Persistence and Triangulation

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(talk IHP; joint with Paul Biran and Jun Zhang)

A. Motivation.

Question 1. How to understand:

rigidity $\longleftrightarrow$ flexibility

in symplectic topology ?

Aim: Endow classes of objects (in our case Lagrangian submanifolds) with certain pseudo-metrics:

$$d^{\xi}(-,-)$$

Study when these are non-degenerate/degenerate.

2. In a variety of contexts we compare objetcs by type and in many others by size.

Type (homotopy)

$S^2 =$

$T^2 =$

Size (volume)

$S^2(r) =$

$S^2(R) =$

Question 2: How to compare objects by taking into account the two points of view at the same time?

We will discuss a solution based on homological algebra.

Types of objects

Objects with sizes

Homological functors

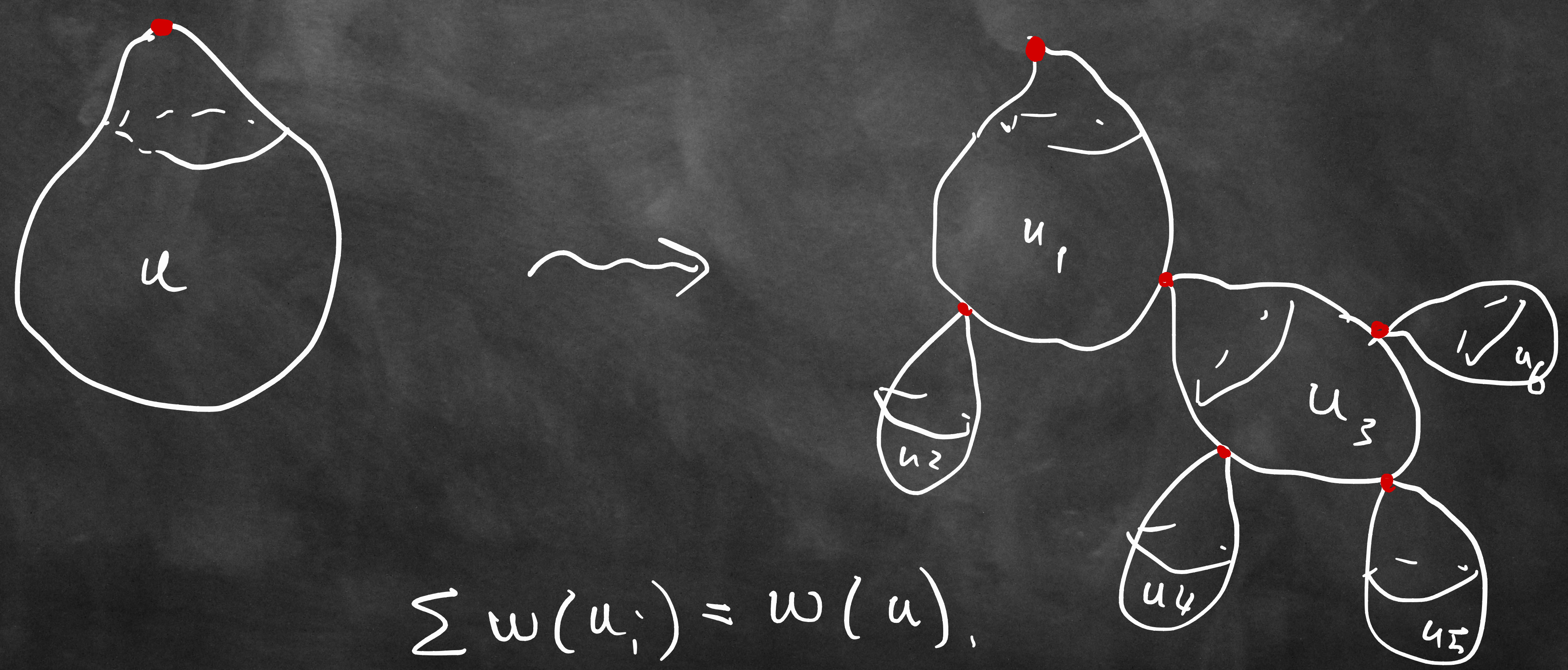
Persistence theory (modules etc)

Homological algebra

Relevant notion: triangulated persistence category (TPC).

Remark. There is a fundamental reason why this is natural in symplectic topology:

Gromov compactness involves simultaneous, controlled changes in size (symplectic area) and of type (bubble trees).



$$\sum \omega(u_i) = \omega(u).$$

B. A brief introduction to TPCs

1. Triangular weights.

$\mathcal{C}$ = triangulated category (Verdier and Puppe early '60's).

$\mathcal{C} = (\mathcal{C}, T, Tr)$, T = translation functor

Tr = distinguished class of triangles

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow TA$$

with functoriality properties similar to cone attachments in topology.

$$(C \sim Cone(f))$$

Remark: Triangulated categories are a bit "primitive" (more sophisticated notions: Quillen model categories; Waldhausen catgories; stable categories).

In a triangulated category $\mathcal{C}$ we will consider iterated distinguished triangles to build an object Y out of another X:

$$\eta = \left\{ \begin{array}{l} \Delta_1 : A_1 \longrightarrow X \longrightarrow X_1 \longrightarrow TA_1 \\ \\ \Delta_2 : A_2 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow TA_2 \\ \quad - \; - \; - \; - \; - \; - \; - \; - \; - \; - \\ \Delta_k : A_k \longrightarrow X_{k-1} \longrightarrow X_k \longrightarrow TA_k \\ \qquad\qquad\qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\qquad\qquad\quad Y \end{array} \right.$$

$\eta =$ iterated cone decomposition of Y from X with linearization $\ell(\eta) = (A_1, A_2, \dots, A_k)$

Remark: Iterated cone decompositions are useful (and have been extensively used) to estimate "type" complexity.

Examples: Morse inequalities; Lusternik-Schnirelman inequality; Arnold conjecture inequality and more.

Method:

- choose a family of objects $\mathcal{F}$
- define:

$$c^{\mathcal{F}}(X, O) = \inf\{ k \in \mathbb{N}^* \mid \exists\, \eta \text{ iterated cone decomposition of } X \text{ from } O \text{ with linearization } (A_1, A_2, \ldots, A_k) \text{ and } A_i \in \mathcal{F} \}$$

For instance:

- if $\mathcal{C}$ is $DFuk_{ex}(D^*N)$ and $\mathcal{F}$ is the module represented by one fiber $F_x \subset D^*N$, then:

$$\# \mathrm{Crit}(f : L \to \mathbb{R}, \mathrm{Morse}) \sim c^{\mathcal{F}}(L, 0) \geq \sum b_i(L)$$

Question ?

By the Arnold conj + FSS

Definition (triangular weight): A triangular weight w on a triangulated category $\mathcal{C}$ is a function :

$$w : Tr(\mathcal{C}) \longrightarrow [0, +\infty)$$

that satisfies

i) a weighted octahedral axiom,

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] \\
D \arrow[r] & D' \arrow[r] & E
\end{tikzcd}$$

Δ_1 Δ_2 Δ_3 Δ_4

$$\Delta_1, \Delta_2 \in Tr$$
$$\Downarrow$$
$$\exists\, \Delta_3, \Delta_4 \in Tr$$

and $$w(\Delta_3) + w(\Delta_4) \leq w(\Delta_1) + w(\Delta_2)$$

ii) a normalization axiom,

$$\exists\, w_0 \geq 0, \quad w(\Delta) \geq w_0 \quad \forall \quad \Delta \in T_r$$

and $\forall\, X$ $\quad w(\Delta_X) = w_0$

where $\Delta_X : 0 \longrightarrow X \xrightarrow{id} X \longrightarrow 0$

Example: The flat weight - $w_{fl}(\Delta) = 1, \ \forall\, \Delta \in T_r$

$$\delta_w^{\mathcal{F}}(Y, X) = \inf \Big\{ \sum_1^k w(\Delta_i) - w_0 \ \Big| \ \eta = (\Delta_1, \Delta_2, \ldots, \Delta_k)$$

iterated cone decomposition of Y from 0

with $\ell(\eta) = (F_1, \ldots, T^{-1}X, \ldots, F_k), \ F_i \in \mathcal{F} \Big\}$

Weighted octahedral axiom $\Longrightarrow$

$$\delta^{\mathcal{F}}(X,Y) \leq \delta^{\mathcal{F}}(X,Z) + \delta^{\mathcal{F}}(Z,Y).$$

We can symmetrize:

$$d^{\mathcal{F}}(X,Y) = \max\left\{ \delta^{\mathcal{F}}(X,Y),\ \delta^{\mathcal{F}}(Y,Z) \right\}$$

Thus $d^{\mathcal{F}}(X,Y)$ is a pseudo-metric.

Crucial question: Are there non-trivial triangular weights?

2. Persistence categories.

Definition. A category $\mathcal{C}$ is a persistence category if for each two objects $A, B \in \mathrm{Ob}(\mathcal{C})$ the morphisms $\mathrm{Mor}(A,B)$ have the structure of persistence modules

$$\mathrm{Mor}(A,B) = \left(\{ \mathrm{Mor}^r(A,B) \}_{r \in \mathbb{R}}, \; i_{r,r'} \right)$$

$$i_{r,r'} : \mathrm{Mor}^r(A,B) \longrightarrow \mathrm{Mor}^{r'}(A,B), \; \forall \; r \leq r'$$

with the usual compatibility conditions and such that persistence is compatible with composition:

$$\mathrm{Mor}^r(A,B) \otimes \mathrm{Mor}^s(B,C) \longrightarrow \mathrm{Mor}^{r+s}(A,C)$$

Example: The elements of a filtered algebra / $\mathbb{Z}/2$ can be viewed as the morphisms of a persistence category with a single object.

Simple notions associated to a PC.

- r-equivalent morphisms $f, g \in Mor^{\alpha}(A, B)$

$$f \simeq_r g \text{ if } i_{\alpha,\alpha+r}(f-g) = 0$$

- r-acyclic objects $X \in Ob(\mathcal{C})$, $X \simeq_r 0$

$$\text{if} \quad id_X \in Mor^{0}(X, X), \quad i_{0,r}(id_X) = 0$$

- 0 and ∞ slices. Categories $\mathcal{C}_0$, $\mathcal{C}_\infty$ with the same objects as $\mathcal{C}$ and with morphisms

$$Mor_{\mathcal{C}_0} = Mor^{0}_{\mathcal{C}} \; ; \quad Mor_{\mathcal{C}_\infty} = \lim_{\alpha \to \infty} Mor^{\alpha}_{\mathcal{C}}$$

We will also need the notion of a shift functor.

$$\Sigma = \{ \Sigma^r \}_{r \in \mathbb{R}} ,$$

$$\Sigma^r : \mathcal{C} \longrightarrow \mathcal{C}$$ functor such that:

$$\mathrm{Mor}^s(A, B) \cong \mathrm{Mor}^{s-r}(\Sigma^r A, B) \cong$$
$$\cong \mathrm{Mor}^{s+r}(A, \Sigma^r B) , \quad \forall r, s$$

There are natural transformations η_r giving these isomorphisms and everything is compatible with the persistence structure.

$$\begin{tikzcd}
\Sigma^r X \arrow[r, "\Sigma^r f"] \arrow[d, "\eta^r_X"'] & \Sigma^r Y \arrow[d, "\eta^r_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd} , \quad \eta^r \in \mathrm{Mor}^0$$

3. Triangulated persistence categories.

Definition. A TPC is a persistence category $\mathcal{C}$ together with a shift functor Σ such that the 0-slice $\mathcal{C}_0$ is triangulated, each $\eta_X^r : \Sigma^r X \longrightarrow X$ has an r-acyclic cone and the functors Σ^r are exact (+ compatibilities).

The most important property of a TPC is that we can define weighted triangles.

The definition is in two steps:

- a map $A \xrightarrow{f} B$ is an $\overbrace{\text{r-isomorphism}}^{\cong_r}$ if it embeds in an exact triangle in $\mathcal{C}_0$

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow TA$$

such that C is r-acyclic.

- a triangle $A \longrightarrow B \longrightarrow C \longrightarrow T\Sigma^{-r}A$ is exact of weight r if there exists a distinguished triangle in $\mathcal{C}_0$ and a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & C' \arrow[r] \arrow[d, "\cong_r"] & TA \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & T\Sigma^{-r}A
\end{tikzcd} \quad (+)$$

Upshot: if $\mathcal{C}$ is a TPC, then $\mathcal{C}_\infty$ is triangulated and the weighted exact triangles as before induce a triangular weight on $\mathcal{C}_\infty$.

Thus, for a family of objects $\mathcal{F}$

We have the associated pseudo-metrics:

$$d^{\mathcal{F}}(\ ,\ )$$

on the objects of $\mathcal{C}$

C. Examples.

- metric spaces + Lipschitz maps *

- topological spaces with additional structures *
inducing a filtration (such a real valued function)

- Tamarkin categories

- homotopical category of filtered chain complexes
(main algebraic example)

- homological category of a filtered, pre-triangulated
dg-category

Remark: Some of these examples are not quite TPC's but
naturally map to one ($\mathcal{C}_0$ is not quite triangulated).

- DFuk (M)

(M, ω) exact symplectic manifold, $\omega = d\lambda$

$$\mathrm{Ob}(\mathrm{Fuk}(M)) = \{ (L, f) \mid L \subset M \text{ Lagrangian}, f : L \to \mathbb{R}, df = \lambda|_L \}$$

$\mathrm{Fuk}(M) =$ Filtered A_∞ - category

[Not quite: it is only weakly filtered. Will neglect the distinction here.]

$$\mathrm{Fuk}(M) \xrightarrow{\mathcal{Y}} \mathcal{M}od_{\mathrm{Fuk}(M)}$$

filtered modules $/ \mathrm{Fuk}(M)$

Complete $\mathcal{Y}(\mathrm{Fuk}(M))$ by cones :

$$\rightsquigarrow \mathrm{Fuk}(M)^{\Delta}$$

$$D\mathrm{Fuk}(M) := H(\mathrm{Fuk}(M)^{\Delta})$$

$\boxed{\text{Corollary : } D\mathrm{Fuk}(M) \text{ is a TPC}}$

<u>Remark:</u> The category $\left(D\,Fuk(M)\right)_\infty$ coincides with the usual derived Fukaya category (forgetting filtrations)

Put $\mathcal{C} = D\,Fuk(M)$. (finite)

Assume:

- $\mathcal{F} \subset Ob(\mathcal{C})$ generates $\mathcal{C}_\infty$
- $\mathcal{F}'$ generic Hamiltonian perturbation of $\mathcal{F}$

Theorem [BCS]: $D = d^{\mathcal{F},\mathcal{F}'}(-,-) = \max\{d^{\mathcal{F}}, d^{\mathcal{F}'}\}$ is non-degnererate.

$K(\mathcal{C}_0)$ = Grothendieck group of the 0-slice of $\mathcal{C} = D\mathrm{Fuk}(M)$

- It is a huge abelian group
- It is endowed with a $\overbrace{\text{pseudo-metric}}^{\text{(translation invariant)}}$ $\hat{D}$ induced by D:

$$\hat{D}(a,b) = \inf\{ D(A,B) \mid [A] = a,\ [B] = b \}$$

Theorem (in progress) i) $\hat{D}$ is non-degenerate

ii) $\mathrm{Ham}(M) \xrightarrow{\Psi} GL(K(\mathcal{C}_0))$ induced by

$$\phi(a) = [\phi(A)],\quad [A] = a$$

is injective.

iii) The representation Ψ induces a bi-invariant metric on $\mathrm{Ham}(M)$ that is bounded above by the Hofer norm.

Remark: There are examples when $\hat{D}$ is bounded and strictly smaller than the Hofer norm.

Thank You !

(and sorry not to be with you)