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Lagrangian cobordism: Rigidity and flexibility aspects

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Abstract. We survey recent work (Biran and Cornea (2013, 2014), Charette and Cornea (to appear in *Israel J. Math.*)) that relates Lagrangian cobordism to the triangulated structure of the derived Fukaya category as well as the background and a number of consequences.

Contents

1. Introduction

The development of modern symplectic topology is articulated around the interplay of two seemingly opposite points of view: the first, "soft", with roots in classical differential topology, centers on flexibility phenomena. That flexibility is present in symplectic geometry is easy to expect given that, by the Darboux and Weinstein theorems, local symplectic geometry is trivial. The second point of view, "hard", originating in algebraic geometry and analysis, emphasizes rigidity. The rigid perspective is also natural but for a more subtle reason, namely the discovery by Gromov [**16**] that almost complex complex structures that are compatible with the symplectic form share many properties with true complex structures and, at the same time, are abundant.

The dichotomy *rigidity-flexibility* is a useful perspective also in what concerns the topology of Lagrangian submanifolds that is our focus in this paper. There are two techniques that establish relations among Lagrangians: the first, originating in the flexible camp, is based on cobordism, a notion central to differential topology since the work of Thom in the '50's and introduced in the Lagrangian setting by Arnold [**1**]; the second, fundamentally rigid, originates in the work of Gromov and Floer [**14**] and is based on symplectic intersection theory.

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Given a symplectic manifold, (M^{2n}, ω), a typical output of the first technique is the cobordism group $G_{cob}(M)$. As in the smooth case, $G_{cob}(M)$ is defined as the quotient of a free group generated by the Lagrangian submanifolds in M modulo relations given by Lagrangian cobordisms.

The second, "rigid", perspective also leads to a group, $K_0(D\mathcal{F}uk(M))$, the Grothedieck group of the derived Fukaya category of M. The derived Fukaya category $D\mathcal{F}uk(M)$ is a canonical triangulated completion of the Donaldson category of M, $Don(M)$. Its detailed construction appears in Seidel's book [**32**]. In turn, $Don(M)$ has as objects the Lagrangian submanifolds $L \subset M$ and as morphisms the Floer homology groups $\mathrm{Mor}_{Don^*(M)}(L, L') = HF(L, L')$. The relation with intersection theory comes from the fact that $HF(L, L')$ is the homology of a chain complex generated (generically) by the intersection points of L and L'.

For the Floer homology groups and the Fukaya categories etc to be defined, the Lagrangians involved have to be submitted to certain constraints. We denote by $\mathcal{L}^*(M)$ the appropriate class of Lagrangians. In this paper, this is a certain class of monotone Lagrangians, see §2.1 and §2.2 (see also Remark 5.1). We add an * to the notation to indicate that all involved Lagrangians belong to this class. This applies to $K_0(D\mathcal{F}uk^*(M))$ as well as to $G^*_{cob}(M)$ etc.

Once this constraint is imposed, the two groups are related by a surjective morphism [**9**]:

$$\Theta : G^*_{cob}(M) \to K_0(D\mathcal{F}uk^*(M)) \ . \tag{1}$$

The existence of Θ follows from the fact that there is [**9**] a functor

$$\widetilde{\mathcal{F}} : \mathcal{C}ob^*(M) \to T^S D\mathcal{F}uk^*(M) \tag{2}$$

relating a cobordism category $\mathcal{C}ob^*(M)$ and an enrichment, $T^S D\mathcal{F}uk^*(M)$, of the derived Fukaya category $D\mathcal{F}uk^*(M)$. The morphism Θ can be viewed as a sort of an analogue of the classical Thom morphism relating smooth cobordism groups to the homotopy groups of certain universal spaces, now called Thom spaces.

The purpose of this paper is to review the the main properties of $\widetilde{\mathcal{F}}$ and Θ and to survey the background. The main constructions are sketched and we provide some ideas of proofs. For more details we refer to [**8**–**10**].

2. Background

2.1. Basic definitions. We consider in this paper a fixed symplectic manifold (M^{2n}, ω) that is closed (or tame at infinity [**4**]). We recall that ω is a 2-form that is closed and non-degenerate. A submanifold $L^n \subset M$ is Lagrangian if $\omega|_{TL} \equiv 0$. Given such a Lagrangian L, there are two natural morphisms

$$\mu : \pi_2(M, L) \longrightarrow \mathbb{Z} \ , \ \omega : \pi_2(M, L) \longrightarrow \mathbb{R}$$

the first called the Maslov index and the second given by integration of ω. We will also need another standard convention in the subject: we put

$$N_L = \inf\{\mu(\alpha) : \alpha \in \pi_2(M, L), \omega(\alpha) > 0\} \ .$$

This number is considered $= \infty$ if there is no class α with $\omega(\alpha) > 0$. A Lagrangian L is called *monotone* if there exists $\rho > 0$ so that the two morphisms above are proportional with constant of proportionality ρ, $\rho\mu(\alpha) = \omega(\alpha)$, $\forall\, \alpha \in \pi_2(M, L)$ and $N_L \geq 2$.

REMARK 2.1. A simple way to think about monotonicity is as a form of symmetry. For instance, the sphere S^2 with the standard volume form is a symplectic manifold and any equator (that is a circle that divides the sphere in two parts of equal area) is a monotone Lagrangian submanifold of S^2. A circle on S^2 that is not an equator (in this sense) is not monotone.

A particular class of monotone Lagrangians are *exact* ones. In this case the symplectic form ω admits a primitive, η, $d\eta = \omega$ and the 1-form $\eta|_L$ is itself exact. The number N_L is $= \infty$ for exact Lagrangians L.

The next definition is a variant of a notion first introduced by Arnold [**1**, **2**].

Endow $\mathbb{R}^2$ with the symplectic structure $\omega_0 = dx \wedge dy$, $(x, y) \in \mathbb{R}^2$ and $\mathbb{R}^2 \times M$ with the symplectic form $\omega_0 \oplus \omega$. Let $\pi : \mathbb{R}^2 \times M \to \mathbb{R}^2$ be the projection. For a subset $V \subset \mathbb{R}^2 \times M$ and $S \subset \mathbb{R}^2$ we let $V|_S = V \cap \pi^{-1}(S)$.

DEFINITION 2.2. Let $(L_i)_{1\leq i\leq k_-}$ and $(L'_j)_{1\leq j\leq k_+}$ be two families of closed Lagrangian submanifolds of M. We say that that these two (ordered) families are Lagrangian cobordant, $(L_i) \simeq (L'_j)$, if there exists a smooth compact cobordism $(V; \coprod_i L_i, \coprod_j L'_j)$ and a Lagrangian embedding $V \subset ([0, 1] \times \mathbb{R}) \times M$ so that for some $\epsilon > 0$ we have:

$$(3)\qquad \begin{aligned} V|_{[0,\epsilon)\times\mathbb{R}} &= \coprod_i ([0, \epsilon) \times \{i\}) \times L_i \\ V|_{(1-\epsilon,1]\times\mathbb{R}} &= \coprod_j ((1 - \epsilon, 1] \times \{j\}) \times L'_j \ . \end{aligned}$$

The manifold V is called a Lagrangian cobordism from the Lagrangian family (L'_j) to the family (L_i). We denote such a cobordism by $V : (L'_j) \rightsquigarrow (L_i)$ or $(V; (L_i), (L'_j))$.

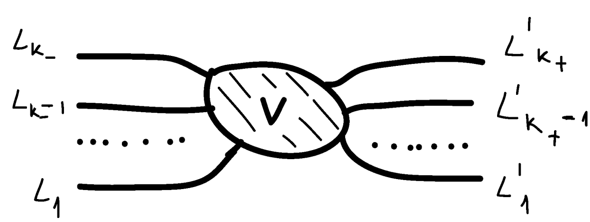

FIGURE 1. A cobordism $V : (L'_j) \rightsquigarrow (L_i)$ projected on $\mathbb{R}^2$.

A cobordism is called *monotone* if

$$V \subset ([0, 1] \times \mathbb{R}) \times M$$

is a monotone Lagrangian submanifold.

We mostly view cobordisms as embedded in $\mathbb{R}^2 \times M$. Given a cobordism $V \subset ([0, 1] \times \mathbb{R}) \times M$ as above we can extend trivially its negative ends towards $-\infty$ and its positive ends to $+\infty$ thus getting a Lagrangian $\overline{V} \subset \mathbb{R}^2 \times M$. We do not distinguish between V and $\overline{V}$. If $V \in \mathbb{C} \times M$ is a cobordism, then, outside a large enough compact set, V equals a union of its negative ends, of the form $(-\infty, -a] \times \{i\} \times L_i$, and its positive ends, of the form $[a, \infty) \times \{i\} \times L'_i$.

There is also an associated notion of isotopy for cobordisms [**8**]: two cobordisms $V, V' \subset \mathbb{C} \times M$ are *horizontally isotopic* if there exists a hamitonian isotopy ϕ_t, $t \in [0,1]$ of $\mathbb{C} \times M$ sending V to V' and so that, outside of a compact, $\phi_t(V)$ has the same ends as V for all $t \in [0,1]$ (in other words, the ends can slide along but their image in $\mathbb{C} \times M$ - outside a large compact set - remains the same; the hamiltonian isotopy is not necessarily with compact support).

2.2. Rigidity: Floer theory and the Fukaya category. Starting from Gromov's [**16**] breakthrough, rigidity properties are extracted from the behaviour of moduli spaces of J-holomorphic curves $u : \Sigma \to M$ (see [**24**] for a modern, thorough treatment of the subject). Here Σ is a Riemann surface, in our case of genus 0, possibly with boundary. The almost complex structure J on M is compatible with the form ω (in the sense that $\omega(-, J-)$ is a Riemannian metric) and the fact that u is J-holomorphic means $du \circ i = J \circ du$. In case Σ has boundary $\partial\Sigma = \bigcup C_i$, then u maps the boundary components to Lagrangians $L_i \subset M$, $u(C_i) \subset L_i$.

In our setting, the first important moduli space $\mathcal{M}(\alpha, J)$ consists of J-holomorphic disks $u : (D^2, S^1) \to (M, L)$ so that $[u] = \alpha \in \pi_2(M, L)$ modulo reparametrizations of the domain. The notation means, in particular that $u(S^1) \subset L$. Here, as above, L is a Lagrangian submanifold of M. The virtual dimension of this moduli space is $= \mu(\alpha) + n - 3$. If L is monotone and α is so that $\mu(\alpha) = 2$, then, for generic J, this moduli space is a manifold of dimension $n - 1$, without boundary. The fact that there is no boundary follows from $\mu(\alpha) = 2$ and $N_L \geq 2$. Considering now the J-holomorphic disks u as before but together with one marked point $P \in \partial D^2$ we obtain a moduli space $\mathcal{M}^1(\alpha, J)$ of J-holomorphic disks with one marked boundary point. It has dimension n and is again a manifold without boundary. This moduli space is endowed with an evaluation map $ev : \mathcal{M}^1(\alpha, J) \to L$, $ev(u) = u(P)$. Let $d_L = deg_{\mathbb{Z}_2}(ev)$. It is easy to see, again due to the monotonicity condition, that d_L is actually independent of J and is thus a simple enumerative invariant of L: it counts (mod 2) the number of J-holomorphic disks through a generic point.

We now briefly describe the most fundamental tool in modern symplectic topology: Floer homology. In our context it is defined (following [**14**], [**26, 27**]) for two Lagrangian submanifolds L, L' both monotone with the same monotonicity constant ρ and, additionally, so that $d_L = d_{L'}$. We also suppose that the two inclusion morphisms $\pi_1(L) \to \pi_1(M)$, $\pi_1(L') \to \pi_1(M)$ have a torsion image. We also assume that L and L' intersect transversely and that they are both closed. The Floer complex $CF(L, L'; J)$ is given by

$$CF(L, L'; J) = (\mathbb{Z}_2 < L \cap L' >, d)$$

with the differential defined as follows. For two intersection points $x, y \in L \cap L'$ consider the moduli space of J-holomorphic curves $u : \mathbb{R} \times [0,1] \to M$ with $u(\mathbb{R} \times \{0\}) \subset L$ and $u(\mathbb{R} \times \{1\}) \subset L'$, and that originate in x, $lim_{s \to -\infty} u(s,t) = x$, and arrive in y, $\lim_{s \to \infty} u(s,t) = y$. Such curves are called Floer strips. For generic J this moduli space, $\mathcal{M}(x, y; J)$, decomposes into connected components each of

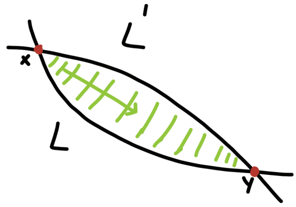

FIGURE 2. A Floer strip relating the intersection points x and y of L and L'.

which is a manifold. The dimensions of the different components is not necessarily the same, but, nevertheless, we put $dx = \sum_y \#(\mathcal{M}(x,y;J))y$ where $\#(\mathcal{M}(x,y;J))$ is the count (mod 2) of the 0-dimensional components. It is a consequence of the Gromov compactness theorem, one of the keystones of the subject, that the sum before is finite and that $d^2 = 0$. Further, the resulting Floer homology $HF(L, L')$ is independent of J and is invariant with respect to Hamiltonian deformations of L and L' in the sense that $HF(L, L') \cong HF(\phi(L), L') \cong HF(L, \phi(L'))$ where $\phi : M \to M$ is a Hamiltonian isotopy. Additionally, if L is exact and $\phi(L)$ is transverse to L, then $HF(L, \phi(L)) \cong H(L; \mathbb{Z}_2)$.

REMARK 2.3. a. Floer homology for monotone Lagrangians has been introduced by Oh [**26**]. Compared to the rather simplified setting discussed here a number of extensions are available. For instance, under additional assumptions there are variants that admit $\mathbb{Z}$ gradings and are defined over $\mathbb{Z}$. There are also far reaching extensions beyond the monotone case [**15**].

b. The condition $d_L = d_{L'}$ is necessary for the following reason. By using the Gromov compactness theorem together with a gluing argument (gluing holomorphic disks to intersection points of L and L') one can show that the Floer "differential" d verifies in general $d^2 x = (d_L - d_{L'})x$. The condition on π_1 (introduced in [**26**]) can be dropped by working over certain Novikov rings but, in the current formalism, where we count Floer trajectories directly over $\mathbb{Z}_2$, it is necessary to insure that the sums appearing in the Floer differential are finite.

By viewing the strips that give the Floer differential as examples of polygons with punctures on the boundary - in this case with two sides and two punctures - one is easily led to more complicated moduli spaces and higher associated structures. These higher structures are assembled in the Fukaya A_∞-category. We only sketch here the definition of this much richer structure and we refer to Seidel's fundamental monograph [**32**] for details on the construction.

First, we define more precisely the class of Lagrangians $\mathcal{L}^*(M)$ that we will work with: for this we fix $\rho > 0$, $d \in \mathbb{Z}_2$. We denote the class of Lagrangians under consideration by $\mathcal{L}^{\rho,d}(M)$. It consists of monotone Lagrangians $L \subset M$ with monotonicity constant ρ and so that additionally:

$$d_L = d,\ \pi_1(L) \to \pi_1(M) \text{ is null and } HF(L, L) \neq 0\ . \tag{4}$$

As before, the condition on π_1 is required to insure the finiteness of certain algebraic sums. The condition $HF(L, L) \neq 0$ (which in the language of [**7**] means that L is not narrow) is imposed here because all the techniques described below basically do not "see" in any way those Lagrangians L so that $HF(L, L) = 0$. Thus, in essence, this condition gets rid of information that is irrelevant for our discussion. We also point out that there exists a meaningful definition of $HF(L,L')$ even if two Lagrangians L and L' are not transversal, for instance when $L' = L$.

To shorten the notation we will continue to put $\mathcal{L}^*(M) = \mathcal{L}^{\rho,d}(M)$.

The first step is to construct the Donaldson category, $\mathcal{D}on^*(M)$. This is a category whose objects are the elements of $\mathcal{L}^*(M)$ and the morphisms are defined as $\mathrm{Mor}(L,L') = HF(L,L')$. The composition, also called the Donaldson triangle product,

$$* : HF(L,L') \otimes HF(L',L'') \to HF(L,L'') \tag{5}$$

is defined by using J-holomorphic polygons $u : D^2 \backslash \{P,Q,R\} \to M$ with three edges C_1, C_2, C_3 that meet at the three punctures $\{P,Q,R\} \subset \partial D^2$, so that $\partial C_1 = \{R,P\}, \partial C_2 = \{P,Q\}, \partial C_3 = \{Q,R\}$; further, the edges C_i are mapped to the Lagrangians L, L', L'' as follows: $u(C_1) \subset L$, $u(C_2) \subset L'$, $u(C_3) \subset L''$ and, assymptotically, the punctures go to intersection points of the Lagrangians involved.

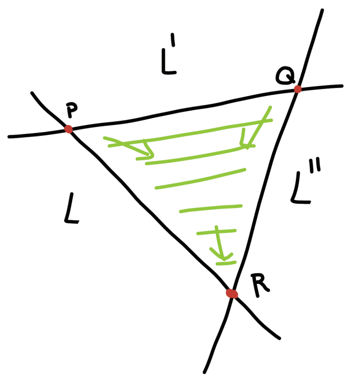

FIGURE 3. A triangle contributing to the Donaldson product.

It is a non-trivial fact that this does indeed produce a product

$$\mu^2 : CF(L,L') \otimes CF(L',L'') \to CF(L,L'')$$

that is associative in homology. The lack of associativity at the chain level leads to the existence of higher operations:

$$\mu^k : CF(L_1,L_2) \otimes CF(L_2,L_3) \ldots \otimes CF(L_k,L_{k+1}) \to CF(L_1,L_{k+1})$$

that are defined using moduli spaces of polygons with $k+1$ edges. For coherence of notation, we rename the Floer differential as $\mu^1 : CF(L,L') \to CF(L,L')$. With appropriate choices of auxiliary data - alsmost complex structures, Hamiltonian perturbations etc (technically these are quite complicated - see [**32**]) the μ^k's satisfy relations of the type:

$$\sum_{i+j=m} \mu^i(-,-,\ldots -,\mu^j,-,\ldots,-) = 0 \ . \tag{6}$$

In other words, the objects in $\mathcal{L}^*(M)$ together with the operations μ^k form an A_∞-category called the Fukaya category $\mathcal{F}uk^*(M)$. While it is very difficult to work directly with $\mathcal{F}uk^*(M)$, one can use this A_∞ category to construct a triangulated completion of $\mathcal{D}on^*(M)$. Roughly, the construction is as follows. There exists a notion of module over an A_∞ category. Specializing to our case, such a module $\mathcal{M}$ associates to each object $L \in \mathcal{L}^*(M)$ a chain complex $\mathcal{M}(L)$ and higher operations

$$\mu^k_{\mathcal{M}} : CF(L_1, L_2) \otimes \ldots \otimes CF(L_{k-1} \otimes L_k) \otimes \mathcal{M}(L_k) \to \mathcal{M}(L_1)$$

that satisfy relations similar to (6). It is easy to define morphisms $\phi : \mathcal{M} \to \mathcal{M}'$. They consist of chain morphisms $\phi_L : \mathcal{M}(L) \to \mathcal{M}'(L)$ together with appropriate higher components for each $L \in \mathcal{L}^*(M)$. As a consequence, modules form themselves an A_∞-category, $\mathcal{M}od^*(M)$. There is a functor

$$\mathcal{Y} : \mathcal{F}uk^*(M) \to \mathcal{M}od^*(M) \tag{7}$$

called the Yoneda functor that is basically an inclusion and sends each object $N \in \mathcal{L}^*(M)$ to its associated Yoneda module defined by $\mathcal{M}_N(L) = CF(L, N)$ (and appropriate higher operations).

Given a morphism $\phi : \mathcal{M} \to \mathcal{M}'$ it is possible to construct the cone over it, $C(\phi)$. This is a module so that on each object L it coincides with the cone - in the category of chain complexes - over the chain map ϕ_L. Any sequence quasi-isomorphic to the sequence $N \xrightarrow{\phi} N' \to C(\phi)$ is called exact.

With this preparation, the derived Fukaya category $D\mathcal{F}uk^*(M)$ is obtained from $\mathcal{F}uk^*(M)$ in two steps: first, we complete, inside $\mathcal{M}od^*(M)$, the image of the Yoneda functor with respect to exact sequences thus getting a new A_∞ category $\mathcal{F}uk^*(M)^\wedge$; secondly, we put $D\mathcal{F}uk^*(M) = H(\mathcal{F}uk^*(M)^\wedge)$. In other words $D\mathcal{F}uk^*(M)$ has the same objects as $\mathcal{F}uk^*(M)^\wedge$ but its morphisms are the homological images of the morphisms in $\mathcal{F}uk^*(M)^\wedge$.

The key property of $D\mathcal{F}uk^*(M)$ is that it is triangulated, with the exact triangles being the image of the exact triangles from $\mathcal{F}uk^*(M)^\wedge$. Clearly, the Donaldson category is contained in $D\mathcal{F}uk^*(M)$, however the latter category contains, *a priori*, many more objects than the former. Basically, richer are the morphisms in $\mathcal{F}uk^*(M)$, more objects are added to those in $\mathcal{D}on^*(M)$.

As $D\mathcal{F}uk^*(M)$ is triangulated, it is possible to decompose objects $L \in \mathcal{L}^*(M)$ with respect to others $L_1, L_2 \ldots \in \mathcal{L}^*(M)$. In the presence of such a decomposition one can recover properties of L from those of the L_i's. At the same time, one of the difficulties with this construction comes from the rather algebraic description of the exact triangles in $D\mathcal{F}uk^*(M)$ which makes them hard to detect in practice.

We now use the triangulated structure of $D\mathcal{F}uk^*(M)$ to associate to it the Grothendieck group

$$K_0(D\mathcal{F}uk^*(M))$$

which is - in our non-oriented and ungraded case - the $\mathbb{Z}_2$-vector space generated by the objects of $D\mathcal{F}uk^*(M)$ modulo the relations $M + M'' = M'$ whenever $M \to M' \to M''$ is an exact sequence.

2.3. Flexibility: h-principle and surgery. Most of the flexibility phenomena in symplectic topology are based on Gromov's h(omotopy)-principle (see [**17**],[**13**],[**23**]). The particular application of the h-principle that is relevant for us here concerns Lagrangian immersions, see [**4**] for this form:

(H) There is a weak homotopy equivalence between the space of Lagrangian immersions $L \to M$ and the space of bundle maps $\Phi : TL \to TM$ that map each fibre T_xL to a Lagrangian subspace of T_xM and are so that the map $\phi : L \to M$, induced on the base, satisfies $[\phi^*\omega] = 0 \in H^2(L; \mathbb{R})$. In particular, deciding whether a map $f : L \to M$ is homotopic to a Lagrangian immersion $f' : L \to M$ reduces to an algebraic-topological verification.

We also need an additional "flexible" construction which is called Lagrangian surgery (see [**21**], [**29**]).

We start by describing the local picture. Fix the following two Lagrangians: $L_1 = \mathbb{R}^n \subset \mathbb{C}^n$ and $L_2 = i\mathbb{R}^n \subset \mathbb{C}^n$ and consider the curve $H \subset \mathbb{C}$, $H(t) = a(t) + ib(t)$, $t \in \mathbb{R}$, with the following properties (see also Figure 4):

- H is smooth.
- $(a(t), b(t)) = (t, 0)$ for $t \in (-\infty, -1]$.
- $(a(t), b(t)) = (0, t)$ for $t \in [1, +\infty)$.
- $a'(t), b'(t) > 0$ for $t \in (-1, 1)$.

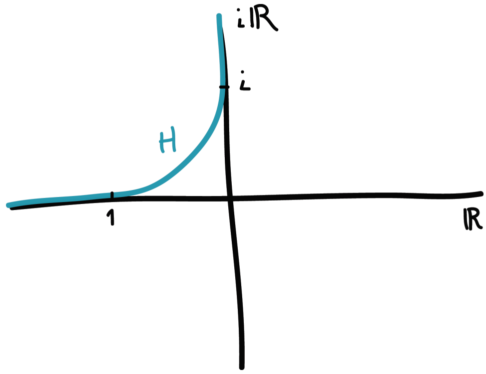

FIGURE 4. The curve $H \subset \mathbb{C}$.

Let

$$L = \left\{ \left((a(t) + ib(t))x_1, \ldots, (a(t) + ib(t))x_n \right) \mid t \in \mathbb{R}, \sum x_i^2 = 1 \right\} \subset \mathbb{C}^n .$$

It is easy to see that L as defined above is Lagrangian. We will denote it by $L = L_1 \# L_2$ (with an abuse of nation as we omitted the handle). Moreover, it is also not difficult to construct [**8**] a cobordism $V : L \rightsquigarrow (L_1, L_2)$.

In case L_1 and L_2 intersect in a single point, then L is diffeomorphic to the connected sum of L_1 and L_2 and one can see (as in [**8**]) that the cobordism V above is homotopy equivalent to the wedge $L_1 \vee L_2$.

By using the Weinstein neighbourhood theorem, the local picture can be implemented globally without difficulty. A few consequences of this construction are relevant here:

(S1) If $L \subset M$ is an immersed Lagrangian with transversal double points, then by surgery at each double point of L we obtain an embedded Lagrangian $L' \subset M$.

(S1) Similarly to the first point: if $V : (L_i) \rightsquigarrow (L'_j)$ is an immersed Lagrangian cobordism with transversal double points but so that the L_i's and the

L_j's are embedded (same definition as in 2.2 but V is immersed, not necessarily embedded), then by surgery at the double points of V, we obtain an embedded cobordism $V' : (L_i) \rightsquigarrow (L'_j)$.

(S3) if $L_1, L_2 \in \mathcal{L}^{\rho,d}(M)$ intersect in a single point, then $L = L_1 \# L_2 \in \mathcal{L}^{\rho,d}(M)$ is cobordant to (L_1, L_2) by a cobordism V so that $V \in \mathcal{L}^{\rho,d}(\mathbb{C} \times M)$. To verify the last condition we use the cobordism constructed in [**8**] so that, as mentioned above, $V \simeq L_1 \vee L_2$. Given that L_1 and L_2 intersect in a single point, this leads to a simple description of the group $\pi_2(\mathbb{C} \times M, V)$ and as the monotonicity constant ρ is the same for both L_1 and L_2 we deduce that V is also monotone with the same monotonicity constant. Interestingly, as we shall discuss later, $d_{L_1} = d_{L_2} = d$ is not required here.

By the notation $V \in \mathcal{L}^*(\mathbb{C} \times M)$ we mean that V is monotone with respective constants (ρ, d) and that $\pi_1(V) \to \pi_1(M)$ is trivial. There is no Floer homology condition imposed to V (this is in contrast to (4)).

We now define the Lagrangian cobordism groups associated to M. The simplest such cobordism group, $\mathcal{G}_{cob}(M)$, is defined as the free group generated by all closed, connected Lagrangian submanifolds $L \subset M$ modulo the relations given by $L_1 \cdot L_2 \cdot \ldots \cdot L_k = 1$ if there is a cobordism $V : \emptyset \rightsquigarrow (L_1, L_2, \ldots, L_k)$.

There are, of course, many variants of this definition but the one of main interest to us is the *monotone* cobordism group, $\mathcal{G}^*_{cob}(M)$, which defined by first fixing $* = (\rho, d)$ and using the same definition as above but now with $L_i \in \mathcal{L}^*(M)$, $V \in \mathcal{L}^*(\mathbb{C} \times M)$.

It is also useful to consider the abelianizations of these groups $G_{cob}(M)$ and, respectively, $G^*_{cob}(M)$.

REMARK 2.4. i. Because we work in a non-oriented setting the two groups $G_{cob}(M)$ and $G^*_{cob}(M)$ are actually $\mathbb{Z}_2$-vector spaces. Moreover, it is easy to see that $\mathcal{G}_{cob}(M)$ is actually abelian so that $\mathcal{G}_{cob}(M) = G_{cob}(M)$. Indeed, consider two curves $\gamma_{1,2}$ and $\gamma_{2,1}$ in the plane so that they are both horizontal at $\pm\infty$ and so that $\gamma_{1,2}$ is constant equal to 1 at $+\infty$ and constant equal to 2 at $-\infty$ while $\gamma_{2,1}$ is constant to 2 at $+\infty$ and equal to 1 at $-\infty$. We assume that the two curves intersect transversely in one point. For any two Lagrangians L_1, L_2 we then define $V : (L_1, L_2) \to (L_2, L_1)$ by $V = \gamma_{1,2} \times L_1 \cup \gamma_{2,1} \times L_2$. This V is obviously not embedded (except if L_1 and L_2 are disjoint) but by a small perturbation we may assume that it is immersed with only double points and then, as explained above, we can surger the double points and get an embedded cobordism $V' : (L_1, L_2) \to (L_2, L_1)$ so that L_1 and L_2 commute in $\mathcal{G}_{cob}(M)$. Notice also that if $L_1, L_2 \in \mathcal{L}^*(M)$ and L_1 and L_2 are either disjoint or intersect in a single point, then - again, by the surgery argument - they commute in $\mathcal{G}^*_{cob}(M)$.

ii. There are clearly even more refined variants of these cobordism groups that take into account orientations and possibly spin structures etc.

The property (S2) together with the h-principle for Lagrangian immersions as stated at (H) above imply that general cobordism is quite flexible and that the "general" cobordism groups can be computed by algebraic-topological methods:

essentially, one uses the h-principle to compute a group defined as above but by using immersed Lagrangians V and not embedded ones; one then shows, by the point (S2), that this group coincides with $\mathcal{G}_{cob}(M)$. Such calculations have been pursued by Eliashberg [**12**] and Audin [**3**].

3. Cobordism categories and the category $T^S D\mathcal{F}uk^*(M)$

3.1. Cobordism categories. The modern perspective on cobordism treats manifolds as objects in a category and the cobordisms relating them as morphisms in an appropriate category. This point of view is quite useful in our setting (see also [**25**] for an alternative approach).

The category of main interest for us here is $\mathcal{C}ob^*(M)$ (see [**9**] where it is denoted by $\mathcal{C}ob_0^d(M)$). The objects of $\mathcal{C}ob^*(M)$ are families $(L_1, L_2, \ldots, L_r)$ with $r \geq 1$, $L_i \in \mathcal{L}^*(M)$.

Given two such families $(L_1, L_2, \ldots, L_r)$ and $(K_1, \ldots K_s)$ a morphism

$$W : (K_1, \ldots, K_s) \to (L_1, L_2, \ldots, L_r)$$

is an ordered family $(W_1, \ldots, W_s)$ where each W_i is a horizontal isotopy class of a cobordism $V_i \in \mathcal{L}^*(\mathbb{C} \times M)$ so that $V_1 : K_1 \rightsquigarrow (L_1, \ldots L_{i_1})$, $V_2 : K_2 \to (L_{i_1+1}, \ldots, L_{i_2})$,..., $V_s : K_s \rightsquigarrow (L_{i_s}, \ldots, L_r)$ (for a more precise description see [**9**]). In particular, each of the V_j's has a single positive end that coincides with K_j. It is easy to see how to embedd the union $(V_1 \cup \ldots \cup V_s)$ as a Lagrangian in $\mathbb{C} \times M$ so that it provides a cobordism $(K_1, \ldots K_s) \rightsquigarrow (L_1, L_2, \ldots, L_r)$ and W can be viewed as the horizontal isotopy class of this cobordism. At the same time, notice that the horizontal isotopy class of an arbitrary cobordism $U : (K_1, \ldots K_s) \rightsquigarrow (L_1, L_2, \ldots, L_r)$ is not in general a morphism in our category (for instance if U is connected and $K_1, K_2 \neq \emptyset$).

Intuitively, a good way to view a basic morphism in our category:

$$V : K \rightsquigarrow (L_1, \ldots, L_i)$$

is as a "formula" that decomposes the Lagrangian K into the pieces $L_1, \ldots, L_i$.

The composition of morphisms is induced by concatenation from right to left: $V \# V'$ is obtained by gluing the negative ends of V to the positive ends of V'.

Remark 3.1. The reason why concatenation does not leave the class $\mathcal{L}^*(M)$ is precisely that each morphism is a union of cobordisms with a single positive end.

With a little more care in defining all of this, it is easy to see that $\mathcal{C}ob^*(M)$ has the structure of a monoidal category so that the operation on objects is given by

$$(L_1, \ldots, L_r), (K_1, \ldots, K_s) \to (L_1, \ldots, L_r, K_1, \ldots, K_s)$$

and similarly for morphisms.

We will also use another category that is a simpler version of $\mathcal{C}ob^*(M)$ and is denoted by $S\mathcal{C}ob^*(M)$. Its objects are Lagrangians $L \in \mathcal{L}^*(M)$ and its morphisms $L \to L'$ are horizontal isotopy classes of cobordisms $V : L \rightsquigarrow (L_1, \ldots, L_i, L')$, $V \in \mathcal{L}^*(M)$. In other words, a morphism from L to L' is represented by a cobordism with a single positive end that coincides with L and with possibly many negative ends but so that the "last" negative end is L'. Composition is again induced by concatenation: if $V' : L' \to (K_1, \ldots, K_r, L'')$ represents a second morphism

$L' \to L''$, then the composition $L \to L' \to L''$ is represented by the cobordism $V\#V' : L \rightsquigarrow (L_1, \ldots, L_i, K_1, \ldots, K_r)$ defined by gluing V' to V along L' and extending the ends $L_1, \ldots, L_i$ trivially in the negative direction.

There is a functor $\mathcal{P} : \mathcal{C}ob^*(M) \to S\mathcal{C}ob^*(M)$ that is defined at the level of objects by $(L_1, \ldots, L_k) \to L_k$ and similarly for morphisms.

3.2. Cone-decompositions in the derived Fukaya category. The purpose of the paper is to explain how the cobordism perspective on Lagrangian submanifolds, as reflected in the categories $\mathcal{C}ob^*(M)$ and $S\mathcal{C}ob^*(M)$, is related to to the "rigid" invariants encoded in the derived Fukaya category, $D\mathcal{F}uk^*(M)$. There is however an immediate obstacle: the most important structural property of $D\mathcal{F}uk^*(M)$ is that it is triangulated while neither one of $\mathcal{C}ob^*(M)$ and $S\mathcal{C}ob^*(M)$ are so, with the consequence that a functor from one of the cobordism categories to $D\mathcal{F}uk^*(M)$ will neglect precisely this triangulated structure.

This is the issue that we deal with here, following [**9**]. Namely, we describe briefly a rather formal construction that shows how to extract, out of a triangulated category, $\mathcal{C}$, another category $T^S\mathcal{C}$ whose morphisms parametrize the various ways to decompose an object by iterated exact triangles in $\mathcal{C}$.

We apply this construction to $D\mathcal{F}uk^d(M)$ thus getting the category $T^S D\mathcal{F}uk^*(M)$ that is the target of the functor $\widetilde{\mathcal{F}}$ from (2).

We recall [**35**] that a triangulated category $\mathcal{C}$ is an additive category together with a translation automorphism $T : \mathcal{C} \to \mathcal{C}$ and a class of triangles called *exact triangles*

$$T^{-1}X \xrightarrow{u} X \xrightarrow{v} Y \xrightarrow{w} Z$$

that satisfy a number of axioms due to Verdier and to Puppe (see e.g. [**35**]).

A cone decomposition of length k of an object $A \in \mathcal{C}$ is a sequence of exact triangles:

$$T^{-1}X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Y_{i+1} \xrightarrow{w_i} X_i$$

with $1 \leq i \leq k$, $Y_{k+1} = A$, $Y_1 = 0$. (Note that $Y_2 \cong X_1$.) Thus A is obtained in k steps from $Y_1 = 0$. To such a cone decomposition we associate the family $l(A) = (X_1, X_2, \ldots, X_k)$ and we call it the *linearization* of the cone decomposition. This definition is an abstract form of the familiar iterated cone construction in case $\mathcal{C}$ is the homotopy category of chain complexes. In that case T is the suspension functor $TX = X[-1]$ and the cone decomposition simply means that each chain complex Y_{i+1} is obtained from Y_i as the mapping cone of a morphism coming from some chain complex, in other words $Y_{i+1} = \text{cone}(X_i[1] \xrightarrow{u_i} Y_i)$ for every i, and $Y_1 = 0$, $Y_{k+1} = A$. There is also a rather obvious equivalence relation among cone-decompositions.

We will now define the category $T^S\mathcal{C}$ called the *category of (stable) triangle (or cone) resolutions over* $\mathcal{C}$. The objects in this category are finite, ordered families $(x_1, x_2, \ldots, x_k)$ of objects $x_i \in \mathcal{O}b(\mathcal{C})$.

We will first define the morphisms in $T^S\mathcal{C}$ with domain being a family formed by a single object $x \in \mathcal{O}b(\mathcal{C})$ and target $(y_1, \ldots, y_q)$, $y_i \in \mathcal{O}b(\mathcal{C})$. For this, consider triples (ϕ, a, η), where $a \in \mathcal{O}b(\mathcal{C})$, $\phi : x \to T^s a$ is an isomorphism (in $\mathcal{C}$) for some index s and η is a cone decomposition of the object a with linearization

$(T^{s_1}y_1, T^{s_2}y_2, \ldots, T^{s_{q-1}}y_{q-1}, y_q)$ for some family of indices $s_1, \ldots, s_{q-1}$. A morphism $\Psi : x \longrightarrow (y_1, \ldots, y_q)$ is an equivalence class of triples (ϕ, a, η) as before up to a natural equivalence relation. We now define the morphisms between two general objects. A morphism

$$\Phi \in \mathrm{Mor}_{T^S\mathcal{C}}((x_1, \ldots x_m), (y_1, \ldots, y_n))$$

is a sum $\Phi = \Psi_1 \oplus \cdots \oplus \Psi_m$ where $\Psi_j \in \mathrm{Mor}_{T^S\mathcal{C}}(x_j, (y_{\alpha(j)}, \ldots, y_{\alpha(j)+\nu(j)}))$, and $\alpha(1) = 1$, $\alpha(j+1) = \alpha(j) + \nu(j) + 1$, $\alpha(m) + \nu(m) = n$. The sum $\oplus$ means here the obvious concatenation of morphisms. With this definition this category is strict monoidal, the unit element being given by the void family. See again [**9**] for more details as well as for the definition of the composition of morphisms (basically, this comes down to the refinement of cone resolutions).

There is a projection functor

$$\mathcal{P} : T^S\mathcal{C} \longrightarrow \Sigma\mathcal{C} \tag{8}$$

Here $\Sigma\mathcal{C}$ stands for the *stabilization category* of $\mathcal{C}$: $\Sigma\mathcal{C}$ has the same objects as $\mathcal{C}$ and the morphisms in $\Sigma\mathcal{C}$ from a to $b \in \mathcal{O}b(\mathcal{C})$ are morphisms in $\mathcal{C}$ of the form $a \to T^s b$ for some integer s. The definition of $\mathcal{P}$ is as follows: $\mathcal{P}(x_1, \ldots x_k) = x_k$ and on morphisms it associates to $\Phi \in \mathrm{Mor}_{T^S\mathcal{C}}(x, (x_1, \ldots, x_k))$, $\Phi = (\phi, a, \eta)$, the composition:

$$\mathcal{P}(\Phi) : x \xrightarrow{\phi} T^s a \xrightarrow{w_k} T^s x_k$$

with $w_k : a \to x_k$ defined by the last exact triangle in the cone decomposition η of a,

$$T^{-1}x_k \longrightarrow a_k \longrightarrow a \xrightarrow{w_k} x_k \ .$$

In this paper we take $\mathcal{C} = D\mathcal{F}uk^*(M)$. We will work here in an ungraded and non-oriented setting so that $T = id$ and all the indexes s_i above equal 1.

4. The functor $\widetilde{\mathcal{F}}$ and some of its properties

4.1. The main theorem and a few corollaries. With the preparation of the last section we can now state the main result surveyed in this paper.

THEOREM 4.1. [**9**] *There exists a monoidal functor,*

$$\widetilde{\mathcal{F}} : \mathcal{C}ob^*(M) \longrightarrow T^S D\mathcal{F}uk^*(M),$$

with the property that $\widetilde{\mathcal{F}}(L) = L$ for every Lagrangian submanifold $L \in \mathcal{L}^*(M)$.

In the remainder of this section we "unwrap" this statement and discuss its consequences.

COROLLARY 4.2. *If $V : L \rightsquigarrow (L_1, \ldots, L_k)$ is a Lagrangian cobordism, then there exist k objects $Z_1, \ldots, Z_k$ in $D\mathcal{F}uk^*(M)$ with $Z_1 = L_1$ and $Z_k \simeq L$ which fit into $k-1$ exact triangles as follows:*

$$L_i \to Z_{i-1} \to Z_i \ \forall\, 2 \leq i \leq k.$$

In particular, L belongs to the triangulated subcategory of $D\mathcal{F}uk^*(M)$ generated by $L_1, L_2, \ldots, L_k$.

This follows directly from Theorem 4.1: given that V represents a morphism in $\mathcal{C}ob(M)$ and in view of the definition of $T^S(-)$, the sequence of exact triangles in the statement is provided by $\widetilde{\mathcal{F}}(V)$.

There exists a simplified version

$$\mathcal{F} : \mathcal{SC}ob^*(M) \to D\mathcal{F}uk^*(M)$$

of $\widetilde{\mathcal{F}}$ that can be made explicit easily. At the level of objects $\mathcal{F}(L) = L$ for each $L \in \mathcal{L}^*(M)$. Concerning morphisms, for each cobordism $V : L \to (L_1, \ldots, L_{k-1}, L')$ that represents a morphism ϕ in $\mathcal{SC}cob^*(M)$ we define

$$\mathcal{F}([V]) \in \hom_{D\mathcal{F}uk}(L, L') = HF(L, L')$$

to be the image of the unity in $HF(L, L)$ (induced by the fundamental class of L) through a morphism

$$\phi_V : HF(L, L) \to HF(L, L') \ , \ \mathcal{F}([V]) = \phi_V([L]) \ . \tag{9}$$

In turn, ϕ_V is given by counting Floer strips $u : \mathbb{R} \times [0, 1] \to \mathbb{R}^2 \times M$ with boundary conditions $u(\mathbb{R} \times \{0\}) \subset \gamma \times L$, $u(\mathbb{R} \times \{1\}) \subset V$, where $\gamma \subset \mathbb{R}^2$, V are as in Figure 5 (with $L' = L_k$).

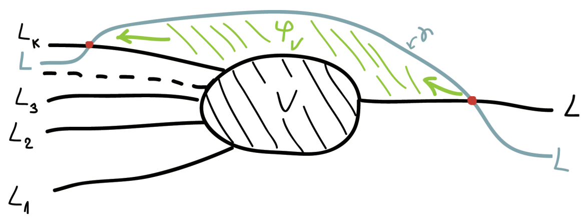

FIGURE 5. A cobordism $V \subset \mathbb{R}^2 \times M$ with a positive end L and with $L' = L_k$ together with the projection of the J-holomorphic strips that define the morphism ϕ_V.

The fact that $\widetilde{\mathcal{F}}$ determines $\mathcal{F}$ results from the commutativity of the diagram (10) which is itself a simple consequence of the construction of $\widetilde{\mathcal{F}}$.

$$\begin{tikzcd} \mathcal{C}ob^*(M) \arrow[r, "\widetilde{\mathcal{F}}"] \arrow[d, "\mathcal{P}"'] & T^S D\mathcal{F}uk^*(M) \arrow[d, "\mathcal{P}"] \\ \mathcal{SC}ob^*(M) \arrow[r, "\mathcal{F}"] & D\mathcal{F}uk^*(M) \end{tikzcd} \tag{10}$$

The functor $\mathcal{F}$ is particularly useful to state another simple consequence of Theorem 4.1.

Corollary 4.3. *Consider the Lagrangian cobordism $V : L \rightsquigarrow (L_1, L_2)$. If $L, L_1, L_2 \in \mathcal{L}^*(M)$ and $V \in \mathcal{L}^*(\mathbb{C}\times M)$, then there is an exact triangle in $D^*\mathcal{F}uk(M)$*

$$\begin{tikzcd} L_2 & \\ & L \arrow[ul, "\mathcal{F}(V)"'] \\ L_1 \arrow[ur, "\mathcal{F}(V')"'] \arrow[from=uu, "\mathcal{F}(V'')"'] & \end{tikzcd} \tag{11}$$

where V' and V'' are the cobordisms obtained by bending the ends of V as in Figure 6 below.

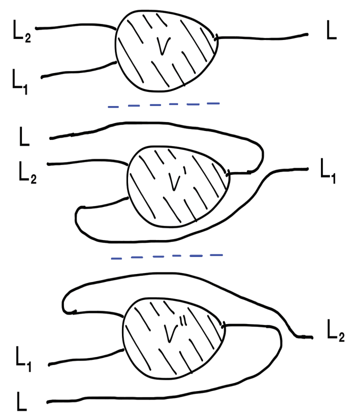

Figure 6. The cobordisms V and V', V'' obtained by bending the ends of V as indicated.

To unwrap the meaning of $\widetilde{\mathcal{F}}$ further, fix $N \in \mathcal{L}^*(M)$. Consider the functor

$$h_N : D\mathcal{F}uk^*(M) \xrightarrow{\hom(N,-)} (\mathcal{V}, \times)$$

where $(\mathcal{V}, \times)$ is the monoidal category of ungraded vector spaces over $\mathbb{Z}_2$, with the monoidal structure $\times$ being direct product. We put $\mathcal{H}F_N = h_N \circ \mathcal{F}$ so that we have the commutative diagram (12).

$$\begin{tikzcd} S\mathcal{C}ob^*(M) \arrow[r, "\mathcal{F}"] \arrow[dr, "\mathcal{H}F_N"'] & D\mathcal{F}uk^*(M) \arrow[d, "\hom(N,-)"] \\ & (\mathcal{V}, \times) \end{tikzcd} \tag{12}$$

The functor $\mathcal{H}F_N$ exhibits Floer homology $HF(N,-)$ as a vector space valued functor defined on a cobordism category. Here are some properties of $\mathcal{H}F_N$ that follow easily from Theorem 4.1.

Corollary 4.4. *For any $N \in \mathcal{L}$ the Floer homology functor*

$$\mathcal{H}F_N : S\mathcal{C}ob(M) \to (\mathcal{V}, \times)$$

defined above verifies:

i. *For each* $L \in \mathcal{L}^*(M)$, $\mathcal{H}F_N(L) = HF(N, L)$. *If* $V : L \rightsquigarrow (L_1, \ldots, L_{k-1}, L')$ *represents a morphism in* $\mathcal{SC}cob^*(M)$, *then* $\mathcal{H}F_N([V])$ *is the morphism*

$$(-) * \phi_V([L])) : HF(N, L) \to HF(N, L')$$

*given by the Donaldson product (*5*) with the element* $\phi_V([L])$ *where* ϕ_V *is as in (*9*).*

ii. *If* V *has just two negative ends* L_1, L_2 *and* V', V'' *are as in Corollary* 4.3, *then there is a long exact sequence that only depends on the horizontal isotopy type of* V

$$\ldots \longrightarrow \mathcal{H}F_N(L_2) \xrightarrow{\mathcal{H}F_N(V'')} \mathcal{H}F_N(L_1) \xrightarrow{\mathcal{H}F_N(V')} \mathcal{H}F_N(L) \xrightarrow{\mathcal{H}F_N(V)} \mathcal{H}F_N(L_2) \longrightarrow \ldots$$

and this long exact sequence is natural in N. *In particular,* $\phi_V([L]) * \phi_{V''}([L_2]) = 0$ *and, similarly,* $\phi_{V''}(L_2]) * \phi_{V'}([L_1]) = 0$.

iii. *More generally, if* V *has negative ends* $L_1, L_2, \ldots, L_k$ *with* $k \geq 2$, *then there exists a spectral sequence* $\mathcal{E}_N(V)$ *so that:*

a. *the* E_2 *term of the spectral sequence satisfies:*

$$(\mathcal{E}_N(V))_2 = \oplus_i \mathcal{H}F_N(L_i)$$

b. *from* E_2 *on, the terms of the spectral sequence only depend on the horizontal isotopy type of* V.

c. $\mathcal{E}_N(V)$ *converges to* $\mathcal{H}F_N(L)$ *and is again natural in* N.

To end the section notice that Corrolary 4.2 and the definition of $K_0(-)$ directly imply that the mapping $\mathcal{L}^*(M) \to K_0(D\mathcal{F}uk^*(M))$ given by $L \to L$ induces an epimorphism

$$\Theta : G^*_{cob}(M) \to K_0 D\mathcal{F}uk^*(M)$$

as stated in equation (1). Recent results of Haug [**18**] show that a version of Θ (defined for a suitable class $\mathcal{L}^*(-)$) and for $M = \mathbb{T}^2$ is an isomorphism. Interestingly, his proof makes use of homological mirror symmetry for the elliptic curve.

4.2. Further related properties.

4.2.1. *Lagrangian suspension and Seidel's representation.* We begin by recalling two important constructions in symplectic topology.

The first one is Seidel's representation $S : \pi_1(Ham(M)) \to QH(M)^*$ of the Hamiltonian diffeomorphism group with values in the invertible elements of the quantum homology of the ambient manifold [**31**]. There also exists a Lagrangian version of Seidel's representation ([**19**],[**20**],[**22**]). As noticed in [**10**], after convenient "categorification", this version of Seidel's representation can be viewed as an action of the fundamental groupoid $\Pi(Ham(M))$ on $D\mathcal{F}uk^*(M)$. This action induces an action of $\Pi(Ham(M))$ on $T^S D\mathcal{F}uk^*(M)$:

$$\widetilde{S} : \Pi(Ham(M)) \times T^S D\mathcal{F}uk^*(M) \to T^S \mathcal{F}uk^*(M) \ . \tag{13}$$

The second construction is Lagrangian suspension [**30**]. This too gives rise [**10**] to an action of $\Pi(Ham(M))$, this time on $\mathcal{C}ob^*(M)$,

$$\Sigma : \Pi(Ham(M)) \times \mathcal{C}ob^*(M) \to \mathcal{C}ob^*(M) \ . \tag{14}$$

It turns out that these two actions are interchanged by $\widetilde{\mathcal{F}}$. In fact, we have the following commutative diagram that "categorifies" Seidel's representation:

Theorem 4.5. [**10**] *The following diagram of categories and functors commutes:*

$$\begin{tikzcd}
\pi_1(Ham(M)) \arrow[r, "S"] \arrow[d, "i"'] & QH(M)^* \arrow[d, "*"] \\
\Pi(Ham(M)) \arrow[r, "\widetilde{S}"] \arrow[d, "\Sigma"'] & \mathcal{E}nd(T^S D\mathcal{F}uk^*(M)) \arrow[d, "\widetilde{\mathcal{F}}^*"] \\
\mathcal{E}nd(\mathcal{C}ob^*(M)) \arrow[r, "\widetilde{\mathcal{F}}_*"'] & fun(\mathcal{C}ob^*(M), T^S D\mathcal{F}uk^*(M))
\end{tikzcd} \tag{15}$$

The categories and functors in the top square are strict monoidal as is the functor Σ.

Here the functor S is Seidel's representation [**31**] viewed as a monoidal functor and the action $*$ is a refinement of the module action of quantum homology on Lagrangian Floer homology [**7**]. The functors $\widetilde{\mathcal{F}}_*$ and $\widetilde{\mathcal{F}}^*$ are induced respectively by composition and pre-composition with $\widetilde{\mathcal{F}}$. Recall that an action $\mathcal{M} \times \mathcal{C} \to \mathcal{C}$ of a monoidal category $\mathcal{M}$ on a category $\mathcal{C}$ can be viewed as a strict monoidal functor $\mathcal{M} \to \mathcal{E}nd(\mathcal{C}, \mathcal{C})$ and thus the commutativity of the bottom square in (15) means that $\widetilde{\mathcal{F}}$ is equivariant with respect to $\widetilde{S}$ from (13) and Σ as in (14).

A good part of the geometric content in Theorem 4.5 is reflected in the following particular case. Assume V is obtained by Lagrangian suspension with respect to a loop of Hamiltonian diffeomorphisms, $\mathbf{g} = \{g_t\}$, $g_0 = g_1 = id$. This means that we consider a time dependent Hamiltonian $G : \mathbb{R} \times M \to \mathbb{R}$ that generates $\mathbf{g}$ (so that G is null for $|t|$ large) and we put $V = (t, G(t,x), \phi_t^G(x)) \subset \mathbb{R} \times \mathbb{R} \times M$. In this case, the class $\phi_V([L])$, with ϕ_V from (9), coincides with $S([\mathbf{g}]) * [L]$ where $*$ is the module action $* : QH(M) \otimes HF(L,L) \to HF(L,L)$.

4.2.2. *Lagrangian quantum homology.* Let $L \subset M$ be a montone Lagrnagian. Denote by $\Lambda = \mathbb{Z}_2[t^{-1}, t]$ the ring of Laurent polynomials in t, graded so that $|t| = -N_L$. (In case L is weakly exact, i.e. $\omega(A) = 0$ for every $A \in \pi_2(M, L)$ we put $\Lambda = K$.)

The Lagrangian quantum homology $QH(L)$ is the homology of a complex, $\mathcal{C}(\mathscr{D})$, called the pearl complex (see [**5**–**7**] for details). It is associated to a triple of auxiliary structures $\mathscr{D} = (f, (\cdot,\cdot), J)$ where $f : L \longrightarrow \mathbb{R}$ is a Morse function on L, $(\cdot,\cdot)$ is a Riemannian metric on L and J is an ω-compatible almost complex structure on M. With these structures fixed we have

$$\mathcal{C}(\mathscr{D}) = \mathbb{Z}_2\langle \mathrm{Crit}(f) \rangle \otimes \Lambda$$

and the differential of this complex counts so called pearly trajectories that consist of negative gradient flow lines of f with a finite number of points "replaced" with non-constant J-holomorphic disks as in Figure 7. The pearl complex is $\mathbb{Z}$-graded,

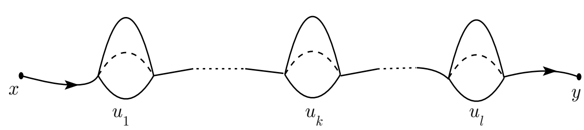

Figure 7. A pearly trajectory contributing to the differential dx of the pearl complex.

the degree corresponding to the critical points of f being given by their Morse index. The homology $H_*(\mathcal{C}(\mathscr{D}), d)$ is independent of $\mathscr{D}$ (up to canonical isomorphisms) and is denoted by $QH_*(L)$. Obviously this homology is also $\mathbb{Z}$-graded. A monotone Lagrangian L is called narrow if $QH(L) = 0$ and it is called wide if $QH(L) \cong H(L; K) \otimes \Lambda$ see [**7**].

It is possible to define a version of Floer homology $HF(L, L; \Lambda)$ with coefficients in Λ and there is an isomorphism, essentially due to Piunikin-Salamon-Schwartz [**28**],

$$PSS : QH_*(L) \cong HF_*(L, L; \Lambda) \ .$$

Further, in case L is exact, then $QH_*(L) \cong H_*(L; \mathbb{Z}_2)$.

Thus $QH(L)$ is just a variant of Floer homology. At the same time, this variant is well-adapted to studying "individual" cobordisms. Indeed, let $V : (L_1, \ldots, L_i) \rightsquigarrow (L'_1, \ldots, L'_k)$ be a cobordism. Consider a Morse function $f : V \to \mathbb{R}$ so that the function is linear along the ends of V. Assume, for instance, that the negative gradient of f (with respect to some metric on V) points "in" along the positive ends and points "out" along the negative ends. This is the typical picture of a function on a cobordism and the resulting Morse complex computes the singular homology $H(V; L'_1 \cup \ldots \cup L'_k; \mathbb{Z}_2)$. It is shown in [**8**] that by choosing an appropriate almost complex structure on $\mathbb{C} \times M$ one can define a pearl complex, again over Λ, associated to this Morse function f. The resulting quantum homology is denoted by $QH(V; L'_1 \cup \ldots \cup L'_k)$. Certainly, one can define similarly also the quantum homology $QH(V; L_1 \cup \ldots \cup L_i)$ as well as, by taking f so that its negative gradient points "in" along all the ends of V, $QH(V)$, and, if f points "out" along all the ends, $QH(V, \partial V)$. All these quantum homologies verify the expected dualities and other properties, just like their Morse counterparts, but more has to be true. Indeed, by Theorem 4.1 and its corollaries we know that the Floer homologies of the ends of a cobordism are related by a series of exact sequences. Given that Floer homology is related - via the PSS morphism - to quantum homology, the quantum homologies of the ends have to satisfy some stronger constraints compared to the respective Morse homologies. This is indeed the case and a prototypical example of this sort is next.

THEOREM 4.6. [**8**] *Let $L, L', L'' \in \mathcal{L}^*(M)$.*

i. *If $V : L \rightsquigarrow L'$ is a cobordism with $V \in \mathcal{L}^*(\mathbb{C} \times M)$, then $QH(V, L) = 0 = QH(V, L')$ and moreover $QH(L)$ and $QH(L')$ are isomorphic (via an isomorphism that depends on $[V]$) as rings. If additionally L and L' are wide, then the singular homology inclusions $H_1(L; \mathbb{Z}_2) \to H_1(V; \mathbb{Z}_2)$ and $H_1(L'; \mathbb{Z}_2) \to H_1(V; \mathbb{Z}_2)$ have the same image. When* $\dim(L) = 2$, *both these inclusions are injective and thus $H_1(L; \mathbb{Z}_2) \cong H_1(L'; \mathbb{Z}_2)$.*
ii. *Assume that $W : L \rightsquigarrow (L', L'')$ is a cobordism with $W \in \mathcal{L}^*(M)$. If $QH(L)$ is a field (in other words, each element in $QH(L)$ admits an inverse with respect to the quantum multiplication), then the inclusion $QH(L) \to QH(V)$ is injective. Moreover, for each k we have the inequality:*

$$rk(QH_k(L)) \leq |rk(QH_k(L_1)) - rk(QH_k(L_2))| \ .$$

REMARK 4.7. For this result the condition on π_1 in the definition of $\mathcal{L}^*$ is not actually necessary.

An interesting particular case is when all Lagrangians and the cobordisms relating them are exact. In that case all the quantum homologies coincide with the respective singular homologies so that, for instance, the first point means that if V is an exact cobordism with a single exact positive end L and a single exact negative end L', then $L \to V$ and $L' \to V$ are homology equivalences over $\mathbb{Z}_2$. In case this homology equivalence could be extended over $\mathbb{Z}$ and assuming in addition that L, L', V are simply connected and $n \geq 5$, we deduce from the h-cobordism theorem that V is diffeomorphic to a trivial cobordism. A Lagrangian cobordism that is diffeomorphic to a cylinder is called a Lagrangian pseudo-isotopy. All of this sugests the following conjecture: an exact Lagrangian cobordism with one positive end that is exact and one negative end, also exact, is a pseudo-isotopy. An important step in this direction has been made recently by Suarez [**33**]: she shows that an exact Lagrangian cobordism as before that is also spin and so that the maps $\pi_1(L) \to \pi_1(V)$, $\pi_1(L') \to \pi_1(V)$ are isomorphisms is indeed a pseudo-isotopy. Besides adjusting the arguments in Theorem 4.6 i so as to take into account orientations, her proof makes use of the Floer-theoretic Whitehead torsion introduced in [**34**] and of the s-cobordism theorem.

An even stronger conjecture seems believable (but is, for the moment, intractable): a Lagrangian cobordism $V : L \rightsquigarrow L'$ with V, L, L' exact, is horizontally isotopic to a Lagrangian suspension.

5. Sketch of the construction of $\widetilde{\mathcal{F}}$

We divide the presentation in two subsections: in the first we explain the basic principles that are behind the machinery involved here; in the second subsection we list the main steps of the proof of Theorem 4.1.

5.1. Ingredients in elementary form.

5.1.1. *Compactness and the open mapping theorem.* The first indication that rigidity can be expected to play a significant role in the study of Lagrangian cobordisms - *under the assumption of monotonicity* - appeared in a paper of Chekanov [**11**]. His result is the following: assume that $V : (L_1, \ldots, L_k) \rightsquigarrow (L'_1, \ldots, L'_s)$ is a montone cobordism so that V is connected. Then all the L_i's and L'_j's are monotone with the same monotonicity constant ρ, and moreover, they all have the same invariant d_L (see §2.2).

The monotonicity part of the claim is easy because the two morphisms: $\omega, \mu : \pi_2(M, L_i) \to \mathbb{R}, \mathbb{Z}$ are both seen to factor via $\omega, \mu : \pi_2(\mathbb{C} \times M, V) \to \mathbb{R}, \mathbb{Z}$. The equality of the d_L's is much more interesting. For instance, it implies that if two monotone Lagrangians L, L' with the same monotonicity constant have $d_L \neq d_{L'}$, then they can not intersect in a single point. Indeed, by the surgery results from §2.3 two such Lagrangians are the end of a cobordism obtained as the "trace" of the surgery in the single intersection point.

Here is the argument for the equality of the d_L's. First, fix some almost complex structure $\widetilde{J}$ on $\mathbb{C} \times M$ so that, outside a set $K \times M$ where $K \subset \mathbb{C}$ is compact, the projection $\pi : \mathbb{C} \times M \to \mathbb{C}$ is $\widetilde{J} - i$ holomorphic. We take K large enought so that $\pi(V)$ equals a union of horizontal lines outside of K as in Figure 8. Recall that d_L counts the number $\in \mathbb{Z}_2$ of J-holomorphic disks of Maslov 2 through any (generic) point of L, in particular this number is independent of the point in L chosen to estimate it. We apply this remark to V and $\widetilde{J}$. Pick one point P that belongs to

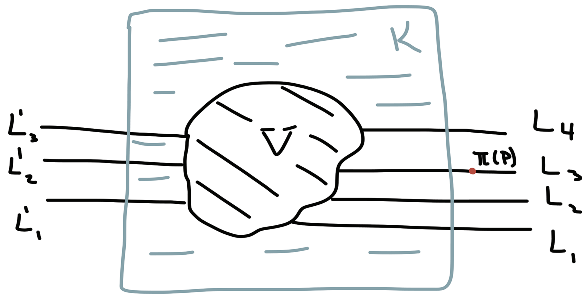

FIGURE 8. The projection $\pi : \mathbb{C} \times M \to M$ is $\widetilde{J} - i$ holomorphic outside of K.

an end of V: $P \in [a, +\infty) \times \{i\} \times L_i \subset V$, and is so that $p = \pi(P) \notin K$ (see §2.1). Consider a $\widetilde{J}$-holomorphic disk with boundary on V, $u : (D^2, S^1) \to (\mathbb{C} \times M, V)$ with $P \in u(S^1)$. Put $v = \pi \circ u$. There is an open set $U \subset D^2$ whose image by v avoids K. Let $v' = v|_U : U \to \mathbb{C} \backslash K$. In particular, v' is holomorphic. As it goes through $p \notin K$ and $\pi(V)$ is a union of horizontal lines outside of K, it is easy to see that, by the open mapping theorem, v' is constant. But this implies that v is constant and thus u has values in the fiber over p. Thus, u is actually a map $u : (D^2, S^1) \to (M, L_i)$. Assuming that the restriction of $\widetilde{J}$ to the fibre over p is regular (which is easy to arrange) the conclusion is that $d_{L_i} = d_V$.

Refinements of this argument are crucial in all the results discussed in this paper. The basic idea is to use again specific almost complex structures as $\widetilde{J}$ before so as to restrict the admissible behaviour of the $\widetilde{J}$-holomorphic curves that are used in the definition of the Floer differential as well as in the other μ^k's. This serves two purposes: it establishes compactness for the respective moduli spaces and, secondly, gives a particular form to the algebraic structures in question. As an example consider again Figure 5. Here is briefly how the definition of $\phi_V : HF(L, L) \to HF(L, L')$ follows from these types of arguments. First we pick $\widetilde{J}$ so that π is $\widetilde{J} - i$ holomorphic outside of a compact set K very close to the "bulb" of V in the picture. In particular, the intersection points of $\gamma \times L$ with V are outside of $K \times M$. We then define the Floer complex $CF(\gamma \times L, V)$. The only issue with this definition is to make sure that the moduli spaces of $\widetilde{J}$-holomorphic strips is compact. But our choice of $\widetilde{J}$ together with a simple application of the open mapping theorem, as before, implies easily this compactness. As a vector space $CF(\gamma \times L, V)$ is isomorphic to $CF(L, L) \oplus CF(L, L')$. The differential in this complex is therefore a matrix:

$$D = \begin{pmatrix} d_1 & \phi \\ \psi & d_2 \end{pmatrix} . \tag{16}$$

If u is a holomorphic Floer strip contributing to D we let $v = \pi \circ u$ and notice that v is holomorphic outside K. In particular, it is holomorphic around the points where γ intersects $\pi(V)$. In view of this, using the open mapping theorem again as well as easy orientation arguments it is easy to deduce that d_1 is the differential in $CF(L, L)$, d_2 is the differential in $CF(L, L')$ and $\psi = 0$. Therefore, $D^2 = 0$ implies that ϕ is a chain morphism and we put $\phi_V = H(\phi)$.

5.1.2. *Using Hamiltonian deformations lifted from* $\mathbb{C}$. The second basic principle behind many of our proofs is that the algebraic structures defined here - in particular $HF(-,-)$ - are invariant with respect to horizontal isotopy and that by using various horizontal isotopies lifted from $\mathbb{C}$ one can get a variety of interesting relations.

To exemplify how this principle is applied in practice we focus again on a situation similar to that in Figure 5 but this time in a simpler situation, when $k = 1$. In other words, we have a cobordism $V : L \rightsquigarrow L'$, $V \in \mathcal{L}^*(M)$ and we would like to notice that in this case the morphism $\phi_V : HF(L, L) \to HF(L, L')$ is in fact an isomorphism (this is, of course, a very particular case of Theorem 4.1). For this purpose consider a second curve γ' as in Figure 9. It is clear that γ' and

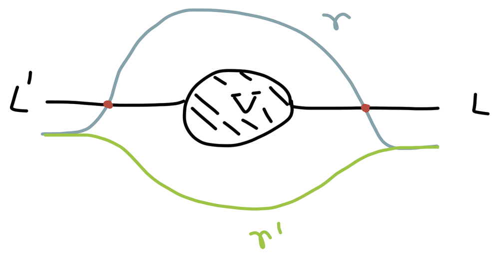

FIGURE 9. $\gamma \times L$ and $\gamma' \times L$ are horizontally isotopic.

γ are horizontally isotopic in the plane. Therefore, $\gamma \times L$ is horizontally isotopic to $\gamma' \times L$. We deduce $HF(\gamma \times L, V) \cong HF(\gamma' \times L, V) = 0$ because $\gamma' \times L \cap V = \emptyset$. But this means that the component ϕ of D in (16) is a quasi-isomorphism.

5.2. Outline of the proof of Theorem 4.1.

5.2.1. *The Fukaya category of cobordisms.* The fundamental step, and the one of highest technical difficulty, is to define a Fukaya category of cobordisms in $\mathbb{R}^2 \times M$ which we denote $\mathcal{F}uk^*_{cob}(\mathbb{R}^2 \times M)$. The objects in this category are therefore cobordisms $V \in \mathcal{L}^*(\mathbb{C} \times M)$ and the morphisms Floer chains $CF(V, V')$. The construction follows the machinery in Seidel's book [**32**] that is truly fundamental here. In particular, to deal with cobordisms that are non-transversal we use moduli spaces of curves verifying Cauchy-Riemann equations perturbed by Hamiltonian terms. One difference with the construction in [**32**] is that we work in a monotone setting and not an exact one. However, by arguments such as in, for instance, [**7**], the resulting issues are easily disposed off. A much more serious difficulty has to do with the compactness of the relevant moduli spaces, basically in continuation of the discussion in §5.1.1. The key issue is seen by looking to the presumtive morphisms from a cobordism V to itself. Thus we are considering the Floer chains $CF(V, V)$. Clearly, to be able to define such chains we need to use Hamiltonian perturbations that are non-compact. But this means that the curves u in our moduli spaces do not have the property that $v = \pi \circ u$ is holomorphic away from a compact set. Indeed, these v's satisfy themselves some perturbed Cuachy-Riemann equations and the open mapping theorem does not apply to them directly. There are probably a variety of solutions to this issue but the one found in [**9**] is to pick very carefully the Hamiltonian perturbations so that the curves v can still be transformed by a change of variable - away from a large compact set - to holomorphic curves to which the open mapping theorem again applies.

5.2.2. *Inclusion, triangles and* $\widetilde{\mathcal{F}}$. Once the category $\mathcal{F}uk^*_{cob}(\mathbb{R}^2 \times M)$ is defined the proof proceeds as follows. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a curve in the plane with horizontal ends. There is an induced functor of A_∞-categories:

$$\mathcal{I}_\gamma : \mathcal{F}uk^*(M) \to \mathcal{F}uk^*_{cob}(\mathbb{R}^2 \times M)$$

defined on objects by $\mathcal{I}_\gamma(L) = \gamma \times L$.

Fix a cobordism $V : L \rightsquigarrow (L_1, \ldots, L_k)$ as in Figure 10. Let $\mathcal{M}_V$ be the Yoneda module associated to V as in (7) but for the category $\mathcal{F}uk^*_{cob}(\mathbb{R}^2 \times M)$. By using the functor $\mathcal{I}_\gamma$ we can pull back this module to a module $\mathcal{M}^\gamma_V$ over $\mathcal{F}uk^*(M)$, $\mathcal{M}^\gamma_V = \mathcal{I}^*_\gamma(\mathcal{M}_V)$. At the derived level, this module only depends on the horizontal isotopy classes of V and γ. We consider a particular set of curves $\alpha_1, \ldots, \alpha_k \subset \mathbb{R}^2$ basically as in Figure 10. Therefore, we get a sequence of modules $\mathcal{M}_{V,i} := \mathcal{M}^{\alpha_i}_V$, $i = 1, \ldots, k$.

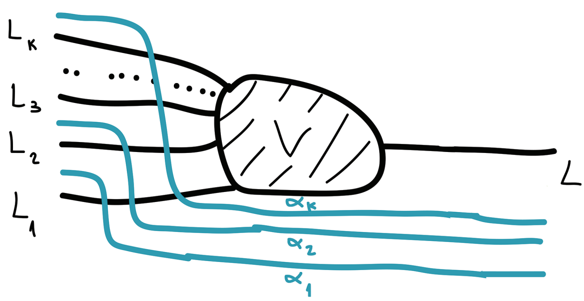

FIGURE 10. A cobordism V together with curves of the type α_i's.

We then show that these modules are related by exact triangles (in the sense of triangulated A_∞ categories):

$$T^{-1}\mathcal{M}_{L_s} \to \mathcal{M}_{V,s-1} \to \mathcal{M}_{V,s} \to \mathcal{M}_{L_s} \ \ \forall\, 2 \le s \le k. \tag{17}$$

and that, moreover, there is a quasi-isomorphism $\phi_V : \mathcal{M}_L \to \mathcal{M}_{V,k}$. This point is certainly the heart of the proof and we will not attempt to explain it here besides indicating that, in essence, the exact triangles are deduced from arguments that eliminate certain behaviour of J-holomorphic polygons, somewhat similarly to how we noticed that the application ψ from (16) vanishes.

Once these exact triangles are established, the definition of $\widetilde{\mathcal{F}}$ is relatively direct, by translating the preceeding structures to the derived setting.

REMARK 5.1. It is an open question at this time how much the results described here - in particular, the construction of the functor $\widetilde{\mathcal{F}}$ and the morphism Θ from (1) - can be extended beyond the montone case. Certainly, there are major technical difficulties with such an extension but this is not only a technical issue. Indeed, Theorem 4.1 implies that, for instance, if V is a monotone cobordism $V : L \rightsquigarrow L'$, then L and L' verify $HF(L, L) \cong HF(L', L')$. Assuming that a reasonable notion of Floer homology $HF(-,-)$ is defined in full generality the same argument would apply even if V is not monotone. But, as seen in our "flexibility" subsection §2.3, constructing general cobordisms V is easy without requiring monotonicity. As a consequence significantly different Lagrangians L and L' would have the same

$HF(-,-)$. In short, we are here in front of an example of precisely the tension *rigidity-flexibility* that was mentioned at the beginning of the paper: any invariant of type HF that is defined in great generality can be expected to be quite weak.

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