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Bounds on the Lagrangian spectral metric in cotangent bundles

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Abstract. Let N be a closed manifold and $U \subset T^*(N)$ a bounded domain in the cotangent bundle of N, containing the zero-section. A conjecture due to Viterbo asserts that the spectral metric for Lagrangian submanifolds in U that are exact-isotopic to the zero-section is bounded. In this paper we establish an upper bound on the spectral distance between two such Lagrangians L_0, L_1, which depends linearly on the boundary depth of the Floer complexes of (L_0, F) and (L_1, F), where F is a fiber of the cotangent bundle.

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1. Introduction and main results

Let N be a closed manifold and $T^*(N)$ its cotangent bundle, endowed with its standard symplectic structure. A domain $U \subset T^*(N)$ is called bounded if there is a Riemannian metric g on N such that U is contained inside the unit-ball cotangent bundle of $T^*(N)$ with respect to the metric associated to g on the fibers of $T^*(N)$. More specifically,

$$U \subset \{v \in T^*(N) \mid |v| \leq 1\},$$

where $|\cdot|$ is the norm on the fibers of $T^*(N)$ corresponding to the metric g via the isomorphism $T^*(N) \cong T(N)$ induced by g. Since N is compact, the boundedness of U is independent of the choice of g.

For a domain $W \subset T^*(N)$ we denote by $\mathcal{L}_{\text{ex}}(W)$ the collection of closed exact Lagrangian submanifolds of W (exactness is considered here with respect to the canonical Liouville form) and by $\mathcal{L}_{\text{ex},N}(W) \subset \mathcal{L}_{\text{ex}}(W)$ the collection of Lagrangians that are exact isotopic (within $T^*(N)$) to the zero section $N \subset T^*(N)$.

There are several Ham-invariant metrics on $\mathcal{L}_{\text{ex},N}(T^*(N))$. For example, the Hofer metric on $\text{Ham}(T^*(N))$ descends to a non-degenerate metric d_{Hof} on $\mathcal{L}_{\text{ex},N}(T^*(N))$. Another important metric, due to Viterbo, is the spectral metric. This was originally defined for $\mathcal{L}_{\text{ex},N}(T^*(N))$, but thanks to more recent

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developments can be extended to the entire of $\mathcal{L}_{\mathrm{ex}}(T^*(N))$; see Remarks 2.2.3 (3) and 2.2.2 (2) for more on this. The spectral distance $\gamma(L_0, L_1)$ between two elements $L_0, L_1 \in \mathcal{L}_{\mathrm{ex}}(T^*(N))$ is defined as:

$$\gamma(L_0, L_1) = c\big([N]; L_0, L_1\big) - c\big([\mathrm{pt}]; L_0, L_1\big), \tag{1}$$

where $c([N]; L_0, L_1)$ and $c([\mathrm{pt}]; L_0, L_1)$ stand for the spectral invariants associated to (L_0, L_1), for the fundamental class $[N] \in H_n(N)$ and for the class of a point $[\mathrm{pt}] \in H_0(N)$, correspondingly; see Section 2 (and more specifically Section 2.2 and Section 2.2.4) below for the precise definitions.

It is well known that $\gamma(L_0, L_1) \leq d_{\mathrm{Hof}}(L_0, L_1)$ for all $L_0, L_1 \in \mathcal{L}_{\mathrm{ex},N}(T^*(N))$. However, beyond this inequality, little is known about the relation between these two metrics.

Let $U \subset T^*(N)$ be a bounded domain. It is also known that, at least for some N's, the Hofer metric on $\mathcal{L}_{\mathrm{ex},N}(U)$ is unbounded. This has been proved for several cases like $N = S^1$ by Khanevsky [20] and is conjectured to hold for all N's.

In contrast to the Hofer metric, there is the following conjecture regarding the spectral metric:

Conjecture (Viterbo). *The spectral metric on $\mathcal{L}_{\mathrm{ex},N}(U)$ is bounded.*

This was conjectured by Viterbo in [38] for the case $N = \mathbb{T}^n$, and is expected to hold for all closed manifolds N. Recently Shelukhin [33, 34] proved this conjecture for several classes of manifolds N (including $\mathbb{T}^n$).

Our main result, which applies to all closed manifolds N, is the following.

Theorem A. *Let N be any closed manifold and $U \subset T^*(N)$ a bounded domain. There exist constants $A, B > 0$ that depend only on U such that for every $L_0, L_1 \in \mathcal{L}_{\mathrm{ex},N}(U)$ we have:*

$$\gamma(L_0, L_1) \leq B\Big(\beta\big(CF(L_0, F_q)\big) + \beta\big(CF(L_1, F_q)\big)\Big) + A. \tag{2}$$

Here $F_q = T^*_q(N)$ is the fiber of the cotangent bundle at an arbitrary point $q \in N$, viewed as a Lagrangian submanifold of $T^*(N)$, and $\beta(CF(L_i, F_q))$ is the boundary depth of the Floer complex of the pair (L_i, F_q), $i = 0, 1$, defined with coefficients in $\mathbb{Z}_2$; see Section 2.1 for the definition of boundary depth.

Remarks 1.0.1. (1) Clearly, the above conjecture of Viterbo would follow from Theorem A if we can show that the boundary depth $\beta(CF(L, F_q))$ is uniformly bounded in $L \in \mathcal{L}_{\mathrm{ex},N}(U)$.

(2) The converse to the statement made at point (1) above turns out to be also true. Namely, if the conjecture of Viterbo holds true then the boundary depth $\beta(CF(L, F_q))$ is uniformly bounded in $L \in \mathcal{L}_{\mathrm{ex},N}(U)$. This follows by a relatively simple argument that we summarize in Section 6.2.

(3) The chain complex $CF(F_q, L_i)$ depends on the point $q \in N$ and so does its boundary depth $\beta(CF(L_i, F_q))$. However, as we will see in Section 6, Lemma 6.1.1, the difference

$$|\beta(CF(L, F_{q'})) - \beta(CF(L, F_{q''}))|$$

is bounded, uniformly in $q', q'' \in N$, $L \in \mathcal{L}_{\mathrm{ex},N}(U)$. Therefore, the formulation of inequality (2) with constants A, B that do not depend on q, makes sense.

(4) While the chain complex $CF(L, F_q)$ may be complicated and have arbitrary large rank, its homology is very simple:

$$HF(L, F_q) \cong \mathbb{Z}_2$$

for every $q \in N$, $L \in \mathcal{L}_{\mathrm{ex},N}(U)$.

1.1. Strategy and main ideas in the proof. The starting point of the proof is borrowed from [14] – we embed a tubular neighborhood $\mathcal{U}$ of the zero section of $T^*(N)$ into a real affine algebraic manifold E which also serves as the total space of a Lefschetz fibration $\pi: E \to \mathbb{C}$ endowed with a real structure. The embedding can be arranged such that the zero section is sent to (one of the components of) the real part of E.

The second step appeals to our previous work [6] which establishes canonical presentations of Lagrangians K in Lefschetz fibrations as iterated cone decompositions with standard factors. These iterated cone decompositions take place in the category of modules over the Fukaya category of E and hold up to quasi-isomorphisms. The factors in the decomposition of K consist of the Yoneda modules of certain Lefschetz thimbles emanating from the critical points of π along N, as well as some factors that involve the Floer complexes of pairs of thimbles and pairs of the type (Thimble, K). This makes it possible to express $CF(L, K)$ for every exact Lagrangians L, as an iterated cone involving chain complexes of the types $CF(L, \text{Thimble})$, $CF(\text{Thimble}, K)$ and CF of pairs of thimbles. Note that the second and third types do not involve L.

By specializing to the case $K = N$ and taking L to correspond to a Lagrangian in the neighborhood $\mathcal{U}$ of the zero-section, the previous cone decomposition of $CF(L, N)$ reduces now to terms of the type $CF(L, F_q)$, for different critical points $q \in N$ of π, and some other fixed chain complexes that do not depend on L. The terms F_q appear here because the previously mentioned thimbles coincide within $\mathcal{U}$ with the fibers F_q of the cotangent bundle. A “local to global” argument in Floer theory shows that replacing the thimble emanating from a critical point $q \in N$ of π by F_q does not change the respective Floer complexes.

The next step is to analyze the spectral metric using the above cone decompositions. This requires a refinement of the cone decomposition in the framework of filtered Floer theory. It turns out that the above cone decomposition continues to hold in the filtered sense up to a bounded action shift. Therefore, in principle one can recover (up to a bounded shift) the filtered Floer homology of (L, N) from the filtered Floer homology of the factors mentioned above and the knowledge of the chain maps between the factors which form the cones. In practice this is not so effective, as these chain maps are in general hard to describe explicitly. Fortunately, this obstacle can be overcome by algebraic means which are described next.

The next step in the proof is purely algebraic. Here we obtain a coarse uniform upper bound on the spectral range of filtered mapping cones

$$C = \mathrm{Cone}(C' \xrightarrow{f} C'')$$

between two filtered chain complexes C' and C''. By "spectral range" of a filtered chain complex we mean the difference between the highest and the lowest spectral invariants of that complex. It turns out that one can derive such a bound on the spectral range of C which involves only the following pieces of data: the spectral ranges of C' and C'', the boundary depths of C' and C'' and the amount of filtration shift of the map f. A crucial point here is that our bound is uniform in f in the sense that it does not involve specific information on the map f, except of the extent by which it shifts the filtrations. We also establish an analogous upper bound for the boundary depth of C. Having these two algebraic ingredients at hand, we can derive similar upper bounds for the spectral range and boundary depth of iterated cones.

The final step puts the geometry and algebra together. We apply the algebraic estimates on the spectral range to the previously mentioned cone decomposition of $CF(L, N)$. While it is possible to describe relatively precisely the chain maps between the terms in this decomposition, this is delicate. Fortunately, this is not needed here as we can easily bound the amount by which these maps shift filtrations. Consequently we obtain an upper bound on the spectral range of $CF(L, N)$ as the sum of two terms: one of them is a constant A that comes from the spectral ranges of the factors in our cone decomposition (these are straightforward to determine) and some uniformly bounded errors that come from our coarse estimates. This constant depends on $\mathcal{U}$ but *not* on L since the only appearance of L in the cone decomposition of $CF(L, N)$ is in terms of the type $CF(L, F_q)$. However, the spectral range of such terms is 0 because $HF(L, F_q)$ is 1-dimensional. The second summand in our bound looks like $B\beta(CF(L, F_q))$, where B is a constant and $\beta(CF(L, F_q))$ is the boundary depth of $CF(L, F_q)$. Our main result now easily follows from these bounds.

The above is only an outline of the main ideas in the proof. Along the way there are several additional ingredients required for the proof to work. These have to do with technicalities in Floer theory, Lefschetz fibrations and filtered homological algebra.

1.2. Organization of the paper. The rest of the paper is organized as follows. Section 2 reviews necessary preliminaries on filtered Floer theory in the framework of exact Lagrangian submanifolds in Liouville manifolds. We also prove in Section 2.4

a general "local vs. global" result, comparing the Lagrangian Floer persistent homologies in a Liouville subdomain with the same type of homology in the entire Liouville manifold.

Section 3 is devoted to Lefschetz fibrations and their relevance to our problem. We go over real Lefschetz fibrations in general and then review a construction from [14] which gives an embedding of a neighborhood of the zero-section in $T^*(N)$ into a real Lefschetz fibration E. We then go over a construction coming from [6] which alters the Lefschetz fibration E into an extended Lefschetz fibration E' containing a collection of matching spheres that will be useful for our purposes. Part of this section is devoted to showing that the construction of E' can be made while preserving a geometric setting amenable to Floer theory like exactness etc.

Section 4 is dedicated to a comparison between the filtered Floer theory inside E and the same theory viewed in E'. In particular we show there that the matching spheres from E', constructed in Section 3, correspond in E to some Lefschetz thimbles emanating from N. These in turn coincide near N with cotangent fibers of $T^*(N)$. We show that these correspondences hold also in a Floer-theoretic sense.

Section 5 is central for the proof of the main theorem. There we discuss iterated cone decompositions in the Fukaya categories of E and E'. In particular we show how to represent Lagrangian submanifolds in E' as iterated cones with standard terms coming from the matching spheres from Section 3. Moreover, in Section 5.3 and 5.4 we extend these decompositions to the realm of Fukaya categories endowed with action filtrations. In particular we also derive a filtered version of the Seidel exact triangle associated to a Dehn-twist.

Section 6 combines the geometric contents of the previous sections together with some filtered homological algebra (developed in Section 7) to conclude the proof of the main theorem. We also sketch the argument for the converse of this theorem.

The algebraic ingredients necessary for the paper are gathered in Section 7. This is a purely algebraic section in which we study spectral invariants and boundary depth of filtered chain complexes. Special attention is given to filtered mapping cones and we establish estimates on the spectral range and boundary depth in that case.

The paper can be read linearly, with the exception of Section 7 which is the last one, but is being referred to at many instances along the paper. At the same time, Section 7 is independent of the rest the paper and can be read separately.

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2. Lagrangian Floer theory and spectral invariants

Here we briefly recall the definitions of spectral invariants, boundary depth and the spectral metric on the space of Lagrangian submanifolds. We refer the reader to [10, 19, 22, 23, 26, 28–30, 35–37, 39] for more details on the general theory of these concepts.

2.1. Filtered chain complexes and their invariants. Fix a unital ring R and let C be a chain complex of R-modules. By a filtration on C we mean an increasing filtration of subcomplexes of R-modules, indexed by the real numbers. More specifically, for every $\alpha \in \mathbb{R}$ we are given a subcomplex $C^{\leq \alpha} \subset C$ of R-modules and for every $\alpha \leq \beta$ we have $C^{\leq \alpha} \subset C^{\leq \beta}$. For simplicity we will assume from now on that the filtration on C is exhaustive, i.e., $\cup_{\alpha \in \mathbb{R}} C^{\leq \alpha} = C$.

The inclusions $C^{\leq \alpha} \subset C^{\leq \beta}$, $\alpha \leq \beta$, and $C^{\leq \alpha} \subset C$ induce maps in homology, which we denote by:

$$i^{\beta,\alpha}\colon H_*(C^{\leq \alpha}) \to H_*(C^{\leq \beta}), \quad i^{\alpha}\colon H_*(C^{\leq \alpha}) \to H_*(C).$$

Given a homology class $a \in H_*(C)$ we define its spectral invariant $\sigma(a) \in \mathbb{R} \cup \{-\infty\}$ to be

$$\sigma(a) := \inf\{\alpha \in \mathbb{R} \mid a \in \operatorname{image} i^{\alpha}\}. \tag{3}$$

Note that $\sigma(0) = -\infty$.

Another important measurement for our purposes is the boundary depth $\beta(C)$ of a filtered chain complex C, which is defined as follows:

$$\beta(C) := \inf\big\{r \geq 0 \mid \ \forall \alpha, \forall c \in C^{\leq \alpha} \text{ which is a boundary in } C, \\ \exists\, b \in C^{\leq \alpha + r} \text{ s.t. } c = d(b)\big\}.$$

We will elaborate more on spectral invariants, boundary depth and other measurements of filtered chain complexes in Section 7.

2.2. Filtered Lagrangian Floer theory. In what follows all symplectic manifolds and their Lagrangian submanifolds will be implicitly assumed to be connected, unless otherwise mentioned. All Hamiltonian functions will be implicitly assumed to be compactly supported.

2.2.1. Liouville and Stein manifolds. In the following we will be mainly concerned with symplectic manifolds of two types: Liouville domains and manifolds that are Stein at infinity. We refer the reader to [9] for the foundations of the theory of such manifolds and much more. Below we briefly recall the basic notions needed for our purposes.

A compact Liouville domain $(W, \omega = d\lambda)$ consists of a compact manifold W with boundary ∂W and an exact symplectic structure ω, with a given primitive

1-form λ (called the Liouville form) such that the following holds: the Liouville vector field X_λ, defined by $i_{X_\lambda}\omega = \lambda$, is outward transverse to ∂W. Under this assumption the restriction $\lambda_{\partial W} := \lambda|_{\partial W}$ is a contact form and we denote by $\xi_\lambda := \ker \lambda_{\partial W}$ the contact structure defined by $\lambda_{\partial W}$ on ∂W. We write $\psi_t : W \to W$, $t \leq 0$, for the flow of X_λ (which exists for all $t \leq 0$). We have $\psi_t^* \lambda = e^t \lambda$ and $\psi_t^* \omega = e^t \omega$.

For a Liouville domain $(W, \omega = d\lambda)$, consider the embedding

$$\Psi : (-\infty, 0] \times \partial W \to W, \quad (s, x) \mapsto \psi_s(x).$$

We have $\Psi^* \lambda = e^s \lambda_{\partial W}$ and $\Psi^* \omega = d(e^s \lambda_{\partial W})$. Define an almost complex structure J^λ on $(-\infty, 0] \times \partial W$ as follows. Fix an almost complex structure J_{ξ_λ} on ξ_λ which is compatible with $\omega|_{\xi_\lambda}$. Denote by $R_{\lambda_{\partial W}} \in T(\partial W)$ the Reeb vector field corresponding to $\lambda_{\partial W}$. Define $J^\lambda|_{\xi_\lambda} := J_{\xi_\lambda}$ and $J^\lambda\big(\frac{\partial}{\partial s}\big) := R$. Note J^λ is compatible with $\Psi^* \omega$ and moreover the function

$$\phi : (-\infty, 0] \times \partial W \to \mathbb{R}, \quad \phi(s, x) := e^s,$$

is a potential for $\Psi^* \omega$, i.e., $\Psi^* \omega = -dd^{J^\lambda} \phi$ (in fact we have $d^{J^\lambda} \phi = -e^s \lambda$). In particular, ϕ is J-plurisubharmonic (or J-convex). Using the map Ψ we can endow image(Ψ) with the almost complex structure $\Psi_*(J^\lambda)$ which, by abuse of notation, will also be denoted by J^λ. (Note that in general J^λ does not extend from image(Ψ) to the entire of W.)

Sometimes it will be useful to work with the completion $(\widehat{W}, \widehat{\omega} = d\widehat{\lambda})$ of a compact Liouville domain $(W, \omega = d\lambda)$. More precisely, set

$$\widehat{W} := W \cup_\Psi \big([-\varepsilon, \infty) \times \partial W\big),$$

where the gluing identifies $[-\varepsilon, 0] \times \partial W$ with a collar neighborhood of ∂W in W via the map Ψ. The Liouville form $\widehat{\lambda}$ is defined by extending λ from W to the cylindrical part $[0, \infty) \times \partial W$ by $\widehat{\lambda} = e^s \lambda_{\partial W}$, where $s \in [0, \infty)$. We denote the corresponding symplectic structure by $\widehat{\omega} := d\widehat{\lambda}$.

All the previous structures, like X_λ, ψ_t, ϕ and J^λ, extend in an obvious way to the completion. More specifically, the Liouville vector field $X_{\widehat{\lambda}}$ (defined by $i_{X_{\widehat{\lambda}}}\widehat{\omega} = \widehat{\lambda}$) extends X_λ by $\frac{\partial}{\partial s}$ along the cylindrical part. We denote the flow of $X_{\widehat{\lambda}}$ by $\widehat{\psi}_t$. Note that this flow is complete (i.e., exists for all times t, both positive and negative). Next, we extend the almost complex structure J^λ from image Ψ to an almost complex structure $\widehat{J}^\lambda$ on

$$(\text{image}\,\Psi) \cup_\Psi \big([-\varepsilon, \infty) \times \partial W\big) \subset \widehat{W}$$

by the same recipe defining J^λ, namely $\widehat{J}^\lambda := J^\lambda$ on image Ψ, and $\widehat{J}^\lambda_{(s,x)}|_{\xi_\lambda} := J_{\xi_\lambda}$, $J^\lambda_{(s,x)}\big(\frac{\partial}{\partial s}\big) := R_{\lambda_{\partial W}}$, for every $(s, x) \in [0, \infty) \times \partial W$ (where here we view $\xi_\lambda \subset T_{(s,x)}(s \times \partial W)$). Finally, note that the plurisubharmonic function ϕ: image $\Psi \to \mathbb{R}$ extends to the cylindrical part $[0, \infty) \times \partial W$ by

$$\widehat{\phi}(s, x) = e^s, \quad \text{and} \quad \widehat{\lambda} = -d^{\widehat{J}^\lambda} \widehat{\phi}, \quad \widehat{\omega} = -dd^{\widehat{J}^\lambda} \widehat{\phi}.$$

Another type of symplectic manifolds that we will encounter are Stein manifolds, which are very much related to the above. By a Stein manifold we mean a triple (V, J_V, φ), where (V, J_V) is an open complex manifold (with integrable J_V) and $\varphi\colon V \to \mathbb{R}$ is an exhaustion plurisubharmonic function. Exhaustion means that φ is proper and bounded from below, and plurisubharmonic means that the 2-form $\omega_\varphi := -dd^{J_V}\varphi$ is compatible with J_V i.e.,

$$\omega_\varphi(u, J_V u) > 0,\ \forall u \quad \text{and} \quad \omega_\varphi(J_V u, J_V v) = \omega_\varphi(u, v),\ \forall u, v.$$

Denote $\lambda_\varphi := -d^{J_V}\phi$ and for $R \in \mathbb{R}$,

$$V_{\varphi \leq R} := \{x \in V \mid \varphi(x) \leq R\}.$$

(Similarly, we have $V_{\varphi<R}$, $V_{\varphi \geq R}$, etc.) Below we will implicitly assume that (V, J_V, φ) is of finite type, namely that φ has a finite number of critical points. Note that if R is a regular value of φ then $(V_{\varphi \leq R}, \omega_\varphi = d\lambda_\varphi)$ is a compact Liouville domain.

Another variant is symplectic manifolds that are Stein at infinity: $(V, J_V, R_0, \varphi, \omega)$. Here V is a symplectic manifold, endowed with a (possibly non-exact) symplectic structure ω. Next we have $\varphi\colon V \to \mathbb{R}$, an exhaustion function with finitely many critical points. The parameter $R_0 \in \mathbb{R}$ is a regular value of φ, and J_V is an integrable complex structure defined on $V_{\varphi \geq R_0}$, and the following holds along $V_{\varphi \geq R_0}$: $\omega = -dd^{J_V}\varphi$ is compatible with J_V. Thus, φ is J_V-convex on $V_{\varphi \geq R_0}$.

Symplectic manifolds that are Stein at infinity admit a slightly different variant of completion, which we now briefly recall; see [2, 9, 11] for more details. Let $(V, J_V, R_0, \varphi, \omega)$ be a symplectic manifold which is Stein at infinity. Let $R \geq R_0$ and assume that $\mathrm{Crit}(\varphi) \subset V_{\varphi<R}$. Then there exists a function $\varphi_R\colon V \to \mathbb{R}$ with the following properties:

(1) φ_R is an exhaustion function and $V_{\varphi_R \leq R} = V_{\varphi \leq R}$. Moreover, $\varphi_R = \varphi$ on $V_{\varphi \leq R}$.

(2) φ_R has no critical points in $V_{\varphi_R \geq R}$.

(3) φ_R is plurisubharmonic on $V_{\varphi \geq R_0}$, i.e., $-dd^{J_V}\varphi_R$ is compatible with J_V along $V_{\varphi \geq R_0}$.

(4) Define the 1-form $\widehat{\lambda}_R := -d^{J_V}\varphi_R$ on $V_{\varphi \geq R_0}$. Define $\widehat{\omega}_R$ on V by setting it to be ω on $V_{\varphi \leq R_0}$ and $\widehat{\omega}_R := d\widehat{\lambda}_R$ on $V_{\varphi \geq R_0}$. Let $X_{\widehat{\lambda}_R}$ be the Liouville vector field, defined along $V_{\varphi \geq R_0}$ by $i_{X_{\widehat{\lambda}_R}}\widehat{\omega}_R = \widehat{\lambda}_R$. Then the flow

$$\widehat{\psi}_t^R\colon V_{\varphi \geq R_0} \to V_{\varphi \geq R_0}$$

of $X_{\widehat{\lambda}_R}$ exists for all $t \geq 0$.

We call $(V, J_V, R_0, \varphi_R, \widehat{\omega}_R)$ a completion of $(V, J_V, R_0, \varphi, \omega)$.

Finally, we will also need the notion of Liouville manifolds that are Stein at infinity. These are symplectic manifolds that are Stein at infinity, $(V, J_V, \varphi, R_0, \omega = d\lambda)$, but now we assume in addition that the symplectic structure ω is globally exact with a prescribed primitive λ. Moreover, λ is assumed to satisfy $\lambda = -d^{J_V}\varphi$ along $V_{\varphi \geq R_0}$.

Note that, as for the case of Stein manifolds, if $R \geq R_0$ is a regular value of φ then $(V_{\varphi \leq R}, \omega = d\lambda)$ is a compact Liouville domain.

Note also that for the completion of Liouville manifolds that are Stein at infinity, the Liouville vector field $X_{\widehat{\lambda}_R}$ is defined all over V and moreover, its flow exists for all $t \in \mathbb{R}$.

2.2.2. Floer theory. We will work here with Floer homology and singular homology, both taken with coefficients in $\mathbb{Z}_2$. We will generally omit the $\mathbb{Z}_2$ from the notation (e.g., writing $H_*(L)$ for $H_*(L; \mathbb{Z}_2)$). Our setting is almost identical to [32, Chapter III, Section 8], with two slight differences. Firstly, we work with homological conventions rather than with cohomological ones. Secondly, we work in an ungraded setting.

Let $(V, \omega = d\lambda)$ be an exact symplectic manifold with a given primitive λ for the symplectic structure. We assume further that this symplectic manifold is of one of the following three types:

(1) $(V, d\lambda)$ is a compact Liouville domain.

(2) $(V, d\lambda)$ is the completion $(\widehat{V}', \widehat{\omega}' = d\widehat{\lambda}')$ of a compact Liouville domain $(V', \omega' = d\lambda')$.

(3) $(V, d\lambda)$ can be endowed with a structure $(V, J_V, R_0, \varphi, \omega = d\lambda)$ of a Liouville manifold which is Stein at infinity. In that case we also fix the additional structures J_V, φ, R_0.

We denote by $\operatorname{Int} V$ the interior of V. (Note that only in case (1), we have $\operatorname{Int} V \subsetneq V$.) Denote by $\mathcal{J}_V$ the space of ω-compatible almost complex structures on V which coincide with, J^λ near the boundary of V in case (1), or with $\widehat{J}^\lambda$ at infinity in case (2), or coincide with J_V on $V_{\varphi \geq R}$ for some $R \geq R_0$ in case (3).

Let $L_0, L_1 \subset \operatorname{Int} V$ be two closed exact Lagrangian submanifolds. (Exactness of a Lagrangian L will be generally considered with respect to the given Liouville form λ. In case we want to emphasize the form with respect to which L is exact we will call L a λ-exact Lagrangian.) We fix primitive functions

$$h_{L_i} : L_i \to \mathbb{R}$$

to $\lambda|_{L_i}$, $i = 0, 1$.

Let $H : [0,1] \times V \to \mathbb{R}$ be a Hamiltonian function. Write $H_t(x) = H(t,x)$. Henceforth, we will implicitly assume that there exists a compact subset $K \subset \operatorname{Int} V$ such that for all $t \in [0,1]$, the function H_t is constant outside of K. The Hamiltonian vector field $X_t^H = X^{H_t}$ of H is given by $\omega(X_t^H, \cdot) = -dH_t(\cdot)$.

Denote by

$$\mathcal{P}_{L_0,L_1} = \{\gamma : [0,1] \to V \mid \gamma(0) \in L_0,\ \gamma(1) \in L_1\}$$

the space of paths with end points on L_0, L_1. The action functional $\mathcal{A}_H : \mathcal{P}_{L_0,L_1} \to \mathbb{R}$ is defined as follows:

$$\mathcal{A}_H(\gamma) := \int_0^1 H(t,\gamma(t))\,dt - \int_0^1 \lambda(\dot{\gamma}(t))\,dt + h_{L_1}(\gamma(1)) - h_{L_0}(\gamma(0)). \tag{4}$$

Denote by $\mathcal{O}(H) = \mathcal{O}_{L_0,L_1}(H) \subset \mathcal{P}_{L_0,L_1}$ the set of Hamiltonian chords with endpoints on (L_0, L_1), namely the set of orbits $\gamma : [0,1] \to V$ of X_t^H with $\gamma(0) \in L_0$, $\gamma(1) \in L_1$.

Let $\mathcal{D} = (H, J)$ be a regular Floer datum, consisting of a Hamiltonian function $H : [0,1] \times V \to \mathbb{R}$ and a time-dependent almost complex structure $J = \{J_t\}_{t\in[0,1]}$, with $J_t \in \mathcal{J}_V$ for every t. Sometimes we will write $\mathcal{O}_{L_0,L_1}(\mathcal{D})$ (or $\mathcal{O}(\mathcal{D})$) for $\mathcal{O}_{L_0,L_1}(H)$.

The negative gradient flow of $\mathcal{A}_H$ (with respect to a metric on $\mathcal{P}_{L_0,L_1}$ induced by J) gives rise to the Floer equation associated to $\mathcal{D}$:

$$\begin{gathered} u : \mathbb{R} \times [0,1] \to M, \quad u(\mathbb{R}\times 0) \subset L_0,\ u(\mathbb{R}\times 1) \subset L_1, \\ \partial_s u + J_t(u)\partial_t u = J_t X_t^H(u), \\ E(u) := \int_{-\infty}^{\infty}\int_0^1 |\partial_s u|^2\,dt\,ds < \infty, \end{gathered} \tag{5}$$

where $(s,t) \in \mathbb{R} \times [0,1]$. The quantity $E(u)$ in the last line of (5) is the energy of a solution u and we consider only finite energy solutions. (Note also that the norm $|\partial_s u|$ in the definition of $E(u)$ is calculated with respect to the Riemannian metric associated to ω and J_t.) Solutions u of (5) are also called Floer trajectories.

For $\gamma_-, \gamma_+ \in \mathcal{O}(H)$ we have the space of *parametrized* Floer trajectories u connecting γ_- to γ_+:

$$\mathcal{M}(\gamma_-, \gamma_+; \mathcal{D}) = \bigl\{u \mid u \text{ solves (5) and } \lim_{s\to\pm\infty} u(s,t) = \gamma_\pm(t)\bigr\}. \tag{6}$$

Note that $\mathbb{R}$ acts on this space by translations along the s-coordinate. This action is generally free, with the only exception being $\gamma_- = \gamma_+$ and the stationary solution $u(s,t) = \gamma_-(t)$ at γ_-.

Whenever $\gamma_- \neq \gamma_+$ we denote by

$$\mathcal{M}^*(\gamma_-, \gamma_+; \mathcal{D}) := \mathcal{M}(\gamma_-, \gamma_+; \mathcal{D})/\mathbb{R} \tag{7}$$

the quotient space (i.e., the space of non-parametrized solutions).

For a generic choice of Floer datum $\mathcal{D}$ the space $\mathcal{M}^*(\gamma_-, \gamma_+; \mathcal{D})$ is a smooth manifold (possibly with several components having different dimensions). Moreover, its 0-dimensional component $\mathcal{M}^*_0(\gamma_-, \gamma_+; \mathcal{D})$ is compact hence a finite set.

The Floer complex $CF(L_0, L_1; \mathcal{D})$ is the vector space, over $\mathbb{Z}_2$, with a basis formed by the set $\mathcal{O}(H)$:

$$CF(L_0, L_1; \mathcal{D}) = \bigoplus_{\gamma\in\mathcal{O}(H)} \mathbb{Z}_2\gamma. \tag{8}$$

Its differential $d: CF(L_0, L_1; \mathcal{D}) \to CF(L_0, L_1; \mathcal{D})$ is defined by counting solutions of the Floer equation:

$$d(\gamma_-) := \sum_{\gamma_+ \in \mathcal{O}(H)} \#_{\mathbb{Z}_2} \mathcal{M}_0^*(\gamma_-, \gamma_+; \mathcal{D})\gamma_+, \quad \forall\, \gamma_- \in \mathcal{O}(H), \tag{9}$$

and extending linearly over $\mathbb{Z}_2$. The homology of $CF(L_0, L_1; \mathcal{D})$ is denoted by $HF(L_0, L_1; \mathcal{D})$, and is called the Floer homology of (L_0, L_1).

The Floer homology is independent of the choice of the Floer datum in the sense that for every two regular choices of Floer data $\mathcal{D}$, $\mathcal{D}'$ there is a quasi-isomorphism, canonical up to chain homotopy,

$$\psi_{\mathcal{D},\mathcal{D}'}: CF(L_0, L_1; \mathcal{D}) \to CF(L_0, L_1; \mathcal{D}'),$$

called a continuation map. The (now canonical) isomorphisms induced in homology

$$H(\psi_{\mathcal{D},\mathcal{D}'}): HF(L_0, L_1; \mathcal{D}) \to HF(L_0, L_1; \mathcal{D}')$$

form a directed system and we can regard the collection of vector spaces $HF(L_0,L_1;\mathcal{D})$, parametrized by regular Floer data $\mathcal{D}$, as one vector space and denote it by $HF(L_0, L_1)$.

2.2.3. PSS and naturality. Given a Hamiltonian function

$$F: [0,1] \times V \to \mathbb{R},$$

denote by

$$\bar{F}(t,x) := -F(t, \phi_t^F(x)) \quad \text{and} \quad \hat{F}(t,x) = -F(1-t, x).$$

The flows of these functions are $\phi_t^{\bar{F}} = (\phi_t^F)^{-1}$ and $\phi_t^{\hat{F}} = \phi_{1-t}^F \circ (\phi_1^F)^{-1}$, respectively. Note that both these flows have the same time-1 map:

$$\phi_1^{\bar{F}} = \phi_1^{\hat{F}} = (\phi_1^F)^{-1}.$$

For two Hamiltonian functions $F, G: [0,1] \times V \to \mathbb{R}$, denote by $G\#F: [0,1] \times V \to \mathbb{R}$ the function

$$(G\#F)(t,x) = G(t,x) + F(t, (\phi_t^G)^{-1}(x)).$$

Its Hamiltonian flow is $\phi_t^{G\#F} = \phi_t^G \circ \phi_t^F$. Given a Floer datum $\mathcal{D} = (F, J)$ and a Hamiltonian flow ϕ_t^G generated by G we denote by $\phi_*^G \mathcal{D} := (G\#F, \phi_*^G J)$ the push-forward Floer datum, where $(\phi_*^G J)_t := D\phi_t^G \circ J_t \circ (D\phi_t^G)^{-1}$.

Let $L_0, L_1 \subset \operatorname{Int} V$ be two exact Lagrangians and assume that the Floer datum $\mathcal{D} = (F, J)$ is regular. Let G be another Hamiltonian function. There is a *naturality map*

$$\begin{gathered} \mathcal{N}_G: CF(L_0, L_1; \mathcal{D}) \to CF(L_0, \phi_1^G(L_1); \phi_*^G \mathcal{D}), \\ \mathcal{N}_G(\gamma)(t) := \phi_t^G \gamma(t), \quad \forall \gamma \in \mathcal{O}_{L_0,L_1}(F). \end{gathered} \tag{10}$$

The map $\mathcal{N}_G$ is a chain isomorphism.

Consider now a Lagrangian L_1' which is exact isotopic to L_1. Fix a Hamiltonian function G such that $\phi_1^G(L_1) = L_1'$. The map induced in homology by $\mathcal{N}_G$ is compatible with the homological maps induced by continuation. Therefore, $\mathcal{N}_G$ induces a well defined isomorphism

$$HF(L_0, L_1) \to HF(L_0, L_1').$$

Moreover, by [17] (see also [8, 16]) this isomorphism is independent of the choice of G (among Hamiltonian functions G with $\phi_1^G(L_1) = L_1'$). We thus obtain a system of canonical isomorphisms

$$\mathcal{N}_{L_1',L_1}^{L_0}: HF(L_0, L_1) \to HF(L_0, L_1'),$$

defined for every pair of exact isotopic Lagrangians L_1, L_1'. Moreover,

$$\mathcal{N}_{L_1,L_1}^{L_0} = \mathrm{id}, \quad \mathcal{N}_{L_1'',L_1'}^{L_0} \circ \mathcal{N}_{L_1',L_1}^{L_0} = \mathcal{N}_{L_1'',L_1}^{L_0}.$$

Remarks 2.2.1. (1) For the latter statement to hold it is important that the Lagrangians are exact, or more generally weakly exact. Indeed, in the presence of holomorphic disks (e.g., for monotone Lagrangians) the isomorphisms $\mathcal{N}_{L_1',L_1}^{L_0}$ might depend on the homotopy class of the path between L_1 and L_1' inside the space of exact Lagrangians.

(2) Denote by $*: HF(L_0, L_1) \otimes HF(L_1, L_1') \to HF(L_0, L_1')$ the product induced by the chain level μ_2-operation. Then there exists a class $c_{L_1,L_1'} \in HF(L_1, L_1')$ such that

$$\mathcal{N}_{L_1',L_1}^{L_0}(a) = a * c_{L_1,L_1'}$$

for every $a \in HF(L_0, L_1)$. In fact,

$$c_{L_1,L_1'} = \mathcal{N}_{L_1',L_1}^{L_1}(e_{L_1}),$$

where $e_{L_1} \in HF(L_1, L_1)$ is the unity.

Similarly to the maps $\mathcal{N}_{L_1',L_1}^{L_0}$ we also have canonical isomorphisms

$$\mathcal{N}_{L_1}^{L_0',L_0}: HF(L_0, L_1) \to HF(L_0', L_1),$$

defined in an analogous way.

We now turn to the PSS isomorphism. Let $L \subset \mathrm{Int}\, V$ be an exact Lagrangian. Let $\mathfrak{m} = (f, \rho)$ be a Morse datum, consisting of a Morse function $f: L \to \mathbb{R}$ and a Riemannian metric ρ on L. Denote by $\mathcal{C}(L; \mathfrak{m})$ the Morse complex associated to $\mathfrak{m}$. Let $\mathcal{D} = (H, J)$ be a regular Floer datum for the pair (L, L). The PSS map is a quasi-isomorphism

$$PSS_{\mathfrak{m},\mathcal{D}}: \mathcal{C}(L; \mathfrak{m}) \to CF(L, L; \mathcal{D}) \tag{11}$$

canonical up to chain homotopy. Moreover, the maps $PSS_{\mathfrak{m},\mathcal{D}}$, defined for different $\mathfrak{m}$, $\mathcal{D}$, are compatible with the corresponding continuation maps up to chain homotopy. Consequently, the isomorphism induced by PSS in homology

$$PSS\colon H_*(L) \to HF(L,L),$$

which we also denote by PSS, is independent of the data $\mathfrak{m}$, $\mathcal{D}$. Moreover, this map is multiplicative (with respect to the intersection product on $H_*(L)$ and the triangle product induced by μ_2 on $HF(L,L)$) and it sends the fundamental class $[L]$ to the unit $e_L \in HF(L,L)$. We refer the reader to [1, 18] for the definition and properties of this map.

Remarks 2.2.2. (1) Let $L_0, L_1 \subset \operatorname{Int} V$ be two exact Lagrangians that are exact isotopic. Choose any exact isotopy

$$\phi_t\colon L_0 \to \operatorname{Int} V, \quad t \in [0,1],$$

with $\phi_0 =$ inclusion of $L_0 \subset V$ and $\phi_1(L_0) = L_1$. By a result of Hu–Lalonde–Leclercq [17], the map

$$\phi_{1*}\colon H_*(L_0;\mathbb{Z}_2) \to H_*(L_1;\mathbb{Z}_2),$$

induced in homology by ϕ_1, is independent of the choice of the isotopy $\{\phi_t\}$. Therefore, there is a canonical map

$$\phi_*\colon H_*(L_0;\mathbb{Z}_2) \to H_*(L_1;\mathbb{Z}_2)$$

between any two exact isotopic exact Lagrangians in $\operatorname{Int} V$. The map ϕ_* is compatible with Floer theory in the following sense. First note that if $\{\phi_t\}$ is an exact isotopy as above, its time-1 map induces a map in Floer homology

$$\phi_1^{HF}\colon HF(L_0,L_0) \to HF(L_1,L_1).$$

Moreover, this map is independent of the choice of the isotopy. In fact,

$$\phi_1^{HF} = \mathcal{N}_{L_1}^{L_1,L_0} \circ \mathcal{N}_{L_1,L_0}^{L_0}.$$

Write $\phi^{HF} := \phi_1^{HF}$. Standard arguments then show that ϕ_* equals the composition

$$H_*(L_0;\mathbb{Z}_2) \xrightarrow{PSS} HF(L_0,L_0) \xrightarrow{\phi^{HF}} HF(L_1,L_1) \xrightarrow{PSS^{-1}} H_*(L_1;\mathbb{Z}_2).$$

(2) In general the space of exact Lagrangians in V might be disconnected (and even contain Lagrangians of different topological types). However, in certain situation this is not expected to be so. For example, a version of the nearby Lagrangian conjecture asserts that if $V = T^*(N)$ is the cotangent bundle of a closed manifold N then all exact Lagrangians are exact isotopic to the zero-section. While this is still open in

general, a result of Fukaya–Seidel–Smith [14, 15] and independently of Nadler [25], says that under mild topological assumptions on N the following holds. Every exact Lagrangian $L \subset T^*(N)$ is canonically isomorphic, when viewed as an objects in the (compact) derived Fukaya category of $T^*(N)$, to the zero-section. Moreover, this isomorphism induces the same map $HF(L, L) \to HF(N, N)$ as the one induced by the projection pr: $T^*(N) \to N$ on homology $H_*(L) \to H_*(N)$, under the canonical identifications $HF(L, L) \cong H_*(L)$ and $HF(N, N) \cong H_*(N)$.

2.2.4. Action filtrations and Floer persistent homology. We begin by recalling the fundamentals of filtered Lagrangian Floer theory in the exact setting. Much of the general theory has been developed in [10, 19, 22, 23, 26, 27], though in somewhat different frameworks like monotone (and weakly exact) Lagrangians. The essence however remains the same and a considerable part of these papers applies with minor changes to the exact case too.

In order to define the action functional and its induced filtrations in Floer theory we need to endow each exact Lagrangian L with a primitive $h_L: L \to \mathbb{R}$ of the exact form $\lambda|_L$. We will refer to h_L as a *marking of* L and to the pair (L, h_L) as a *marked Lagrangian*. However, for simplicity of notation we will often continue to denote marked Lagrangians by a single letter, e.g., L, with the understanding that the primitive h_L has been fixed.

Let $L_0, L_1 \subset \operatorname{Int} V$ be two marked Lagrangians. Let $\mathcal{D} = (H, J)$ be a regular Floer datum for (L_0, L_1). For $\alpha \in \mathbb{R}$, denote

$$CF^{\leq \alpha}(L_0, L_1; \mathcal{D}) := \bigoplus_{\gamma \in \mathcal{O}(H),\, \mathcal{A}_H(\gamma) \leq \alpha} \mathbb{Z}_2 \gamma. \tag{12}$$

For convenience, we extend $\mathcal{A}_H$ to all elements of $CF(L_0, L_1; \mathcal{D})$ by defining it on $\lambda = \sum_{i=1}^{k} a_i \gamma_i$, $a_i \in \mathbb{Z}_2$ to be:

$$\mathcal{A}_H(\lambda) = \max\left\{\mathcal{A}_H(\gamma_i) \mid a_i \neq 0\right\} = \inf\left\{\alpha \mid \lambda \in CF^{\leq \alpha}(L_0, L_1; \mathcal{D})\right\}.$$

Here we use the convention that $\max \emptyset = -\infty$, so that $\mathcal{A}_H(0) = -\infty$.

The subspaces $CF^{\leq \alpha} \subset CF$ are in fact subcomplexes. This is so because for every Floer trajectory $u \in \mathcal{M}(\gamma_-, \gamma_+; \mathcal{D})$ we have the following action-energy relation:

$$\mathcal{A}_H(\gamma_+) = \mathcal{A}_H(\gamma_-) - E(u) \leq \mathcal{A}_H(\gamma_-).$$

Therefore, $\mathcal{A}_H(d\gamma) \leq \mathcal{A}_H(\gamma)$, hence

$$d(CF^{\leq \alpha}(L_0, L_1; \mathcal{D})) \subset CF^{\leq \alpha}(L_0, L_1; \mathcal{D}).$$

We write $HF^{\leq \alpha}(L_0, L_1; \mathcal{D}) := H_*(CF^{\leq \alpha}(L_0, L_1; \mathcal{D}))$ and for $\alpha \leq \beta \leq \infty$ we denote by

$$i_{\beta,\alpha}: HF^{\leq \alpha}(L_0, L_1; \mathcal{D}) \to HF^{\leq \beta}(L_0, L_1; \mathcal{D})$$

the map induced by the inclusion $CF^{\leq\alpha}(L_0, L_1; \mathcal{D}) \subset CF^{\leq\beta}(L_0, L_1; \mathcal{D})$. For $\beta = \infty$ we abbreviate $i^{\alpha} := i^{\infty,\alpha}$.

The homologies $HF^{\leq\alpha}(L_0, L_1; \mathcal{D})$, $\alpha \in \mathbb{R}$, and the maps $i_{\beta,\alpha}$, $\alpha \leq \beta$, fit together into a persistence module, which we denote by $HF^{\leq\bullet}(L_0, L_1; \mathcal{D})$ and call the Floer persistent homology.

Next, we briefly discuss to what extent the Floer persistent homology depends on the Floer data. The continuation maps $\psi_{\mathcal{D}',\mathcal{D}}$ do not preserve action-filtrations in general, hence there is no meaning to write $H(CF^{\leq\alpha}(L_0, L_1))$ without specifying the Floer datum. Nevertheless, if $\mathcal{D}' = (H, J')$ and $\mathcal{D}'' = (H, J'')$ are two regular Floer data with the same Hamiltonian function H, then one can choose the continuation map

$$\psi_{\mathcal{D}'',\mathcal{D}'}: CF(L_0, L_1; \mathcal{D}') \to CF(L_0, L_1; \mathcal{D}'')$$

to be action preserving. Moreover, for such Floer data, the chain homotopies between $\psi_{\mathcal{D}',\mathcal{D}''} \circ \psi_{\mathcal{D}'',\mathcal{D}'}$ and id can be also chosen to preserve action. It follows that $\psi_{\mathcal{D}'',\mathcal{D}'}$ induces an isomorphism between the persistence modules $HF^{\leq\bullet}(L_0, L_1; \mathcal{D}')$ and $HF^{\leq\bullet}(L_0, L_1; \mathcal{D}'')$. Moreover, standard arguments imply that this isomorphism is canonical (in the sense that there is a preferred such isomorphism). Thus the Floer persistent homology of (L_0, L_1) depends only on the Hamiltonian function in the Floer data, hence will sometimes be denoted by $HF^{\leq\bullet}(L_0, L_1; H)$. In case $L_0 \pitchfork L_1$ we can take the Hamiltonian function to be 0, and the Floer persistent homology using this choice will be abbreviated as $HF^{\leq\bullet}(L_0, L_1)$.

The persistence modules $HF^{\leq\bullet}(L_0, L_1; \mathcal{D})$ give rise to a variety of numerical invariants. The most important ones for us will be spectral invariants and boundary depth.

Given $a \in HF(L_0, L_1; \mathcal{D})$ we denote by $\sigma(a; L_0, L_1; \mathcal{D})$ the spectral invariant of a, defined by the recipe in (3) of Section 2.1 for the chain complex $CF(L_0, L_1; \mathcal{D})$. By the preceding discussion the spectral invariants $\sigma(a; L_0, L_1; (H, J))$ as well as boundary depth $\beta(CF(L_0, L_1; (H, J))$ do not depend on J, hence we will sometimes denote them by $\sigma(a; L_0, L_1; H)$ and $\beta(CF(L_0, L_1; H))$, respectively.

Next we discuss the version of spectral invariants involved in the definition of the spectral metric, namely $c(a; L_0, L_1)$, where $L_0 \subset \operatorname{Int} V$ is a marked exact Lagrangian, $a \in H_*(L_0)$, and $L_1 \subset \operatorname{Int} V$ is another marked Lagrangian which is exact isotopic to L_0. (Here the marking on L_1 is arbitrary and is not assumed to be related in any way to the given marking of L_0 via any isotopy going from L_0 to L_1.) Consider the following composition of isomorphisms

$$H_*(L_0) \xrightarrow{PSS} HF(L_0, L_0) \xrightarrow{\mathcal{N}^{L_0}_{L_1,L_0}} HF(L_0, L_1).$$

Assume first that L_1 intersects L_0 transversely. Choose an almost complex structure J such that the Floer datum $(0, J)$ is regular. Consider the chain complex

$$CF\big(L_0, L_1; (0, J)\big)$$

endowed with the action filtration, as defined at (12). Consider also the class

$$\mathcal{N}^{L_0}_{L_0,L_1} \circ PSS(a)$$

viewed as an element of $H_*(CF(L_0, L_1; (0, J))) = HF(L_0, L_1)$. We then define

$$c(a; L_0, L_1) := \sigma\big(\mathcal{N}^{L_0}_{L_1,L_0} \circ PSS(a); L_0, L_1; 0\big). \tag{13}$$

In case L_0 and L_1 do not intersect transversely, we define

$$c(a; L_0, L_1) = \lim_{\|H\|\to 0} \sigma\big(\mathcal{N}^{L_0}_{L_1,L_0} \circ PSS(a); L_0, L_1; H\big),$$

where

$$\|H\| := \int_0^1 \big(\max_{x\in V} H(t,x) - \min_{x\in V} H(t,x)\big)\, dt,$$

and $\|H\| \to 0$ through Hamiltonian functions for which $\phi_1^H(L_0) \pitchfork L_1$. The fact that the limit exists and is finite follows from Lipschitz continuity of the spectral invariants σ with respect to the Hofer norm (see, e.g., [22]).

Finally, given an exact Lagrangian $L \subset \operatorname{Int} V$ we define the spectral distance γ on the space $\mathcal{L}_{\mathrm{ex},L}(\operatorname{Int} V)$ of Lagrangians in $\operatorname{Int} V$ which are exact isotopic to L by

$$\gamma(L_0, L_1) = c\big([L_0]; L_0, L_1\big) - c\big([\mathrm{pt}]; L_0, L_1\big), \quad \forall L_0, L_1 \in \mathcal{L}_{\mathrm{ex},L}(\operatorname{Int} V). \tag{14}$$

Remarks 2.2.3. (1) The primitives $h_{L_i}: L_i \to \mathbb{R}$ for the exact 1-forms $\lambda|_{L_i}$, $i = 1, 2$, are uniquely determined only up to additions of constants. Similarly, one can add to the Hamiltonian function H a (time dependent) constant $C(t)$. Different such choices have no effect on Floer complex $CF(L_0, L_1; \mathcal{D})$ and its homology, but they do add a constant to the action functional, hence shift the filtration on $CF(L_0, L_1; \mathcal{D})$ by an overall constant. Consequently, the spectral numbers $\sigma(a; L_0, L_1; H)$ and $c(b; L_0, L_1)$ get shifted by a constant which is independent of a, b. Let $a', a'' \in HF(L_0, L_1)$, $b', b'' \in H_*(L_0)$. It follows that each of the differences

$$\sigma(a''; L_0, L_1; H) - \sigma(a'; L_0, L_1; H), \quad c(b''; L_0, L_1) - c(b'; L_0, L_1)$$

is *independent* of the preceding choices. In particular, the spectral distance $\gamma(L_0, L_1)$ is independent of any choice of marking on L_0 and L_1.

(2) The action functional and the spectral invariants depend on the choice λ of the Liouville form. However, altering λ by an exact 1-form has no effect on these quantities. More specifically, let $f: V \to \mathbb{R}$ be a smooth function and consider $\lambda' = \lambda + df$. The latter is also a primitive of the symplectic form ω.

Clearly, a Lagrangian in V is λ-exact if and only if it is λ'-exact. Let $L_0, L_1 \subset V$ be two λ-exact Lagrangians and fix primitives $h_{L_i}: L_i \to \mathbb{R}$ for $\lambda|_{L_i}$, $i = 0, 1$. Then $h'_{L_i} := h_{L_i} + f|_{L_i}$ is a primitive for $\lambda'|_{L_i}$. Denote by $\mathcal{A}'_H: \mathcal{P}_{L_0,L_1} \to \mathbb{R}$ the action functional defined using λ' and the primitives h'_{L_i}, and by $\mathcal{A}_H$ the one defined using λ and the h_{L_i}'s. A simple calculation shows that $\mathcal{A}'_H = \mathcal{A}_H$. It follows that the spectral invariants σ and c remain the same when replacing λ by λ' (provided we use the primitives h'_{L_i} as above). Consequently, the spectral metric γ remains unchanged too (the latter does not even depend on the choices of the primitive functions h_{L_i} or h'_{L_i}).

(3) In case $V = T^*(N)$ is the cotangent bundle of a closed manifold N, one can extend the definition of the spectral invariants $c(a; L_0, L_1)$ as well as the spectral metric $\gamma(L_0, L_1)$ to arbitrary pairs of exact Lagrangians (i.e., including also pairs that are, hypothetically, not isotopic one to the other). This follows from point (2) of Remark 2.2.2.

Another source of numerical invariants comes from the barcode

$$\mathcal{B}\big(HF^{\leq \bullet}(L_0, L_1; \mathcal{D})\big)$$

of the persistence module $HF^{\leq \bullet}(L_0, L_1; \mathcal{D})$, see [28] for the definition. Of main interest for our considerations is the boundary depth $\beta(L_0, L_1; \mathcal{D})$, which by definition is the length of the longest finite bar in the barcode $\mathcal{B}(HF^{\leq \bullet}(L_0, L_1; \mathcal{D}))$. We will discuss this invariant in more detail in Section 7.1, and give alternative equivalent definitions of it.

2.3. Weakly filtered Fukaya categories. Occasionally it will be convenient to view all exact Lagrangian submanifold as objects in a Fukaya category, taking into account action filtrations.

Denote by $\mathcal{F}uk(V)$ the Fukaya category whose objects are the closed marked Lagrangian submanifolds $L \subset V$ (see the beginning of Section 2.2.4 for the definition). Note that each underlying Lagrangian appears in this category with all its possible markings. Here $\mathcal{F}uk(V)$ is an A_∞-category whose realization requires additional auxiliary structures, namely Floer data for all pairs of objects as well as coherent perturbation data for every tuple of objects. We will suppress these choices from the notation, whenever these choices are clear (or irrelevant). We refer to [32] for the foundations of Fukaya categories. In contrast to this (and most) references on the subject, our Fukaya categories (and all Floer complexes in general) will be ungraded.

The Fukaya category $\mathcal{F}uk(V)$ has the structure of a so called weakly filtered A_∞-category. This means that

$$\hom_{\mathcal{F}uk(V)}(L_0, L_1) = CF(L_0, L_1)$$

between every pair of objects (L_0, L_1) is a filtered chain complex, and moreover each of the higher order operations μ_d, $d \geq 2$, preserves these filtrations up to a uniformly bounded error (i.e., the error for μ_d depends only on d, and not on the objects involved in it). We refer the reader to [7, §2] for more details on this theory.

2.4. Local and global Floer theory. Let $(V, J_V, \varphi, R_0, \omega = d\lambda)$ be a Liouville manifold which is Stein at infinity. Let $W_0 \subset V$ be a compact Liouville subdomain, endowed with the structures λ and ω coming from V. Let $L_0, L_1 \subset \operatorname{Int} W_0$ be two closed marked λ-exact Lagrangian submanifolds. Consider Hamiltonian functions $H\colon [0,1] \times W_0 \to \mathbb{R}$, compactly supported in $[0,1] \times \operatorname{Int} W_0$, such that $\phi_1^H(L_0) \pitchfork L_1$. We will view these also as Hamiltonian functions on V by extending them to be 0 outside W_0.

The following proposition compares the local and global Floer invariants of (L_0, L_1). It says that the Floer homologies as well as filtered numerical invariants of $(L_0, L_1; H)$, when viewed either in W_0 ("local") or in V ("global"), coincide.

Proposition 2.4.1. *There exist isomorphisms of persistence modules*

$$j^{\leq \bullet}\colon HF^{\leq \bullet}\big(L_0, L_1; H; (W_0, \omega = d\lambda)\big) \to HF^{\leq \bullet}\big(L_0, L_1; H; (V, \omega = d\lambda)\big)$$

defined for every pair of closed marked Lagrangians (L_0, L_1) and H as above. Moreover, the corresponding isomorphisms

$$j := j^{\leq \infty}\colon HF\big(L_0, L_1; (W_0, \omega = d\lambda)\big) \to HF\big(L_0, L_1; (V, \omega = d\lambda)\big)$$

on the total homologies are independent of H and have the following further properties:

(1) *They are compatible with the triangle products.*

(2) *They are compatible with the naturality maps $\mathcal{N}^{L_0}_{L_1', L_1}$ from Section 2.2.3 (in case L_1' and L_1 are exact-isotopic) as well as with PSS (in case $L_0 = L_1$).*

(3) *They preserve spectral invariants, namely*

$$\sigma\big(j(a); L_0, L_1; H; (V, \lambda)\big) = \sigma\big(a; L_0, L_1; H; (W_0, \lambda)\big), \qquad \forall a \in HF(L_0, L_1; (W_0, \omega)).$$

Remark 2.4.2. Proposition 2.4.1 does not hold without the assumption that L_0, L_1 are exact. For example, take $L_0 = L_1$ to be a circle in $V = \mathbb{R}^2$ endowed with the standard symplectic structure ω_{std}, and let W_0 be a small tubular neighborhood of this circle. Then, $HF(L_0, L_1; W_0, \omega_{\mathrm{std}}) \cong H_*(S^1)$, but $HF(L_0, L_1; V, \omega_{\mathrm{std}}) = 0$.

Proof of Proposition 2.4.1. The main idea in the proof is based on a rescaling (or shrinking) argument from [14, Section 5] which we adapt here to our setting.

We will assume without loss of generality that $L_0 \pitchfork L_1$ and that $H \equiv 0$. This simplifies notation and the proof of the general case is very similar to the one we will present below.

Fix $R > R_0$ such that $R > \max_{\overline{W}_0} \varphi$ (so that $\overline{W}_0 \subset V_{\varphi < R}$). Consider the completion

$$(V, J_V, R_0, \varphi_R, \widehat{\omega}_R = d\widehat{\lambda}_R)$$

of $(V, J_V, R_0, \varphi, \omega = d\lambda)$, as described in Section 2.2.1. Put $\lambda_0 := \lambda|_{\overline{W}_0}$.

Denote by $\widehat{\psi}_t: V \to V$ the Liouville flow corresponding to the completion, and by $\widehat{\psi}: \mathbb{R} \times \partial W_0 \to V$ the embedding $\Psi(s,x) := \widehat{\psi}_s(x)$. Note that $\widehat{\psi}_t|_{W_0} = \psi_t|_{W_0}$ for every $t \leq 0$, where ψ_t is the Liouville flow corresponding to the uncompleted Liouville manifold. For an interval $I \subset \mathbb{R}$ we write

$$\mathcal{S}(I) := \widehat{\psi}(I \times \partial W_0) \subset V.$$

Recall from Section 2.2.1 the model almost complex structure $\widehat{J}^{\lambda_0}$ on $\mathbb{R} \times \partial W_0$. Consider now its push forward $\widehat{\psi}_* \widehat{J}^{\lambda_0}$ defined on $\mathcal{S}(\mathbb{R})$. Slightly abusing notation we will continue to denote this almost complex structure by $\widehat{J}^{\lambda_0}$. Note that $\mathcal{S}(\mathbb{R})$ is invariant under the flow $\widehat{\psi}_t$, and moreover $\widehat{\psi}_t$ is $\widehat{J}^{\lambda_0}$-holomorphic along $\mathcal{S}(\mathbb{R})$.

Fix $\delta > 0$ small enough such that $L_0, L_1 \subset W_0 \setminus \mathcal{S}([-\delta, 0])$. For every $T > 0$, consider the space $\mathcal{J}_{(T)}$ of almost complex structures J on V that have the following properties:

(1) J is compatible with $\widehat{\omega}_R$.

(2) $J = \widehat{J}^{\lambda_0}$ on $\mathcal{S}([-\delta, T])$.

(3) $J = J_V$ at infinity.

Denote the space of time-dependent almost complex structure $J = \{J_t\}_{t\in[0,1]}$ with $J_t \in \mathcal{J}_{(T)}$ for every t, by $\mathcal{J}_{(T)}^{[0,1]}$.

Lemma 2.4.3. *There exists $T_0 > 0$ such that the following holds for every $T \geq T_0$: for every regular Floer datum $\mathcal{D} = (0, J)$ with $J \in \mathcal{J}_{(T)}^{[0,1]}$ and every Floer strip $u: \mathbb{R} \times [0,1] \to V$ corresponding to $(L_0, L_1; \mathcal{D})$ we have* $\operatorname{image} u \subset W_0$.

Proof of Lemma 2.4.3. Consider the Lagrangian submanifolds

$$L_0^{-T} := \psi_{-T}(L_0), \quad L_1^{-T} := \psi_{-T}(L_1)$$

of W_0. Note that L_0^{-T}, L_1^{-T} are both λ-exact and $L_0^{-T} \pitchfork L_1^{-T}$. For $x \in L_0 \cap L_1$ write

$$x_{-T} := \psi_{-T}(x).$$

Denote by $\mathcal{A}^{(L_0,L_1)}$ and by $\mathcal{A}^{(L_0^{-T},L_1^{-T})}$ the action functionals of (L_0, L_1) and (L_0^{-T}, L_1^{-T}), respectively, both defined with the Hamiltonian perturbation term $H \equiv 0$. A simple calculation shows that

$$\mathcal{A}^{(L_0^{-T},L_1^{-T})}(x_{-T}) = e^{-T} \mathcal{A}^{(L_0,L_1)}(x). \tag{15}$$

Let $u: \mathbb{R} \times [0,1] \to \mathbb{R}$ be a Floer strip associated to $(L_0, L_1; (0, J))$ with $J \in \mathcal{J}_{(T)}^{[0,1]}$. Put $v_{-T} := \widehat{\psi}_{-T} \circ u$. Then v_{-T} is a Floer strip corresponding to

$$\big(L_0^{-T}, L_1^{-T}; (0, (\widehat{\psi}_{-T})_* J)\big).$$

Note that $(\widehat{\psi}_{-T})_* J$ is compatible with $\widehat{\omega}_R$. Moreover, by the definition of $\mathcal{J}_{(T)}$ we have $(\widehat{\psi}_T)_* J = J$ on $\mathcal{S}([-\delta - T, 0])$. Recall also that by definition $J \equiv J^{\lambda_0}$ on $\mathcal{S}([\delta, 0])$.

Denote the energy of Floer strips by E. We have:

$$E(v_{-T}) = e^{-T} E(u) \leq e^{-T}\big(\max_{x \in L_0 \cap L_1} \mathcal{A}^{(L_0,L_1)}(x) - \min_{y \in L_0 \cap L_1} \mathcal{A}^{(L_0,L_1)}\big). \tag{16}$$

By a standard energy-length (a.k.a. monotonicity) estimate for pseudo-holomorphic curves (see, e.g., [14, Section 5.a]) we have that image $v_{-T} \subset W_0$ provided that the right-hand side of (16) is small enough, which in turn can be assured by taking T to be large enough.

Now $L_0^{-T}, L_1^{-T} \subset W_0 \setminus \mathcal{S}([-\delta - T, 0])$, hence by the maximum principle (applied to the J^{λ_0}-convex function $\phi\colon \mathcal{S}(\mathbb{R}) \to \mathbb{R}$, $\phi(s,x) = e^s$) we in fact have:

$$\text{image}\, v_{-T} \subset W_0 \setminus \mathcal{S}\big([-\delta - T, 0]\big).$$

It follows that $u = \widehat{\psi}_T \circ v_{-T}$ has its image inside $W_0 \setminus \mathcal{S}\big([-\delta, 0]\big) \subset W_0$. This concludes the proof of Lemma 2.4.3. □

We proceed now with the proof of Proposition 2.4.1. Fix J^{λ_0} on $\mathcal{S}([-\delta, 0])$. Consider Floer data of the type $\mathcal{D} = (H \equiv 0, J)$ with $J \in \mathcal{J}_{(T)}^{[0,1]}$. By standard transversality arguments, for every $T > 0$ there exists J as above which makes $\mathcal{D}$ regular. Lemma 2.4.3 implies that there exists $T_0 > 0$ such that for every $T \geq T_0$ and every $J \in \mathcal{J}_{(T)}^{[0,1]}$ with $(0, J)$ regular, the identity map

$$i\colon CF\big(L_0, L_1; (0, J|_{W_0}); (W_0, \omega = d\lambda)\big) \to CF\big(L_0, L_1; (0, J); (V, \widehat{\omega}_R = d\widehat{\lambda}_R)\big) \tag{17}$$

is a chain map. Clearly i preserves action, hence induces an isomorphism of persistence modules

$$i^{\leq \bullet}\colon HF^{\leq \bullet}\big(L_0, L_1; H = 0; (W_0, \omega)\big) \to HF^{\leq \bullet}\big(L_0, L_1; H = 0; (V, \widehat{\omega}_R)\big).$$

Finally, note that by the maximum principle the persistence modules

$$HF^{\leq \bullet}\big(L_0, L_1; H = 0; (V, \widehat{\omega}_R = d\widehat{\lambda}_R)\big), \quad HF^{\leq \bullet}\big(L_0, L_1; H = 0; (V, \omega = d\lambda)\big)$$

coincide. Thus, the isomorphism $i^{\leq \bullet}$ induces the isomorphism $j^{\leq \bullet}$ claimed by the proposition. It implies also the statement at point (3).

As mentioned at the beginning of the proof, the arguments above can be easily adapted to the case of Floer data of the type $\mathcal{D} = (H, J)$ with $J \in \mathcal{J}_{(T)}^{[0,1]}$ and $H\colon [0,1] \times V \to \mathbb{R}$ compactly supported inside $[0,1] \times W_0 \setminus \mathcal{S}([-\delta, 0])$.

Moreover, very similar arguments to the above imply that if

$$L_0, \dots, L_d \subset W_0 \setminus \mathcal{S}\big([-\delta, 0]\big)$$

are exact Lagrangians then there exists $T_0 > 0$ such that for every disk S (with $(d+1)$ boundary punctures) and for every choice of perturbation data $\mathcal{D}_{L_0,\dots,L_d} = (K, J)$ with Hamiltonian term K such that K_z is compactly supported in $W_0 \setminus \mathcal{S}([-\delta, 0])$ for every $z \in S$ and with almost complex structure $J = \{J\}_{z\in S}$ such that $J_z \in \mathcal{J}_{(T)}$ for every $z \in S$, the following holds: every Floer polygon $u: S \to V$ corresponding to $(L_0, \dots, L_d; (K, J))$ satisfies image $u \subset W_0$.

The statements of points (1) and (2) readily follow. □

3. Cotangent bundles and real Lefschetz fibration

3.1. Real Lefschetz fibrations. In this paper we will adopt the following definition of Lefschetz fibrations, essentially as in [14]. By a Lefschetz fibration $\pi: E \to \mathbb{C}$ we mean a symplectic manifold E, endowed with a symplectic structure ω_E as well as an ω_E-compatible almost complex structure J_E such that the following holds:

(1) π is (J_E, i)-holomorphic and has a finite number of critical points. Moreover, we assume that every critical value of π corresponds to precisely one critical point of π. We denote the set of critical points of π by $\mathrm{Crit}(\pi)$ and by $\mathrm{Critv}(\pi) \subset \mathbb{C}$ the set of critical values of π. For every $z \in \mathbb{C}$ we denote by $E_z = \pi^{-1}(z)$ the fiber over z.

(2) All critical points of π are ordinary double points in the following sense. For every $p \in \mathrm{Crit}(\pi)$ there exist a J_E-holomorphic chart around p (hence J_E is integrable on this chart) with respect to which π is a holomorphic Morse function.

(3) There exists and exhaustion function $\varphi_E: E \to \mathbb{R}$ and $R_0 \in \mathbb{R}$ such that $(E, J_E, R_0, \varphi_E, \omega_E)$ is a symplectic manifold which is Stein at infinity; see Section 2.2.1.

(4) We assume that for every compact subset $K \subset \mathbb{C}$ there exists $R_K \geq R_0$ such that each level set $\varphi_E^{-1}(R)$, $R \geq R_K$, intersects each fiber E_z, $z \in K$, transversely. Note that this implies that for every $z \in K$,

$$\mathrm{Crit}(\varphi_E|_{E_z}) \subset E_{\varphi_E \leq R}.$$

Thus, $(E_z, J_E|_{E_z}, R_K, \varphi_E|_{E_z}, \omega_E|_{E_z})$ is a symplectic manifold which is Stein at infinity, for every $z \in K \setminus \mathrm{Critv}(\pi)$.

(5) Denote by Γ the symplectic connection on $E \setminus \mathrm{Crit}(\pi)$, associated to ω_E. (Recall that the horizontal distribution of this connection is the ω_E-complement of the tangent spaces of the fibers of π.) Let $\gamma: [0, 1] \to \mathbb{C}$ be a smooth curve. Then the parallel transport $\Pi_\gamma: E_{\gamma(0)} \to E_{\gamma(1)}$ along γ is well defined at infinity.

We now turn to real Lefschetz fibrations. By a real structure on a Lefschetz fibration $\pi: E \to \mathbb{C}$ we mean an involution $c_E: E \to E$ which is anti ω_E-symplectic and covers (with respect to π) the standard complex conjugation $c_{\mathbb{C}}: \mathbb{C} \to \mathbb{C}$. We will assume in addition that c_E is anti J_E-holomorphic. We denote by $E_{\mathbb{R}} \subset E$ the fixed locus of c_E and call it the *real part of* E. Note that $E_{\mathbb{R}}$ is automatically a smooth Lagrangian submanifold of E (of course, it might be void).

It turns out that every smooth connected closed manifold can be realized as the real part of a Lefschetz fibration. This is proved in [14, Section 3]. More precisely, in that paper the following is proved. Given a connected closed n-manifold N and a Morse function $f: N \to \mathbb{R}$ with the property that the level set of each critical value contains precisely one critical value, there exist the following:

(1) a smooth affine variety E, endowed with a complex structure denote by J_E;

(2) a proper holomorphic function $\pi: E \to \mathbb{C}$;

(3) a plurisubharmonic function $\varphi: E \to \mathbb{R}$ which is proper and bounded below. Denote by

$$\omega_E = -dd^{\mathbb{C}}\varphi$$

the associated symplectic structure on E. Put also

$$\lambda_E = -d^{\mathbb{C}}\varphi,$$

so that $\omega_E = d\lambda_E$. (Here and in what follows, for a real valued function φ on a complex manifold with complex structure J we denote by $d^{\mathbb{C}}\varphi$ the 1-form $dh \circ J$.);

(4) an anti-J_E-holomorphic involution $c_E: E \to E$ with the following properties:

- (i) The function φ is c_E-invariant. In particular c_E is anti-ω_E-symplectic.
- (ii) $\pi \circ c_E = c_{\mathbb{C}} \circ \pi$, i.e., c_E covers the standard complex conjugation $c_{\mathbb{C}}$.
- (iii) $\pi: E \to \mathbb{C}$ is a Lefschetz fibration (in the sense of the definition from the beginning of Section 3.1) with respect to the structures ω_E and J_E. Moreover, when endowed with c_E, $\pi: E \to \mathbb{C}$ is a real Lefschetz fibration according to the preceding definition.
- (iv) The real part $E_{\mathbb{R}} \subset E$ (with respect to c_E) is diffeomorphic to N.

Moreover, E and its associated structures above can be chosen such that there is a diffeomorphism $\vartheta: N \to E_{\mathbb{R}}$ with $\pi|_{E_{\mathbb{R}}} \circ \vartheta: N \to \mathbb{R}$ arbitrarily close to f in the C^2-topology.

Note that $\mathrm{Critv}(\pi)$ is invariant under the conjugation $c_{\mathbb{C}}$, hence the points of $\mathrm{Critv}(\pi) \setminus \mathbb{R}$ come in pairs of conjugate points. Further, we have $\pi(E_{\mathbb{R}}) \subset \mathbb{R}$ and

$$\mathrm{Critv}(\pi|_{\mathbb{R}}) = \mathrm{Critv}(\pi) \cap \mathbb{R}.$$

For a point $x \in \mathrm{Critv}(\pi) \cap \mathbb{R}$ denote by $T_x^{\uparrow} \subset E$ the Lefschetz thimble associated to the curve $[0,\infty) \ni t \mapsto itx \in \mathbb{C}$.

3.2. Embedding the ball cotangent bundle into a real Lefschetz fibration. A simple calculation shows that $\lambda_E|_{E_{\mathbb{R}}} = 0$, hence $E_{\mathbb{R}} \subset E$ is a λ_E-exact Lagrangian submanifold.

Fix a Riemannian metric on N and denote by $|\cdot|$ the norm on the fibers of $T^*(N)$ corresponding to the Riemannian metric via the isomorphism $T^*(N) \cong T(N)$ induced by the same metric. We denote

$$T^*_{\leq r}(N) = \{v \in T^*(N) \mid |v| \leq r\}$$

the radius-r ball cotangent bundle. Similarly we have $T^*_{<r}(N)$, $T^*_{\geq r}(N)$ etc. and more generally for any subset $I \subset \mathbb{R}$ we write

$$T^*_I(N) = \{v \in T^*(N) \mid |v| \in I\}.$$

Denote by $\lambda_{\mathrm{can}} = p\,dq$ the standard Liouville form on $T^*(N)$ and let $\omega_{\mathrm{can}} = d\lambda_{\mathrm{can}}$ be the canonical symplectic structures. We identify N with the zero section of $T^*(N)$.

Fix a diffeomorphism $\vartheta\colon N \to E_{\mathbb{R}}$ as provided by the previous construction of the real Lefschetz fibration $\pi\colon E \to \mathbb{C}$. By the Darboux–Weinstein theorem there exists $r_0 > 0$ and a symplectic embedding $\kappa\colon T^*_{\leq r_0}(N) \to E$ such that $\kappa(x) = \vartheta(x)$ for every $x \in N$. Moreover, by possibly decreasing $r_0 > 0$ we can arrange that the embedding κ sends the cotangent fibers

$$T^*_x(N) \cap T^*_{\leq r_0}(N), \quad x \in \vartheta^{-1}(\mathrm{Critv}(\pi|_{\mathbb{R}})),$$

to the thimbles $T^{\uparrow}_{\vartheta(x)} \cap \mathrm{image}(\kappa)$ in E. We write from now on

$$\mathcal{U}_{\leq r} = \kappa\big(T^*_{\leq r}(N)\big) \quad \text{for } r \leq r_0,$$

and as before we have the analogous subsets $\mathcal{U}_{<r}$, $\mathcal{U}_{>r}$ and $\mathcal{U}_I$.

Next, as explained in [14] the symplectic embedding κ is exact. More precisely, there exists a function $f\colon T^*_{\leq r_0}(N) \to \mathbb{R}$ such that $\kappa^*\lambda_E = \lambda_{\mathrm{can}} + df$. Moreover, we may assume that $f|_N = 0$. (These statements follow from the fact that $\kappa^*\lambda_E - \lambda_{\mathrm{can}}$ is closed and vanishes along the zero-section $N \subset T^*_{\leq r_0}(N)$.) In view of point ((2)) of Remark 2.2.3 we can replace λ_{can} by $\lambda := \kappa^*\lambda_E$ and work from now on with the form λ_E for defining the action functional, spectral invariants and the spectral metric for exact Lagrangians in $\mathcal{U}_{\leq r_0} \subset E$.

Henceforth. we will identify $T^*_{\leq r_0}(N)$ with $\mathcal{U}_{\leq r_0}$ and write $T^*_{\leq r_0}(N)$ and $\mathcal{U}_{\leq r_0}$ (resp. N and $E_{\mathbb{R}}$) interchangeably for the same thing.

In the following we will need a slight extension of Proposition 2.4.1 that holds also for the thimbles $T^{\uparrow}_{x_j}$. Clearly the thimbles $T^{\uparrow}_{x_j}$ are λ_E-exact Lagrangians and we fix a marking for them. Note that $\mathcal{U}_{\leq r_0} \subset E$ is a compact Liouville subdomain. The following shows that Proposition 2.4.1 essentially holds also for pairs of Lagrangians of the type $(L, T^{\uparrow}_{x_j})$. For simplicity in this proposition we take the Hamiltonian terms in the Floer data to be 0.

Proposition 3.2.1. *There exist isomorphisms of persistence modules*

$$j^{\leq \bullet}: HF^{\leq \bullet}\big(L, T^{\uparrow}_{x_j}; (\mathcal{U}_{\leq r_0}, \omega_E)\big) \to HF^{\leq \bullet}\big(L, T^{\uparrow}_{x_j}; (E, \omega_E)\big)$$

defined for all closed marked λ_E-exact Lagrangians $L \subset \mathcal{U}_{\leq r_0}$. Moreover, the corresponding isomorphisms

$$j := j^{\leq \infty}: HF\big(L, T^{\uparrow}_{x_j}; (\mathcal{U}_{\leq r_0}, \omega_E)\big) \to HF\big(L, T^{\uparrow}_{x_j}; (E, \omega_E)\big)$$

on the total homologies have the following properties:

(1) *They are compatible with the triangle products (among closed Lagrangians).*

(2) *They are compatible with the naturality maps $\mathcal{N}^{L_0', L_0}_{T^{\uparrow}_{x_j}}$ from Section 2.2.3 (in case L_0' and L_0 are exact-isotopic).*

(3) *They preserve spectral invariants, namely*

$$\sigma\big(j(a); L, T^{\uparrow}_{x_j}; (\mathcal{U}_{\leq r_0}, \lambda_E)\big) = \sigma\big(a; L, T^{\uparrow}_{x_j}; (E, \lambda_E)\big),$$
$$\forall a \in HF\big((L, T^{\uparrow}_{x_j}); (\mathcal{U}_{\leq r_0}, \omega_E)\big).$$

Completely analogous statements to the above continue to hold also for pairs of the type $(T^{\uparrow}_{x_j}, L)$ with $L \subset \mathcal{U}_{\leq r_0}$ closed λ_E-exact Lagrangians.

We will omit the proof, as it is based on very similar ideas as the proof of Proposition 2.4.1.

3.3. The extended Lefschetz fibration. In order to use the theory developed in [6] we consider yet another Lefschetz fibration $\pi': E' \to \mathbb{C}$, which we call the extended fibration of E. The construction is taken from [6] and goes as follows. Write the critical values of π as $\mathrm{Critv}(\pi) = \{x_1, \dots, x_k, z_1, \overline{z}_1, \dots, z_l, \overline{z}_l\}$, where $x_i \in \mathbb{R}$ are the real critical values and $z_j, \overline{z}_j$ are pairs of non-real complex conjugate critical values of π. Let $p_i \in E_{x_i}$ be the critical point corresponding to x_i. Let $\nu > 0$ be large enough such that $\nu > |\operatorname{Im} z_j|$ for every j.

Proposition 3.3.1. *There exists a Lefschetz fibration $\pi': E' \to \mathbb{C}$ with the following properties:*

(1) *$(E', \pi', J_{E'}, \omega_{E'})$ coincides with (E, π, J_E, ω_E) over $\{z \in \mathbb{C} \mid -\nu < \operatorname{Im} z\}$. Moreover,*

$$\mathrm{Critv}(\pi') = \{x_1, \dots, x_k, x_1', \dots, x_k', z_1, \overline{z}_1, \dots, z_l, \overline{z}_l\},$$

namely every real critical value x_i, has now a corresponding critical value x_i' (which is not assumed to be real anymore). The new critical values x_i' have $\operatorname{Im} x_i' < -\nu$, and they are placed as depicted in Figure 1.

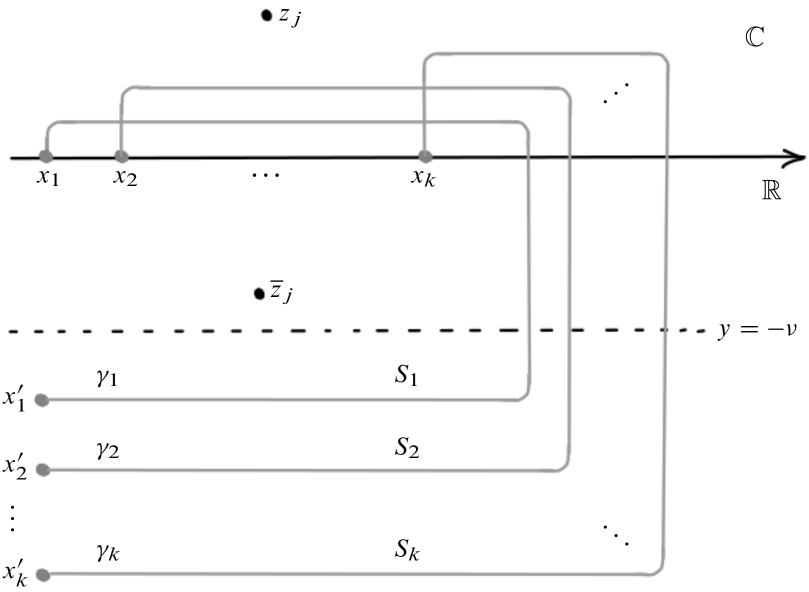

Figure 1. The extended Lefschetz fibration E' and the matching spheres S_j, projected to $\mathbb{C}$.

(2) *Denote by $\gamma_i \subset \mathbb{C}$, $i = 1, \ldots, k$, the paths connecting x_i with x'_i, as in Figure 1 and denote by $p'_i \in E'_{x'_i}$ the critical point corresponding to x'_i. The Lefschetz thimbles emanating from p_i and from p'_i along the two opposite ends of γ_i form a matching sphere $S_i \subset E'$, lying over γ_i. (Put in different words, the vanishing cycles emanating from p_i along γ_i converge over the other end of γ_i to the point p'_i and their union forms a smooth Lagrangian sphere S_i.)*

(3) *The symplectic structure $\omega_{E'}$ is exact. Moreover, it admits a primitive $\lambda_{E'}$ which coincides with λ_E over $E|_{-\nu<\mathrm{Im}\, z}$.*

(4) *There exists an exhaustion function $\varphi' \colon E' \to \mathbb{R}$ and $R_0 \in \mathbb{R}$ such that $(E', J_{E'}, \varphi', R_0, \omega_{E'})$ is a symplectic manifold which is Stein at infinity.*

(5) *The matching spheres S_i from* (2) *are $\lambda_{E'}$-exact.*

Remark 3.3.2. We do not require that the exact 1-form $\lambda_{E'}$ from point (3) of the proposition coincides with $-d^{J_{E'}}\varphi'$ at infinity. While it seems that this can be arranged, we will not need such a statement in the following.

Proof of Proposition 3.3.1. Statements (1), (2), and (4) follow from the theory developed in [32, Sections 15d, 16e].

To prove (3) we begin by showing that $\omega_{E'}$ is exact. Denote $E^+ := E|_{\{-\nu<\mathrm{Im}\, z\}}$. Let $\gamma'_i \subset \mathbb{C}$ be the path obtained from γ_i by chopping a little neighborhood of its second end near x_i', namely $\gamma_i' = \gamma_i \setminus D_i'$, where D_i' is a little open disk around x_i'. Fix also another point $y_i \in \gamma_i \cap E^+$ which is different from x_i.

Denote by $T_{x_i'} \subset E'$ the Lefschetz thimble emanating from p_i along the path γ_i' and by $T_{y_i} \subset T_{x_i'}$ the part of that thimble lying over γ_i', between x_i and y_i. Denote by $\partial T_{x_i'}$ and ∂T_{y_i} the boundaries of these "partial" thimbles. These are Lagrangian spheres in the fibers of E' over x_i' and y_i respectively.

By standard topological arguments there is a canonical isomorphism

$$\kappa\colon H_2(E^+, \cup_{i=1}^k \partial T_{y_i}) \to H_2(E'), \tag{18}$$

where the homologies are taken with any given coefficient group. This isomorphism is induced from the following chain-level map. Let C be a relative cycle of $(E^+, \cup_{i=1}^k \partial T_{y_i})$. For $w \in \gamma_i$ denote by $\Pi_{\gamma_i}^{y_i,w}$ the parallel transport (with respect to the connection induced by $\omega_{E'}$) along γ_i from $E'_{y_i} = E_{y_i}$ to E'_w. Take the part of ∂C lying in ∂T_{y_i} and consider its trail under this parallel transport from y_i to x_i', namely the union of $\Pi_{\gamma_i}^{y_i,w}(\partial C \cap \partial T_{y_i})$, where w runs along γ_i between y_i and x_i'. Note that while $\Pi_{\gamma_i}^{y_i,w}$ is in general not defined for the end point $w = x_i'$, here we apply $\Pi_{\gamma_i}^{y_i,x_i'}$ to $\partial C \cap \partial T_{y_i}$ which yields the point p_i'. Therefore the trail of $\partial C \cap \partial T_{y_i}$ along γ_i between y_i and x_i' is well defined and gives another relative cycle in $(E', \partial T_{y_i})$, which we denote by $\mathrm{Tr}_{y_i,x_i'}(\partial C)$. Note that

$$\partial \mathrm{Tr}_{y_i,x_i'}(\partial C) = -(\partial C \cap \partial T_{y_i}).$$

We can now cap the trails $\mathrm{Tr}_{y_i,x_i'}(\partial C)$, $i = 1, \dots, k$, to C along $\partial C \cap \partial T_{y_i}$, and obtain at the end an absolute cycle C' in E'. The map κ is induced by the chain level map $C \mapsto C'$.

In order to show that $\omega_{E'}$ is exact, we will use the isomorphism κ, with coefficients in $\mathbb{R}$. It is enough to prove that

$$\langle [\omega_{E'}], \kappa(A) \rangle = 0$$

for every $A \in H_2(E^+, \cup_{i=1}^k \partial T_{y_i}; \mathbb{R})$. To this end, note that $\omega_{E'}$ vanishes over each of the trails $\mathrm{Tr}_{y_i,x_i'}(\partial C)$, hence

$$\langle [\omega_{E'}], \kappa(A) \rangle = \langle [\omega_{E'}], A \rangle = \langle [\omega_E], A \rangle,$$

where the last equality holds because $\omega_{E'}|_{E^+} = \omega_E|_{E^+}$. Now $\omega_E = d\lambda_E$, hence

$$\langle [\omega_E], A \rangle = \sum_{i=1}^{k} \langle [\lambda_E|_{\partial T_{y_i}}], \partial_i A \rangle, \tag{19}$$

where $\partial_i A$ is the component of ∂A corresponding to $H_1(\partial T_{y_i}; \mathbb{R})$. But T_{y_i} is clearly a λ_E-exact Lagrangian submanifold, thus the right-hand side of (19) vanishes. This completes the proof that $\omega_{E'}$ is exact.

Next, we prove that $\omega_{E'}$ admits a primitive $\lambda_{E'}$ that extends $\lambda_E|_{E^+}$. We claim that this would follow from the assertion that the map induced by inclusion

$$i_*: H_1(E^+;\mathbb{R}) \to H_1(E';\mathbb{R})$$

is injective. Indeed, fix a small $\varepsilon > 0$ such that $\mathrm{Im}\ x'_j < -(\nu+\varepsilon)$ for all j, and write

$$E^+_\varepsilon = E|_{-(\nu+\varepsilon)<\mathrm{Im}\ z}.$$

Denote by $i^\varepsilon: E^+_\varepsilon \to E'$ the inclusion. Clearly, i_* is injective iff $i^\varepsilon_*: H_1(E^+_\varepsilon;\mathbb{R}) \to H_1(E')$ is injective. Fix any primitive λ' of $\omega_{E'}$ and consider the 1-form

$$\lambda_E|_{E^+} - \lambda'|_{E^+}.$$

This form is closed because $\omega_E|_{E^+} = \omega_{E'}|_{E^+}$. Since i^ε_* is injective, the restriction map

$$(i^\varepsilon)^*: H^1(E';\mathbb{R}) \to H^1(E^+_\varepsilon;\mathbb{R})$$

is surjective, hence there exists a closed 1-form α' on E' and a smooth function $f: E^+_\varepsilon \to \mathbb{R}$ such that

$$\alpha'|_{E^+_\varepsilon} = \lambda_E|_{E^+_\varepsilon} - \lambda'|_{E^+_\varepsilon} + df.$$

Now cut off the function f in between E^+ and E^+_ε to obtain another function $f': E' \to \mathbb{R}$, which coincides with f on E^+ and vanishes outside of E^+_ε. The desired 1-form $\lambda_{E'}$ is then given by

$$\lambda_{E'} := \alpha' + \lambda' - df'.$$

To complete the proof it remains to show that

$$i_*: H_1(E^+;\mathbb{R}) \to H_1(E';\mathbb{R}) \tag{20}$$

is injective. To this end, denote by $F = \pi^{-1}(w)$ the fiber of $\pi: E \to \mathbb{R}$ over a regular value w of π with $w \in \{z \in \mathbb{C} \mid \mathrm{Im}\ z > -\nu\}$.

Assume first that $\dim F > 0$. By standard arguments, the inclusions $F \subset E^+$ and $F \subset E'$ induce isomorphisms $H_1(F) \cong H_1(E^+)$ and $H_1(F) \cong H_1(E')$, where the homologies are taken with arbitrary coefficients. Therefore, $i_*: H_1(E^+) \to H_1(E')$ is an isomorphism.

Assume now that $\dim F = 0$. Choose a small $\varepsilon > 0$ such that all the critical values of π are in $\{\mathrm{Im}\ z > -\nu + \varepsilon\}$ and write

$$E'^- = E'|_{\mathrm{Im}\ z < -\nu+\varepsilon}.$$

Note that $E^+ \cap E'^-$ is homotopy equivalent to F, which is discrete, hence

$$H_1(E^+ \cap E'^-;\mathbb{R}) = 0.$$

By the Mayer–Vietoris sequence for $E' = E^+ \cup E'^-$ it follows that

$$i_*: H_1(E^+;\mathbb{R}) \to H_1(E';\mathbb{R})$$

is injective.

This completes the proof of the injectivity of i_* in (20) for all possible values of $\dim F$, hence also the proof of point (3) of the proposition.

Point (5) is obvious if $\dim F > 0$ (since in that case $\dim(S_i) \geq 2$). Assume that $\dim F = 0$. In this case $N \approx S^1$, and without loss of generality we may assume that the number of real critical values of π is $k = 2$. (This is not really essential for the rest of the proof, it just simplifies a bit the notation.) Let $\lambda_{E'}$ be a 1-form from point (3), whose existence we have just proved. In the course of the argument below we will need to alter this 1-form, so we will denote it by λ'.

Let E^+_ε be as earlier in the proof. Denote by

$$j^\varepsilon_*: H_1(E^+_\varepsilon;\mathbb{R}) \to H_1(E, \partial T_{x'_1} \cup \partial T_{x'_2};\mathbb{R}), \quad i^\varepsilon_*: H_1(E^+_\varepsilon;\mathbb{R}) \to H_1(E';\mathbb{R})$$

the maps induced by the inclusion $E^+_\varepsilon \subset E'$. Similarly to the isomorphism from (18) we have also an isomorphism

$$\kappa: H_1(E, \partial T_{x'_1} \cup \partial T_{x'_2};\mathbb{R}) \to H_1(E';\mathbb{R}),$$

which we continue denoting by κ and which is defined by exactly the same means.

Consider the homology classes $[S_1], [S_2] \in H_1(E';\mathbb{R})$ as well as the subspace $\operatorname{image} i^\varepsilon_* \subset H_1(E';\mathbb{R})$. We claim that no non-trivial linear combination of $[S_1], [S_2]$ belongs to $\operatorname{image} i^\varepsilon_*$. This can be easily seen by looking at the images of

$$\kappa^{-1}[S_1] = [T_{x'_1}] \quad \text{and} \quad \kappa^{-1}[S_2] = [T_{x'_2}]$$

under the connecting homomorphism

$$\partial_*: H_1(E, \partial T_{x'_1} \cup \partial T_{x'_2};\mathbb{R}) \to H_0(\partial T_{x'_1} \cup \partial T_{x'_2};\mathbb{R}) = H_0(\partial T_{x'_1};\mathbb{R}) \oplus H_0(\partial T_{x'_2};\mathbb{R})$$

and noting that $\kappa^{-1}(\operatorname{image} i^\varepsilon_*) = \operatorname{image} j^\varepsilon_*$ is sent to 0 by ∂_*.

In view of the preceding claim we can find a closed 1-form θ on E' such that:

(1) $[\theta] \in H^1(E';\mathbb{R})$ vanishes on $\operatorname{image} i^\varepsilon_*$.

(2) $\langle [\theta], [S_1] \rangle = \int_{S_1} \lambda'$ and $\langle [\theta], [S_2] \rangle = \int_{S_2} \lambda'$.

By the property of θ we have $(i^\varepsilon)^*[\theta] = 0 \in H^1(E^+_\varepsilon;\mathbb{R})$, hence there exists a smooth function $h: E^+_\varepsilon \to \mathbb{R}$ such that $\theta|_{E^+_\varepsilon} = dh$. Now, cutoff h near $\{\operatorname{Im} z = -\nu - \varepsilon\}$ and extend the resulting function to a smooth function $h': E' \to \mathbb{R}$ which vanishes on $\{\operatorname{Im} z \leq -\nu - \varepsilon\}$ and such that $h' = h$ on $E^+ = \{\operatorname{Im} z > -\nu\}$. Replacing the form $\lambda_{E'}$ provided by point (3) of the proposition by the form

$$\lambda'' := \lambda' - \theta + dh'$$

we still obtain a primitive of $\omega_{E'}$ that coincides with λ_E over E^+ and such that the matching spheres S_1, S_2 are λ''-exact. This completes the proof of point (5) of the proposition in case the fibers of $\pi: E \to \mathbb{C}$ are 0-dimensional. □

4. Floer theory in E versus E'

In the sequel we will be primarily interested in Floer theory for Lagrangians in E (or even in smaller domains inside it). However, for the arguments used in Section 5 it would be more convenient to view these Lagrangians as lying in the extended Lefschetz fibration E', since its matching cycles will be used in an essential way. We therefore need to establish a comparison between the Floer homologies in E and E'. The main results in this section show that, at least for Lagrangians situated in suitable domains in E, the two Floer theories coincide.

In Section 4.2 we extend this discussion to the thimbles $T^{\uparrow}_{x_j} \subset E$ and the matching spheres $S_j \subset E'$ and show that the two Floer complexes $CF(L, T^{\uparrow}_{x_j})$ and $CF(L, S_j)$, in E and E' respectively, can be identified for Lagrangians $L \subset E$ whose projection to $\mathbb{C}$ lies in a suitable domain. We also extend these results to the framework of A_∞-modules over the respective Fukaya categories.

The arguments used to prove the statements in this section are mostly slight variations on the methods developed in [3, 4, 6] hence will only be outlined.

4.1. Floer theory for Lagrangians lying in the overlapping parts of E and E'. Recall that the extended Lefschetz fibration $\pi' \colon E' \to \mathbb{C}$ from Section 3.3 has been constructed such that it coincides, together with its associated structures, with the original Lefschetz fibration $\pi \colon E \to \mathbb{C}$ over $\{z \in \mathbb{C} \mid -\nu < \operatorname{Im} z\}$.

Let $L_0, L_1 \subset E'$ be two marked exact Lagrangians and assume that

$$L_0, L_1 \subset E'|_{\{-\nu<\operatorname{Im} z\}} = E|_{\{-\nu<\operatorname{Im} z\}}.$$

By the arguments from [6] the Floer complexes of (L_0, L_1) coincide, when viewed in E and in E', provided we choose the right Floer data. More precisely, let H be a Hamiltonian function compactly supported in $E|_{\{-\nu<\operatorname{Im} z\}}$. Then there exist regular Floer data $\mathcal{D} = (H, J)$ in E and $\mathcal{D}' = (H, J')$ in E', with the same Hamiltonian function H such that all the Floer trajectories for (L_0, L_1) with respect to $\mathcal{D}$ coincide with those for $\mathcal{D}'$ and they all lie inside $E|_{\{-\nu<\operatorname{Im} z\}}$. This easily follows from the open mapping theorem for holomorphic functions, by choosing appropriate compatible almost complex structures J and J' for which the projections π and π' are holomorphic. Consequently we have a chain isomorphism (induced by the identity map on $\mathcal{O}(H)$)

$$CF(L_0, L_1; \mathcal{D}; E) \to CF(L_0, L_1; \mathcal{D}'; E'), \tag{21}$$

which preserves the action filtration. The E and E' in the notation of the Floer complexes in the preceding formula indicate the ambient manifold in which the respective Floer complex is being considered. Consequently, (21) induces an action preserving isomorphism of persistence modules

$$HF^{\leq \bullet}(L_0, L_1; E) \cong HF^{\leq \bullet}(L_0, L_1; E'),$$

hence the spectral invariants and boundary depths of $CF(L_0, L_1)$, viewed either in E or in E', coincide.

The above can be generalized to the Fukaya categories of E and E'. More specifically, denote by $\mathcal{F}uk(E)$ and $\mathcal{F}uk(E')$ the Fukaya categories of E and E', whose objects are the closed marked exact Lagrangian submanifolds in E and E'. Let $\mathcal{F}uk(E; -\nu) \subset \mathcal{F}uk(E)$ be the full subcategory whose objects are closed exact Lagrangians $L \subset E|_{\{-\nu < \operatorname{Im} z\}}$. As explained in [6], it is possible to choose the auxiliary data required for the definitions of $\mathcal{F}uk(E)$ and $\mathcal{F}uk(E')$ in such a way that the inclusion of objects

$$\mathrm{Ob}(\mathcal{F}uk(E; -\nu)) \subset \mathrm{Ob}(\mathcal{F}uk(E'))$$

extends to a (homologically) full and faithful A_∞-functor

$$\mathrm{Inc}\colon \mathcal{F}uk(E; -\nu) \to \mathcal{F}uk(E').$$

Moreover, if we view $\mathcal{F}uk(E; -\nu)$ and $\mathcal{F}uk(E')$ as weakly filtered A_∞-categories, we can assume that the functor Inc is a weakly filtered functor (see Section 2.3 and Section 7.5 for a brief explanation of these concepts, and [7, §2] for the precise definitions and more details).

This has the following consequence for A_∞-modules. Let $L \subset E'$ be a marked exact Lagrangian and assume that $L \subset E|_{\{-\nu < \operatorname{Im} z\}}$. Denote by $\mathcal{L}^{E'}$ the Yoneda module of L, viewed as an A_∞-module over $\mathcal{F}uk(E')$ and by $\mathcal{L}^{E,-\nu}$ the Yoneda module of L over $\mathcal{F}uk(E; -\nu)$. Both modules are weakly filtered in the sense of [7] and with the right choices of auxiliary data for $\mathcal{F}uk(E; -\nu)$, $\mathcal{F}uk(E')$ we have that

$$\mathrm{Inc}^*(\mathcal{L}^{E'}) = \mathcal{L}^{E,-\nu}$$

as weakly filtered $\mathcal{F}uk(E; -\nu)$-modules.

4.2. Floer theory of the matching spheres S_j versus the thimbles $T^{\uparrow}_{x_j}$. Next, we compare the Floer theory of the matching spheres S_j in E' with the Floer theory of the thimbles $T^{\uparrow}_{x_j}$ in E, defined on page 22. Fix a rectangle $\mathcal{R} \subset \mathbb{C}$ of the type

$$\mathcal{R} = \{x + iy \in \mathbb{C} \mid x \in (a, b), -\nu < y < \varepsilon\} \tag{22}$$

such that $S_j \cap \pi'^{-1}(\mathcal{R}) = T^{\uparrow}_{x_j} \cap \pi^{-1}(\mathcal{R})$; see Figure 2.

Let $L \subset E'$ be a marked exact Lagrangian and assume that $\pi'(L) \subset \mathcal{R}$. Let H be a Hamiltonian function compactly supported in $\pi^{-1}(\mathcal{R})$. Then there exist almost complex structures J on E and J' on E', compatible with ω_E and $\omega_{E'}$ respectively, making the Floer data $\mathcal{D} = (H, J)$ and $\mathcal{D}' = (H, J')$ regular and such that the Floer trajectories for $(L, S_j; \mathcal{D}')$ in E' and the Floer trajectories of $(L, T^{\uparrow}_{x_j}; \mathcal{D})$ in E coincide and moreover all these trajectories lie inside $\pi^{-1}(\mathcal{R})$. This follows again from an open mapping theorem argument as in [6].

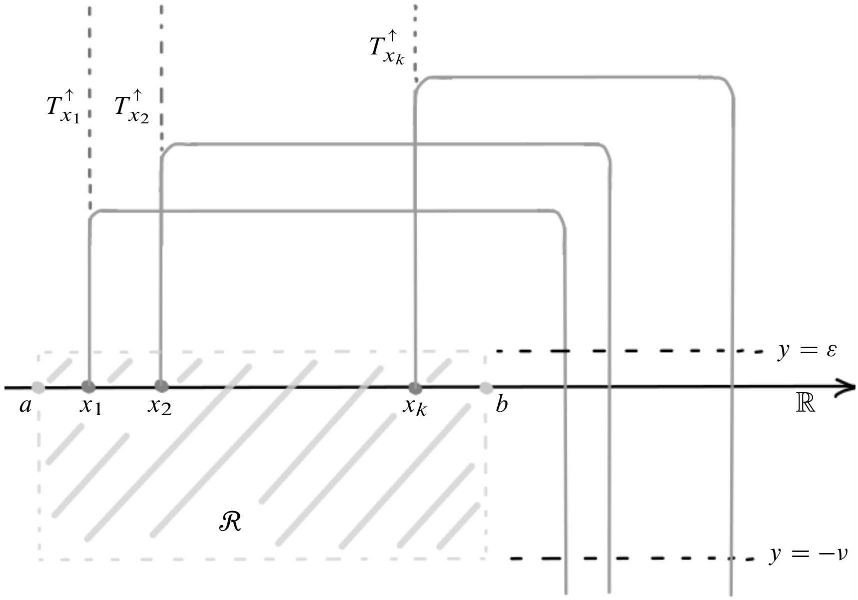

Figure 2. The rectangle $\mathcal{R}$ and the projection to $\mathbb{C}$ of the thimbles $T^{\uparrow}_{x_j}$.

It follows that the identity map on $\mathcal{O}(H)$ gives an action preserving chain isomorphism

$$CF(L, S_j; \mathcal{D}'; E') \to CF(L, T^{\uparrow}_{x_j}; \mathcal{D}; E).$$

Here we view $T^{\uparrow}_{x_j} \subset E$ as a marked exact Lagrangian with primitive function adjusted such that it coincides with the given primitive function of S_j along

$$S_j \cap \pi'^{-1}(\mathcal{R}) = T^{\uparrow}_{x_j} \cap \pi^{-1}(\mathcal{R}).$$

Denote by $\mathcal{F}uk(E; \mathcal{R}) \subset \mathcal{F}uk(E')$ the full subcategory whose objects are marked exact Lagrangians L with $\pi(L) \subset \mathcal{R}$. Similarly to Inc we have weakly filtered inclusion A_∞-functors

$$\mathrm{Inc}_{\mathcal{R},-\nu} \colon \mathcal{F}uk(E; \mathcal{R}) \to \mathcal{F}uk(E; -\nu) \quad \text{and} \quad \mathrm{Inc}_{\mathcal{R},E'} \colon \mathcal{F}uk(E; \mathcal{R}) \to \mathcal{F}uk(E')$$

with $\mathrm{Inc}_{\mathcal{R},E'} = \mathrm{Inc} \circ \mathrm{Inc}_{\mathcal{R},-\nu}$.

Putting all these constructions together we deduce:

Lemma 4.2.1. *Let $\mathcal{S}_j$ be the Yoneda module of S_j and let $\mathcal{T}^{\uparrow}_{x_j}$ be the Yoneda module of $T^{\uparrow}_{x_j}$, the latter being viewed as a module over $\mathcal{F}uk(E; -\nu)$. With the appropriate choice of auxiliary data, we have*

$$\mathrm{Inc}^*_{\mathcal{R},E'}(\mathcal{S}_j) = \mathrm{Inc}^*_{\mathcal{R},-\nu}(\mathcal{T}^{\uparrow}_{x_j}) \tag{23}$$

as weakly filtered $\mathcal{F}uk(E; \mathcal{R})$-modules.

5. Cone decompositions in Lefschetz fibrations

Recall from [6] that the Yoneda modules associated to closed Lagrangian submanifolds (or more generally Lagrangian cobordisms), satisfying appropriate exactness or monotonicity conditions, in a Lefschetz fibration E can be represented as iterated cones of modules involving the matching spheres S_j in the extended Lefschetz fibration E'. We will apply these results below, to the fibrations E and E' constructed in Sections 3.1–3.3 above, while also keeping track of the action filtrations.

5.1. Decomposing Lagrangians in E in terms of matching spheres in E'. Consider a real Lefschetz fibration $\pi\colon E \to \mathbb{R}$ with critical values $x_1, \dots, x_k, z_1, \overline{z}_1, \dots, z_l, \overline{z}_l$ and let $\pi'\colon E' \to \mathbb{C}$ be its associated extended Lefschetz fibration, as in Section 3.3. Fix $\varepsilon > 0$ with $\varepsilon < |\operatorname{Im} z_j|$ for every j. Let $K \subset E$ be a closed λ_E-exact Lagrangian submanifold and assume that $K \subset E|_{\{|\operatorname{Im}| z<\varepsilon\}}$. Consider the matching spheres $S_j \subset E'$ and denote by $\tau_{S_j}\colon E' \to E'$ the Dehn-twist around S_j, supported in a small neighborhood of S_j. Note that τ_{S_j} is well defined up to Hamiltonian isotopy (supported near S_j) since the sphere S_j, being a matching sphere, has a canonical smooth identification with S^n ($2n = \dim_{\mathbb{R}} E$) up to smooth isotopy.

Put $K^{(0)} := K$, $K^{(j)} := \tau_{S_j}(K^{(j-1)})$, $j = 1, \dots, k$. We view these Lagrangians as objects of the $\lambda_{E'}$-exact Fukaya category $\mathcal{F}uk(E')$ of E'. Denote by $\mathcal{K}^{(j)}$ the Yoneda modules associated to $K^{(j)}$, $j = 0, \dots, k$. Write also $\mathcal{K} := \mathcal{K}^{(0)}$ for the Yoneda module of K and denote by $\mathcal{S}_j$, $j = 1, \dots, k$, the Yoneda modules associated to the matching spheres S_j.

The main goal of this section is to obtain an iterated cone decomposition of K in terms of the Yoneda modules $\mathcal{S}_j$ of the spheres S_j. In order to state the result we need a bit of notation. Let $1 \le j \le k$ and $1 \le d \le j-1$. Denote by $\mathcal{I}_{d,j-1}$ the set of all multi-indices $\underline{i} = (i_1, \dots, i_d)$ with $1 \le i_1 < i_2 < \dots < i_d \le j-1$. We order the elements of $\mathcal{I}_{d,j-1}$ by the lexicographic order. For each multi-index $\underline{i} \in \mathcal{I}_{d,j-1}$ put

$$\mathcal{C}_{\underline{i},j} := \mathcal{S}_j \otimes CF(S_j, S_{i_d}) \otimes CF(S_{i_d}, S_{i_{d-1}}) \otimes \dots \otimes CF(S_{i_2}, S_{i_1}) \otimes CF(S_{i_1}, K). \tag{24}$$

Let $m_{d,j-1} := \#\mathcal{I}_{d,j-1}$ and order the elements of $\mathcal{I}_{d,j-1} = \{\underline{i}^{(1)}, \dots, \underline{i}^{(m_{d,j-1})}\}$ in such a way that $\underline{i}^{(1)} \precneqq \underline{i}^{(2)} \precneqq \dots \precneqq \underline{i}^{(m_{d,j-1})}$.

Proposition 5.1.1. *$\mathcal{K}$ is quasi-isomorphic, in the A_∞-category of modules over $\mathcal{F}uk(E')$, to an iterated cone of $\mathcal{F}uk(E')$-modules of the following type:*

$$\mathcal{K} \cong \left[\mathcal{B}_1 \to \dots \to \mathcal{B}_k \to \mathcal{K}^{(k)}\right], \tag{25}$$

where each of the modules $\mathcal{B}_j$, $j = 1, \dots, k$, has itself an iterated cone decomposition of the following type:

$$\mathcal{B}_j = \left[\mathcal{S}_j \otimes CF(S_j, K) \to \mathcal{B}_{j,1} \to \mathcal{B}_{j,2} \to \dots \to \mathcal{B}_{j,j-1}\right], \tag{26}$$

and the modules $\mathcal{B}_{j,d}$, $1 \le d \le j-1$, *that appear in* (26) *are of the form*

$$\mathcal{B}_{j,d} = \big[\mathcal{C}_{\underline{i}^{(1)},j} \to \mathcal{C}_{\underline{i}^{(2)},j} \to \cdots \to \mathcal{C}_{\underline{i}^{(m_{d,j-1})},j}\big]. \tag{27}$$

The proof of Proposition 5.1.1 follows from the main results in [6], and will be outlined in Section 5.2 below.

Before we go on, here is a concrete example showing how the cone decomposition of $\mathcal{K}$ looks like in case the number of real critical values of π is $k = 3$:

$$\begin{aligned}\mathcal{K} \cong \big[&\mathcal{S}_1 \otimes CF(S_1, K)\\ &\to \mathcal{S}_2 \otimes CF(S_2, K) \to \mathcal{S}_2 \otimes CF(S_2, S_1) \otimes CF(S_1, K)\\ &\to \mathcal{S}_3 \otimes CF(S_3, K) \to \mathcal{S}_3 \otimes CF(S_3, S_2) \otimes CF(S_1, K)\\ &\to \mathcal{S}_3 \otimes CF(S_3, S_2) \otimes CF(S_2, K)\\ &\to \mathcal{S}_3 \otimes CF(S_3, S_2) \otimes CF(S_2, S_1) \otimes CF(S_1, K) \to \mathcal{K}^{(3)}\big]\end{aligned}$$

Having established a cone decomposition of the module $\mathcal{K}$ over the A_∞-category $\mathcal{F}uk(E')$ we consider its pull-back to Fukaya categories associated to E and also address properties of the first module $\mathcal{K}^{(k)}$ in the cone decomposition (25).

Recall from Section 4 that we have the Fukaya categories $\mathcal{F}uk(E;\mathcal{R})$ and $\mathcal{F}uk(E;-\nu)$. We take the rectangle $\mathcal{R}$ from (22) to be wide enough such that it contains $\pi(K)$. Recall also the inclusion functor

$$\mathrm{Inc}_{\mathcal{R},E'}\colon \mathcal{F}uk(E;\mathcal{R}) \to \mathcal{F}uk(E')$$

that factors as the composition $\mathrm{Inc}_{\mathcal{R},E'} = \mathrm{Inc} \circ \mathrm{Inc}_{\mathcal{R},-\nu}$ of the two functors

$$\mathrm{Inc}_{\mathcal{R},-\nu}\colon \mathcal{F}uk(E;\mathcal{R}) \to \mathcal{F}uk(E;-\nu), \quad \mathrm{Inc}\colon \mathcal{F}uk(E;-\nu) \to \mathcal{F}uk(E')$$

Proposition 5.1.2. *By pulling back the cone decomposition* (25) *via* $\mathrm{Inc}^*_{\mathcal{R},E'}$ *we obtain a similar cone decomposition for* $\mathcal{K}$ *(now viewed as a module over* $\mathcal{F}uk(E;\mathcal{R})$*), where the modules* $\mathcal{S}_j$ *in* (26) *and* (24) *are replaced by* $\mathrm{Inc}^*_{\mathcal{R},-\nu}(\mathcal{T}^{\uparrow}_{x_j})$*; see* (23)*. (Note that the terms involving the Floer complexes of* S_j *and of* S_{i_l} *remain unchanged.)*

Moreover, the pullback $\mathrm{Inc}^*_{\mathcal{R},E'}\mathcal{K}^{(k)}$ *of the module* $\mathcal{K}^{(k)}$ *which appears in* (25) *is acyclic.*

The rest of Section 5 is organized as follows. In Section 5.2 we outline the proofs of Propositions 5.1.1 and 5.1.2. Then in Section 5.3 and Section 5.4 we will refine these results to take into account also the action filtrations.

5.2. Exact triangles associated to Dehn twists. Let $(X^{2n}, \omega = d\lambda)$ be a Liouville domain and $S^n \xrightarrow{\approx} S \subset X$ a parametrized Lagrangian sphere. In the case $n = 1$, we additionally assume that S is λ-exact. Let $\tau := \tau_S\colon X \to X$ be a symplectomorphism, supported in $\mathrm{Int}\, X$, which represents the symplectic mapping class of the

Dehn twist around S. Note that τ is an exact symplectomorphism, and hence sends exact Lagrangians to exact Lagrangians.

A well known result of Seidel [31,32] says that for every exact Lagrangian $Q \subset X$ there is the following distinguished triangle in the derived Fukaya category $\mathcal{F}uk(X)$:

$$\begin{tikzcd} \mathcal{S} \otimes CF(S, Q) \arrow[r] & \tau(\mathcal{Q}) \arrow[d] \\ & \mathcal{Q} \arrow[ul] \end{tikzcd} \tag{28}$$

Here, $\mathcal{S}$, $\mathcal{Q}$, and $\tau(\mathcal{Q})$ stand for the A_∞-modules corresponding to S, Q, and $\tau(Q)$ under the Yoneda embedding.

The above distinguished triangle implies that, up to a quasi-isomorphism of modules, $\mathcal{Q}$ can be expressed as the following mapping cone:

$$\mathcal{Q} \cong \big[\mathcal{S} \otimes CF(S, Q) \to \tau(\mathcal{Q})\big]. \tag{29}$$

By rotating (28) we obtain also the following quasi-isomorphism:

$$\tau(\mathcal{Q}) \cong \big[\mathcal{Q} \to \mathcal{S} \otimes CF(S, Q)\big]. \tag{30}$$

Note that here and in what follows we work in an ungraded setting, hence no grading shifts appear in any of (28)–(30).

We now turn to the cone decomposition (25), and assume that $(X, d\lambda) = (E', \lambda_{E'})$ as in Section 3.3. The decomposition (25) follows by successively applying (29) and (30). Specifically, we begin with $\mathcal{K}^{(1)} = \tau_{S_1}(\mathcal{K})$ and obtain from (29):

$$\mathcal{K} \cong \big[\mathcal{S}_1 \otimes CF(S_1, K) \to \mathcal{K}^{(1)}\big]. \tag{31}$$

By the same argument we also have

$$\mathcal{K}^{(1)} \cong \big[\mathcal{S}_2 \otimes CF(S_1, K^{(1)}) \to \mathcal{K}^{(2)}\big],$$

which together with (31) gives

$$\mathcal{K} \cong \big[\mathcal{S}_1 \otimes CF(S_1, K) \to \mathcal{S}_2 \otimes CF(S_2, K^{(1)}) \to \mathcal{K}^{(2)}\big]. \tag{32}$$

But by (30) we have $\mathcal{K}^{(1)} \cong [\mathcal{K} \to \mathcal{S}_1 \otimes CF(S_1, K)]$. Substituting this into (32) yields

$$\begin{aligned} \mathcal{K} \cong \big[\mathcal{S}_1 \otimes CF(S_1, K) &\to \mathcal{S}_2 \otimes CF(S_2, K) \\ &\to \mathcal{S}_2 \otimes CF(S_2, S_1) \otimes CF(S_1, K) \to \mathcal{K}^{(2)}\big]. \end{aligned} \tag{33}$$

Continuing in a similar vein, decomposing $\mathcal{K}^{(2)}$, $\mathcal{K}^{(3)}$, etc., we obtain the cone decomposition (25) with items as described in (26)–(27). This concludes the outline of the proof of Proposition 5.1.1.

We now turn to Proposition 5.1.2. The first statement in the proposition is straightforward. It remains to address the acyclicity of the module $\mathrm{Inc}^*_{\mathcal{R},E'}\mathcal{K}^{(k)}$. (Recall that $K^{(k)} = \tau_{S_k} \cdots \tau_{S_1}(K)$). This follows from [6, §4.4], where it is proved that there is a Hamiltonian diffeomorphism $\phi\colon E' \to E'$ such that $\phi(K^{(k)}) \subset E'|_{\{\mathrm{Im}\, z \leq -\nu\}}$; see also [5] for more details. In particular, for every Lagrangian submanifold $L \subset \pi'^{-1}(\mathcal{R})$ we have $CF(L, \phi(K^{(k)})) = 0$.

5.3. Taking filtrations into account. We now go back to the cone decomposition (25) and review it from the perspective of action filtrations.

From now on we assume all the exact Lagrangian submanifolds to be marked, unless otherwise stated. By a slight abuse of notation, we now redefine the objects of the Fukaya categories $\mathcal{F}uk(E)$, $\mathcal{F}uk(E')$, as well as $\mathcal{F}uk(E;\mathcal{R})$, $\mathcal{F}uk(E;-\nu)$, to be *marked* exact Lagrangians, subject to the additional constraints in each of these categories. These categories now become weakly filtered A_∞-categories, where the filtrations are induced by the action functional. We refer the reader to [7, §2] for the definitions and basic theory of weakly filtered A_∞-categories and weakly filtered modules over such.

Below we will take the exact Lagrangian $K \subset E|_{\{|\mathrm{Im}|\, z<\varepsilon\}}$ to have an arbitrary marking. This marking induces a marking on $K^{(j)} = \tau_{S_j} \cdots \tau_{S_1}(K)$, $j = 1, \dots, k$, see Section 5.4, page 38. The Lagrangian spheres S_j are also assumed to be marked in advance.

Note that all the items in the cone decomposition (25), as detailed in (26)–(27) are weakly filtered modules. This is so because the $\mathcal{S}_j$'s and $\mathcal{K}^{(k)}$ are Yoneda modules over a weakly filtered A_∞-category, and the chain complexes $CF(S_{i_l}, S_{i_{l-1}})$ and $CF(S_j, K)$ are filtered.

Next, we claim that all the maps in the iterated cones (25), (26), and (27) are weakly filtered maps. This means, in particular, that when evaluating these iterated cones modules on a given exact Lagrangian L, each of these maps specializes to a filtered chain map that shifts filtrations by an amount bounded from above *uniformly in* L. More specifically:

Proposition 5.3.1. *In the iterated cone* (27)

$$\begin{aligned}\mathcal{B}_{j,d} = \Big[\mathcal{C}_{\underline{i}^{(1)},j} \xrightarrow{\varphi_{1,j}} \Big[\mathcal{C}_{\underline{i}^{(2)},j} \xrightarrow{\varphi_{2,j}} \big[\cdots \\ \longrightarrow \big[\mathcal{C}_{\underline{i}^{(m_{d,j-1}-1)},j} \xrightarrow{\varphi_{m_{d,j-1}-1,j}} \mathcal{C}_{\underline{i}^{(m_{d,j-1})},j}\big]\cdots\big]\Big]\Big],\end{aligned} \tag{34}$$

each of the module homomorphisms $\varphi_{l,j}$ is weakly filtered, and shifts action by $\leq s_{\varphi_{l,j}}$, for some $s_{\varphi_{l,j}} \geq 0$.

This implies that the right-hand side of (34) is filtered using the filtrations of the factors $\mathcal{C}_{\underline{i}^{(l)},j}$ and the recipe (53).

In particular, for every exact Lagrangian L, the module homomorphism $\varphi_{l,j}$ specializes to an $s_{\varphi_{l,j}}$-filtered chain map (still denoted by $\varphi_{l,j}$):

$$\varphi_{l,j}\colon \mathcal{C}_{\underline{i}^{(l)},j}(L) \to \Big[\mathcal{C}_{\underline{i}^{(l+1)},j}(L) \xrightarrow{\varphi_{l+1,j}} \big[\cdots \\ \to \big[\mathcal{C}_{\underline{i}^{(m_{d,j-1}-1)},j}(L) \xrightarrow{\varphi_{m_{d,j-1}-1,j}} \mathcal{C}_{\underline{i}^{(m_{d,j-1})},j}(L)\big]\cdots\big]\Big].$$

A crucial point for us will be that the filtration-shifts $s_{\varphi_{l,j}}$ are independent of L.

Having filtered the modules $\mathcal{B}_{j,d}$, the preceding statements apply also to the maps in the iterated cone of (26), and finally also to the right-hand side of (25). We will prove Proposition 5.3.1 in Section 5.4 below.

Furthermore, we claim that the module quasi-isomorphism at (25) between $\mathcal{K}$ and the (now weakly filtered) iterated cone on the right-hand side is filtered in the following sense.

Proposition 5.3.2. *There exist $s_{\mathcal{K}} \geq 0$ and weakly-filtered module homomorphisms*

$$\varphi\colon \mathcal{K} \to \big[\mathcal{B}_1 \to \cdots \to \mathcal{B}_k \to \mathcal{K}^{(k)}\big], \quad \psi\colon \big[\mathcal{B}_1 \to \cdots \to \mathcal{B}_k \to \mathcal{K}^{(k)}\big] \to \mathcal{K}$$

that shift filtrations by $\leq s_{\mathcal{K}}$ and such that

$$\varphi \circ \psi = \mathrm{id} + \mu_1^{\mathrm{mod}}(h'), \quad \psi \circ \varphi = \mathrm{id} + \mu_1^{(\mathrm{mod})}(h'')$$

for weakly filtered pre-module homomorphisms h', h'' that shift filtrations by $\leq s_{\mathcal{K}}$.

The proof of this statement is again postponed to Section 5.4. The constant $s_{\mathcal{K}}$ depends on K (and its marking) as well as on the marking on the spheres $S_1, \ldots, S_k$.

In particular, the above implies that for every exact Lagrangian L we have chain maps

$$\begin{aligned} \varphi_L\colon & CF(L,K) \to \big[\mathcal{B}_1(L) \to \cdots \to \mathcal{B}_k(L) \to CF(L,K^{(k)})\big], \\ \psi_L\colon & \big[\mathcal{B}_1(L) \to \cdots \to \mathcal{B}_k(L) \to CF(L,K^{(k)})\big] \to CF(L,K), \end{aligned} \tag{35}$$

which are $s_{\mathcal{K}}$-filtered and such that $\varphi_L \circ \psi_L$ and $\psi_L \circ \varphi_L$ are chain homotopic to the identities via chain homotopies that shift filtrations by $\leq s_{\mathcal{K}}$. Once again, it is important to stress that the bound on the action shift $s_{\mathcal{K}}$ is independent of L.

Phrased in the terminology of Definition 7.5.3, the above says that the module $\mathcal{K}$ (resp., filtered chain complex $CF(L,K)$) and the module on the right-hand side of (25) (resp., the filtered chain complex $[\mathcal{B}_1(L) \to \cdots \to \mathcal{B}_k(L) \to CF(L,K^{(k)})]$) are at distance $\leq s_{\mathcal{K}}$ one from the other.

Finally, recall that the pullback module $\mathrm{Inc}^*_{\mathcal{R},E'}\mathcal{K}^{(k)}$ is acyclic. We claim that this acyclicity holds also in the filtered sense. Namely, there exists a constant $s_C = s_C(K)$, which depends on K, and a weakly filtered pre-module homomorphism

$$h\colon \mathrm{Inc}^*_{\mathcal{R},E'}\mathcal{K}^{(k)} \to \mathrm{Inc}^*_{\mathcal{R},E'}\mathcal{K}^{(k)}$$

that shifts action by $\leq s_C$ such that in $\hom_{\mathrm{mod}_{\mathcal{F}uk(E;\mathcal{R})}}(\mathrm{Inc}^*_{\mathcal{R},E'}\mathcal{K}^{(k)}, \mathrm{Inc}^*_{\mathcal{R},E'}\mathcal{K}^{(k)})$ we have $\mathrm{id} = \mu_1^{\mathrm{mod}}(h)$. In particular, for every exact Lagrangian $L \subset \pi^{-1}(\mathcal{R})$ we have:

$$\beta\big(CF(L, \mathcal{K}^{(k)})\big) \leq s_C. \tag{36}$$

Here, $\beta(CF(L, \mathcal{K}^{(k)}))$ is the boundary depth of the acyclic filtered chain complex $CF(L, \mathcal{K}^{(k)})$.

The inequality (36) follows from the last paragraph of Section 5.2 on page 35. Indeed, by standard Floer theory we can take $s_C = 2\rho_{\mathrm{Hof}}(\mathrm{id}, \phi)$, where $\phi: E' \to E'$ is a Hamiltonian diffeomorphism that sends $K^{(k)}$ to $E'|_{\{\mathrm{Im}\, z \leq -\nu\}}$, and ρ_{Hof} stands for the Hofer metric on the group of Hamiltonian diffeomorphisms.

Remark 5.3.3. The constant s_C appearing in (36) depends a priori on K (though not on L). A more careful argument, based on [6, §4.4], shows that the Hamiltonian diffeomorphisms ϕ, mentioned above, can be taken to be at a uniformly bounded (in K) Hofer-distance from id, as long as we restrict to Lagrangians $K \subset E|_{\{|\mathrm{Im}\,| z < \varepsilon\}}$. Consequently the constant s_C can be assumed to be independent of K.

However, this additional information will not be used in the rest of the paper. The reason is that we will use the filtered cone decomposition (25) only for one Lagrangian K, namely $K = N$ - the zero-section of $T^*(N)$ viewed as a Lagrangian in E.

5.4. Proof of the statements from Section 5.3. We continue to assume here all exact Lagrangian submanifolds (and cobordisms) to be marked.

We begin with a brief digression on inclusion and product functors. Let $(Y, d\lambda_Y)$ be a Liouville manifold as in Section 2.2.2. Let $\gamma: \mathbb{R} \to \mathbb{R}^2$ be a smooth proper embedding sending the ends of $\mathbb{R}$ to horizontal rays in $\mathbb{R}^2$. By abuse of notation we denote by γ also the image of this embedding. By the results of [4, 7] there is a weakly filtered A_∞-functor (called in [4] "inclusion functor")

$$\mathcal{I}_\gamma: \mathcal{F}uk(Y) \to \mathcal{F}uk_{\mathrm{cob}}(\mathbb{R}^2 \times Y),$$

which sends the object $L \subset Y$ to $\mathcal{I}_\gamma(L) = \gamma \times L \subset \mathbb{R}^2 \times Y$. Here $\mathcal{F}uk(Y)$ stands for the Fukaya category of closed λ_Y-exact Lagrangians in Y and $\mathcal{F}uk_{\mathrm{cob}}(\mathbb{R}^2 \times Y)$ for the Fukaya category of exact cobordisms in $\mathbb{R}^2 \times Y$, with respect to the 1-form $xdy \oplus \lambda_Y$.

Let $(X, \omega = d\lambda)$ be a Liouville manifold as in Section 2.2.2. We denote by X^- the manifold X endowed with the symplectic structure $-\omega$. Take $Y = X \times X^-$, endowed with the symplectic structure $\omega \oplus -\omega$ and Liouville form $\widetilde{\lambda} := \lambda \oplus -\lambda$ (playing the role of λ_Y). Fix $\widetilde{\lambda}' := xdy \oplus \lambda \oplus -\lambda$ as the primitive of $\omega_{\mathbb{R}^2} \oplus \omega \oplus -\omega$.

Fix an exact Lagrangians $Q \subset X$. A slight variation on the inclusion functor $\mathcal{I}_\gamma$ is the A_∞-functor

$$\mathcal{I}_{\gamma,Q}: \mathcal{F}uk(X) \to \mathcal{F}uk_{\mathrm{cob}}(\mathbb{R}^2 \times X \times X^-),$$

which sends an exact Lagrangian $L \subset X$ to $\mathcal{I}_{\gamma,Q}(L) := \gamma \times L \times Q$. The construction of this functor is very similar to the construction of $\mathcal{I}_\gamma$ (for the case $Y = X \times X^-$), as detailed in [4, Section 4.2]. In fact, $\mathcal{I}_{\gamma,Q}$ factors as $\mathcal{I}_{\gamma,Q} := \mathcal{I}_\gamma \circ \mathcal{P}_Q$, where

$$\mathcal{P}_Q : \mathcal{F}uk(X) \to \mathcal{F}uk(X \times X^-)$$

is the obvious functor that sends $L \subset X$ to $L \times Q \subset X \times X^-$.

The main ingredient to show Propositions 5.3.1 and 5.3.2 is to establish a filtered version of the Seidel's Dehn-twist triangle (28) (or more precisely (29)). We pursue this now.

Lemma 5.4.1. *The mapping cone in equation* (29) *admits a filtered version.*

In the course of the proof we will indicate more precisely the relevant shifts involved and their dependence on the choices involved in the construction.

Proof. Let $(X^{2n}, \omega = d\lambda)$ be a Liouville manifold as in Section 2.2.2 and $S \subset X$, $\tau = \tau_S : X \to X$ be as at the beginning of Section 5.2. It is possible to choose τ (a representative of the Dehn-twist symplectic mapping class) such that τ is supported near S, and moreover such that $\tau^*\lambda = \lambda + dh_\tau$, where $h_\tau : X \to \mathbb{R}$ is a smooth function compactly supported near S. (The latter easily follows from the fact that given any neighborhood of the zero-section in $T^*(S^n)$, there is a model Dehn-twist $T^*(S^n) \to T^*(S^n)$ supported in that neighborhood which is λ_{can}-exact, and the fact that the sphere S is λ-exact.) Note that we have

$$(\tau^{-1})^*\lambda = \lambda - d(h_\tau \circ \tau^{-1}).$$

Let $Q \subset X$ be a marked exact Lagrangian with primitive $h_Q : Q \to \mathbb{R}$ for $\lambda|_Q$. Then $\tau(Q)$ is also a marked exact Lagrangian. Indeed, $h_{\tau(Q)} : \tau(Q) \to \mathbb{R}$ defined by

$$h_{\tau(Q)}(x) := h_Q(\tau^{-1}(x)) + h_\tau(\tau^{-1}(x))$$

is a primitive of $\lambda|_{\tau(Q)}$. We will use this function to mark $\tau(Q)$.

We now get back to Dehn-twists, from the perspective of Lagrangian cobordism. By a result of Mak–Wu [24], there exists an exact Lagrangian cobordism

$$W \subset \mathbb{R}^2 \times X \times X^-$$

with two negative ends and one positive end, as follows. The upper negative end is $S \times S$ and the lower negative end is the graph $\Gamma_{\tau^{-1}}$ of τ^{-1}. The positive end is the graph of the identity map (i.e., the diagonal in $X \times X^-$); see Figure 3.

Let $\gamma \subset \mathbb{R}^2$ be the curve depicted in Figure 3, and denote by $\mathcal{W}$ the Yoneda module corresponding to $W \in \mathrm{Ob}(\mathcal{F}uk_{\mathrm{cob}}(\mathbb{R}^2 \times X \times X^-)$. Denote also by $\mathcal{S} \times \mathcal{S}$, $\tau(\mathcal{Q})$ the Yoneda modules (over $\mathcal{F}uk(X \times X^-)$ corresponding to the Lagrangians $S \times S$ and $\tau(\mathcal{K})$, respectively. Ignoring filtrations for the moment, a straightforward calculation

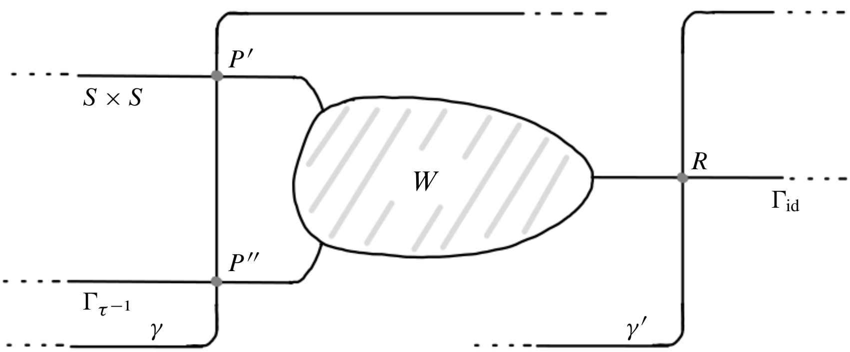

Figure 3. Projection to $\mathbb{R}^2$ of the Mak–Wu cobordism $W \subset \mathbb{R}^2 \times X \times X^-$, and the curves γ, γ'.

(based on the theory from [4]) shows that the pullback module $\mathcal{I}^*_{\gamma,Q}\mathcal{W}$ coincides with a mapping cone

$$\mathcal{I}^*_{\gamma,Q}\mathcal{W} = \big[\mathcal{S} \otimes CF(S,Q) \xrightarrow{\varphi} \tau(\mathcal{Q})\big] \tag{37}$$

for some module homomorphism $\varphi\colon \mathcal{S} \otimes CF(S,Q) \to \tau(\mathcal{Q})$.

Consider now the curve $\gamma' \subset \mathbb{R}^2$ from Figure 3. Ignoring filtrations again, it is easy to see that $\mathcal{I}^*_{\gamma',Q}\mathcal{W} = \mathcal{Q}$, the Yoneda module corresponding to $Q \subset X$.

The curves γ and γ' are isotopic via a Hamiltonian isotopy which is horizontal at infinity. Therefore, the modules $\mathcal{I}^*_{\gamma,Q}\mathcal{W}$ and $\mathcal{I}^*_{\gamma',Q}\mathcal{W}$ are quasi-isomorphic (in the category $\mathrm{mod}_{\mathcal{F}uk(X)}$). Thus we have a quasi-isomorphism

$$\mathcal{Q} \cong \big[\mathcal{S} \otimes CF(S,Q) \xrightarrow{\varphi} \tau(\mathcal{Q})\big]. \tag{38}$$

Our goal now is to derive a coarse filtered version of (38). More specifically, we will have to address two thing: explain why the module homomorphism φ is filtered, and then show that the quasi-isomorphism in (38) is weighted in the sense of Definition 7.5.3.

Note that $\widetilde{\lambda}'$ coincides with $\widetilde{\lambda}$ along each horizontal end of W (because xdy vanishes along horizontal rays). We also have

$$\widetilde{\lambda}|_{\Gamma_{\mathrm{id}}} = 0, \quad \widetilde{\lambda}|_{S\times S} = \lambda_S \oplus -\lambda_S, \quad \widetilde{\lambda}|_{\Gamma_{\tau^{-1}}} = d(h_\tau \circ \tau^{-1}),$$

where $\lambda_S := \lambda|_S$. Let $h_W\colon W \to \mathbb{R}$ be a primitive of $\widetilde{\lambda}'|_W$. By the above, h_W restricts along each of the ends of W to a primitive function for the restriction of $\widetilde{\lambda}$ to the Lagrangian corresponding to that end. We will use these functions, denoted by $h_{W,\Gamma_{\mathrm{id}}}$, $h_{W,S\times S}$, and $h_{W,\Gamma_{\tau^{-1}}}$, for primitives of $\widetilde{\lambda}|_{\Gamma_{\mathrm{id}}}$, $\widetilde{\lambda}|_{S\times S}$, and $\widetilde{\lambda}|_{\Gamma_{\tau^{-1}}}$, respectively. Note that $h_{W,\Gamma_{\mathrm{id}}}$ is constant, and by subtracting this constant from h_W we may assume without loss of generality that $h_{W,\Gamma_{\mathrm{id}}} \equiv 0$. (Note that the exact Lagrangian cobordism W does not come with a preferred marking, and we are free to choose h_W as we wish.)

Pick any marking on S, i.e., a primitive function $h_S\colon S \to \mathbb{R}$ for λ_S. We have:

$$\begin{aligned} h_{W,S\times S}(x,y) &= h_S(x) - h_S(y) + C_{W,S\times S}, \quad && \forall\,(x,y) \in S \times S,\\ h_{W,\Gamma_{\tau^{-1}}}(x,\tau^{-1}(x)) &= h_\tau(\tau^{-1}(x)) + C_{W,\Gamma_{\tau^{-1}}}, && \forall x \in X, \end{aligned} \tag{39}$$

for some constants $C_{W,S\times S}$, $C_{W,\Gamma_{\tau^{-1}}}$. Fix a primitive $h_\gamma\colon \gamma \to \mathbb{R}$ of $(xdy)|_\gamma$. Note that h_γ is constant along the positive and negative ends of γ. Given any marked exact Lagrangian $L \subset X$, with a primitive function $h_L\colon L \to \mathbb{R}$ for $\lambda|_L$, we will use the function $h_{\gamma\times L\times Q} := h_\gamma + h_L - h_Q$ as a primitive for $\widetilde{\lambda}'|_{\gamma\times L\times Q}$.

Consider the Floer complex $CF(\gamma \times L \times Q, W)$ with Floer data consisting of a zero Hamiltonian and any regular almost complex structure. (We assume here without loss of generality that $(L \times Q) \pitchfork S \times S$ and $L \times Q \pitchfork \Gamma_{\tau^{-1}}$.)

Given two exact Lagrangians L', L'' in a Liouville manifold $(Y, d\lambda_Y)$, endowed with primitives

$$h_{L'}\colon L' \to \mathbb{R}, \quad h_{L''}\colon L'' \to \mathbb{R}$$

for $\lambda_Y|_{L'}$ and $\lambda_Y|_{L''}$, and given a Floer datum for (L', L'') we denote by $\mathcal{A}(-;(L',L''))$ the action functional associated to the given Floer datum and the choices of primitives $h_{L'}$, $h_{L''}$. Here "$-$" stands for a path connecting a point from L' to a point in L''.

We will now examine the action functional $\mathcal{A}$ for the pairs $(\gamma \times L \times Q, W)$, (L, S) and (S, Q). As before, we use here Floer data with zero Hamiltonian terms. We begin with calculating $\mathcal{A}$ on the intersection points of $(\gamma \times L \times Q) \cap W$ (viewed as constant paths). These intersection points fall into two types:

(1) (P', x_1, x_2), where $P' \in \mathbb{R}^2$ is as depicted in Figure 3 and $x_1, x_2 \in S$.

(2) (P'', x_1, x_2), where $P'' \in \mathbb{R}^2$ is as in Figure 3 and $x_1 \in L \cap \tau(Q)$, $x_2 = \tau^{-1}(x_1)$.

For points of the first type we have:

$$\begin{aligned} &\mathcal{A}(P', x_1, x_2; (\gamma \times L \times Q, W))\\ &\quad = h_S(x_1) - h_S(x_2) + C_{W,S\times S} - h_\gamma(p') - (h_L(x_1) - h_Q(x_2))\\ &\quad = (h_S(x_1) - h_L(x_1)) + (h_Q(x_2) - h_S(x_2)) + (C_{W,S\times S} - h_\gamma(P'))\\ &\quad = \mathcal{A}(x_1; (L,S)) + \mathcal{A}(x_2; (S,Q)) + (C_{W,S\times S} - h_\gamma(P')). \end{aligned} \tag{40}$$

Note that the sum of the first two terms in the last equality is precisely the action-level of the generator $x_1 \otimes x_2 \in CF(L,S) \otimes CF(S,Q)$.

Turning to the intersection points of the second type, we have:

$$\begin{aligned} &\mathcal{A}(P'', x_1, x_2; (\gamma \times L \times Q)\\ &\quad = h_\tau(\tau^{-1}(x_1)) + C_{W,\Gamma_{\tau^{-1}}} - h_\gamma(P'') - h_L(x_1) + h_Q(\tau^{-1}(x_1))\\ &\quad = h_{\tau(Q)}(x_1) - h_L(x_1) + C_{W,\Gamma_{\tau^{-1}}} - h_\gamma(P'')\\ &\quad = \mathcal{A}(x_1; (L, \tau(Q))) + (C_{W,\Gamma_{\tau^{-1}}} - h_\gamma(P'')). \end{aligned} \tag{41}$$

Now recall from (37) that

$$CF(\gamma \times L \times Q, W) = \big[CF(L,S) \otimes CF(S,Q) \xrightarrow{\varphi} \tau(Q)\big],$$

and that by the results of [4] counts Floer strips going from the intersection points of type 1 to points of type 2.

From the standard action-energy identity we obtain the following: if the generator $x \in L \cap \tau(Q)$ of $CF(L, \tau(Q))$ participates in $\varphi(x_1 \otimes x_2)$, then

$$\begin{aligned} &\mathcal{A}(x_1; (L,S)) + \mathcal{A}(x_2; (S,Q)) + C_{W,S\times S} - h_\gamma(P') \\ &\quad \geq \mathcal{A}(x; (L, \tau(Q))) + C_{W,\Gamma_{\tau^{-1}}} - h_\gamma(P''). \end{aligned} \tag{42}$$

It follows that φ shifts action by

$$s_\varphi \leq h_\gamma(P'') - h_\gamma(P') + C_{W,S\times S} - C_{W,\Gamma_{\tau^{-1}}}. \tag{43}$$

The latter quantity is a constant which is independent of Q and L.

Next, consider the curve γ' from Figure 3 and

$$\mathcal{I}_{\gamma',Q}\colon \mathcal{Fuk}(X) \to \mathcal{Fuk}_{\mathrm{cob}}(\mathbb{R}^2 \times X \times X^-).$$

Recall that up to a filtration shift we have $\mathcal{I}^*_{\gamma',Q}\mathcal{W} = \mathcal{Q}$, and therefore

$$CF(L \times Q, \Gamma_{\mathrm{id}}) \cong CF(L, Q),$$

again up to a filtration shift. We will now determine this shift. To this end, recall first that $h_{W,\Gamma_{\mathrm{id}}} \equiv 0$. The intersection points of $(\gamma' \times L \times Q) \cap W$ are of the type (R, x, x), $x \in L \cap Q$. Calculating the action on such points we get:

$$\begin{aligned} \mathcal{A}(R, x, x; (\gamma' \times L \times Q, W)) &= -h_{\gamma'}(R) - h_L(x) + h_Q(x) \\ &= \mathcal{A}(x; (L,Q)) - h_{\gamma'}(R). \end{aligned} \tag{44}$$

Therefore, the identification $\mathcal{I}^*_{\gamma',Q}\mathcal{W} = \mathcal{Q}$ holds up to an action shift of the constant $h_{\gamma'}(R)$.

Finally, there exists a constant $S(W) \geq 0$ that depends only on W and another constant $C(h_\gamma, h_{\gamma'}) \geq 0$ that depends only on the choices of the primitives h_γ, $h_{\gamma'}$ such that the following holds. There exist weakly filtered module homomorphisms

$$\phi\colon \mathcal{I}^*_{\gamma,Q}\mathcal{W} \to \mathcal{I}^*_{\gamma',Q}\mathcal{W} \quad \text{and} \quad \phi'\colon \mathcal{I}^*_{\gamma',Q}\mathcal{W} \to \mathcal{I}^*_{\gamma,Q}\mathcal{W}$$

that shift filtrations by $\leq S(W) + C(h_{\gamma'})$ such that

$$\phi' \circ \phi = \mathrm{id} + \mu_1^{\mathrm{mod}}(H) \quad \text{and} \quad \phi \circ \phi' = \mathrm{id} + \mu_1^{\mathrm{mod}}(H')$$

for some weakly filtered pre-module homomorphisms H, H' that shift filtrations by $\leq 2(S(W) + C(h_{\gamma'}))$. We refer the reader to [7, §5] for more details on this.

The constant $S(W)$ is the *shadow* of the cobordism W — namely, the area of the domain in $\mathbb{R}^2$ consisting of the projection of W to $\mathbb{R}^2$ together with all the bounded connected components of the complement of this projection.

As a result, we obtain a weakly filtered quasi-isomorphism

$$\mathcal{Q} \cong \left[\mathcal{S} \otimes CF(S, Q) \xrightarrow{(\varphi, s_\varphi)} \tau(\mathcal{Q})\right], \tag{45}$$

of weight bounded from above by a constant that depends only on W and γ, γ'; see Definition 7.5.3. As seen above at (43) the amount of shift of φ is bounded from above by a constant s_φ which does not depend on Q. This concludes the construction of the filtered version of the Seidel exact triangle. □

Remark 5.4.2. There are a number of other ways to construct Seidel's exact triangle associated to a Dehn twist. Certainly, Seidel's original construction in [31] and also the method in [6]. These methods can also be used to deduce filtered versions of the exact triangle. We used here the method in [24] as it appears to provide the fastest approach in our context.

Propositions 5.3.1 and 5.3.2 now follow by applying the procedure indicated at the end of Section 5.2, but now using the filtered version of (29) and (30), as in Lemma 5.4.1, in conjunction with the algebraic remarks contained in Proposition 7.2.3 and the statement from the beginning of Section 7.3.

Remark. The weight of the quasi-isomorphism at (45) as well as s_φ do depend (also) on W (hence on the specific choice of the representative τ of the symplectic mapping class of the Dehn-twist), however these choices are made in advance, once and for all. The dependencies of this weight and of s_φ on γ, γ' and $h_{\gamma'}$, $h_{\gamma'}$ can in fact be eliminated by estimating more sharply the shifts in ϕ, ϕ', H, H' above. However this is not needed for our purposes.

6. Proof of the main theorem

This section contains two parts. The first, and main part, provides the proof of Theorem A. The second is concerned with the converse of the statement, as indicated in Remark 1.0.1 (1).

6.1. The spectral norm bound in equation (2). For the proof of the main theorem we will need the following lemma. Fix a tubular neighborhood $\mathcal{V} = T^*_{\leq r_0}(N)$ of the zero-section. For $q \in N$, denote by $F_q = T^*_q(N) \cap \mathcal{V}$ the part of the cotangent fiber over q that lies inside $\mathcal{V}$. We endow the exact Lagrangians F_q with the 0 function as a primitive of λ_{can}. Note that for every marked exact Lagrangian $L \subset \mathcal{V}$ and every $q \in N$ we have $HF(L, F_q) \cong \mathbb{Z}_2$, and hence

$$\sigma_+(CF(L, F_q)) = \sigma_-(CF(L, F_q)).$$

We denote this number by $\sigma(CF(L, F_q))$.

Lemma 6.1.1. *There exist constants $C = C(\mathcal{V}) > 0$ and $C' = C'(\mathcal{V}) > 0$, that depend only on $\mathcal{V}$, such that for every marked exact Lagrangian $L \subset \mathrm{Int}(\mathcal{V})$ and every $q', q'' \in N$ we have*

$$|\sigma(CF(L, F_{q'})) - \sigma(CF(L, F_{q''}))| \leq C,$$
$$|\beta(CF(L, F_{q'})) - \beta(CF(L, F_{q''}))| \leq C'.$$

Proof. The proof is based on standard arguments, hence we will only outline it.

The statements in the lemma follow from the following somewhat stronger statement (in conjunction with Lemma 7.1.2): *All the $\mathcal{F}uk(\mathcal{V})$-modules corresponding to F_q, $q \in N$, are at a bounded distance from one to the other in the sense of Definition* 7.5.3.

Here is an outline of the proof of the stronger statement. Since N is compact, it is enough to prove the statement locally for $q \in N$. Fix $q_0 \in N$ and let $B' \subset N$ be a ball chart around q_0 and $\overline{B} \subset B'$ a smaller closed ball around q_0.

We claim that there exists $r_0' > r_0$, a compact subset $K \subset B'$ and a family of Hamiltonian functions

$$H^{(q)}: [0,1] \times T^*(N) \to \mathbb{R},$$

parametrized by $q \in \overline{B}$, such that the following holds:

(1) All the functions $H^{(q)}$, $q \in \overline{B}$, are compactly supported in

$$\mathcal{V}' := T^*_{<r_0'}(N) \cap \pi^{-1}(K),$$

where $\pi: T^*(N) \to N$ is the projection.

(2) The family $H^{(q)}$ depends smoothly on $q \in \overline{B}$. In particular, the Hofer norm of the elements of the family is uniformly bounded in q:

$$\sup_{q \in \overline{B}} \int_0^1 \|H_t^{(q)}\|_{\mathrm{osc}}\, dt < \infty.$$

Here $H_t^{(q)}(x) := H^{(q)}(t,x)$ and for a compactly supported $H: T^*(N) \to \mathbb{R}$, $\|H\|_{\mathrm{osc}}$ stands for its L^∞-oscillation norm $\|H\|_{\mathrm{osc}} := \max H - \min H$.

(3) $\phi_t^{H^{(q)}}(\mathcal{V}) = \mathcal{V}$ for every $t \in [0,1]$, $q \in \overline{B}$.

(4) $\phi_1^{(q)}(F_{q_0}) = F_q$ for every $q \in \overline{B}$.

The existence of a family $H^{(q)}$ with the above properties is straightforward.

Let $q \in \overline{B}$ and $L \subset \mathrm{Int}\,\mathcal{V}$ a marked exact Lagrangian. Without loss of generality assume that $L \pitchfork F_{q_0}$ and $L \pitchfork F_q$. Pick a regular almost complex structure J as in Section 2.2.2. For a domain $\mathcal{U} \subset T^*(N)$ and two transverse marked exact Lagrangians $L', L'' \subset \mathcal{U}$ we denote by $CF(L', L''; (0, J); \mathcal{U})$ the Floer complex of (L', L'') with Floer data $(H \equiv 0, J)$ inside the domain $\mathcal{U}$, whenever well defined.

By standard arguments in Floer theory there is a quasi-isomorphism

$$\varphi_{(q)}: CF\big(L, T^*_{q_0}; (0, J); T^*(N)\big) \to CF\big(L, \phi_1^{H^{(q)}}(T^*_{q_0}(N)); (0, J); T^*(N)\big), \tag{46}$$

of weight $\leq 2C_1(q) + C_2(q)$, where

$$C_1(q) = \int_0^1 \|H_t^{(q)}\|_{\mathrm{osc}}\, dt$$

and $C_2(q)$ is a constant that depends only on $\mathcal{V}'$ and on the C^2-size of $H^{(q)}$ in a continuous way. See Definition 7.5.3 (and the discussion after it) for weighted quasi-isomorphisms. Here we endow the exact Lagrangian $\phi_1^{H^{(q)}}(T^*_{q_0}(N))$ with a primitive function that is 0 along $\phi_1^{H^{(q)}}(T^*_{q_0}(N)) \cap \mathcal{V} = F_q$.

The quasi-isomorphisms φ_q and its homotopy inverse can be constructed either by counting solutions of the Floer equation with moving boundary conditions, or alternatively, by applying the standard continuation map (comparing the 0-Hamiltonian with $H^{(q)}$) followed by a naturality map as in (10). (The generalization in terms of A_∞-modules corresponding to $T^*_{q_0}(N)$ and $\phi_1^{H^{(q)}}(T^*_{q_0}(N))$ can be established by similar methods.) The bound on the weight of φ_q follows from standard action-energy estimates in Floer theory.

An important point about the previous weight is that it does not depend on L, and moreover that

$$\sup_{q \in \bar{B}} (2C_1(q) + C_2(q)) < \infty.$$

By choosing J appropriately near the boundary of $\mathcal{V}$ (and along $T^*(N) \setminus \mathcal{V}$) an argument based on the maximum principle (or alternatively, arguing as in the proof of Proposition 2.4.1) shows that all the Floer trajectories contributing to any of the chain complexes

$$CF\big(L, T^*_{q_0}; (0, J); T^*(N)\big) \quad \text{and} \quad CF\big(L, \phi_1^{H^{(q)}}(T^*_{q_0}(N)); (0, J); T^*(N)\big)$$

must be entirely contained inside $\mathcal{V}$. (An analogous statement holds also for Floer polygons contributing to the higher order operations of the modules corresponding to $T^*_{q_0}(N)$ and $\phi_1^{H^{(q)}}(T^*_{q_0}(N))$ as long as we view them as modules over the Fukaya category of $\mathcal{V}$.)

Since $\phi_1^{H^{(q)}}(T^*_{q_0}(N)) \cap \mathcal{V} = F_q$ and $T^*_{q_0}(N) \cap \mathcal{V} = F_{q_0}$, the statement we wanted to prove follows. □

We are now ready to prove the main theorem.

Proof of Theorem A. Fix a small $r_0 > 0$ and tubular neighborhood $\mathcal{V} = T^*_{\leq r_0}(N)$ of N. Recall from Section 3.2 the symplectic embedding $\kappa: \mathcal{V} \to E$ and its image $\mathcal{U} := \kappa(\mathcal{V}) \subset E$.

We now appeal to the cone decomposition (25) from Section 5 of Yoneda modules over $\mathcal{F}uk(E')$. We apply this to the Lagrangian $K = N$ (i.e., the zero section) and its Yoneda module $\mathcal{N}$. Let $L \subset \mathcal{U}$ be any exact Lagrangian. The filtered cone decomposition of $\mathcal{N}$, as described in Propositions 5.3.1 and 5.3.2, gives a filtered cone decomposition of the chain complex $CF(L, N; E')$, which by the formulas (25)–(27) involves the following types of filtered chain complexes as well as their tensor products:

(1) $CF(L, S_i; E'), CF(S_i, N; E'), i = 1, \dots, k$.

(2) $CF(S_{j''}, S_{j'}; E'), 1 \leq j' < j'' \leq k$.

(3) $CF(L, N^{(k)}; E')$.

The chain complexes in (2) do not depend on L. In particular their spectral invariants and boundary depths are independent of L.

Formulas (25)–(27) together with Proposition 7.6.1 and Lemma 7.4.1 imply that there are constants $A_1, B_1, C_1 > 0$, that do not depend on L, such that

$$\rho\big(CF(L, N; E')\big) \leq A_1 \widetilde{\rho}\big(CF(L, S_1; E'), \dots, CF(L, S_k; E')\big) + B_1 \sum_{i=1}^{k} \beta\big(CF(L,S_i); E'\big) + C_1.$$

Passing from E' to E, as described in Section 4, we have action preserving chain isomorphisms

$$CF(L, N; E) \cong CF(L, N; E') \quad \text{and} \quad CF(L, T^{\uparrow}_{x_i}; E) \cong CF(L, S_i; E')$$

for every $1 \leq i \leq k$. Consequently, the spectral invariants and boundary depths of the chain complexes in E coincide with the corresponding ones in E'.

Next we appeal to Proposition 2.4.1 (with $W_0 = \mathcal{U}$, $V = E$, $L_0 = L$, and $L_1 = N$) and to Proposition 3.2.1 and deduce that

$$\rho\big(CF(L, N; \mathcal{U})\big) \leq A_1 \widetilde{\rho}\big(CF(L, F_{q_1}; \mathcal{U}), \dots, CF(L, F_{q_k}; \mathcal{U})\big) + B_1 \sum_{i=1}^{k} \beta\big(CF(L,F_{q_i}); E'\big) + C_1,$$

where $q_i = \kappa^{-1}(x_i) \in N$.

Put $q := q_1$. By Lemma 6.1.1, we have that both

$$|\sigma\big(CF(L, F_q)\big) - \sigma\big(CF(L, F_{q_i})\big)|$$

as well as

$$|\beta\big(CF(L, F_q)\big) - \beta\big(CF(L, F_{q_i})\big)|$$

are uniformly bounded (with respect to L and i), and hence there exist constants $A_2, B_2 > 0$ that do not depend on L, such that

$$\rho\big(CF(L,N;\mathcal{U})\big) \leq A_2 + B_2\beta\big(CF(L,F_q)\big).$$

Now, $\gamma(L,N) \leq \rho(CF(L,N;\mathcal{U}))$, hence

$$\gamma(L,N) \leq A_2 + B_2\beta\big(CF(L,F_q)\big)$$

for all exact Lagrangians $L \subset \mathcal{U}$. The last inequality together with the triangle inequality for γ imply inequality (2) and conclude the proof of Theorem A. □

6.2. Boundedness of the spectral metric implies boundedness of $\beta(CF(L, F_q))$. Here we outline an argument showing the statement at point (2) of Remark 1.0.1. Namely, if the function

$$\mathcal{L}_{\mathrm{ex},N}(U) \ni L \mapsto \gamma(N,L)$$

is bounded, then

$$\mathcal{L}_{\mathrm{ex},N}(U) \ni L \mapsto \beta\big(CF(L,F_q)\big)$$

is bounded too. In other words the conjecture of Viterbo from page 2 implies the boundedness of the boundary depths $CF(-,F_q)$ over the collection of exact Lagrangians $L \subset U$ that are exact isotopic to the zero-section N.

Here is an outline of the proof. Let $L \in \mathcal{L}_{\mathrm{ex},N}(U)$ and assume without loss of generality that $L \pitchfork N$, $L \pitchfork F_q$. Fix an arbitrary marking for L and mark N and F_q by taking their primitive functions to be identically 0. Put

$$\alpha_+ = c\big([N]; N, L\big), \quad \alpha_- = c\big([N]; L, N\big).$$

We have $\alpha_+ + \alpha_- = \gamma(N,L)$. Note that α_+ and α_- depend on the marking of L but their sum $\alpha_+ + \alpha_-$ does not. Also note that $\beta(CF(N,L))$ is independent of the marking of L.

We will now need to carry out a chain-level calculation with Floer complexes. To this end we take the Floer complexes $CF(N,L)$, $CF(L,N)$, $CF(N,F_q)$, and $CF(L,F_q)$ with Floer data having 0 Hamiltonian terms. We also fix a Floer datum for (L,L) whose Hamiltonian term is induced from a C^2-small Morse function $L \to \mathbb{R}$ with a unique critical point of top index, so that the unity $e_L \in HF(L,L)$ has a unique representing cycle in $CF(L,L)$.

Let $\varepsilon > 0$. Choose perturbation data for each of the tuples (L,N,L,F_q), (N,L,N), (L,N,L), (N,L,F_q), and (L,N,F_q), which are compatible with each other as well as with the previous choices of Floer data and such that

(1) The associated μ_3-operation

$$\mu_3 \colon CF(L,N) \otimes CF(N,L) \otimes CF(L,F_q) \to CF(L,F_q)$$

shifts action by $\leq \varepsilon$.

(2) The μ_2-operations corresponding to any of the triplets above shift action by $\leq \varepsilon$.

This is possible in view of the action-energy relation, as described in [32, Chapter II (8g)] for example; see also a more detailed calculation of the action-energy difference in [4, pp. 1769–71] which is carried out for monotone Lagrangians but can be easily adapted to the exact case. Note that in our case, we have *fixed* L earlier in the proof, hence here we are dealing with μ_2 and μ_3 operations involving only a finite number of Lagrangians.

Let $a \in CF^{\leq \alpha_+}(N, L)$ and $b \in CF^{\leq \alpha_-}(L, N)$, be cycles representing the Floer homology classes $\mathcal{N}^{N}_{L,N}([N])$ and $\mathcal{N}^{N,L}_{N}([N])$; see Section 2.2.3. Consider the following two filtered chain maps:

$$\begin{aligned} \varphi\colon CF(L, F_q) &\to CF(N, F_q), \quad \varphi(x) := \mu_2(a, x), \\ \phi\colon CF(N, F_q) &\to CF(L, F_q), \quad \phi(y) := \mu_2(b, y). \end{aligned} \tag{47}$$

By our choices of data, φ shifts action by $\leq \alpha_+$ and ϕ by $\leq \alpha_-$. Note that $CF(N, F_q) = \mathbb{Z}_2 q$ and it is easy to see that $\varphi \circ \phi = \mathrm{id}$. We claim that $\phi \circ \varphi$ is chain homotopic to the identity via a chain homotopy H that shifts action by $\leq \gamma(N, L) + \varepsilon$.

Before proving the last claim, let us see how it implies the main statement we want to prove. For this purpose we would like to use Lemma 7.1.2 which compares the boundary depths of two chain complexes that are chain homotopy equivalent (specifically in our case, $CF(L, F_q)$ and $CF(N, F_q)$). However in order to employ Lemma 7.1.2 we need the shifts of each of φ and ϕ to be non-negative and we also need to relate each of these shifts to the shift of the chain homotopy H which is claimed to be $\gamma(N, L) + \varepsilon$. The "problem" is that φ and ϕ have shifts of $\leq \alpha_+$ and $\leq \alpha_-$, respectively, and we do not have information on the size of each of them alone – we only know that $\alpha_+ + \alpha_- = \gamma(N, L)$.

To go about this technical problem we proceed as follows. We shift the marking of L by a constant such that $\alpha_- = 0$. Consequently, α_+ will now become equal to $\gamma(N, L)$. We thus assume from now on that $\alpha_- = 0$ and $\alpha_+ = \gamma(N, L)$. Under these circumstances we can now apply Lemma 7.1.2 and obtain that

$$|\beta\big(CF(L, F_q)\big) - \beta\big(CF(N, F_q)\big)| \leq 2\gamma(N, L) + 2\varepsilon.$$

Since $\beta(CF(N, F_q)) = 0$ and by assumption $\gamma(N, -)$ is bounded, the main statement follows.

It remains to show the existence of the required chain homotopy H between $\phi \circ \varphi$ and the id. Consider the tuple of Lagrangians (L, N, L, F_q). Choose Floer perturbation data for this tuple, which is compatible with the previous choices of Floer data, and such that

$$\mu_2(\mu_2(b, a), x) = x, \quad \text{for every } x \in CF(L, F_q).$$

Note that by our choices of Floer data, $\mu_2(b, a) \in CF(L, L)$ is the unique cycle representing the unity $e_L \in HF(L, L)$.

By standard A_∞-identities (applied with $\mathbb{Z}_2$-coefficients) we have for every $x \in CF(L, F_q)$:

$$\begin{aligned} \phi \circ \varphi(x) &= \mu_2(b, \mu_2(a, x)) \\ &= \mu_2(\mu_2(b, a), x) + \mu_3(b, a, \mu_1(x)) + \mu_1\mu_3(b, a, x) \\ &= x + \mu_3(b, a, \mu_1(x)) + \mu_1\mu_3(b, a, x). \end{aligned} \tag{48}$$

The required homotopy $H: CF(L, F_q) \to CF(L, F_q)$ is then $H(x) := \mu_3(b, a, x)$. Since, by construction, the μ_3-operation in (48) is assumed to shift action by $\leq \varepsilon$, it follows that H shifts action by $\leq \gamma(N, L) + \varepsilon$. $\square$

Remark 6.2.1. A similar argument appears, for a different purpose, in [21]. At a conceptual level these arguments are a reflection of a Yoneda type lemma in the filtered setting that allows translation of relations among morphisms of Yoneda modules (over the A_∞ Fukaya category) in terms of μ^k-operations. Such a result, called there the λ-lemma, appears in [7].

7. Filtered homological algebra

The purpose of this section is to establish a number of algebraic results that allow control of the spectral range and boundary depth of filtered complexes through cone-attachments.

7.1. Background on filtered complexes. We consider here filtered modules C over a ring R. We assume the filtration to be indexed by the reals and increasing, namely for every $\alpha \in \mathbb{R}$ we have a submodule $C^{\leq \alpha} \subset C$ and $C^{\leq \alpha} \subset C^{\leq \alpha'}$ for $\alpha \leq \alpha'$. For simplicity we will always assume that the filtration is exhaustive, i.e., $\cup_{\alpha \in \mathbb{R}} C^{\leq \alpha} = C$.

The shift of order $s \in \mathbb{R}$ of a filtered module C is the filtered module $C[s]$ defined by $(C[s])^{\leq \alpha} = C^{\leq \alpha + s}$. (Despite the similarity in notation, this has nothing to do with grading-shifts. In fact, in this paper we work in an ungraded setting.) An R-linear map $f: C \to C'$ between two filtered modules is called *s-filtered* if $f(C^\alpha) \subset (C')^{\leq \alpha + s}$ for all $\alpha \in \mathbb{R}$. We will refer to such a number s as an admissible shift for the map f. We will also say that f shifts action by $\leq s$, or sometimes that f is filtered of shift s. Notice that if f is s-filtered then it is also s'-filtered for all $s' \geq s$. An R-linear map $f: C \to C'$ is called *filtered* if it is s-filtered for some $s \geq 0$. For reasons of convenience we will consider only shifts s that are non-negative. There is no loss of generality in doing that as any map that shifts action by a negative amount can be viewed as 0-filtered. A slight drawback of this convention is that some of the estimates on invariants of filtered chain complexes developed below will be less sharp. Since our applications are concerned with coarse estimates this will not play an important role in our considerations.

Let C be a filtered chain complex or R-modules. This means that C is a filtered module and the differential d of C preserves the filtration, i.e., $d(C^{\leq\alpha}) \subset C^{\leq\alpha}$ for every α. To such a chain complex we can associate a persistence module $H^{\leq\bullet}(C)$ consisting of the homologies of the subcomplexes of C prescribed by the filtration:

$$H^{\leq\alpha}(C) = H(C^{\leq\alpha}), \quad i^{\beta,\alpha}\colon H^{\leq\alpha}(C) \to H^{\leq\beta}(C), \quad \alpha \leq \beta,$$

where the maps $i^{\beta,\alpha}$ are induced by the inclusions $C^{\leq\alpha} \subset C^{\leq\beta}$. We also have the maps $i^{\alpha}\colon H^{\leq\alpha}(C) \to H(C)$ induced by the inclusions $C^{\leq\alpha} \subset C$.

The boundary depth of the filtered complex C is defined as:

$$\beta(C) = \inf\left\{b \in [0,\infty) \mid \forall\alpha \in \mathbb{R},\ \ker(i^{\alpha}) = \ker(i^{\alpha+b,\alpha})\right\}.$$

For every $a \in H(C)$ we define the spectral invariant $\sigma(a)$ by

$$\sigma(a) = \inf\left\{\alpha \mid a \in \operatorname{image} i^{\alpha}\right\}.$$

We also define

$$\begin{aligned} \sigma_+(C) &= \inf\left\{r \in \mathbb{R} \mid t \geq r \Rightarrow \operatorname{Coker}(i^t) = 0\right\}, \\ \sigma_-(C) &= \sup\left\{s \in \mathbb{R} \mid t \leq s \Rightarrow i^t = 0\right\}, \\ \rho(C) &= \sigma_+(C) - \sigma_-(C). \end{aligned}$$

As the notation suggests $\sigma_+(C)$ is the top (or supremal) spectral invariant of C and $\sigma_-(C)$ is the bottom (or infimal) one. We call $\rho(C)$ the *spectral range* of C.

Remark 7.1.1. The notions above can easily be reformulated in terms of the modern terminology of *barcodes* [28]. For instance, $\beta(C)$ is the length of the longest finite bar of C. Further, if the bar code associated to $H^{\leq\bullet}(C)$ is the collection $\{[i_k, j_k)\}$, then $\sigma_+(C)$ is the maximal i_k among all bars with $j_k = \infty$ and $\sigma_-(C)$ is the minimal i_k among the same (infinite) bars.

We now describe the behavior of σ and β with respect to some operations with filtered chain complexes. We begin with the simple remark that if $f\colon C \to C'$ is a quasi-isomorphism and is s-filtered then we have:

$$\sigma_-(C) \geq \sigma_-(C') - s, \quad \sigma_+(C) \geq \sigma_+(C') - s. \tag{49}$$

In particular, if f admits an s-filtered homological inverse, we deduce

$$|\sigma_\pm(C) - \sigma_\pm(C')| \leq s, \quad |\rho(C) - \rho(C')| \leq 2s.$$

In order to relate the boundary depth of two quasi-isomorphic chain complexes we will need the notion of boundary depth of a map. Let $f\colon C \to C'$ be a filtered chain map and let $s \geq 0$ be an admissible shift for f. The map f induces a map of persistence modules

$$f_*^{\bullet}\colon H^{\leq\bullet}(C) \to H^{\leq\bullet}(C')[s], \quad f_*^{\bullet} = \{f_*^{\alpha}\}$$

with $f_*^\alpha : H^{\leq\alpha}(C) \to H^{\leq\alpha+s}(C')$ induced by f. We define the boundary depth of f, viewed as an s-filtered map, by:

$$\beta_s(f) = \inf\big\{b \in [0,\infty) \mid \forall\alpha \in \mathbb{R},\ \mathrm{image}(f_*^\alpha) \cap \ker(i^{\alpha+s}) \subset \ker(i^{\alpha+s+b,\alpha+s})\big\}.$$

Clearly, $\beta(C) = \beta_0(\mathrm{id}_C)$, $\beta_s(f) \leq \beta(C')$ and for $s \leq s'$,

$$\beta_{s'}(f) = \max\big\{0, \beta_s(f) - s' + s\big\}.$$

Assume now that $f : C \to C'$, $g : C' \to C$ are s-filtered chain maps with $g_* \circ f_* =$ id in homology. We have the inequality:

$$\beta(C) \leq \max\big\{\beta(C') + 2s, \beta_{2s}(g \circ f - \mathrm{id}_C)\big\}. \tag{50}$$

The simplest way to control the boundary depth of maps as above is by using filtered homotopies. Let $f, f' : C \to C'$ be two s-filtered maps that are homotopic with a homotopy $h : f \simeq f'$ which is s'-filtered, then:

$$\beta_s(f - f') \leq \min\{0, s' - s\}. \tag{51}$$

Assume now that $f : C \to C'$, $g : C' \to C$ are s-filtered chain maps such that there is an s-filtered homotopy $h : g \circ f \simeq \mathrm{id}_C$. In this case, $\beta_{2s}(g \circ f - \mathrm{id}_C) = 0$ and we deduce that

$$\beta(C) \leq \beta(C') + 2s.$$

Summing up:

Lemma 7.1.2. *If $f : C \to C'$ and $g : C' \to C$ are s-filtered and there are s-filtered chain homotopies $h : g \circ f \simeq \mathrm{id}_C$ and $h' : f \circ g \simeq \mathrm{id}_{C'}$, then we have:*

$$|\beta(C) - \beta(C')| \leq 2s, \quad |\sigma_\pm(C) - \sigma_\pm(C')| \leq s, \quad |\rho(C) - \rho(C')| \leq 2s. \tag{52}$$

7.2. Mapping cones. Let A, B be filtered chain complexes and $f : A \to B$ an s-filtered chain map. The filtered mapping cone

$$[A \xrightarrow{(f,s)} B]$$

of f is the mapping cone of f endowed with the following filtration:

$$[A \xrightarrow{(f,s)} B]^{\leq\alpha} = A^{\leq\alpha-s} \oplus B^{\leq\alpha}. \tag{53}$$

Of course, this choice of filtration is somewhat ad-hoc and there are other possibilities. Firstly, one can shift the above filtration by any real number. The reason for the specific choice in (53) is to make the inclusion

$$B \to [A \xrightarrow{(f,s)} B]$$

filtration preserving. Secondly, the filtration in (53) depends on s (and therefore this parameter appears in the notation).

We will now estimate the boundary depth and spectral range of the mapping cone in terms of the invariants of its factors.

Lemma 7.2.1. *Let* $C := [A \xrightarrow{(f,s)} B]$. *We have the following inequalities:*

$$\sigma_-(C) \geq \min\{\sigma_-(B) - \beta(A), \sigma_-(A) + s\}, \tag{54}$$

$$\sigma_+(C) \leq \max\{\sigma_+(B), \sigma_+(A) + \beta(B) + s\}, \tag{55}$$

and

$$\beta(C) \leq \beta(A) + \beta(B) + \max\{0, \sigma_+(A) - \sigma_-(B) + s\}. \tag{56}$$

Note that the estimates in the lemma do not depend on the chain map f (though they do depend on the amount of shift s of f).

Proof of Lemma 7.2.1. The basic ingredient in the proof is provided by the long exact sequences:

$$\cdots \to H^{\leq \alpha - s}(A) \xrightarrow{f} H^{\leq \alpha}(B) \xrightarrow{h} H^{\leq \alpha}(C) \xrightarrow{p} H^{\leq \alpha - s}(A) \to \cdots,$$

where h is induced by inclusion and p by the projection. The maps i^{α}, $i^{\beta,\alpha}$ relate functorially these exact sequences.

To see (54) let $c \in H^{\leq \alpha}(C)$,

$$\alpha < \min\{\sigma_-(B) - \beta(A), \sigma_-(A) + s\}.$$

Then, $p(c) \in H^{\leq \alpha - s}(A)$ and as $\alpha - s < \sigma_-(A)$, then $i^{\alpha - s}(p(c)) = 0$. This implies that

$$i^{\alpha + b - s, \alpha - s}(p(c)) = 0 \quad \text{for all } b > \beta(A).$$

We take b sufficiently small such that $\alpha < \sigma_-(B) - b$. Thus, there is $c' \in H^{\leq \alpha + b}(B)$ such that

$$h(c') = i^{\alpha + b, \alpha}(c).$$

But we also have that $\alpha + b < \sigma_-(B)$, so that

$$i^{\alpha + b}(c') = 0.$$

Therefore, $i^{\alpha}(c) = 0$, which shows the first inequality.

The proof of (55) is similar. Indeed, if

$$\alpha > \max\{\sigma_+(B), \sigma_+(A) + \beta(B) + s\}$$

and $c \in H(C)$, then fix $b > \beta(B)$ very close to $\beta(B)$ such that

$$\alpha > \sigma_+(A) + b + s.$$

There exists $c' \in H^{\leq \alpha-s-b}(A)$ such that $i^{\alpha-s-b}(c') = p(c)$, and moreover

$$f(i^{\alpha-s,\alpha-s-b}(c')) = 0.$$

Let $c'' = i^{\alpha-s,\alpha-s-b}(c')$. There is $c''' \in H^{\leq\alpha}(C)$ such that $p(c''') = c''$. Now,

$$p(i^{\alpha}(c''') - c) = 0,$$

and therefore there is $\widetilde{c} \in H(B)$ such that $h(\widetilde{c}) = i^{\alpha}(c''') - c$. But $\alpha > \sigma_+(B)$, and hence there is $\widetilde{c}' \in H^{\leq\alpha}(B)$ such that $i^{\alpha}(\widetilde{c}') = \widetilde{c}$. It follows that

$$i^{\alpha}(c''' - h(\widetilde{c}')) = c,$$

and thus $i^{\alpha}\colon H^{\leq\alpha}(C) \to H(C)$ is surjective.

Finally, to show (56), assume

$$r > \beta(A) + \beta(B) + \max\big\{0, \sigma_+(A) - \sigma_-(B)\big\}$$

and let $c \in H^{\leq\alpha}(C)$ such that $i^{\alpha}(c) = 0$. We want to show that $i^{\alpha+r,\alpha}(c) = 0$. Note that

$$i^{\alpha+b-s,\alpha-s}(p(c)) = 0 \quad \text{for } b > \beta(A).$$

Let $c' = i^{\alpha+b,\alpha}(c)$. Therefore, there exists $c'' \in H^{\leq\alpha+b}(B)$ with $h(c'') = c'$. In the case $\alpha + b < \sigma_-(B)$, then $i^{\alpha+b}(c'') = 0$, and thus for $b' > \beta(B)$ we have

$$i^{\alpha+b+b',\alpha+b}(c'') = 0.$$

This implies that $i^{\alpha+b+b',\alpha}(c) = 0$ and, by taking b, b' small enough, this shows that $i^{\alpha+r,\alpha}(c) = 0$. The other possibility to consider is when $\alpha + b \geq \sigma_-(B)$. In this case let $\widehat{c} = i^{\alpha+b}(c'')$. As $h(\widehat{c}) = 0$ there is $\widehat{c}' \in H(A)$ such that $f(\widehat{c}') = \widehat{c}$. Now consider,

$$k > \max\big\{0, \sigma_+(A) + s - \sigma_-(B)\big\}.$$

There exists $\widehat{c}'' \in H^{\leq\alpha+b-s+k}(A)$ such that $i^{\alpha+b-s+k}(\widehat{c}'') = \widehat{c}'$. Now,

$$f(\widehat{c}'') \in H^{\leq\alpha+b+k}(B) \quad \text{and} \quad i^{\alpha+b+k}\big(i^{\alpha+b+k,\alpha+b}(c'') - f(\widehat{c}'')\big) = 0.$$

Thus,

$$i^{\alpha+b+b'+k,\alpha+b+k}\big(i^{\alpha+b+k,\alpha+b}(c'') - f(\widehat{c}'')\big) = 0,$$

which combined with $h(f(\widehat{c}'')) = 0$ implies that $i^{\alpha+b+b'+k,\alpha}(c) = 0$, which shows our claim by taking b, b', k small enough. □

From inequalities (54) and (55) we deduce a simpler (but rougher) estimate for the spectral range of $C = [A \xrightarrow{(f,s)} B]$:

$$\rho(C) \leq \max\big\{\sigma_+(A), \sigma_+(B)\big\} - \min\big\{\sigma_-(A), \sigma_-(B)\big\} + \beta(A) + \beta(B) + s \quad (57)$$

It is important to note that one can not, in general, eliminate the boundary depth from estimates such as (54), (55), or (57), nor can one eliminate the spectral values σ_+, σ_- from an estimate like (56).

Remarks 7.2.2. (1) Above we have considered s-morphisms with $s \geq 0$. Occasionally it makes sense to consider also the case $s < 0$ (such maps not only preserve filtrations but in fact shift them downwards by $(-s)$). The estimates (49)–(57) can be easily adjusted to the case $s < 0$. However, for the applications needed in this paper it is enough to assume $s \geq 0$.

(2) Let A, B be two filtered chain complexes and $f\colon A \to B$ an s-filtered chain map, where we allow here any $s \in \mathbb{R}$ (also $s < 0$). Let $s' \geq s$. Then f is also an s'-filtered chain map. We can now endow the mapping cone of f with two different filtrations, following (53), once using the shift s and once the shift s'. Denote the corresponding filtered mapping cones by

$$C := [A \xrightarrow{(f,s)} B] \quad \text{and} \quad C' := [A \xrightarrow{(f,s')} B].$$

It easily follows (e.g., from Lemma 7.1.2) that

$$|\sigma_\pm(C') - \sigma_\pm(C)| \leq s' - s, \quad |\rho(C') - \rho(C)|, |\beta(C') - \beta(C)| \leq 2(s' - s). \tag{58}$$

Next we analyze equivalences of mapping cones, taking into account filtrations. Consider the following diagram:

$$\begin{tikzcd} A' \arrow[r, "{(f',s_{f'})}"] \arrow[d, "{(\psi',s_{\psi'})}"'] & B' \arrow[d, "{(\phi',s_{\phi'})}"] \\ A'' \arrow[r, "{(f'',s_{f''})}"'] & B'' \end{tikzcd} \tag{59}$$

where A', B', A'', B'' are filtered chain complexes and the notations on the edges of the square are pairs consisting of a filtered chain map and an admissible shift. (e.g., $(f', s_{f'})$ means that $f'\colon A' \to B'$ is an $s_{f'}$-filtered chain map etc.)

We assume that (59) commutes up to an $s_{h'}$-filtered chain homotopy $h'\colon A' \to B'$ (i.e., $\phi' \circ f' - f'' \circ \psi' = dh + hd$) for some $s_{h'} \geq s_{f'}, s_{\phi'}, s_{\psi'}, s_{f''}$. Further, assume that ψ' and ϕ' have filtered homotopy inverses, i.e., there exists an $s_{\psi''}$-filtered chain map $\psi''\colon A'' \to A'$ and an $s_{\phi''}$-filtered chain map $\phi''\colon B'' \to B'$ with

$$\begin{aligned} {}'' \circ \psi' &= dk' + k'd, \quad & \psi' \circ \psi'' &= dk'' + k''d, \\ \phi'' \circ \phi' &= dr' + r'd, \quad & \phi' \circ \phi'' &= dr'' + r''d, \end{aligned} \tag{60}$$

where $k'\colon A' \to A'$, $k''\colon A'' \to A''$, $r'\colon B' \to B'$, $r''\colon B'' \to B''$ are filtered linear maps. We denote by $s_{k'}$, $s_{k''}$, $s_{r'}$, $s_{r''}$ admissible shifts for these maps.

Denote by

$$C(f', s_{f'}) := [A' \xrightarrow{(f',s_{f'})} B'] \quad \text{and} \quad C(f'', s_{f''}) := [A'' \xrightarrow{(f'',s_{f''})} B'']$$

the filtered mapping cones of $(f', s_{f'})$ and $(f'', s_{f''})$, respectively.

Proposition 7.2.3. *There exist filtered chain maps $\varphi' \colon C(f', s_{f'}) \to C(f'', s_{f''})$ and $\varphi'' \colon C(f'', s_{f''}) \to C(f', s_{f'})$ that fit into the following diagrams:*

$$
\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "\psi'"'] & B' \arrow[r] \arrow[d, "\phi'"] & C(f', s_{f'}) \arrow[r] \arrow[d, "\varphi'"] & A' \arrow[d, "\psi'"'] \\
A'' \arrow[r, "f''"'] & B \arrow[r] & C(f'', s_{f''}) \arrow[r] & A''
\end{tikzcd}
$$
$$
\begin{tikzcd}
A' \arrow[r, "f'"] & B' \arrow[r] & C(f', s_{f'}) \arrow[r] & A' \\
A'' \arrow[r, "f''"'] \arrow[u, "\psi''"] & B \arrow[r] \arrow[u, "\phi''"'] & C(f'', s_{f''}) \arrow[r] \arrow[u, "\varphi''"'] & A'' \arrow[u, "\psi''"]
\end{tikzcd}
\tag{61}
$$

where the unmarked horizontal maps in both diagrams are the canonical chain maps associated to cones. These maps are filtered. The left-hand square in the second diagram commutes up to a filtered chain homotopy h''. The second and third squares, in each diagram, commute. The compositions $\varphi'' \circ \varphi'$ and $\varphi' \circ \varphi''$ are chain homotopic to the identities via filtered chain homotopies H' and H''. Moreover, there exist admissible shifts $s_{\varphi'}$, $s_{\varphi''}$, $s_{H'}$, $s_{H''}$, $s_{h''}$ for φ', φ'', H', H'', h'', and a universal constant C (that depends neither on the initial diagram nor on any of the other maps mentioned above) such that

$$
\begin{aligned}
s_{\varphi'}, s_{\varphi''}, s_{H'}, s_{H''}, s_{h''} \leq C\big(& s_{f'} + s_{f''} + s_{\phi'} + s_{\phi''} + s_{\psi'} + s_{\psi''} \\
& + s_{h'} + s_{k'} + s_{k''} + s_{r'} + s_{r''}\big).
\end{aligned}
\tag{62}
$$

Proof. The existence of φ', φ'', H', H'' is standard homological algebra. In fact, it is straightforward to write down explicit formulae for these maps. For example, φ' can be taken to be $\varphi'(a', b') = (\psi(a'), \phi'(b') + h'(a'))$. One then uses the chain homotopies k', k'', r', r'' to describe explicitly h'', φ'' and H', H''.

The only possibly non-standard ingredients are the statements concerning the actions shifts and inequality (62). These can be easily derived from the formulae for φ', φ'', h'', H', H''. □

Remark 7.2.4. By deriving explicit formulae for $\varphi', \varphi'', H', H'', h''$ it is possible obtain sharper estimates for each of $s_{\varphi'}, s_{\varphi''}, s_{H'}, s_{H''}, s_{h''}$ than the uniform bound (62). In the following we will be interested only in coarse estimates on these shifts, hence we will not need such sharp estimates.

7.3. Iterated cones. Let E, F, G be filtered chain complexes and $f \colon F \to G$ an s_f-filtered chain map. Let

$$g \colon E \to [F \xrightarrow{(f, s_f)} G]$$

be an s_g-filtered chain map and define

$$C = [E \xrightarrow{(g,s_g)} [F \xrightarrow{(f,s_f)} G]].$$

There exists a chain map $g'\colon E \to F$ that shifts action by $\leq s_{g'} := \max\{0, s_g - s_f\}$, and another chain map

$$f'\colon [E \xrightarrow{(g',s_{g'})} F] \to G$$

that shifts action by $\leq s_{f'} := s_f$, such that the chain complex

$$C' = \big[[E \xrightarrow{(g',s_{g'})} F] \xrightarrow{(f',s_{f'})} G\big]$$

is isomorphic to C by the map $C \to C'$ induced from the underlying identity map. Moreover, if $s_g < s_f$ (i.e., $s_{g'} = 0$) then this map shifts action by $\leq (s_f - s_g)$ and if $s_g \geq s_f$ (i.e., $s_{g'} \geq 0$) it shifts action by ≤ 0.

Remark 7.3.1. The asymmetry in the action shifts comes from our convention to consider only non-negative action shifts, i.e., to regard a map that shifts action by a negative amount as shifting action by ≤ 0. If we would have allowed for negative action-shifts then we could take $s_{g'} = s_g - s_f$ and the identity map $C \to C'$ would become action preserving. But as remarked at the beginning of Section 7.1 we will stick to the convention that shifts in action are always non-negative.

It follows from the above that

$$|\sigma_\pm(C') - \sigma_\pm(C)| \leq |s_f - s_g|, \quad |\beta(C') - \beta(C)| \leq 2|s_f - s_g|. \tag{63}$$

In the following we will be interested in coarse bounds on spectral invariants and boundary depths of *iterated* cones. Therefore, by abuse of notation we will often write them as $K = [A_r \to A_{r-1} \to \cdots \to A_1 \to A_0]$, whenever the maps are clear from the context and their action shifts are fixed up to a bounded change. The spectral invariants and boundary depths of K will then be determined up to a bounded error.

7.4. Estimating the spectral range of iterated cones.

Let $A_0, \ldots, A_k$ be a finite collection of filtered chain complexes of R-modules. Assume that each of the A_i's has finite spectral range. Define the following values:

$$\begin{aligned} \widetilde{\sigma}_+(A_k, \ldots, A_0) &:= \max\big\{\sigma_+(A_k), \ldots, \sigma_+(A_0)\big\}, \\ \widetilde{\sigma}_-(A_k, \ldots, A_0) &:= \min\big\{\sigma_-(A_k), \ldots, \sigma_-(A_0)\big\}, \\ \widetilde{\rho}(A_k, \ldots, A_0) &:= \widetilde{\sigma}_+(A_k, \ldots, A_0) - \widetilde{\sigma}_-(A_k, \ldots, A_0). \end{aligned} \tag{64}$$

From inequalities (54)–(57), and using the notation (64), we obtain the following inequalities for the mapping cone

$$C = [A \xrightarrow{(f,s)} B]$$

of an s-filtered chain map $f: A \to B$:

$$\begin{aligned} \sigma_+(C) &\leq \ \ \widetilde{\sigma}_+(A,B) + \beta(B) + s, \\ -\sigma_-(C) &\leq -\widetilde{\sigma}_-(A,B) + \beta(A), \\ \beta(C) &\leq \beta(A) + \beta(B) + \widetilde{\sigma}_+(A,B) - \widetilde{\sigma}_-(A,B) + s. \end{aligned} \tag{65}$$

It follows that both $\rho(C)$ as well as $\beta(C)$ can be bounded from above by the same expression:

$$\rho(C), \beta(C) \leq \widetilde{\rho}(A,B) + \beta(A) + \beta(B) + s. \tag{66}$$

Turning to the case of iterated cones, let $A_0, \ldots, A_r$ be filtered chain complexes. Put $C_0 := A_0$. Let $\varphi_1: A_1 \to C_0$ be an s_1-filtered chain map for some $s_1 \geq 0$. Define

$$C_1 := [A_1 \xrightarrow{(\varphi_1, s_1)} C_0],$$

filtered as described in (53). Continuing inductively, assume that we have constructed already the filtered chain complex C_i for some $1 \leq i \leq r-1$ and let $\varphi: A_{i+1} \to C_i$ be an s_{i+1}-filtered chain map for some $s_{i+1} \geq 0$. Define

$$C_{i+1} = \big[A_{i+1} \xrightarrow{(\varphi_{i+1}, s_{i+1})} C_i\big].$$

We call the final chain complex C_r an iterated cone with attachments $A_0, \ldots, A_r$ and sometime denote it by

$$C_r = \big[A_r \to \big[A_{r-1} \to \cdots \to [A_2 \to [A_1 \to A_0]] \cdots \big]\big],$$

omitting references to the chain maps φ_i and the action-shifts s_i.

The following Lemma follows easily from (65).

Lemma 7.4.1. *There exists (universal) constants $a_r, b_r, e_r > 0$, depending only on r, such that for every iterated cone C_r as above we have:*

$$\rho(C_r) \leq a_r \widetilde{\rho}(A_r, \ldots, A_0) + b_r \sum_{j=0}^{r} \beta(A_j) + e_r \sum_{j=1}^{r} s_j. \tag{67}$$

7.5. Weakly filtered A_∞-categories and modules. Recall that a weakly filtered A_∞-category $\mathcal{C}$ is an A_∞-category such that for every two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ the chain complex $(\hom_{\mathcal{C}}(X,Y), \mu_1^{\mathcal{C}})$ is filtered and additionally each of the higher order operations $\mu_d^{\mathcal{C}}$, $d \geq 2$, preserves filtrations up to a (uniform) bounded error. Similarly, filtered modules $\mathcal{M}$ over such categories are $\mathcal{C}$-modules such that for every object $X \in \mathrm{Ob}(\mathcal{C})$ the chain complex $(\mathcal{M}(X), \mu_1^{\mathcal{M}})$ is filtered, and the higher order operations $\mu_d^{\mathcal{M}}$, $d \geq 2$, preserve filtrations up to (uniform) bounded errors (one for each d). One can define weakly filtered pre-module (resp., module) homomorphisms $f: \mathcal{M} \to \mathcal{N}$ between weakly filtered modules, by analogy to filtered maps (resp., chain maps). The first order component $f_1 \colon \mathcal{M}(X) \to \mathcal{N}(X)$, $X \in \mathrm{Ob}(\mathcal{C})$, of such a map is a filtered linear map (resp., chain map) that shifts filtrations by $\leq s_f$, where s_f is a constant that does not depend on X. An analogous condition is imposed on the higher order f_d components of f. (Sometimes, by abuse of notation we will omit the subscript in f_1 and denote the first order component also by f.) Finally, there is also the notion of weakly filtered A_∞-functors between weakly filtered A_∞-categories (in contrast to module homomorphisms which are allowed to shift filtrations, such functors are assumed to preserve filtrations, up to bounded errors). We refer the reader to [7] for the basic theory and formalism of weakly filtered A_∞-categories.

Remark 7.5.1. A word of caution about terminology differences is in order. The notion "weakly filtered" appears in the literature with two different meanings. In the formalism of [12, 13] "weakly filtered map" stands for a map between filtered chain complexes (or A_∞-algebras) that preserves filtrations up to a shift, whereas in our terminology such maps are called "filtered" or s-filtered if we specify the amount of shift s. Our notion of "weakly filtered" means something else. For example, in the case of weakly filtered categories, the first order operations (i.e., the differentials of the hom's) preserve filtrations, but the higher order operations preserve filtrations only up to uniform errors (which we call in [7] discrepancies), and the wording "weakly" refers to that. Thus, without these discrepancies we would have called such categories "filtered categories". In a similar vein we have weakly filtered functors, modules and (pre)-module homomorphisms.

The contents of the entire section above (Sections 7.1–7.4) applies with minor modifications also to the framework of weakly filtered A_∞-modules over a weakly filtered A_∞-category $\mathcal{C}$ rather than just chain complexes. For example, if one replaces the filtered chain complexes A, B by weakly filtered $\mathcal{C}$-modules $\mathcal{A}$, $\mathcal{B}$ and $f \colon A \to B$ by a module homomorphism, then one can define an A_∞-mapping cone module

$$\mathscr{C} = [\mathcal{A} \xrightarrow{f} \mathcal{B}],$$

which is weakly filtered in a similar way as in (53); see [7, §2.4] for more details. The inequalities from (58) then continue to hold with C' and C replaced by $\mathscr{C}'(X)$ and $\mathscr{C}(X)$ respectively, for every object X in the underlying A_∞-category $\mathcal{C}$. Similar modifications apply to (63) as well as to (52).

It is important to note that in the case of A_∞-modules the preceding inequalities hold uniformly for all objects X, since the shift parameters (s_f, s_g etc.) depend only on the modules and the homomorphisms between them, and not on the choice of a particular object in the A_∞-category.

Remark 7.5.2. Through this paper we appeal several times to the notions of weakly filtered A_∞-categories, functors and modules. However, from a purely formal viewpoint this is not really necessary. Indeed, in this paper we essentially have not used the higher operations associated to A_∞-structures or special features that distinguish such structures from filtered chain complexes. Thus in principle one can "downgrade" the entire algebraic formalism in this paper to filtered chain complexes and their persistent homology. The reason we opted for using a bit of A_∞ formalism is the following. A considerable part of the algebra in this paper is devoted to establishing bounds on invariants of filtered Floer chain complexes, e.g., of the type $CF(-,-)$, which are uniform in the "variables" $(-,-)$, or at least one of them. These variables are Lagrangian submanifolds, hence are objects of a Fukaya category (which is weakly filtered). As explained at several points above, the uniformity of various quantities related to action filtration can be more concisely expressed using the language of A_∞-modules.

We end this section with a useful definition.

Definition 7.5.3. Let $\mathcal{M}$, $\mathcal{N}$ be two weakly filtered A_∞-modules. Let $f\colon \mathcal{M} \to \mathcal{N}$ be a weakly filtered module homomorphism and $w \geq 0$. We say that f is a quasi-isomorphism of weight $\leq w$ if the following holds:

(1) f shifts filtration by $\leq w$.

(2) There exists a weakly filtered module homomorphism $g\colon \mathcal{N} \to \mathcal{M}$ that shifts filtration by $\leq w$ and two weakly filtered pre-module homomorphisms $h\colon \mathcal{M} \to \mathcal{M}$, $k\colon \mathcal{N} \to \mathcal{N}$ that shift filtrations by $\leq w$, such that:
$$g \circ f = \mathrm{id} + \mu_1^{\mathrm{mod}}(h), \quad f \circ g = \mathrm{id} + \mu_1^{\mathrm{mod}}(k). \tag{68}$$

We say that two weakly filtered modules $\mathcal{M}$ and $\mathcal{N}$ are at distance w one from the other if there exists a quasi-isomorphism $f\colon \mathcal{M} \to \mathcal{N}$ of weight $\leq w$.

Remark 7.5.4. Similar notions appear in relation to the so-called bottleneck distance in persistence module theory, for instance in [37], as well as in a somewhat different context in [7].

The same definition can be easily adapted to the case when $\mathcal{M}$ and $\mathcal{N}$ are just filtered chain complexes and $f\colon \mathcal{M} \to \mathcal{N}$ is a w-filtered chain map. In this case, the analogue of condition (68) simply means that $f \circ g$ and $g \circ f$ are chain homotopic to the respective identities via w-filtered chain homotopies. Note that despite being called only a "quasi-isomorphism", f satisfies a stronger condition - it is implicitly assumed to have a homotopy inverse.

7.6. Spectral range and boundary depth of tensor products. Let A, B be finite dimensional filtered chain complexes over a field R. The tensor product (over R) chain complex $A \otimes B$ inherits a filtration from A and B, where $(A \otimes B)^{\leq \alpha} \subset A \otimes B$ is generated by the collection of subspaces $A^{\leq \alpha - s} \otimes B^{\leq s}$, $s \in \mathbb{R}$.

Proposition 7.6.1. *For the tensor product chain complex $A \otimes B$ we have:*
$$\sigma_\pm(A \otimes B) = \sigma_\pm(A) + \sigma_\pm(B), \quad \rho(A \otimes B) = \rho(A) + \rho(B),$$
$$\beta(A \otimes B) \leq \max\{\beta(A), \beta(B)\}.$$

Proof. This follows by direct calculation of the barcode of the persistence module $H_*(A \otimes B)$, using the Künneth formula for persistence modules from [30]. □

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