}-\frac{9}{a^{5/2}}\right)}.$$

This evaluates to $\frac{2}{9}$ as a tends to infinity.