Aspects of Lagrangian Topology.

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Symplectic manifolds and Lagrangian submanifolds.

(M^{2n}, ω) symplectic $\Leftrightarrow$ ω 2-form, $d\omega = 0$, ω non-degenerate.

$L^n \hookrightarrow M$ submanifold - in this talk, compact, closed.

$$L \text{ Lagrangian} \iff \omega|_L \equiv 0 \ .$$

Examples.

a. $\mathbb{C}$; $\omega_0 = dx \wedge dy$; $\mathbb{R} \subset \mathbb{C}$ or $S^1 \subset \mathbb{C}$.

b. $\mathbb{C}^n$; $\omega_0 = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$; $\mathbb{R}^n \subset \mathbb{C}^n$.

c. $\mathbb{R}P^n \hookrightarrow \mathbb{C}P^n$; $\mathbb{T}^n_{cliff} \subset \mathbb{C}P^n$,
$\mathbb{T}^n_{cliff} = \{[z_0 : z_1 : \ldots : z_n] : |z_0| = |z_1 = \ldots = |z_n|\}$.

d. $N \hookrightarrow T^*N$.

Pairs $L \hookrightarrow (M, \omega)$ appear in classical mechanics, string theory, algebraic geometry, complex analysis etc...

Flexibility

LagrangianTopology

Rigidity

Relation with complex analysis (*Gromov* '85):

(M,ω) symplectic $\Rightarrow \exists\ J : TM \to TM$ almost complex structure compatible with ω ($\Leftrightarrow\ J^2 = -Id,\ \omega(-, J-)$ is a Riemannian metric).

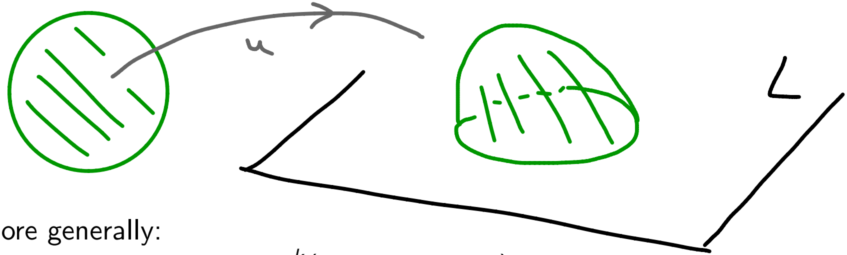
Example. $i : \mathbb{C}^n \to \mathbb{C}^n$.

J a. c. structure $\Rightarrow$ Cauchy-Riemann operator:

$$\bar{\partial}_J(-) = \frac{1}{2}\left[\frac{\partial}{\partial s}(-) + J\frac{\partial}{\partial t}(-)\right] .$$

Moduli spaces: $\alpha \in \pi_2(M, L)$

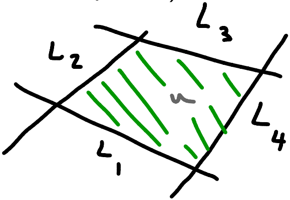
$$\mathcal{M}(\alpha, J; L) = \{u : (D^2, S^1) \to (M, L) : \bar{\partial}_J(u) = 0, [u] = \alpha\}$$



More generally:

$$\mathcal{M}^k(\alpha, J; L_1, \ldots, L_s)$$

formed by u, $[u] = \alpha$ with domain D^2 with k - boundary punctures (or, alternatively, k marked points) and with boundary conditions along $L_1, L_2..., L_s$.



Assuming regularity $\mathcal{M}$ is a manifold $\Rightarrow$ admits Gromov compactification as manifold with boundary $\Rightarrow$ various invariants.

a. <u>Floer Homology</u> (*Floer* '88 using work of *Gromov* '85, continued by *Hofer*, *Salamon*, *Oh*, *Fukaya*, *Fukaya-Oh-Ohta-Ono* etc):

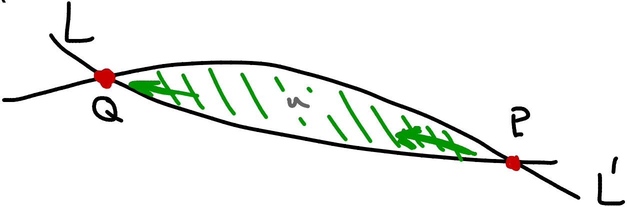
$L, L' \subset M$ Lagrangians, $L \cap L'$ transverse.

$$CF(L, L') = \mathbb{Z}_2 < L \cap L' > \quad \text{with differential}$$

$d : CF(L, L') \to CF(L, L')$ that counts J-holomorphic strips:

$$dP = \sum \# \mathcal{M}^2(\alpha, J; L, L'; P, Q) \ Q$$

α is so that $\mathcal{M}^2(\alpha, J; L, L')$ is 0-dimensional; the punctures are sent to P and Q.



$d^2 = 0$ because of the structure of the compactification:

$$\partial\overline{\mathcal{M}}^2(P,R) = \bigcup \mathcal{M}^2(P,Q) \times \mathcal{M}^2(Q,R) \tag{1}$$

$\Rightarrow \quad HF(L,L') = H(CF(L,L'),d).$

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Two properties make HF a legitimate invariant:

i. $HF(L,L')$ is independent of J

ii. If $\phi : M \to M$ is a Hamiltonian isotopy, then:

$$HF(\phi(L),L') \cong HF(L,\phi(L')) \cong HF(L,L') \ .$$

b. Constraints needed to define $HF(L, L')$.

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ii. Monotone case: $\exists\ \rho > 0, \quad \hat{\omega} = \rho\mu$ and $N_L \geq 2$.
$HF(L, L')$ well defined if both L and L' are monotone with same ρ + additional condition.

Possible that: *L ham isotopic to L' and* $HF(L, L') = 0$.
Example: $S^1 \subset \mathbb{C}$, $HF(S^1, S^1) = 0$.

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iii. General case: the algebra is different (*Fukaya-Oh-Ohta-Ono*'04 -'09, *Lalonde-C.*'05); the analysis is *highly difficult* - still being perfected (*Fukaya-Oh-Ohta-Ono*, *Hofer-Wysocki-Zehnder*, *McDuff-Werheim*).

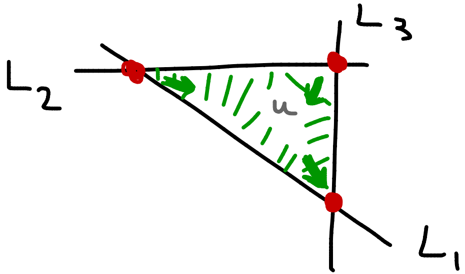
We work from now on in the monotone setting - in a uniform way.

c. The triangle product.

$L_1, L_2, L_3 \subset M$ Lagrangians in general position.

$$* : CF(L_1, L_2) \otimes CF(L_2, L_3) \to CF(L_1, L_3)$$

given by counting J-holomorphic triangles $\in \mathcal{M}^3(J; L_1, L_2, L_3)$



Product *is associative in homology* $\rightsquigarrow$ (due to *Donaldson* '93) the Donaldson category, $\mathcal{D}on(M)$.

$\mathcal{D}on(M)$ has as objects $L \in Lag(M)$ (assuming uniform monotonicity)

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Kontsevich '97: Use $\mathcal{F}uk(M) \quad \rightsquigarrow$

$$\rightsquigarrow \ \textit{triangulated completion of } \mathcal{D}on(M) =$$

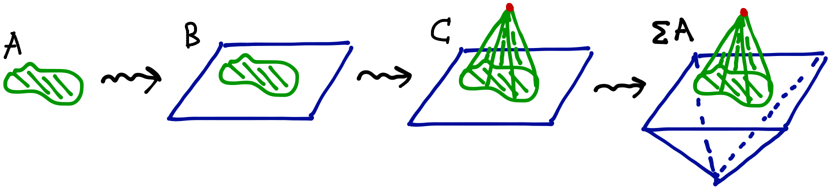
$$= D\mathcal{F}uk(M) \ .$$

These structures are described in *Fukaya-Oh-Ohta-Ono* '09 (and earlier) and *Seidel* '06.

d. Triangulated categories and K_0.

A category $\mathcal{C}$ is triangulated (*Verdier* '63, *Dold-Puppe* '61) if it has a class of *exact (or distinguished) triangles* subject to axioms similar to the properties of cofibrations sequences in topology:

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A, \quad C = B \cup_f CA$$



$\mathcal{C}$ triangulated $\Rightarrow$

- can decompose objects by iterated triangles.

- Grothendieck group

$$K_0(\mathcal{C}) = \mathbb{Z}_2 < O \in \mathcal{O}b(\mathcal{C}) > /\mathcal{R}' \ .$$

Relations $\mathcal{R}'$ are generated by:

$$A \to B \to C \text{ exact triangle} \Rightarrow A - B + C \in \mathcal{R}' \ .$$

Remark

The category $D\mathcal{F}uk(M)$ contains $Don(M)$ but has additional objects. These detect the existence of non-trivial *chain level* morphisms. For instance, if $L \xrightarrow{0} L' \to K$ and $L \xrightarrow{0} L' \to K'$ are exact with $K \neq K'$, then $rk(CF(L, L')) \geq 2$.

e. Some consequences.

i. The Arnold Conjecture (Floer). Aspherical setting - if L and L' are Hamiltonian isotopic then

$$\#(L \cap L') \geq rk\left[H(L;\mathbb{Z}_2)\right] \; .$$

Remark

For *very small, exact deformations*
$L' \equiv$ a graph of a Morse function $f : L \to \mathbb{R}$ and thus
$L \cap L' = Crit(f)$. The estimate follows by the Morse inequalitites.

But for *large* deformations the "smooth" lower bound is $\chi(L)$!

In this case, the estimate follows from:

$$CF(L, L') = \mathbb{Z}_2 < L \cap L' > \; , \; H(CF(L, L'), d) \cong H(L; \mathbb{Z}_2) \; .$$

ii. Some applications of the Fukaya category.

- Cases of *Homological Mirror Symmetry* (mostly since '00): *Seidel*, *Abouzaid*, *Auroux*, *Smith*, *Sheridan* ; many interesting other results by *Perutz*, *Lekili* and others. Much of this work uses *Seidel*'s book ('06) as foundation.

- Nearby Lagrangians: An *exact* Lagrangian $L \subset T^*N$ (under additional constraints) is homologically equivalent to the zero section (*Fukaya-Seidel-Smith* '07, *Nadler* '07 by different methods); further extended to *homotopy equivalent* by *Abouzaid* '10.

Remark

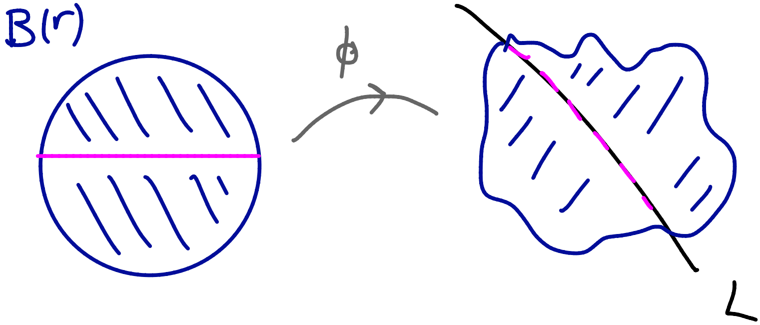
The Arnold nearby Lagrangian conjecture is: An *exact* Lagrangian $L \subset T^*N$ is Hamiltonian isotopic to the 0-section.

iii. Gromov width - a test problem.

Measure the "size" of $L \hookrightarrow (M, \omega)$ by width (*Barraud-C.* '06).

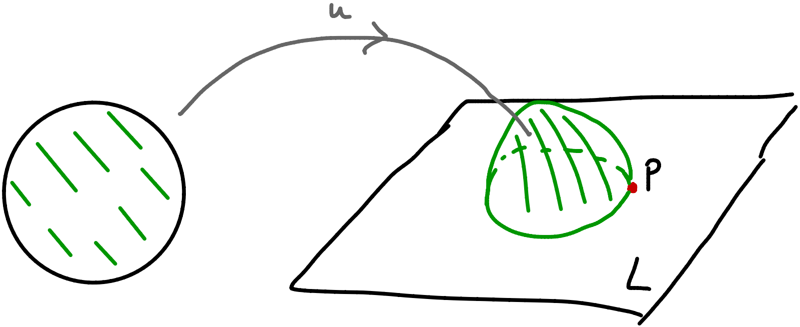
$$w(L) = \sup\{\pi r^2 : \exists\, \phi : B(r) \hookrightarrow M,\ \phi^*\omega = \omega_0, \phi^{-1}(L) = B(r) \cap \mathbb{R}^n\}$$

$(B(r), \omega_0) \subset \mathbb{C}^n$ standard ball of radius r.



L is *uniruled* if $\exists K > 0$ so that for $\forall\ J$ and $\forall\ P \in L$,

$$\exists\ u \in \mathcal{M}(J; L), \text{ with } P \in u(S^1) \text{ and } \omega(u) \leq K\ .$$



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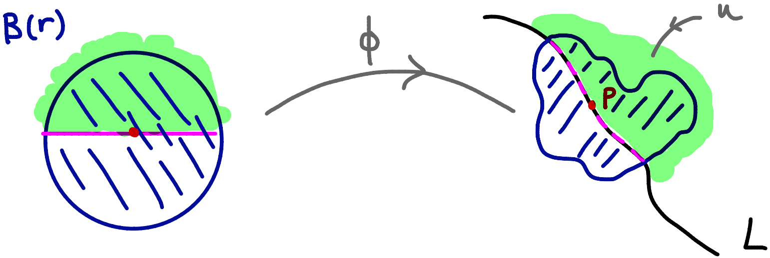


Easy to see: L uniruled $\Rightarrow w(L) \leq 2K$

Remark

Conjectured (Barraud-C. '06): any closed Lagrangian $L \subset \mathbb{C}^n$ has finite Gromov width. Even further, it is uniruled.

Gromov width conjecture is true if:

i. (C.- Lalonde '06, Biran-C. '07) L monotone; uniruling is also true; stronger results by Charette ('12).

ii. (Biran-C. '07, Charette - in progress) L two-dimensional and orientable.

iii. (Borman-McLean '13) L admits a metric of non-positive curvature - proof does not go through uniruling.

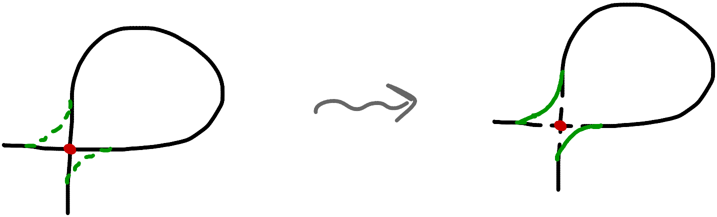
iv. (C.-Lalonde '06, Fukaya '06)* General Lagrangians diffeomorphic to (among other possibilities) $L = S^1 \times S^{2k+1}$.
*This assumes the analysis works in the "general" setting.

First case that all the machinery can not hit: $S^1 \times S^{even}$.

a. The Gromov *h*-principle.(*Gromov*, *Eliashberg* '80's)
Lagrangian *immersions* are governed by the *h*-principle: algebraic topological criteria suffice to decide whether a map can be perturbed to a Lagrangian immersion.

Such an immersion can be further perturbed so that it has only transversal double points.

b. Lagrangian Surgery. (*Lalonde-Sikorav*, *Polterovich* '91) Double points can be removed via surgery $\Rightarrow$ *embedded* Lagrangians



a. How natural is the machinery ?

Floer homology is not a "homology theory" as topologists understand these; the derived Fukaya category is not purely geometric, nor purely algebraic, and the triangular structure is obscure geometrically.

b. Where is the boundary flexibility/rigidity ?

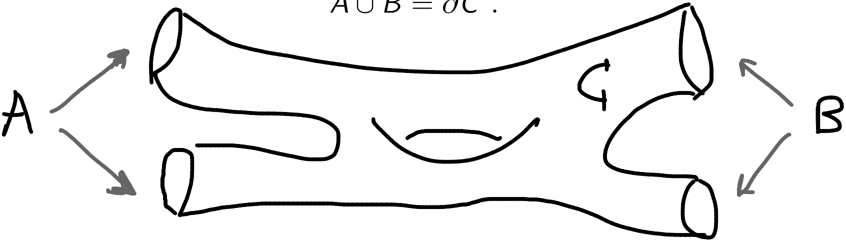
Flexible constructions are often imcompatible with J-holomorphic techniques (surgery destroys monotonicity etc). Is this a reflection of geometry or an artifact of methods that are not efficient enough? Thus, even if the "general" machinery is technically very hard it does not solve a seemingly simple problem such as showing:

$$w(S^1 \times S^{2k}) < \infty \ .$$

Since work of *Thom '54*, cobordism has been central to the study of manifolds.

Two closed (not necessarily connected) manifolds A^n and B^n are cobordant if there is a manifold C^{n+1} so that

$$A \cup B = \partial C \ .$$



Manifolds - up to cobordism - are organized in cobordism groups, operation is $\coprod$.

smooth cobordism groups $\cong$ homotopy groups of Thom spaces

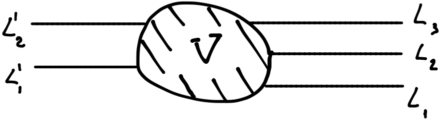
Definition (*Arnold '80*)

(M, ω) symplectic manifold; $(L_1, \ldots, L_k)$, $(L'_1, \ldots, L'_{k'})$ two families of closed, connected Lagrangian submanifolds $\subset M$.
A Lagrangian cobordism:

$$V : (L_i) \to (L'_j) \text{ is a Lagrangian } V \subset (\mathbb{C} \times M, \omega_0 \oplus \omega) \text{ so that}$$

$$V|_{[1,\infty)\times\mathbb{R}\times M} = \cup_i [1, \infty) \times \{i\} \times L_i$$
$$V|_{(-\infty,0]\times\mathbb{R}\times M} = \cup_j (-\infty, 0] \times \{j\} \times L'_j \ .$$

If $\pi : \mathbb{C} \times M \to \mathbb{C}$ is the projection, $\pi(V)$ looks like this:



Form a group: $G^*_{cob}(M) = \mathbb{Z}_2 < L \subset M, \text{ Lagrangian} > /\mathcal{R}_{cob}$.

Relations $\mathcal{R}_{cob}$ generated by:

$$L_1 + \ldots L_k = 0 \text{ if } (L_1, \ldots L_k) \text{ is null} - \text{bordant.}$$

$-^*$ means that the Lagrangians and the cobordisms are restricted - in our case *uniformly monotone*.

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Remark

By surgery we can transform any *immersed* cobordism into an *embedded* one $\Rightarrow$ "general" cobordism is a very flexible equivalence relation $\Rightarrow$ the resulting "general" cobordism groups do not reflect hard rigidity properties. But they are computable. For $M = \mathbb{C}^n$ computations are due to *Audin* '85 and *Eliashberg* '84.

Theorem (*Biran-C.* '11 & '13)

a. $\exists$ *group morphism* : $\hat{\mathcal{F}} : G^*_{cob}(M) \longrightarrow K_0(D\mathcal{F}uk(M))$ *that lifts the natural morphism* $G^*_{cob}(M) \to H_n(M;\mathbb{Z}_2)$.

b. If $V : L \to L'$ *cobordism* $\Rightarrow$ $L \cong L'$ *in* $D\mathcal{F}uk(M)$.
In particular, $HF(L,L) \cong HF(L',L')$.

c. $W : L \to (L_1, L_2)$ *cobordism* $\Rightarrow$
$\exists$ *exact triangle in* $D\mathcal{F}uk(M)$: $L_2 \to L_1 \to L$.

d. $\exists$ *"categorified" versions of b,c* $\Rightarrow$ *view* $HF(N,-)$ *as a functor*
$\mathcal{HF}_N$: Cobordism Category $\to$ Vector Spaces

a. Comments:

- $\hat{\mathcal{F}} : G^*_{\mathcal{C}ob}(M) \to K_0 D\mathcal{F}uk(M)$ is a sort of rigid version of the Thom map mentioned before.
- $K_0(D\mathcal{F}uk(M))$ is known in some cases, mainly surfaces, by work of *Seidel*, *Abouzaid*. It can be "identified" by homological mirror symmetry (when this applies).
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- From c: decompositions by exact triangles in $D\mathcal{F}uk$ are natural - they correspond to "splitting" via cobordisms.

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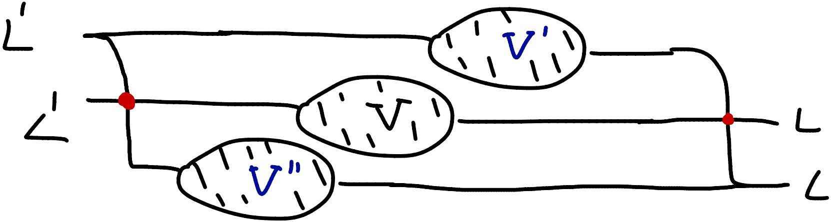
iii. consider $V : L \to L'$ and two copies of it V', V'', $V'' = \phi(V')$ for an appropriate horizontal Hamiltonian isotopy ϕ.



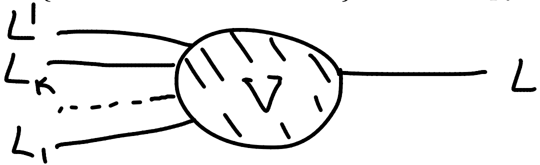
Notice $HF(L, L) \cong HF(V, V') \cong HF(V, V'') \cong HF(L', L')$ □

c. Categorification. The full statement of the theorem follows from a stronger, "categorified" version.

Simple cobordism category $\mathcal{SC}ob^*(M)$:

$$\mathcal{O}b(\mathcal{SC}ob^*(M)) = \{L \in Lag^*(M)\}$$

$$hom_{\mathcal{SC}ob^*(M)}(L, L') = \{V \subset \mathbb{C} \times M \text{ cobordism}\} \text{ mod isotopy.}$$



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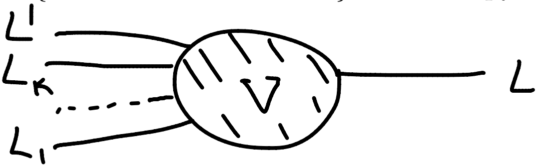

$\exists$ functor $\mathcal{F} : \mathcal{S}\mathcal{C}ob^*(M) \to D\mathcal{F}uk(M)$ whose properties imply the Theorem.

Remark

A categorical formalism for Lagrangian cobordism has been independently introduced by *Nadler-Tanaka* '12

Recall the Puzzles ?

a. Cobordism helps with our first "naturality" puzzle .

- exact triangles are often a reflection of geometric decompositions by cobordism.

- Floer homology has the structure of a functor with properties somewhat similar to a TQFT.

So while all this machinery is complex it is more natural than it might first appear.

b. Boundary between rigidity and flexibility.

- J-holomorphic based Lagrangian invariants such as HF are more invariant than expected from their construction - to cobordism and not only Hamiltonian isotopy.

The downside is that without any “constraints” (like monotonicity or others) these invariants have to be weak - or are not defined - because “general” cobordism is too flexible an equivalence relation.

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...advances in flexibility by *Murphy* '13 have led to examples (*Rizell*'13) of Lagrangians $\approx S^1 \times S^{even}$ of infinite width !!

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