Metrics on spaces of Lagrangian submanifolds.

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(joint work with Paul Biran and Jun Zhang)

ETH - LP 60

Music: LP 33 1/3 = Long Play, thirty three and a third rotations per minute (refers to discs for turntables)

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Mathematics: LP 60 = Leonid Polterovich at 60, many more rotations per minute (dynamics and symplectic topology)

Metrics on spaces of Lagrangian submanifolds.

(M, ω) is a Weinstein domain, $\omega = d\lambda$

$$\underline{\mathcal{L}}ag(M) = \{ (L, f_L) : i_L : L^n \to M^{2n}, \text{ embedding}, (i_L)^* \lambda = df_L \}$$

(L closed)

Main result: If M satisfies a global finiteness condition (to be made explicit later), then

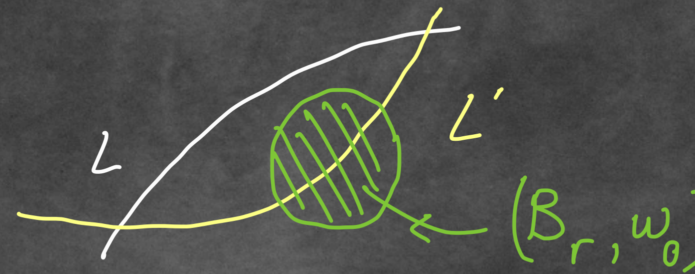
$\underline{\mathcal{L}}ag(M)$ carries a family of metrics d (called fragmentation metrics)

with the following properties:

i. [spectrality] $d(L, L') \leq \gamma(L, L')$ (= the spectral metric)

ii. [quantitative non-degeneracy] $d(L, L') \geq B(L, L')$ (= a relative Gromov width type quantity relative to L and L′)

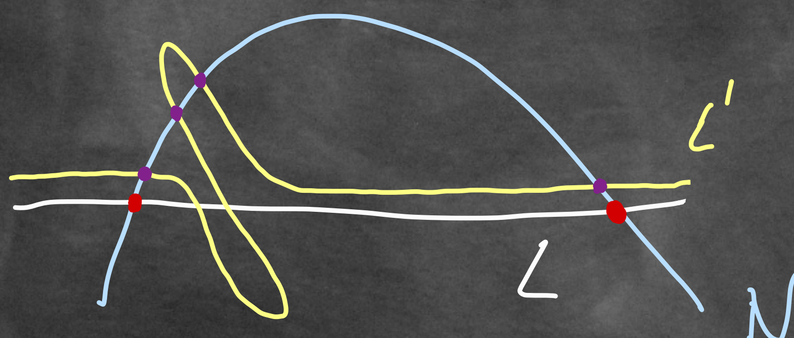
$$B(L, L') \approx \sup \left\{ \frac{\pi r^2}{2} : \right.$$



$\left. \right\}$

iii. [geometricity] $N, L \in \underline{L}ag(M)$, N transverse to L.

$\exists\ \varepsilon_{L,N}$ such that if $L' \in Lag(M)$, $d(L, L') \leq \varepsilon_{L,N}$, then $\#(L' \cap N) \geq \#(L \cap N)$

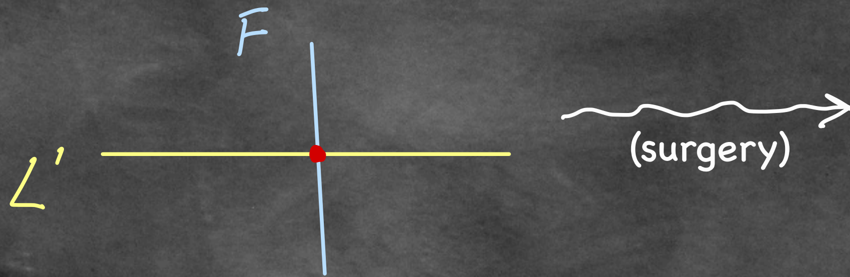


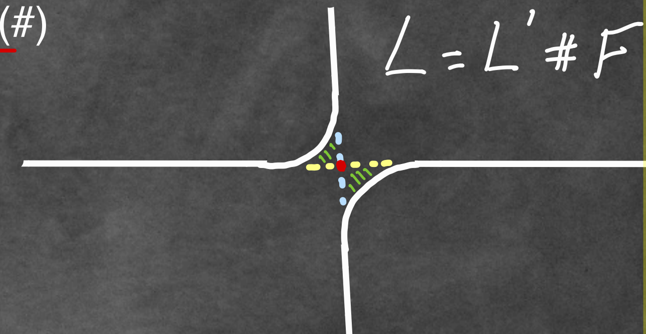
$$\#(L' \cap N) > \#(L \cap N)$$

The name fragmentation is due to:

iv. [surgery] $L = L' \# F$ (where # is Lagrangian surgery),

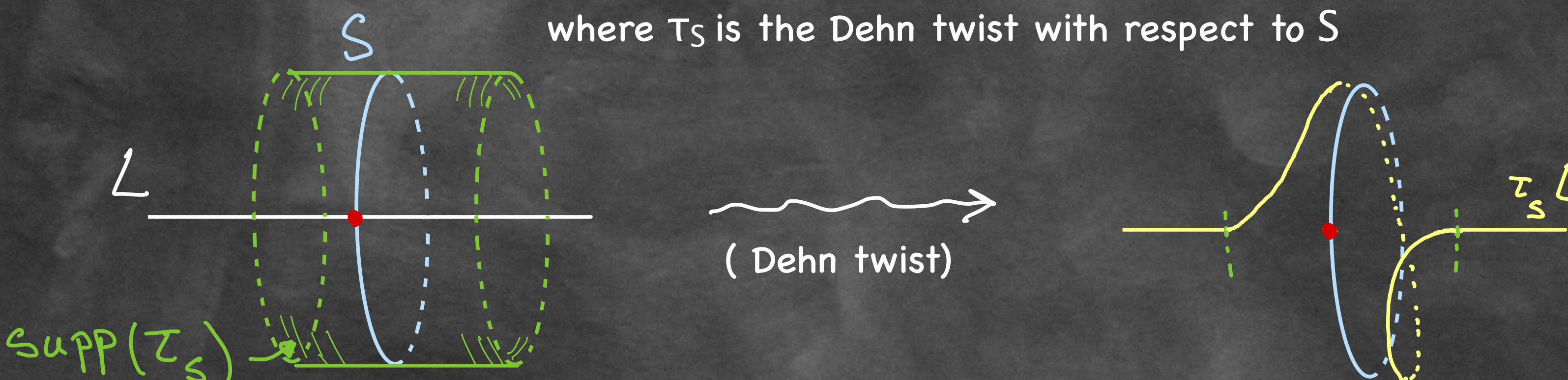
then $d(L,L') \leq d(F,0) + w(\#)$





v. [twist] $d(L, \tau_S L) \leq \#(L \cap S)\, d(S, 0) + w(Supp(\tau_S))$

where τ_S is the Dehn twist with respect to S



Remarks. DFuk(M) = derived Fukaya category with objects $\underline{L}ag(M)$ - as constructed in Seidel's book.

a. Finiteness condition in the statement:

DFuk(M) has a finite set of triangular generators.

b. If $\Delta : A \to B \to C \to TA$ is exact in DFuk(M), then

$$d(B,C) \leq d(A,0) + w(\Delta) \quad \text{(this extends iv, v).}$$

c. If the triangle Δ is associated to a cobordism V with ends A,B,C then

$$w(\Delta) \leq \text{Shadow}(V)$$

c. $\phi \in \text{Symp}(M)$, $\|\phi\| = \sup \{ d(\phi(L), L) : L \in \underline{L}ag(M) \}$ defines a group norm on Symp(M) (in some cases finite).

Aim of the talk is to explain the construction of this class of metrics.

II. Construction of the fragmentation metrics.

Step 1 : geometry (would work just like this in an ideal world).

Will use a larger family: $\underline{L}ag(M) \subset \underline{L}ag^{imm}(M)$

$\underline{L}ag^{imm}(M)$ consists of triples (L, f_L, P_L) :

- $i_L : L \to M$ is an immersion with self-transverse double points, $(i_L)^*\lambda = df_L$

Denote by Σ_L the set of double points of $L \subset L \times L$

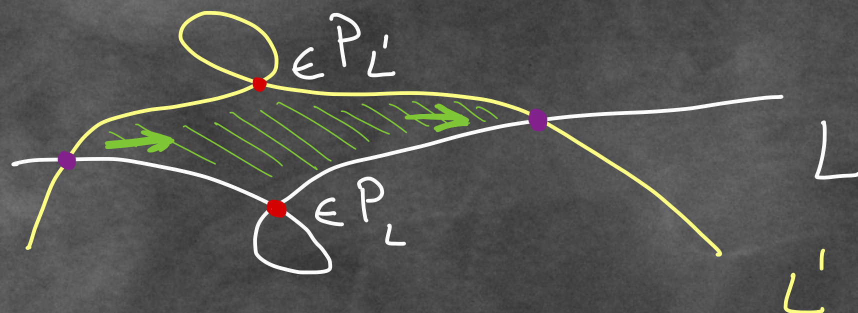
- $P_L \subset \Sigma_L$ is a subset of double points such that L is "unobstructed".

Unobstructed $\Rightarrow$ For $L, L' \in \underline{L}ag^{imm}(M)$ may define the FLoer complex

$$CF(L,L') = (Z_2 < L \cap L' >, \partial)$$

Remarks. a. The definition of the Floer complex depends on additional data (an almost complex structure J, perturbations etc).

b. The differential ∂ counts (perturbed) J - holomorphic strips with possible additional punctures belonging to P_L and to $P_{L'}$.



c. The Floer complex $CF(L,L)$ has as generators the critical points of a Morse function on L together with the points in Σ_L each of them appearing twice.

d. The construction is delicate: Floer, Hofer, Salamon, Oh, Fukaya-Oh-Ohta-Ono, Akaho, Akaho-Joyce, Alston-Bao, Biran-C. [Lagrangian Pictionary, '21]

The complex $CF(L,L')$ is filtered:

$$\dots CF^{\leq\alpha}(L,L') \subset CF^{\leq\beta}(L,L') \subset \dots \qquad \alpha\leq\beta$$

With: $CF^{\leq\alpha}(L,L') = Z_2 < x\in L\cap L' : A(x)\leq\alpha >$ where $A(x) = f_{L'}(x) - f_L(x)$

∂ respects the filtration $\Rightarrow$ $HF(L,L')$ (Floer homology) is a persistence module

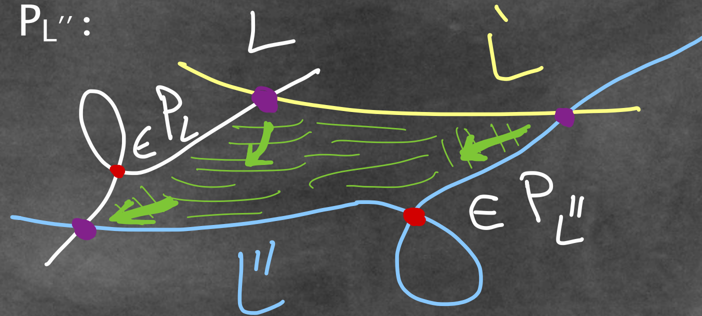
This is a family of vector spaces $M^\alpha, \alpha \in R$ and linear maps $i_{\alpha,\beta} : M^\alpha \to M^\beta$, $\alpha\leq\beta$ such that $i_{\alpha,\alpha} = id$, $i_{\alpha,\gamma} = i_{\beta,\gamma} \circ i_{\alpha,\beta}$.

In our case $M^\alpha = HF^\alpha(L,L') = H(CF^{\leq\alpha}(L,L') , \partial)$

There is also a composition - the Donaldson product:

$$\circ : CF (L, L') \otimes CF (L', L'') \to CF (L, L'')$$

This is defined by counting (perturbed) J-holomorphic :triangles with possible additional punctures in P_L, $P_{L'}$, $P_{L''}$:



$\circ$ is filtered and descends to an associative operation of persistence modules:

$$\circ \ : \ HF^{\alpha}(L,L') \otimes HF^{\beta}(L,L') \ \rightarrow \ HF^{\alpha+\beta}(L,L'')$$

Thus: $\underline{L}ag^{imm}(M)$ are the objects of a category with morphisms persistence modules $Mor(L,L') = HF(L,L')$ and composition $\circ$ that respects the persistence structure.

Such a category C is called a persistence category.

For us: $Obj(C) = \underline{L}ag^{imm}(M)$, $Mor_C(L,L') = HF(L,L')$, $\circ$ is the Donaldson prod.

Remarks. a) This means that filtered algebra is the natural setting for this subject.

b) Filtrations were essential for energy bounds. Gained intrinsic interest through spectral invariants (Viterbo, Schwartz, Oh).

c) Persistence language introduced to symplectic topology by Leonid + Egor Shelukhin, Maia Fraser, Michael Usher, Usher+Zhang.

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d) Special thanks to Leonid for a big contribution - with big impact on the subject and on my own life - students: Paul (my closest collaborator, and friend), Egor (close colleague, and collaborator), Yaron, Frol, Vukasin,...

There are two other categories associated to a persistence category C (category with morphisms persistence modules and composition $\circ$ that respects the persistence structure):

- C_0 has the same objects as C, $Mor_{C_0}(L, L') = Mor_C{}^0(L, L')$

- C_∞ same objects as C, $Mor_{C_\infty}(L,L') = \lim Mor_C{}^\alpha(L,L')$

In our case ($Obj(C) = Lag^{imm}(M)$ etc) C has two additional structures:

i. C_0 is triangulated with triangles induced by:

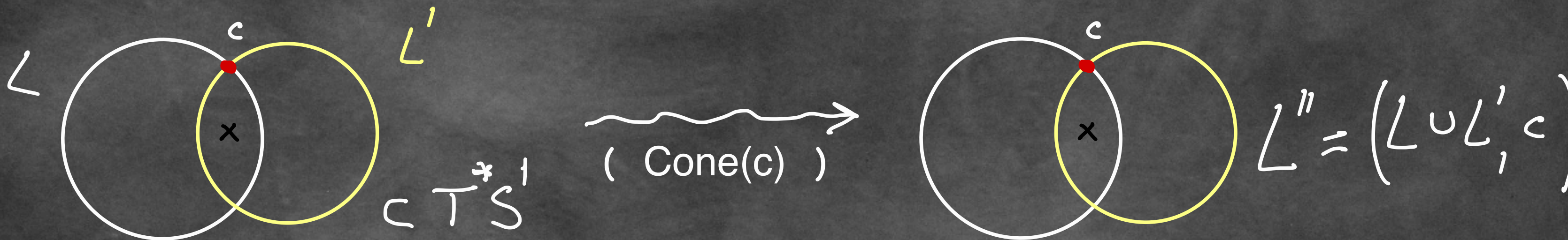
$$L \xrightarrow{[c]} L \to Cone(c) = (\, L \cup L',\, P_L \cup P_{L'} \cup \{c\}\,) = L''$$

Here $[c] \in HF^0(L,L')$, $c \in CF^{\leq 0}(L,L')$ (this is key for L'' to be unobstructed)

$c = P_1+P_2+\ldots P_k$, is a cycle in $CF^{\leq 0}(L,L')$, $P_i \in L \cap L'$, $\{c\} =\{ P_1,P_2,\ldots,P_k \}$

In $L'' = \mathrm{Cone}(c) = (\, L \cup L',\, P_L \cup P_{L'} \cup \{c\}\,)$ we have:

$c = P_1 + P_2 + \ldots P_k$, is a cycle in $CF^{\leq 0}(L, L')$, $P_i \in L \cap L'$, $\{c\} = \{P_1, P_2, \ldots, P_k\}$



ii. C carries shift functors $\Sigma^r : C \rightarrow C$ induced by:

$$\Sigma^r : (L, f_L, P_L) \rightarrow (L, f_L + r, P_L)$$

A persistence category C with C_0 triangulated and with shift functors (+...) is called a triangulated persistence category (TPC).

Theorem: The category C with objects $\underline{L}\mathrm{ag}^{imm}(M)$ as defined before is a TPC.

Step 2: algebra. This part concerns a TPC $\mathcal{C}$

(thus $\mathcal{C}$ is a persistence category such that $\mathcal{C}_0$ is triangulated …)

An r-acyclic $K \in Obj(\mathcal{C})$ is an object so that $id_K \in Mor^0(K, K) \to 0 \in Mor^r(K, K)$

The r-acyclics (for all r) form a sub TPC $\mathcal{AC} \subset \mathcal{C}$.

It is not difficult to see that the Verdier localization $\mathcal{C}_0/\mathcal{AC}_0$ equals $\mathcal{C}_\infty$

$\Rightarrow$ $\mathcal{C}_\infty$ is triangulated.

Theorem: $\mathcal{C}_\infty$ carries a triangular weight w induced by the persistence structure.

A triangular weight w on a triangulated category associates to each exact triangle Δ a weight $w(\Delta) \in [0, \infty)$ such that a weighted version of the octahedral axiom is true.

Assume that a triangulated category D carries a triangular weight w .

Fix $F \subset Obj(D)$. For any two $X, X' \in Obj(D)$ put:

$$\delta^F(X, X') = \inf \{ \sum_i w(\Delta_i) : \ <\Delta_i>_i \text{ exact decomposition of } X \text{ rel } X' \}.$$

Such an exact decomposition is given by exact triangles Δ_i , $1 \leq i \leq m$

$$\Delta_1 : \ X_1 \to Y_0 \to Y_1 \to TX_1$$

. . . .

$$\Delta_i : \ X_i \to Y_{i-1} \to Y_i \to TX_i$$

. . . .

$$\Delta_m : \ X_m \to Y_{m-1} \to Y_m \to TX_m$$

With $Y_0 = 0$, $Y_m = X$, $X_i \in F$ except for one index j for which $X_j = T^{-1}X'$.

The weighted octahedral axiom implies the triangle inequality:

$$\delta^F(X, X'') \leq \delta^F(X, X') + \delta^F(X', X'') \,.$$

Symmetrize δ^F to obtain a "fragmentation" pseudo-metric on $Obj(D)$:

$$d^F(X, X') = \sup\{ \delta^F(X, X'), \delta^F(X', X) \} \,.$$

This is a pseudo-metric and takes finite values if F is a family of triangular generators of D .

Can define additional pseudo-metrics. For instance:

$$d^{F,F'}(X,X') = \sup\{ d^F(X,X') \,,\, d^{F'}(X,X') \}$$

where F' is another family of objects of D .

Step 3: Return to Lagrangians.

The triangulated persistence category $\mathcal{C}$ of interest has the properties:

- Objects $\underline{L}ag^{imm}(M)$
- Morphisms $Mor_{\mathcal{C}}(L,L') = HF(L,L')$
- Composition $\circ$ is the Donaldson product
- In $\mathcal{C}_0$ morphisms preserve filtration
- $DFuk(M) \subset \mathcal{C}_\infty$ as triangulated subcategory

$\Rightarrow$ $DFuk(M)$ carries a triangular weight

Pick $\mathcal{F}$ a finite family of triangular generators and $\mathcal{F}'$ a generic Hamiltonian perturbation.

The metric we are searching for is $d = d^{\mathcal{F},\mathcal{F}'}$ constructed at Step 2.

The metric d can be estimated as follows.

In a TPC it is easy to define the interleaving distance d_{int} (similar to the definition for persistence modules). For $L \in \underline{L}ag(M)$ we put:

$$Q(L) = \inf \{ d_{int}(L,N) \; : \; N = \cup_i F_i, \; F_i \in \mathcal{F}, \; N \in \underline{L}ag^{imm}(M) \}$$

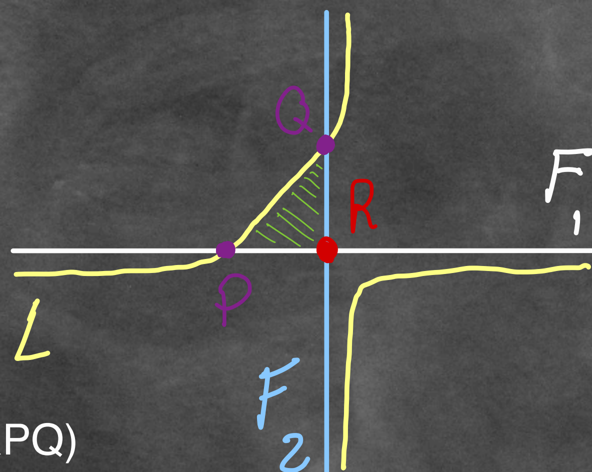
We then have: $\frac{1}{4} Q(L) \leq d^{\mathcal{F}}(L, 0) \leq 4\, Q(L)$

An example:

$\mathcal{F} = \{F_1, F_2\}$

$N = F_1 \cup F_2$

$d_{int}(L,N) \leq 2 \text{ Area } (RPQ)$



Plumbing

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Once upon a time - around 2001 - I was giving a talk and I noticed a fellow in the front row who looked highly skeptical, like whatever I was saying was wrong.

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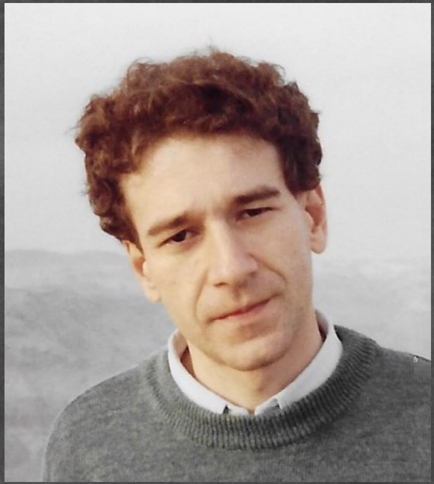

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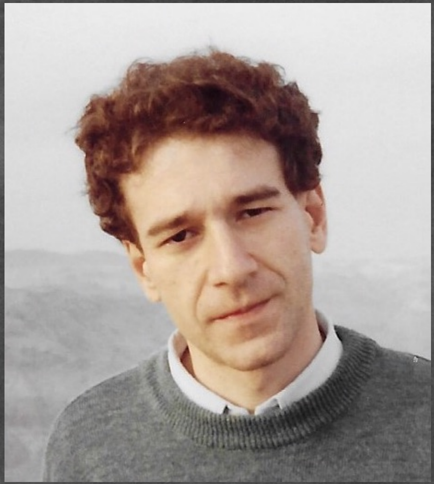

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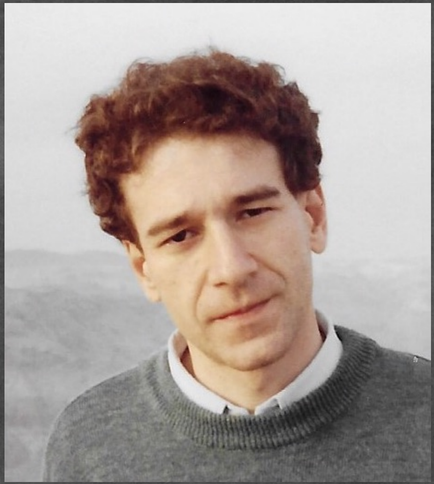

After the talk he wanted to chat and had many interesting things to say, and the other Leonid, who became a friend along the years, showed up:

This brings us back to today's talk: a good dose of skepticism is justified (and I am sure that, under normal circumstances, Leonid would give me the VERY SERIOUS look).

III. How the geometry really works.

Significant difficulties dealing with immersed Lagrangians and establishing the triangulated structure.

Instead:

- construct a filtered Fukaya A_∞ category. (Biran-C.-Zhang, Ambrosioni '23)
    - strict units are required - need to use "cluster" type moduli spaces mixing Morse flow lines and J-curves.
    - need to only start with a finite family X of embedded Lagrangians in generic position - this is used for energy estimates.
    - solve some filtered A_∞ issues, in particular invariance.
    - deduce a TPC associated to X - DFuk(X) (homological category of filtered twisted modules over the family X) and a metric on X.
- extend the metric on all of $\underline{\mathcal{L}}ag(M)$ by enlarging X.