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LAGRANGIAN COBORDISM AND METRIC INVARIANTS

Octav Cornea* & Egor Shelukhin

Abstract

We introduce new pseudo-metrics on spaces of Lagrangian submanifolds of a symplectic manifold (M, ω) by considering areas associated to projecting Lagrangian cobordisms in $\mathbb{C} \times M$ to the "time-energy plane" $\mathbb{C}$. We investigate the non-degeneracy properties of these pseudo-metrics, reflecting the rigidity and flexibility aspects of Lagrangian cobordisms.

1. Introduction

One of the central aims of symplectic topology is to understand the topology of the Lagrangian submanifolds L (in this paper compact) of a given symplectic manifold (M^{2n}, ω) which is closed or tame at infinity. A basic question is whether there is some natural topology, or a distance, on the space $\mathcal{L}(M)$ of all such submanifolds that has some interesting, specifically symplectic, features. Without additional constraints on the class of Lagrangians under consideration, a positive answer to this question is hard to expect for two main reasons: the topological type of the submanifolds L in question is not fixed; symplectic rigidity properties are not preserved by isotopies, even Lagrangian ones, but only by Hamiltonian isotopies.

Recall that the Hamiltonian diffeomorphism group, $\mathrm{Ham}(M, \omega)$, is endowed with a bi-invariant metric introduced by Hofer [**28**] (its non-degeneracy was further studied in [**49**], [**42**], and proved in [**31**] in complete generality). Many recent advances in symplectic topology are related to properties of Hofer's geometry. The Lagrangian Hofer metric is also relevant to our problem: if we fix some $L \in \mathcal{L}(M)$ and consider the orbit $\mathcal{L}^L(M)$ of L under the action of $\mathrm{Ham}(M, \omega)$ (in other words, these are all the Lagrangians in M that are Hamiltonian isotopic to L), then the pseudo-metric on this space naturally induced by Hofer's

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metric is non-degenerate, as shown by Chekanov [17, Theorem 1] and is, therefore, a metric on $\mathcal{L}^L(M)$ (see Remark 1.4 for more details).

The purpose of this paper is to show that there is a natural construction of a family of metrics that are defined and non-degenerate on certain subsets of $\mathcal{L}(M)$ that are, in general, larger than Hamiltonian orbits. Moreover, this construction extends the construction of the Hofer metric on the space of Hamiltonian isotopic Lagrangians. We will see that using such a metric one can compare certain Lagrangians that are not even smoothly isotopic.

1.1. Measuring cobordisms. The construction of the metrics mentioned above employs the Lagrangian cobordism machinery as developed in [12, 13]. The notion of Lagrangian cobordism was introduced by Arnold [5, 6] and we refer to [12] for the variant that we use in this paper. In short, a Lagrangian cobordism V is a non-compact Lagrangian submanifold of $(\mathbb{C} \times M, \Omega = \omega_0 \oplus \omega)$, $\omega_0 = dx \wedge dy$ whose ends are *cylindrical* of two types, positive and negative. The positive ones are of the form $[\beta_+, \infty) \times \{r\} \times L_r$ where $r \in \mathbb{Z}_{>0}$, $\beta_+ \in \mathbb{R}$ and L_r is a Lagrangian in M. Similarly, the negative ends are of the form $(-\infty, \beta_-] \times \{r\} \times L'_r$ with $\beta_- < \beta_+$. We refer to each of the L_j's as a positive end of V and to each of the L'_i's as a negative end. A cobordism like this is sometimes written as $V : (L_j)_{1\leq j\leq k_+} \rightsquigarrow (L'_i)_{1\leq i\leq k_-}$ when we have k_+ positive ends and k_- negative ends. In case $k_+ = k_- = 1$ the cobordism is called *simple*.

Under the canonical projection $\pi : \mathbb{C} \times M \to \mathbb{C}$ a cobordism looks as in Figure 1.

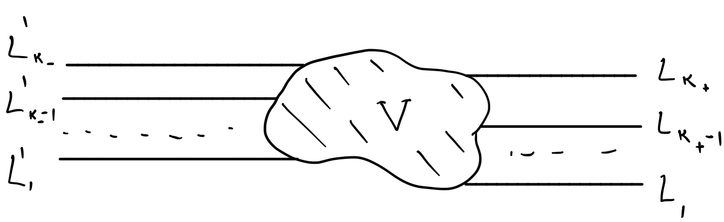

Figure 1. A cobordism $V : (L_j) \rightsquigarrow (L'_i)$ projected on $\mathbb{C}$.

Our construction is based on the following natural measure of Lagrangian cobordisms.

Definition 1.1. Given a Lagrangian cobordism $V \subset \mathbb{C} \times M$, $V : (L_j) \rightsquigarrow (L'_i)$, the *outline*

$$ou_V \subset \mathbb{C},$$

of V is the closed subset of $\mathbb{C}$ given as the complement of the union of the unbounded components of $\mathbb{C}\backslash\pi(V)$. The *shadow* of V is given by:

$$\mathcal{S}(V) = \text{Area}(ou_V).$$

REMARK 1.2. In case $\pi(V)$ is connected, by [**45**] the shadow of V coincides with the Hamiltonian displacement energy of $\pi(V)\cap\{\beta_-\leq Re(z)\leq\beta_+\}\subset\mathbb{C}$, for β_-,β_+ as in the definition of V.

1.2. Statement of the main result. There is a dichotomy

$$\textit{flexibility} \;\leftrightarrow\; \textit{rigidity}$$

in symplectic topology that is very much in evidence in the study of Lagrangian submanifolds and which is also apparent in the main result of the paper, Theorem 1.3 below. This dichotomy is reflected in the way certain topological constraints impact the geometric (symplectic) properties of Lagrangian submanifolds. We work in this paper with certain classes $\mathcal{L}^*(M)$ of closed Lagrangian submanifolds in (M,ω) as well as with the corresponding classes of simple cobordisms $\mathcal{L}^*_{cob}(\mathbb{C}\times M)$. The superscript $(-)^*$ refers to the constraints on the Lagrangians considered. These constraints are essentially standard in the field but we refer to §2 for details. In short, from flexible to rigid, $*$ can take the following values: g for general, or unconstrained Lagrangians and cobordisms; wm for weakly monotone by which we mean monotone without restrictions on the minimal Maslov class (with a fixed monotonicity factor, possibly negative); m for monotone (with fixed positive monotonicity factor); e for exact; finally, L_0 for a fixed closed Lagrangian $L_0\subset M$ (not necessarily belonging to any of the classes above) with the notation $\mathcal{L}^{L_0}(M)$ meaning the Hamiltonian orbit of L_0 (with cobordisms given by Lagrangian suspensions) as in the beginning of the introduction. An inequality such as $*\geq wm$ means that we work with Lagrangians that are at least as rigid as weakly monotone.

Theorem 1.3. *Let M be closed or tame at infinity. For two closed Lagrangians $L,L'\in\mathcal{L}^*(M)$ let:*

$$d^*(L,L')=\inf\ \{\mathcal{S}(V)\mid V:L\rightsquigarrow L'\ ,\ V\in\mathcal{L}^*_{cob}(\mathbb{C}\times M)\}. \tag{1}$$

i. *If $*\geq wm$, then $d^*(L,L')$ defines a metric, possibly infinite, on $\mathcal{L}^*(M)$.*
ii. *If $*=L_0$, for* any *fixed closed Lagrangian $L_0\subset M$, the metric d^{L_0} on the Hamiltonian orbit of L_0 coincides with the Lagrangian Hofer metric* [**17**].
iii. *There are examples of closed symplectic manifolds M and Lagrangians $L,L'\in\mathcal{L}^{wm}(M)$ so that $d^{wm}(L,L')\neq\infty$, L and L' are not smoothly isotopic and $d^m(L,L')=\infty$.*
iv. *For $*=g$, equation (1) defines a degenerate pseudo-metric. More precisely, for any two Lagrangians L,L' the only possible values of $d^g(L,L')$ are 0 and ∞ and $d^g(L,L')=\infty$ iff there are no simple Lagrangian cobordisms relating L to L'.*

REMARK 1.4. a. We emphasize that $d^*(L,L')=\infty$ means that L and L' are not cobordant via a simple cobordism of class $*$ and $d^*(L,L')=0$

means that there are simple cobordisms of arbitrarily small shadow that belong to this class and relate L and L'.

b. Obviously, the pseudo-metric d^* is invariant with respect to the action of the symplectic group in the sense that $d^*(\phi L, \phi L') = d^*(L, L')$ for any symplectic diffeomorphism $\phi : M \to M$ (and all choices of $*$).

c. Let $\phi \in \mathrm{Ham}(M, \omega)$. The Hofer norm of ϕ is given by:

$$||\phi||_H = \inf_{G,\ \phi_1^G = \phi} \int_0^1 (\max_{x \in M} G(t,x) - \min_{x \in M} G(t,x))\, dt,$$

where $G : [0,1] \times M \to \mathbb{R}$ is a Hamiltonian function. For further use we refer to the quantity

$$\mathrm{osc}(G) = \int_0^1 (\max_{x \in M} G(t,x) - \min_{x \in M} G(t,x))\, dt,$$

as the oscillation of G.

Here M is assumed to be compact or ϕ and G are assumed to have compact support. The Lagrangian Hofer metric referred to in the statement of the Theorem is a variant of the associated distance that is relative to a closed Lagrangian $L \subset M$. Assume that L' is a Lagrangian that is Hamiltonian isotopic to L. Then we can define a pseudo-distance:

$$d_H(L, L') = \inf_{G,\ \phi_1^G(L) = L'} [\int_0^1 (\ \max_{z \in \phi_t^G(L)} G(t,z) - \min_{z \in \phi_t^G(L)} G(t,z)\ )\, dt].$$

This pseudo-distance is actually a distance on the Hamiltonian orbit of L – the Hofer distance between Hamiltonian-isotopic Lagrangians. This fact is due to Chekanov [**17**, Theorem 1], and since his work, it has been proven again by many authors in slightly different forms (herein we follow [**9**, **8**, **14**]; see [**48**] for additional references and alternative definitions of this metric). For completeness, in §4.2, we sketch the proof that d_H is non-degenerate. In particular, at point ii. in the Theorem there is no constraint on the Lagrangian L_0 (other than the fact that it is closed).

The structure of the paper is as follows. Section 2 contains the required background on the various constraints $*$. In §3 we state and prove the main technical ingredient needed to establish Theorem 1.3. This result is of some independent interest. Its application in this paper is in §4 where we show Theorem 1.3. The paper ends with a few other comments in §5.

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2. Background: classes of Lagrangians, from flexible to rigid

In this section, we recall a series of standard, more and more strict topological constraints on Lagrangian manifolds and fix the relevant notation.

2.1. $* = g$**.** This is the most flexible choice of constraint and indicates that no restriction is imposed – g comes from *general.* In other words, $\mathcal{L}^g(M)$ are all the Lagrangian submanifolds of M and $\mathcal{L}^g_{cob}(\mathbb{C} \times M)$ are all the simple cobordisms in $\mathbb{C} \times M$.

2.2. $* = wm$**.** This is the *weakly monotone* case. Given a Lagrangian $K \subset M$ there are two morphisms:

$$\mu : \pi_2(M, K) \to \mathbb{Z} \qquad \omega : \pi_2(M, K) \to \mathbb{R},$$

the first being given by the Maslov class and the second by the integration of ω. The Lagrangian K is called weakly monotone if there exists a constant $\rho \in \mathbb{R}$ so that for each class $\alpha \in \pi_2(M, K)$ we have $\omega(\alpha) = \rho\mu(\alpha)$. Notice that there are no restrictions in this case on the constant ρ or on the minimal Maslov class. The Lagrangians in the class $\mathcal{L}^{wm}(M)$ are all weakly monotone with the same monotonicity constant ρ and, similarly, for the cobordisms in the class $\mathcal{L}^{wm}_{cob}(\mathbb{C} \times M)$. To be more precise we may include the constant ρ in the notation in which case we write $\mathcal{L}^{wm(\rho)}(M)$.

2.3. $* = m$**.** This is the *monotone* case. A Lagrangian $K \subset M$ is called monotone if it is weakly monotone and, additionally, the constant $\rho \geq 0$ and, further, the minimal Maslov number:

$$N_K = \min\{\mu(\alpha) \mid \alpha \in \pi_2(M, K)\ ,\ \omega(\alpha) > 0\}$$

is at least 2. In this context Floer homology is well-defined [**36**, **37**]. For simplicity, in this paper we only use Floer invariants over the base ring $\mathbb{Z}_2$, which allows us to disregard questions of orientation. If K is monotone, then for a point $P \in K$ and a generic almost complex structure J on M, the number of Maslov index 2 J-holomorphic disks going through P – these are the maps $u : D^2 \to M$, $u(S^1) \subset K$, $u(+1) = P$, $\bar{\partial}_J u = 0$ modulo reparametrization – is finite. We denote by $d_K \in \mathbb{Z}_2$ the number of these disks modulo 2. The set $\mathcal{L}^m(M)$ indicates in this case all the monotone Lagrangians in M with the same monotonicity constant ρ as well as with the same number d_K. In case we want to indicate explicitly these two constants we will write $\mathcal{L}^{m(\rho,d)}(M)$ to mean the monotone Lagrangians K in M of monotonicity constant ρ

and so that $d_K = d$. We use similar notation for cobordisms so that, for instance, $\mathcal{L}_{cob}^{m(\rho,d)}(\mathbb{C} \times M)$ are the cobordisms in $\mathbb{C} \times M$ that are monotone as Lagrangians in $\mathbb{C} \times M$ with the respective constants (ρ, d). For any two Lagrangians in $\mathcal{L}^{m(\rho,d)}(M)$ the Floer homology of the pair is well-defined [**36**, **37**]. For any two $L, L' \in \mathcal{L}^{m(\rho,d)}$ the Floer chain complex of L and L' will be denoted by $CF(L, L')$. The homology of this complex, $HF(L, L')$, is called the Floer homology of L and L'. If L and L' intersect transversely, then $CF(L, L')$ is basically identified with the $\mathbb{Z}_2$ vector space generated by the intersection points of L and L' and the differential of the Floer complex counts (possibly with weights given by the symplectic area) J-holomorphic strips joining these intersection points. A more precise description appears in §3. Floer homology can also be adapted to the case of cobordisms themselves – as in [**12**, **13**] – so that, given any two cobordisms $V, V' \in \mathcal{L}_{cob}^{m(\rho,d)}(\mathbb{C} \times M)$ the Floer complex $CF(V, V')$ is well-defined (as usual, up to canonical quasi-isomorphisms). There is also an obvious notion of *Lagrangian with cylindrical ends* in $\mathbb{C} \times M$ that is more general than cobordism. This is defined in the same way as in the cobordism case except that the ends are of the type $(-\infty, \beta] \times \{a_i\} \times L_i$, respectively, $[\alpha, \infty) \times \{b_i\} \times L'_j$ for $a_i, b_j \in \mathbb{R}$ while for cobordisms $a_i, b_j \in \mathbb{Z}_{>0}$. Floer homology is again defined for any pair of such Lagrangians. Furthermore, there is also an associated notion of isotopy for cobordisms [**12**] (as well as, more generally, for Lagrangians with cylindrical ends): two cobordisms $V, V' \subset \mathbb{C} \times M$ are *horizontally isotopic* if there exists a Hamiltonian isotopy $\{\phi_t\}_{t\in[0,1]}$ of $\mathbb{C} \times M$ sending V to V' and so that, essentially, outside of a compact set, $\phi_t(V)$ has the same ends as V for all $t \in [0, 1]$ (in other words, the ends can slide along but their image in $\mathbb{C} \times M$ – outside a large compact set – remains the same; the Hamiltonian isotopy is not necessarily with compact support). As shown in [**12**], this type of Hamiltonian isotopy leaves invariant the Floer homology $HF(V, V')$ just as in the usual compactly supported setting. The distinction between cobordisms and Lagrangians with cylindrical ends seems somewhat arbitrary but is relevant, in fact, for the definition of the Fukaya category of Lagrangian cobordisms – such as in [**13**]. As this Fukaya category is not needed in this paper, the two notions will be used interchangeably here.

2.4. $* = e$**.** This is the *exact* case. In this case, we assume that the manifold M is exact, i.e., $\omega = d\lambda$, and, moreover, the primitive λ restricts to an exact form on each of the Lagrangians belonging to $\mathcal{L}^e(M)$, and, similarly, for cobordisms. A useful extension of this notion is the *weakly exact* case, where integrating ω induces the zero map $\pi_2(M, L) \to \mathbb{R}$. The Floer machinery was initially developed in the (weakly) exact case [**19**] and, as mentioned above, this was later extended to the monotone setting (in fact, Floer assumes $\pi_2(M,L)=0$, but actually needs $\omega|_{\pi_2(M,L)}=0$, and in case one wishes a $\mathbb{Z}$-grading, also $\mu=0$).

2.5. $*=L_0$**.** This is the case of Hamiltonian orbits. We fix L_0 a Lagrangian in M and we denote by $\mathcal{L}^{L_0}(M)$ all the Lagrangians in M that are Hamiltonian isotopic to L_0. The cobordisms in $\mathcal{L}^{L_0}_{cob}(\mathbb{C}\times M)$ consist only of Lagrangian suspensions.

We recall the Lagrangian suspension construction. Let $\phi\in \mathrm{Ham}(M,\omega)$ be a Hamiltonian diffeomorphism. Let $G:[0,1]\times M\to\mathbb{R}$ be a time dependent Hamiltonian so that the time-1 diffeomorphism associated to G is ϕ, $\phi^G_1=\phi$. We denote by ϕ^G_t the time-t Hamiltonian diffeomorphism associated to G for all $t\in[0,1]$. In particular, $\phi^G_0=id$. We may assume that G is normalized in such a way so that for some small $\epsilon>0$ it vanishes on $([0,\epsilon]\cup[1-\epsilon,1])\times M$. Such a normalization is easy to achieve by reparametrizing the Hamiltonian flow (see [**43**]): if $b:[0,1]\to[0,1]$ is a smooth function, the Hamiltonian isotopy $\phi^G_{b(t)}$ is generated by the Hamiltonian function $(t,z)\to b'(t)G(b(t),z)$ and we may take $b(t)$ equal to t on $[\frac{3\epsilon}{2},1-\frac{3\epsilon}{2}]$ and constant, equal to 0 on $[0,\epsilon]$ and constant equal to 1 on $[1-\epsilon,1]$. In view of our normalization, we extend G by zero outside of $[0,1]\times M\subset\mathbb{R}\times M$, and view it as a map $G:\mathbb{R}\times M\to\mathbb{R}$. There is a symplectomorphism $\Phi:\mathbb{C}\times M\to\mathbb{C}\times M$ defined by

$$\Phi(x+iy,z)=(x+i(y+G(x,\phi^G_x(z))),\phi^G_x(z)).$$

This symplectomorphism is itself Hamiltonian (but not horizontal). For convenience we will denote the corresponding Hamiltonian isotopy by $\Psi_t:\mathbb{C}\times M\to\mathbb{C}\times M$ so that $\Psi_1=\Phi$ and $\Psi_0=id$. It is also possible to assume (by an appropriate reparametrization) that the path of symplectomorphisms Ψ_t is constant for t near the ends of the interval $[0,1]$. Fix now a connected, closed Lagrangian $L\subset M$ and let its Lagrangian suspension along G (see [**43**] as well as [**15**] where the cobordism perspective on this construction is made explicit) be defined by:

$$L^G=\Phi(\mathbb{R}\times L).$$

Our normalization for G implies that L^G is a Lagrangian cobordism, $L^G:\phi(L)\rightsquigarrow L$. Define $\mathcal{L}^{L_0}_{cob}(\mathbb{C}\times M)$ to be the set of Lagrangian cobordisms in $\mathbb{C}\times M$ that can be written as a Lagrangian suspension L^G for some $L\in\mathcal{L}^{L_0}(M)$ and $G:[0,1]\times M\to\mathbb{R}$.

In case L_0 is monotone, then obviously $\mathcal{L}^{L_0}(M)\subset\mathcal{L}^{m(\rho,d)}(M)$ where ρ is the monotonicity constant of L and $d=d_L$ and, in particular, all the Floer machinery applies. However, if L_0 is not restricted in any way, the Floer homology $HF(L_0,L_1)$ is not defined in general even if L_1 is Hamiltonian isotopic to L_0, that is $L_1=\phi(L_0)$ for some $\phi\in\mathrm{Ham}(M,\omega)$. Despite this, starting with Chekanov's work [**16**], it is now well-known that at least parts of the Floer machinery can be used inside $\mathcal{L}^{L_0}(M)$ for energies under the bubbling threshold. In all cases, from the perspective of this paper, the restriction to just one Hamiltonian orbit is the most rigid constraint in our list.

Given that we have listed the possible choices for $*$ in order, starting from the most flexible to the most rigid we will say, for instance, that a Lagrangian L, or a family of Lagrangians, is at least weakly monotone to mean that they belong to $\mathcal{L}^*(M)$ for some $* \geq wm$ – in this case the value of the constraint $*$ could be wm, m, e or L_0 – and, similarly, for cobordisms.

Remark 2.1. There is yet another condition $*$ that we have not mentioned before to avoid excessively complicating the discussion. This is the case of *rational Lagrangians* and cobordisms with the same rationality constant ρ: a Lagrangian L is said to be rational if the image of the morphism $\pi_2(M, L) \to \mathbb{R}$ given by integrating ω has a discrete image and the positive generator is ρ. We denote this condition by $* = r$. This is weaker than weak monotonicity but the argument used to show Theorem 1.3 i. extends immediately to show that this point of the Theorem remains true for $* \geq r$. Moreover, in the case $* = r$ with rationality constant ρ, an immediate consequence of the proof of Proposition 3.2 below shows that every cobordism between disjoint Lagrangians, or a null-cobordism, must have a shadow at least ρ. We note that, as communicated to us by Paul Biran and Leonid Polterovich, Gromov's figure-8 trick [**25**] as applied in [**42**], gives the lower bound of $\rho/2$ for cobordisms with disjoint ends. Finally, it is easy to see that any cobordism between two rational Lagrangians with proportional rationality constants can be made rational by an arbitrarily small C^0-perturbation. It is interesting to compare this fact with Theorem 1.3 iv.

Remark 2.2. To ensure that the glueing of two cobordisms along a common end preserves the classes of Lagrangians that we consider, one has to impose additional topological assumptions. To this aim we will assume in the paper that in the monotone as well as in the weakly monotone cases the inclusions of all Lagrangians and cobordisms into the respective ambient manifolds induce the zero map on fundamental groups.

3. The main technical result

The proof of the main Theorem is based on a technical result whose statement and proof are contained in this section. To formulate it we need to recall two additional numerical invariants associated to Lagrangian submanifolds.

The first is a positive real number associated to a pair of two Lagrangian submanifolds $L, L' \subset M$. It is called the *Gromov width of* L *relative to* L' (while we use this terminology for the sake of brevity, a more appropriate name could be "the Gromov width of $L \setminus L'$ inside

$M \setminus L'$") and was introduced in [9]:

$$w(L, L') = \sup_{r>0} \{\frac{\pi r^2}{2} \mid \exists\ e : B_r \to M,\ \text{symplectic embedding}, \quad e^{-1}(L) = B_r \cap \mathbb{R}^n,\ e(B_r) \cap L' = \emptyset\}. \tag{2}$$

Here $B_r \subset \mathbb{C}^n$ is the standard ball of radius r and center 0.

The second is a positive number associated to a Lagrangian L and an almost complex structure J on M that is compatible with ω. It is called the *bubbling threshold of L with respect to J*:

$$\delta(L, J) = \inf_u \{\omega(u) \mid \overline{\partial}_J u = 0,\ \omega(u) > 0, \quad u : (D^2, S^1) \to (M, L) \text{ or } u : S^2 \to M\}. \tag{3}$$

Obviously, this represents the energy at which the bubbling of either a J-disk with boundary on L or a J-sphere may occur. It is taken to be ∞ if there are no relevant J-disks or spheres. With this in mind, by [34, Proposition 4.1.4.], $\delta(L, J) > 0$. This definition also makes sense in case the almost complex structure J is time-dependent, $J = \{J_t\}_{t\in[0,1]}$. In this case, we take the infimum in Equation (3) over all disks $u : (D^2, S^1) \to (M, L)$ that are J_1 or J_0-holomorphic and over all spheres $u : S^2 \to M$ that are J_t-holomorphic for some $t \in [0, 1]$.

This number is well-defined also for cobordisms V for the following type of almost complex structures J on $\mathbb{C} \times M$.

Definition 3.1. We call an almost complex structure J on $\mathbb{C} \times M$ *trivial at infinity with negative end J_- and positive end J_+* if J is compatible with Ω and has the following properties:

i. there is a compact family of almost complex structures $\mathcal{J}$ on M compatible with ω and a compact set $K \subset \mathbb{C}$ so that for $z \in \mathbb{C} \setminus K$, J is of the form $i \times J'$ with $J' \in \mathcal{J}$.
ii. for some $\alpha_- > 0$ it restricts to $i \times J_-$, $J_- \in \mathcal{J}$, over the set $(-\infty, -\alpha_-] \times \mathbb{R} \times M$.
iii. for some $\alpha_+ > 0$, J restricts to $i \times J_+$, $J_+ \in \mathcal{J}$ over the set $[\alpha_+, \infty) \times \mathbb{R} \times M$.

For convenience, we will also assume (without loss of generality) that the compact set K above is included in $[-\alpha_-, \alpha_+] \times \mathbb{R}$. With our conventions J is compatible with ω if $\omega(-, J-)$ is a Riemannian metric.

Proposition 3.2. *Consider a cobordism*

$$V : (L_1, \dots, L_{i_0-1}, L, L_{i_0} \dots, L_r) \rightsquigarrow (L'_1, \dots, L'_{j_0-1}, L', L'_{j_0} \dots, L'_s).$$

If $L \cap L_k = \emptyset\ \ \forall\ 1 \le k \le r$ and $L \cap L'_m = \emptyset\ \ \forall\ 1 \le m \le s$, then for each $\epsilon_0 > 0$ there exists a time independent almost complex structure J_- on M (depending on ϵ_0) that is compatible with ω so that

$$\mathcal{S}(V) \ge \min \{w(L, L') - \epsilon_0, \delta(V; J)\},$$

for any time dependent almost complex structure $J = \{J_t\}_{t\in[0,1]}$ *on* $\mathbb{C} \times M$ *with the following three properties:*

a. J *is compatible with* $\omega_0 \oplus \omega$,
b. J_t *is trivial at infinity with* J_- *as its negative end for all* $t \in [0,1]$,
c. $J_0 = i \times J_-$.

REMARK 3.3. a. One reason why this result is of interest is that there are no conditions of any sort imposed on the Lagrangians and cobordisms involved (or, in the terminology of the paper, $* = g$). Thus, in practice, to "measure" (estimate from below) the distance between L and L' using the shadow of cobordisms the whole question comes down to having a uniform lower bound for $\delta(V, J)$ that applies to all the cobordisms V in a given class.

b. The presence of ϵ_0 in the inequality claimed in the Proposition is required for the following reason. The almost complex structure J_- basically depends on the choice of an embedding $e : B_r \hookrightarrow M$, as in the definition of relative width, so that $\frac{\pi r^2}{2} \geq w(L, L') - \frac{\epsilon_0}{2}$. With this choice, J_- extends the standard almost complex structure $e_* i$ outside B_r. In particular, the number $\delta(V, J)$ also depends on the choice of e. Certainly, by picking different embeddings e for increasing values of r we may reduce ϵ_0 arbitrarily close to 0. However, in this process the choices of J_- vary and to eliminate ϵ_0 from the statement we would need to control the convergence of the associated $\delta(V, J)$'s. While this might be possible, we preferred to avoid this additional complication as it is not justified in view of our applications here.

c. The statement of the proposition specialized to the case of simple cobordisms (i.e., $r = 0 = s$), is the only case needed to prove Theorem 1.3. The non-simple case is, in fact, an immediate consequence of the proof in the simple case. Additionally, the conditions on the ends of the cobordism V (different from L and L') are quite stringent and it is not clear whether these precise conditions are necessary. However, it is certain that some conditions that "separate" L and L' from the other ends need to be imposed. To see this consider two Lagrangians $L, L' \subset M$ that are disjoint. We can easily find such Lagrangians, even exact, in certain symplectic manifolds. One example is provided by two homologically non-trivial, disjoint curves on a surface of genus 2 (note that these Lagrangians are weakly exact). Consider two curves $\gamma, \gamma' \subset \mathbb{C}$ so that: $\gamma = \mathbb{R} + 2i$; γ' intersects γ in the single point $2i \in \mathbb{C}$; outside of $[-10, 10] \times \mathbb{R}$ γ' equals $\{[10, +\infty) + 3i\} \cup \{(-\infty, -10] + i\}$. We now consider the Lagrangian $W = (\gamma \times L) \cup (\gamma' \times L')$. As L, L' are disjoint this is a cobordism $W : (L', L) \rightsquigarrow (L, L')$ with vanishing shadow. This example W is not connected but by using L and L' that intersect transversely at a single point (such as the longitude and latitude on a torus) we can start with the Lagrangian W constructed as before – which is now immersed – use Lagrangian surgery ([**32**], [**41**]) to eliminate the single self-intersection point of W and, by using a sufficiently small Lagrangian handle in the surgery, obtain for any $\epsilon > 0$ a connected cobordism $V_\epsilon : (L', L) \rightsquigarrow (L, L')$ of shadow $\leq \epsilon$.

d. For certain choices of $*$, simple cobordisms verifying the constraint $*$ are quite special. For instance, it is conjectured that in the exact case, $* = e$, any simple cobordism is horizontally Hamiltonian isotopic to a Lagrangian suspension. A partial result in this direction is due to Suarez-Lopez [**46**] who showed that under some topological constraints any exact simple cobordism is smoothly trivial. In the monotone case, $* = m$, only very recently there has been constructed – by Haug [**27**] – a simple monotone cobordism (with Maslov class at least 2) with non-homeomorphic ends. It is useful to note that if two Lagrangians $L, L' \in \mathcal{L}^m(M)$ are related by a simple cobordism in $\mathcal{L}^m_{cob}(\mathbb{C} \times M)$, then, by the results in [**13**], L and L' are isomorphic objects in the relevant derived Fukaya category. As a consequence, at least by standard Floer theoretic methods, it is difficult to distinguish between such L and L'. However, Theorem 1.3 i. implies that the d^m-distance between them must be positive.

The Proof of Proposition 3.2 occupies the rest of this section.

3.1. Outline of the proof. We present here the main idea and the organization of the proof. Focus on the case when the cobordism V is simple, $V : L \rightsquigarrow L'$ and L is connected. Fix a close Hamiltonian deformation L_ϵ of L so that L_ϵ intersects transversely both L and L' (this is a technical step necessary to make various Floer complexes well-defined, and on first reading one can replace L_ϵ by L everywhere). Choose a symplectic embedding of a ball $e : B_r \to M \setminus L'$ with half-capacity $\frac{\pi r^2}{2}$ very close to $w(L, L')$, sending the real part of B_r to L_ϵ (by choosing L_ϵ close enough to L we can make $\frac{\pi r^2}{2}$ greater than $w(L, L') - \epsilon_0$ as desired). Take J_- so that it extends the push-forward of the standard complex structure on B_r by e. Consider now any J as in the statement (in particular, J_- is the negative end of J_t for all $t \in [0, 1]$). Assume $\mathcal{S}(V) < \delta(V, J)$. The main part of the argument is to show that for every point $R \in L_\epsilon$ there exists a J_--holomorphic strip of area $\leq \mathcal{S}(V) +$ *arbitrarily small constant* with boundary on L_ϵ and L' passing through R. In case such a strip can be constructed, we may take $R = e(0)$ and the usual monotonicity type inequalities show that $\frac{\pi r^2}{2} \leq \mathcal{S}(V)$ so that we deduce $w(L, L') - \epsilon_0 \leq \mathcal{S}(V)$.

To show the existence of the strip in question we start with some preparatory steps. It is not difficult to see (this is detailed in §3.11) that we may assume that $\pi(V)$ looks like in Figure 2. Using this, in §3.3, we associate to V two auxiliary cobordisms V' and V'' – as in Figure 4 – so that $\pi(V')$ intersects the line $\mathbb{R} + i$ transversely and only at the point P and $\pi(V'')$ intersects $\mathbb{R} + i$ transversely and only at the point Q. Moreover, V' and V'' are Hamiltonian isotopic by a horizontal isotopy induced by a Hamiltonian $H : \mathbb{C} \times M \to \mathbb{R}$ with $\mathrm{osc}(H)$ as close as desired to $\mathcal{S}(V)$. In §3.4 – we introduce another auxiliary cobordism: $W = \mathbb{R} \times \{1\} \times L_\epsilon$ and we also fix the almost complex structure J_- as described above.

With this preparation we can explain the main idea. It starts with the following simple observation. Fix a Morse function f_1 on L_ϵ. Consider the negative gradient flow of f_1 (with respect to some metric). Notice that through any generic point $R \in L_\epsilon$ passes exactly one flow line that starts at the maximum, max, of f_1. This fact can be formalized algebraically by saying that in the Morse homology of L_ϵ the product $[\min] * [\max] \neq 0$ (with our homological conventions $[\max]$ is the unit). The actual definition of the product defined at the level of Morse complex requires using a second Morse function f_2 chosen generically with respect f_1. By choosing f_2 so that its minimum equals R we see that $[\min] * [\max] \neq 0$ implies the existence of a flow line of f_1 through R.

To explain how this elementary remark enters our proof we will assume for a moment that all the relevant Floer complexes are well defined and that no disk or sphere bubbling interferes. With this simplifying assumption, there is a quasi-isomorphism of $CF(L_\epsilon, L)$ with the Morse homology of L_ϵ. This quasi-isomorphism also relates the module structures of $HF(L_\epsilon, L)$ over the Morse homology of L_ϵ with the module structure of the Morse homology of L_ϵ over itself.

The key step in the proof is to use a fact noticed in [**12**]: a Hamiltonian H as above induces a quasi-isomorphism relating the Floer complexes $CF(L_\epsilon, L)$ and $CF(L_\epsilon, L')$. Moreover, this quasi-isomorphism identifies the module structures of $HF(L_\epsilon, L)$ and $HF(L_\epsilon, L')$ over the Morse homology of L_ϵ. As a result, we obtain a relation of the type $[R] * HF(L_\epsilon, L') \neq 0$ which we can reinterpret as saying that there is a Floer trajectory with boundary on L_ϵ and L' that passes through R.

While the proof follows this idea closely, we need to take into account that bubbling may occur, so that the various Floer complexes and comparison maps are not defined as such and, moreover, we need to bound the energy of the Floer trajectory in terms of $\mathcal{S}(V)$ (up to an arbitrarily small constant). The solution to both issues goes through a technique originating in Chekanov's work [**16**] (and appearing often in the subject since then). It comes down to using appropriate truncated Floer complexes. Indeed, if the truncation window is smaller than the bubbling threshold $\delta(V, J)$, then, on one hand, the relevant (truncated) Floer complexes and morphisms are well-defined and, on the other, the Floer trajectories produced by these arguments have energy bounded by the truncation window. Simultaneously, the truncation window cannot be too small as this might make the comparison morphism $CF(L_\epsilon, L) \to CF(L_\epsilon, L')$ non-essential. However, as soon as the truncation window is larger than $\mathrm{osc}(H)$ this morphism suffices for our purposes. As we have assumed $\delta(V, J) > \mathcal{S}(V)$ and as $\mathrm{osc}(H)$ is as close as desired to $\mathcal{S}(V)$, we can take the truncation window larger than $\mathrm{osc}(H)$ and still smaller than $\delta(V, J)$ so that the argument succeeds.

This material is organized as follows. In §3.5 we define the relevant truncated complexes. In §3.6 we make precise the type of holomorphic curves that we are looking for. While we want to produce a Floer curve associated to the complex $CF(L_\epsilon, L') = CF(W, V'')$, action estimates are easier to control for the complex $CF(W, V'; -H)$. The two types of curves and complexes are related by a naturality transformation that is discussed in this subsection. In §3.7 we discuss the comparison maps relating the Floer complexes of the two ends of the cobordism. The module structure of Floer homology over Morse homology is recalled in §3.8. In §3.9 we reduce the problem to an identity in the fiber over the point P and, finally, in §3.10 we show this identity by comparing with Morse homology.

3.2. Basic setup. We will first prove the result under the additional assumption that the cobordism V is simple (i.e., $k = s = 0$) and that L is connected. We assume this from now on. Moreover, to fix ideas, we will assume that the constants $\alpha_\pm$ in the definition of an almost complex structure with negative end trivial at infinity are $\alpha_- = \alpha_+ = 1$. In particular, over $(-\infty, -1] \times \mathbb{R}$ the almost complex structure coincides with $i \times J_-$ and over $[1, \infty) \times \mathbb{R}$ it is of the form $i \times J_+$, $J_\pm \in \mathcal{J}$ (J_- is the time-independent structure in the statement and J_+ is, possibly, time-dependent). Further, again to simplify notation we will assume that the given cobordisms are cylindrical outside the region $[-1, 1] \times \mathbb{R}$, in other words, for the positive ends the constant β_+ is no bigger than 1 and for the negative ends the constant β_- is no smaller than -1 – see §1.1. This condition is very easy to achieve by using an appropriate horizontal isotopy.

3.3. Simplifying assumptions and two auxiliary cobordisms. We start by proving the statement by assuming that V is positioned as in Figure 2 below. We will see at the end of the proof that the same arguments apply to the general case.

More explicitly, the assumption is that:

i. $\pi(V) \cap (\mathbb{C} \setminus [-1, 1] \times \mathbb{R}) = ((-\infty, -1) \cup (1, \infty)) \times \{1\}$.
ii. $\pi(V) \subset \{(x + iy) \in \mathbb{C} \mid \beta(x) \leq y \leq 1\ ,\ x \in \mathbb{R}\}$ where $\beta : \mathbb{R} \to (-\infty, 1]$ is a smooth function so that the support of $1 - \beta(x)$ is inside $[-2, 2]$ and $\beta(x) < 1$ for $x \in (-2, -1] \cup [1, 2)$.
iii. $\int_{-\infty}^{\infty}(1 - \beta(x))dx \leq \mathcal{S}(V) + \delta$ for some $\delta > 0$.

We postpone to the end of the proof, in §3.11, the fact that for any small $\delta > 0$ there is a symplectic transformation that takes the initial cobordism into one of identical shadow that satisfies the three conditions above (relative to the fixed δ).

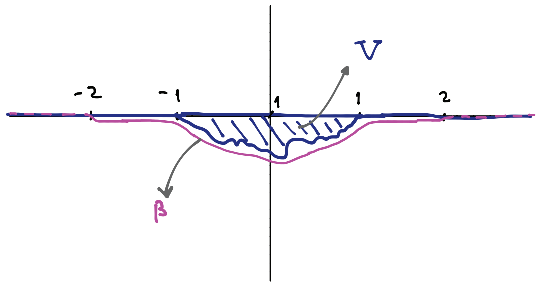

Figure 2. The cobordism V projected onto $\mathbb{C}$ and the graph of β. In this picture as well as the following ones the horizontal axis is at height 1.

In view of this, we now focus on the main step in the proof which is to show (under the assumptions i, ii, iii above) that:

$$\mathcal{S}(V) + \delta \geq \min\{w(L, L') - \epsilon_0, \delta(V, J)\}, \tag{4}$$

for J as in the statement of Proposition 3.2. Using the function β we define a new function $\tilde{\beta} : \mathbb{R} \to \mathbb{R}$ as follows: fix a large positive constant $\Gamma > 2$ and let $\tilde{\beta}(x) = \beta(x)$ for $x \leq \Gamma$ and $\tilde{\beta}(x) = 2 - \beta(x - \Gamma - 3)$ for $x > \Gamma$ – see Figure 3.

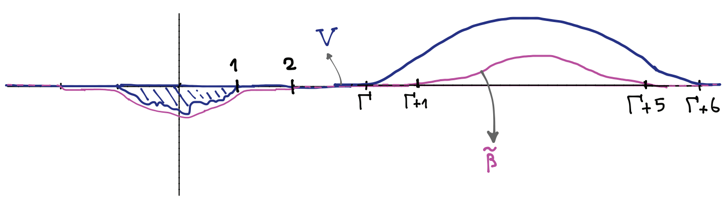

Figure 3. The graph of $\tilde{\beta}$ and the cobordism V transformed by bending its positive end. The "bump" of $\tilde{\beta}$ in between Γ and $\Gamma + 6$ coincides with the graph of the function r.

Consider now the following function $h_0 : \mathbb{C} \to \mathbb{R}$:

$$h_0(x + iy) = \int_{-\infty}^{x} (1 - \tilde{\beta}(t))dt.$$

Obviously, this function only depends on x. It is constant equal to 0 for $x \leq -2$, it is then increasing in the interval $(-2, 2)$, and it remains constant maximal from $x = 2$ till $x = \Gamma + 1$, when it starts decreasing till it reaches 0 again at $x = \Gamma + 5$. It remains equal to 0 after that. In particular, h_0 is a positive function and its maximum is equal to $\int_{-\infty}^{\infty}(1 - \beta(t))dt$.

Let $B > 0$ be a constant so that $-B < \min_x \beta(x) - 2$ and $B \geq \max_x(2 - \beta(x)) + 2$. We remark that B subject to this condition can be chosen to be arbitrarily large – we will use this freedom of choice below. Consider a new function $h : \mathbb{C} \to [0, \infty)$ that is supported inside $[-4, \Gamma + 8] \times [-B - 1, B + 1]$ and that has the following two properties: h is equal to h_0 over the band $\mathbb{R} \times [-B, B]$ and $h \leq h_0$ everywhere over $\mathbb{C}$. Such an h is easy to construct by cutting off h_0 outside $\mathbb{R} \times [-B, B]$.

At this point we need to also modify V in a simple fashion that takes into account the "shape" of $\tilde{\beta}$. The modification only consists in "bending" the positive end of V by replacing the portion of this end given by the region $[\Gamma, \Gamma+6] \times \{1\} \times L$ with $l \times L$ where l is the graph of a smooth function $r : [\Gamma, \Gamma+6] \to [1, B)$ with $r(t) > \tilde{\beta}(t)$, $\forall\, t$ and $r(t) = 1$ for $t \in [\Gamma, \Gamma + \frac{1}{2}] \cup [\Gamma + \frac{11}{2}, \Gamma + 6]$. We will denote the resulting cylindrical Lagrangian still by V – see Figure 3. It obviously has the same shadow as the initial cobordism. Moreover, we assume that Γ is bigger than the constant α_+ associated to the almost complex structure J (as in Definition 3.1). In view of this and by a straightforward application of the open mapping theorem, the quantity $\delta(V, J)$ is not affected by this modification of V.

We now consider the Hamiltonian $\bar{h} : \mathbb{C} \times M \to \mathbb{R}$ defined by $\bar{h} = h \circ \pi$ and list its properties that will play an important role later in the proof:

a. The oscillation of $\bar{h}$ is $\int_{-\infty}^{\infty} (1 - \beta(x)) dx$.
b. The Hamiltonian vector field $X^{\bar{h}}$ is a horizontal lift of the vector field X^h which has compact support.
c. Over the set $\mathbb{R} \times [-B, B]$, we have
$$X^h(x + iy) = i(1 - \tilde{\beta}(x)).$$
d. Because of point c and, due to the perturbation of the positive end of V involving the curve l above, the time one diffeomorphism associated to $X^{\bar{h}}$, $\phi^{\bar{h}}$, has the property that $\pi(\phi^{\bar{h}}(V)) \subset \mathbb{R} \times [1, \infty)$.

We need to construct two further Lagrangians with cylindrical ends that are basically copies of V. The first will be denoted V'. It is obtained by first cutting V along $L_0 = \{\frac{3}{2} + i\} \times L$ into two pieces one with projection to the left of $(\frac{3}{2} + i)$ denoted by V_-, and one with projection to the right of $(\frac{3}{2} + i)$, denoted by V_+. We translate V_- using the transformation $(x + iy, m) \to (x + i(y - \delta'), m)$ for $\delta' > 0$ very small, thus, getting a copy of V_- denoted by V'_-. We translate V_+ using the transformation $(x + iy, m) \to (x + i(y + \delta'), m)$ and denote by V'_+ the resulting submanifold with boundary. We then bend slightly the ends at L_0 of V'_- and V'_+ and glue them together. We denote by V' the resulting cylindrical Lagrangian – as in Figure 4. We do this construction so that the Lagrangian V' has the property that $\pi(V')$ intersects $\mathbb{R} \times \{1\}$ transversely, in just a single point equal to $P = \frac{3}{2} + i \in \mathbb{C}$. The positive

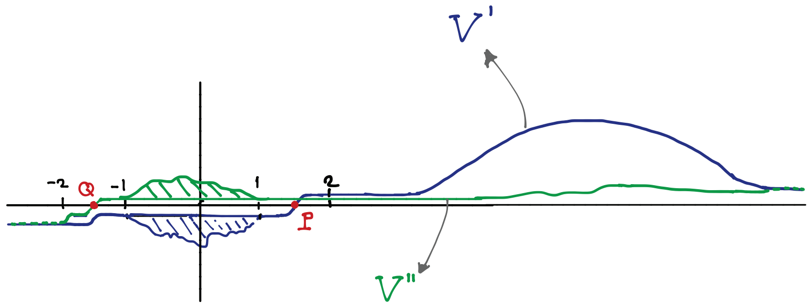

Figure 4. The two Lagrangians with cylindrical ends V' and V'' projected on $\mathbb{C}$.

end of V' is now at height $1+\delta'$. The negative end of V' is at height $1-\delta'$. Finally, we also need to make another slight adjustment to V'. We perturb it so that its projection $\pi(V')$ is the graph of an increasing, non-constant function over the region $[-2,-1]\times\mathbb{R}$ – see Figure 4. It is useful to note that by taking the various constants used in the construction of V' small enough we can make V' as close to V as desired.

The second Lagrangian will be denoted by V''. To define it take yet another small constant δ'' and consider the Hamiltonian $H=(1+\delta'')\bar{h}$ together with the associated Hamiltonian isotopy ϕ_t^H. Now define

$$V''=\phi_1^H(V').$$

First notice that V'' and V' are horizontally isotopic (in fact, the ends of V' remain fixed during the isotopy). Further, recall that $\beta(x)<1$ for $x\in(-2,-1]\cup[1,2)$ so that by taking δ' sufficiently small and possibly adjusting slightly β and V' we may assume that the only intersection between $\pi(V'')$ and $\mathbb{R}\times\{1\}$ is a single point $Q\in(-2,-1)\times\{1\}\subset\mathbb{C}$ and that this intersection is transverse. For convenience we put $q=\mathrm{Re}(Q)\in(-2,-1)$. There is an additional technical assumption on the function β that will be required below.

iv. There is a small constant $\delta'''>0$ so that the function β is constant in the interval $(q-\delta''',q+\delta''')\subset(-2,-1)$.

Due to the behaviour of $\pi(V')$ in the region $[-2,-1]\times\mathbb{R}$ this can be easily achieved by a small perturbation of β. Finally, we choose the cutoff parameter B in h so that the area $A=2\delta'''\cdot(B-3)$ of each of the two rectangles $\{q-\delta'''<x<q+\delta''',\ 2<\pm y<B-1\}$ satisfies $A>\mathrm{osc}(H)$.

To summarize the construction, we have constructed two cylindrical Lagrangians V' and V'' that are as in Figure 4. Moreover, V' and V'' both have ends L and L' and they are horizontally Hamiltonian isotopic via the Hamiltonian H whose oscillation is

$$\mathrm{osc}(H) = (1+\delta'')\int_{-\infty}^{\infty}(1-\beta(t))dt = (1+\delta'')(\mathcal{S}(V)+\delta).$$

To show (4) it is enough to prove that

$$\mathrm{osc}(H) \geq \min\{w(L,L') - \epsilon_0, \delta(V,J)\}. \tag{5}$$

3.4. A comparison Lagrangian and the almost complex structure J_-. For the arguments below we will need a third Lagrangian denoted W that we define now. We consider a small Hamiltonian deformation L_ϵ of L inside M so that L_ϵ intersects L as well as L' transversely and, moreover, $w(L_\epsilon, L')$ does not differ from $w(L,L')$ by more than $\epsilon_0/2$. We may take L_ϵ in a Weinstein neighbourhood of L to be the graph of a form $d\kappa$ where $\kappa : L \to \mathbb{R}$ is a sufficiently small Morse function. With this notation the Lagrangian W is simply

$$W = \mathbb{R} \times \{1\} \times L_\epsilon.$$

In Figure 4 the projection of W on the plane $\mathbb{C}$ coincides with the horizontal line $\mathbb{R}\times\{1\}$. Notice that W intersects transversely both V' and V''. To prove (5) for a simple cobordism, it is enough to show:

$$\mathrm{osc}(H) \geq \min\{w(L_\epsilon,L') - \epsilon_0/2, \delta(V,J)\}. \tag{6}$$

Using the Lagrangians L_ϵ and L' we can now explain our choice of almost complex structure J_-. First consider a symplectic embedding of the standard ball $e : B_r \to M$ so that $e^{-1}(L_\epsilon) = B_r \cap \mathbb{R}^n$, $e(B_r)\cap L' = \emptyset$ and

$$\frac{\pi r^2}{2} > w(L_\epsilon, L') - \epsilon_0/2.$$

Let $R = e(0)$ and let J_- be an almost complex structure on M that is compatible with ω and is an extension to the exterior of $e(B_r)$ of the almost complex structure $e_*(i)$ (i is here the standard complex structure on $\mathbb{C}^n$). Claim (6) follows if we can show that, assuming $\mathrm{osc}(H) < \delta(V,J)$,

$$\mathrm{osc}(H) \geq w(L_\epsilon, L') - \epsilon_0/2. \tag{7}$$

We pursue the proof under the assumption $\mathrm{osc}(H) < \delta(V,J)$ and with the aim to prove (7).

3.5. Filtered Floer complexes. Let J be an almost complex structure as in the statement of the first point of the Proposition 3.2. We will discuss in this subsection the construction of a few truncated Floer complexes together with comparison maps relating them that are the basic tools in the proof. These complexes are of the following form:

$$C_{G,J} = CF_b^a(W, V'; G, J),$$

for $a, b \in \mathbb{R}$, $0 < a - b < \delta(V, J)$, and $G : \mathbb{C} \times M \to \mathbb{R}$ a Hamiltonian such that dG is compactly supported and that $\mathrm{osc}(G) < a - b$ and J is the almost complex structure fixed above.

To start the construction of the complex $C_{G,J}$ we need to review some basic elements of Lagrangian Floer theory ([**19**], we use the same notation and path spaces as in [**8**]). Recall that L_ϵ is the graph of a form $d\kappa$ with $\kappa : L \to \mathbb{R}$ a Morse function. We will assume here that κ has a single minimum m_0 and a single maximum denoted w_0. Notice that we can view (P, m_0) as well as (P, w_0) as (constant) paths from W to V'. Consider the path space $\mathcal{P}(W, V')$ which consists of the smooth paths $\gamma : [0, 1] \to \mathbb{C} \times M$, $\gamma(0) \in W$, $\gamma(1) \in V'$. Let $\mathcal{P}_0$ be the path component of $\mathcal{P}(W, V')$ that contains m_0. Pick m_0 as a basepoint in $\mathcal{P}_0$. Denote by $\widetilde{\mathcal{P}_0}$ the cover of $\mathcal{P}_0$ corresponding to the subgroup $\ker I_\Omega \subset \pi_1(\mathcal{P}_0)$, where $I_\Omega : \pi_1(\mathcal{P}_0) \to \mathbb{R}$ is given by integrating the symplectic form $\Omega = \omega_0 \oplus \omega$. Finally, let $p : \widetilde{\mathcal{P}_0} \to \mathcal{P}_0$ be the projection.

As a vector space the complex $CF(W, V'; G, J)$ is the $\mathbb{Z}_2$ vector space freely generated by the set $\widetilde{\Gamma}_G$ of those γ in $\widetilde{\mathcal{P}_0}$ that project to paths $x = p(\gamma) \in \mathcal{P}_0$ that are Hamiltonian chords for G in the sense that they satisfy

$$\frac{dx}{dt} = X^G(x),$$

where X^G is the Hamiltonian vector field of G (with our conventions, it verifies by $\omega(X, X^G) = dG(X)$ for all vector fields X). We denote by Γ_G all these Hamiltonian chords. Fix also a lift $\tilde{m}_0$ of m_0 (say as a constant path) to $\widetilde{\mathcal{P}_0}$ and define the action of this point $\tilde{m}_0$ by $\mathcal{A}_G(\tilde{m}_0) = G(m_0)$. Further, for any $\gamma \in \widetilde{\mathcal{P}_0}$ define

$$\mathcal{A}_G(\gamma) = \int_0^1 G((p \circ \gamma)(t))dt - \int_{[0,1]\times[0,1]} (p \circ \bar{\gamma})^*(\omega),$$

where $\bar{\gamma}$ is a path in $\widetilde{\mathcal{P}}_0$ that joins, in this order, $\tilde{m}_0$ to γ. We will say that $\tilde{m}_0$ is the basepoint for the action. It is easy to see that the action $\mathcal{A}_G(\gamma)$ is independent of the choice of $\bar{\gamma}$. Notice that we work here in the absence of grading and orientations. The presumptive differential of the Floer complex $d : CF(W, V'; G, J) \to CF(W, V'; G, J)$ is defined by

$$d\tilde{x} = \sum_y \#_2 \mathcal{M}(\tilde{x}, \tilde{y}; G, J)\tilde{y},$$

where $\tilde{x}, \tilde{y} \in \widetilde{\Gamma}_G$, and $\mathcal{M}(\tilde{x}, \tilde{y}; G, J)$ is a moduli space consisting of paths $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{P}_0}$ – modulo reparametrization by the action of $\mathbb{R}$ – that go from $\tilde{x}$ to $\tilde{y}$ and so that the map $u = p(\tilde{u})$ is a Floer strip u from

$x = p(\tilde{x})$ to $y = p(\tilde{y})$. In other words u verifies the Floer equation:

$$
\begin{gathered}
u : \mathbb{R} \times [0,1] \to \mathbb{C} \times M \ , \\
u(\mathbb{R} \times \{0\}) \subset W \ , \ u(\mathbb{R} \times \{1\}) \subset V', \\
\lim_{s \to -\infty} u(s,t) = x(t) \ , \ \lim_{s \to +\infty} u(s,t) = y(t) \ , \\
\frac{\partial u}{\partial s} + J(t,u)\frac{\partial u}{\partial t} + \nabla G(u) = 0,
\end{gathered} \tag{8}
$$

where the gradient $\nabla G(u) = -J(t,u)X^G(u)$ is computed with respect to the metric $\omega(-, J-)$ given by J and ω.

By definition, $\#_2 A$ vanishes whenever the set A is infinite and equals the number (mod 2) of elements of A when A is finite. Recall that the almost complex structure is time dependent which explains the notation $J(t,u)$. The energy of a strip u as before is defined as

$$E(u) = \frac{1}{2}\int_{\mathbb{R}\times[0,1]} (||\frac{\partial u}{\partial s}||^2 + ||\frac{\partial u}{\partial t} - X^G(u)||^2) ds dt,$$

where the norms are taken with respect to the metric $\omega(-, J-)$. This energy is equal to the difference of actions

$$E(u) = \mathcal{A}_G(\tilde{x}) - \mathcal{A}_G(\tilde{y}).$$

The square of the linear map d defined as above does not generally vanish. The action $\mathcal{A}_G$ decreases along Floer trajectories. This means that for $a \in \mathbb{R}$ and with the notation above we can define:

$$CF^a(W, V'; G, J) = \mathbb{Z}_2 < \tilde{x} \in \widetilde{\Gamma}_G \mid \mathcal{A}_G(\tilde{x}) < a >$$

and, further, for some other $b \in \mathbb{R}$, $b \le a$

$$CF^a_b(W, V'; G, J) = CF^a(W, V'; G, J)/CF^b(W, V'; G, J).$$

Recall that outside a large compact $\widetilde{K}_{G,J} \subset \mathbb{C}$ the gradient ∇G vanishes and the almost complex structure J can be written, at each point z, as a product $i \times J_z$. We assume also that Γ_G is contained in $\widetilde{K}_{G,J} \times M$ and that V' is cylindrical outside this compact. Recall also that by diminishing the constants used in the construction of V' we can make V' as close to V as desired.

Lemma 3.4. *Assume that $a - b < \delta(V, J)$. If*

i. *L_ϵ is sufficiently close to L and V' is sufficiently close to V,*
ii. *$\widetilde{G}$ and $\widetilde{J}$ are sufficiently small, generic deformations of J and, respectively, G with the property that $d\widetilde{G}$ is supported inside $\widetilde{K}_{G,J}$ and $\widetilde{J}$ is a product outside $\widetilde{K}_{G,J}$,*

then the linear map $d : CF^a_b(W, V'; G, J) \to CF^a_b(W, V'; G, J)$ is well-defined and a differential.

Proof. The argument is typical for Floer theory with minor twists related to our context. We first discuss compactness of the relevant moduli spaces. There are three different phenomena that can lead to lack of compactness of the 1-dimensional moduli spaces:

a. Breaking along Hamiltonian orbits in $\Gamma_{\widetilde{G}}$. This is precisely the type of non-compactness that appears in showing that $d^2 = 0$.
b. The fact that the target manifold $\mathbb{C} \times M$ is not compact. Outside the compact $\widetilde{K}_{G,J} \subset \mathbb{C}$ the gradient $\nabla\widetilde{G}$ vanishes and the almost complex structure $\widetilde{J}$ can be written, at each point, as a product. As a consequence, if u is a solution of (8) and we put $u' = \pi \circ u$, then, outside of $\widetilde{K}_{G,J}$, the strip u' is holomorphic and an easy application of the open mapping theorem (see also [**12**]) immediately implies that u' does not get out of $\widetilde{K}_{G,J}$.
c. Bubbling of disks or spheres. Fix some small constant $\xi > 0$. The curves that can potentially bubble off are potentially of three types:
   i. $\widetilde{J}_t$-spheres for any $t \in [0,1]$ – they have an area no less than $\delta(V,J) - \xi$ if $\widetilde{J}$ is close enough to J,
   ii. $\widetilde{J}_1$-disks with boundary on V' – by Gromov compactness, these are of area at least $\delta(V,J) - \xi$, when V' is sufficiently close to V and $\widetilde{J}$ sufficiently close to J,
   iii. $\widetilde{J}_0$-disks with boundary on W. By taking $\widetilde{J}$ sufficiently close to J we can ensure that these disks have area at least $\delta(W,J_0) - \frac{\xi}{2}$. Recall that $W = \mathbb{R} \times \{1\} \times L_\epsilon$ and $J_0 = i \times J_-$ so that the area of these disks is at least $\delta(L_\epsilon, J_-)$. By now taking L_ϵ sufficiently close to L we get $\delta(L_\epsilon, J_-) \geq \delta(L, J_-) - \frac{\xi}{2}$. Now L is the positive end of V and, thus, $\delta(L, J_-) \geq \delta(V, J)$.

   To summarize all spheres or disks that can bubble off have energy at least $\delta(V,J) - \xi$.

We now assume that the constant ξ is small enough so that $\delta(V,J) - \xi > a - b$ so that inside our action window no bubling off is actually possible.

The regularity of the moduli spaces considered here can be achieved using generic perturbations $\widetilde{G}$ and $\widetilde{J}$ as in the statement because our Floer strips u are known *a priori* to remain inside $\widetilde{K}_{G,J} \times M$ (by point (b.)). In short, this shows that the usual Floer construction works to both define the differential d and to show $d^2 = 0$ on $CF^a_b(W, V'; \widetilde{G}, \widetilde{J})$. q.e.d.

For the continuation of the argument we possibly diminish still the constant ξ that has appeared in the proof of the Lemma so that

$$\delta'(V,J) = \delta(V,J) - \xi > \mathrm{osc}(H),$$

where H is the particular Hamiltonian constructed in §3.3.

Fix an additional constant $\zeta > 0$ so that

$$\operatorname{osc}(H) + 2\zeta < \delta'(V, J). \tag{9}$$

With these conventions we now can fix the constants a and b that we use further in the proof:

$$a = \zeta \ , \ b = -\operatorname{osc}(H) - \zeta. \tag{10}$$

Finally, we can define the three truncated Floer complexes that are at the center of the argument. They are

$$C = CF_b^a(W, V'; -H, \widetilde{J}) \ \text{ and } \ C_i = CF_b^a(W, V'; H_i, \widetilde{J}),$$

for $i = 1, 2$ where the Hamiltonians H_i are as follows: H_1 is the constant Hamiltonian

$$H_1 \equiv -\operatorname{osc}(H) - \delta'' \text{ with } \ \zeta > \delta'' > 0$$

and H_2 is the constant Hamiltonian equal to 0.

The almost complex structure $\widetilde{J}$ is a small enough generic perturbation of J that will be fixed as follows. First, we slightly perturb J_- to a possibly time dependent almost complex structure $\widetilde{J}_-$ on M and J_+ (which is, in general, already time-dependent) to a structure $\widetilde{J}_+$ so that the data $(L_\epsilon, L; 0, \widetilde{J}_+)$ and $(L_\epsilon, L'; 0, \widetilde{J}_-)$ are Floer regular. We then extend this perturbation of the ends of J to a generic perturbation $\widetilde{J}$ of J itself that remains in the class of almost complex structures trivial at infinity and so that $\widetilde{J}$ has $\widetilde{J}_-$ as negative end and has $\widetilde{J}_+$ as positive end and the associated constants for these ends remain $\alpha_- = 1 = \alpha_+$. We also assume from now on that V' is close enough to V, L_ϵ sufficiently close to L and $\widetilde{J}$ is close enough to J so that Lemma 3.4 applies to the complexes C, C_1 and C_2. Given that the intersection between W and V' is transverse this class of perturbations of $\widetilde{J}$ is enough to achieve the regularity required to define the complexes C_i. Similarly, because the intersection of W and V'' is transverse this is also sufficient to define C (in other words, we do not need to perturb H and H_i). Finally, we remark that one can choose a $\widetilde{J}$ that achieves regularity for all three complexes at the same time, because an intersection of three co-meager sets is again co-meager. In short, by the Lemma these three complexes are well-defined.

3.6. Reduction of (7) to the existence of certain Floer strips. The purpose of this subsection is to notice our claim (7) is implied by the following statement:

$$\begin{gathered}\exists \ u : \mathbb{R} \times [0,1] \to \mathbb{C} \times M, \\ u \text{ verifies (8) with } G = -H, J = \widetilde{J} \\ \text{and } u(0,0) \in \mathbb{R} \times \{1\} \times R \ , \ E(u) \leq a - b = \operatorname{osc}(H) + 2\zeta\end{gathered} \tag{11}$$

Indeed, given such a u let $v(s,t) = \phi_t^H(u(s,t))$. It is easy to see that v is a solution of

$$
\begin{gathered}
v : \mathbb{R} \times [0,1] \to \mathbb{C} \times M, \\
v(\mathbb{R} \times \{0\}) \subset W,\ v(\mathbb{R} \times \{1\}) \subset V'', \quad \bar{\partial}_{J'} v = 0 \text{ and} \\
v(0,0) \in \mathbb{R} \times \{1\} \times R\ ,\ \Omega(v) \leq \mathrm{osc}(H) + 2\zeta.
\end{gathered} \tag{12}
$$

for an almost complex structure $J' = (\phi_t^H)_*(\widetilde{J})$.

Due to condition (iv) in the construction of the Hamiltonian H in §3.3, the almost complex structure J'_t is of the form $i \times \widetilde{J}_-$ in a rectangle of the form $(q - \delta''', q + \delta''') \times [-B+1, B-1]$ where $Q = (q,1)$ is the only point of intersection of $\pi(V'')$ and $\pi(W)$. This is because over that strip ϕ_t^H is just a translation in the imaginary direction in this rectangle (in $\mathbb{C}$). Consider now a curve v given as in (12) and let $v' = \pi \circ v$. This is a curve that is holomorphic over $(q - \delta''', q + \delta''') \times [-B+1, B-1]$, and is asymptotic to Q at both infinite ends of $\mathbb{R} \times [0,1]$. By first using an orientation argument we deduce that unless v' is constant, it must pass through one of the regions of $[(q - \delta''', q + \delta''') \times \mathbb{R}] \setminus [\pi(W) \cup \pi(V'')]$. But due to the open mapping theorem, and the condition $2\delta''' \cdot (B-3) > \mathrm{osc}(H)$, this is not possible (for reasons of symplectic area). We deduce that v' is constant and equal to Q and, therefore, v is contained in $\{Q\} \times M$, and is a solution of:

$$
\begin{gathered}
w : \mathbb{R} \times [0,1] \to M,\ w(\mathbb{R} \times \{0\}) \subset L_\epsilon,\ w(\mathbb{R} \times \{1\}) \subset L',\ \bar{\partial}_{\widetilde{J}_-} w = 0 \\
\text{and } R = w(0,0), \omega(w) = \int_w \omega \leq \mathrm{osc}(H) + 2\zeta.
\end{gathered} \tag{13}
$$

The almost complex structure $\widetilde{J}_-$ can be taken as close as needed to J_- so that, by Gromov compactness, we deduce the existence of a holomorphic curve w' passing through the center of B_r with boundary on ∂B_r and on $\mathbb{R}^n \cap B_r$ and with $\omega(w') \leq \omega(w)$. The existence of w' implies, by a standard argument based on the Lelong inequality (see [**24**]), the inequality

$$
\omega(w') \geq \frac{\pi r^2}{2}.
$$

As ζ can be taken as close to 0 as desired this implies the inequality (7).

Thus, from now on our aim is to show (11).

3.7. Comparison of the complexes C, C_1, C_2. Returning now to our Hamiltonians, notice that $H_2 \geq -H \geq H_1$. As usual in Floer theory, to compare two Floer complexes associated to different Hamiltonians, we use homotopies that relate these Hamiltonians. Such homotopies induce quasi-isomorphisms between the complexes associated to the two Hamiltonians. The construction of these maps is now standard and can be found, for instance, in [7]. In our case, because we deal with truncated complexes, it is useful to use (decreasing) monotone homotopies. Such homotopies have been introduced in the study of comparison maps in Floer theory in [21]. They are also used, with exactly the same sign conventions as here, in [18]. More recent references on filtered Floer complexes and the associated structures, that cover all the material in this section (actually, in a more refined way than the one we need) are, for example, [38] and §2 in [47] (notice, however, that the sign conventions in [38] and [47] are different from those used here, with the conventions there to deduce the results in this paper one should use *increasing* homotopies).

To start the construction consider monotone decreasing homotopies $G_2 : [0,1] \times \mathbb{C} \times M \to \mathbb{R}$ from H_2 to $-H$ (in the sense that $(G_2)_0 = H_2$, $(G_2)_1 = -H$) and $G_1 : [0,1] \times \mathbb{C} \times M \to \mathbb{R}$ from $-H$ to H_1 that have the property that for each value of the parameter $s \in [0,1]$, the function $G_i(s,-)$ is constant outside the large compact $\widetilde{K}_{H,J} \subset \mathbb{C}$ (as described in point c.i. above). Moreover, we assume here that G_i has been extended to a smooth function on $\mathbb{R} \times \mathbb{C} \times M$ by $G_i(0,-)$ for $s < 0$ and by $G_i(1,-)$ for $s > 1$. Such a homotopy is called *monotone decreasing* if $\partial_s G_i \leq 0$.

Similarly, consider a monotone decreasing homotopy $G_{2,1} : \mathbb{R} \times \mathbb{C} \times M$ from H_2 to H_1 so that $G_{2,1}(s,-) = H_2$, $s \leq 0$, $G_{2,1}(s,-) = H_1$, $s \geq 1$ and, for fixed s, $G_{2,1}(s,-)$ is a constant function (recall that H_2 and H_1 are constant functions, $H_2 \equiv 0$ and $H_1 \equiv -\operatorname{osc}(H) - \delta''$). In the language of [47] we could say that $G_{2,1}$ is an interpolating homotopy from H_2 to H_1.

Lemma 3.5. *There are Floer comparison maps induced respectively by G_1, G_2 and $G_{2,1}$:*

$$\psi_{G_1} : C = CF_b^a(W, V'; -H, \widetilde{J}) \to CF_b^a(W, V'; H_1, \widetilde{J}) = C_1, \tag{14}$$

$$\psi_{G_2} : C_2 = CF_b^a(W, V'; H_2, \widetilde{J}) \to CF_b^a(W, V'; -H, \widetilde{J}) = C \tag{15}$$

and

$$\psi_{2,1} : CF_b^a(W, V'; H_2, \widetilde{J}) \to CF_b^a(W, V'; H_1, \widetilde{J}),$$

so that $\psi_{2,1}$ is chain homotopic to $\psi_{G_1} \circ \psi_{G_2}$.

Proof. The comparison maps ψ_{G_i} are given by counting solutions of an equation similar to (8):

$$\bar{\partial}_{\widetilde{J}} u(s,t) + \nabla G_i(s, u(s,t)) = 0. \tag{16}$$

The boundary conditions for these solutions are identical to those in (8). For ψ_{G_2} the assymptotic conditions are such that $x(t) = \lim_{s \to -\infty} u(s,t)$ is an orbit of H_2 and $y(t) = \lim_{s \to \infty} u(s,t)$ is an orbit of $-H$ and, similarly, for ψ_{G_1}, where $x = p(\tilde{x})$, $y = p(\tilde{y})$. One difference compared with the definition of the Floer differential is that we no longer divide by the $\mathbb{R}$ action (as $\mathbb{R}$ does no longer act on these moduli spaces). Further, an easy calculation (that can be found with the same sign conventions, for instance, in §2.1.2 of [18]) shows that the monotonicity of G_2 implies that if u is such a solution joining $\tilde{x}$ to $\tilde{y}$ as above, then $\mathcal{A}_{-H}(\tilde{y}) \leq \mathcal{A}_{H_2}(\tilde{x})$ and, similarly, for G_1 (more generally, a monotone homotopy induces a comparison chain morphism that decreases action). The compactness arguments in the proof of Lemma 3.4 apply here too. To ensure regularity, by the standard method in Floer theory (see also §2 of [47]), we need to potentially replace $\widetilde{J}$ in (16) by an appropriate generic s-dependent homotopy from $\widetilde{J}$ to $\widetilde{J}$ (which can be taken as a sufficiently small perturbation of the constant homotopy). Therefore, the maps ψ_{G_i} are well-defined and are chain maps. The definition of the chain morphism $\psi_{2,1}$ induced by $G_{2,1}$ is perfectly similar.

Comparison maps in Floer theory have a couple of important additional properties. First, any two homotopies relating the same data induce chain homotopic comparison morphisms (cf. [7, Chapter 11]). One considers two homotopies h^0 and h^1 relating the Hamiltonians (and almost complex structures) that are compared. These homotopies induce, as described above, comparison morphisms ψ_{h_0} and ψ_{h_1} between the relevant Floer complexes. One then considers a smooth one-parametric family of homotopies relating the same data, h^λ, $\lambda \in [0,1]$ between h^0 and h^1 (such an h^λ is usually called a homotopy of homotopies). The moduli spaces associated to a homotopy of homotopies consist of pairs (λ, u) where u verifies (16) for the data h^λ (instead of G_i, $\tilde{J}$). Of course, for regularity purposes, the homotopy of homotopies of almost complex structures that are part of h^λ has to be slightly perturbed in a generic way. In case h^0, h^1 are monotone decreasing, it is easily seen (as, for instance, in [47]) that all this machinery is compatible with the action filtrations as soon as the homotopies h^λ are monotone (decreasing, in our case) for each $\lambda \in [0,1]$. In this case, the resulting chain homotopy between ψ_{h_0} and ψ_{h_1} is also action decreasing.

A second property of these comparison maps – we refer again to [7, Chapter 11] for an accessible reference as well as to [47] – is that the comparison morphism $\psi_{h\#h'}$ associated to a composition $h\#h'$ (also called the concatenation) of homotopies h, h' is chain homotopic to the composition $\psi_{h'} \circ \psi_h$ of the comparison morphisms associated to h' and to h. Here, of course, we assume that the "end" data h_1 of h coincides with the "starting" data h'_0 of h' so that concatenation is possible. The definition of the concatenation of h and h' actually depends on a splicing parameter that reflects the length ρ of an interval $I_\rho \subset \mathbb{R}$ where the concatenation homotopy restricts to the constant homotopy, equal to $h_1 = h'_0$ for all $s \in I_\rho$ (see [7, Chapter 11] for the formulas of the concatenated homotopies depending on the splicing parameter). Varying ρ provides a one parametric family of concatenated homotopies, $h\#_\rho h'$, with $h\#h'$ occuring for $\rho = 0$. The composition $\psi_{h'} \circ \psi_h$ is obtained by making $\rho \to \infty$. The chain homotopy above is obtained by counting solutions of equation (16) where G_i is replaced by $h\#_\rho h'$, $\rho \in [0, \infty)$ and with $\widetilde{J}$ replaced by a generic, small (s and ρ dependent) perturbation. In our case, if h and h' are monotone decreasing, the concatenation $h\#_\rho h'$ is again monotone decreasing (for any splicing parameter) and, as a consequence, the chain homotopy between $\psi_{h'} \circ \psi_h$ and $\psi_{h\#h'}$ is action decreasing.

We now use the compactness arguments from the proof of Lemma 3.4 together with this general machinery. We deduce that for monotone decreasing homotopies such as $G_1, G_2, G_{2,1}$ the induced comparison maps, as defined in our setting for action intervals with $a - b < \delta(V, J)$, have the property that $\psi_{G_1} \circ \psi_{G_2}$ is chain homotopic to the comparison map $\psi_{G_2\#G_1}$ and, moreover, this is chain homotopic to the comparison map $\psi_{2,1}$ induced by $G_{2,1}$. Further, the chain morphism induced by $G_{2,1}$ is canonical up to chain homotopy (in particular, there is no abuse of notation in omitting $G_{2,1}$ from the notation of the morphism $\psi_{2,1}$). q.e.d.

Remark 3.6. The morphism $\psi_{2,1}$ is easiest to understand at the chain level if one takes the homotopy of almost complex structures to be constant equal to $\widetilde{J}$. Given that $G_{2,1}(s, -)$ is a constant function for each s (with $H_2 = G_{2,1}(0, -) = 0$, $H_1 = G_{2,1}(1, -) = -\operatorname{osc}(H) - \delta''$), this is sufficient for regularity purposes and, in this case, the Floer trajectories contributing to $\psi_{2,1}$ are all constant. Thus, this morphism only reflects how the action window is applied to the same orbits when they are assigned the actions associated to H_1 or to H_2.

3.8. Truncated Floer homology as a module over Morse homology. Assume for a moment that we are given two Lagrangian submanifolds N, N' so that Floer homology $HF(N, N')$ is defined and no bubbling off is possible (possibly, by imposing appropriate exactness conditions). In this case, there is a module multiplication making $HF(N, N')$ a module, over the Morse homology of N viewed as an algebra with the intersection product (recall that, at the chain level, the Morse product is of the form $CM(f_1) \otimes CM(f_2) \to CM(f_3)$ with f_1, f_2, f_3 in generic position; the homologies of any two Morse complexes $CM(f_i)$ are identified canonically and each is also canonically identified with the singular homology of N; the product itself is canonically identified with the intersection product [**44**]).

In case bubbling is present, but in a controlled way, for instance, under the assumption of monotonicity, Morse homology needs to be replaced with pearl homology [**11**]: $HF(N, N')$ is in this case a module over $QH(N)$. Structures of this type appear often in the literature, see, in particular, §3.5 in [**14**] where the author uses this module product, in the monotone case, for a purpose similar to our aim here – to detect Floer type trajectories through a point.

In our application we will actually only need to use some very basic properties of the module product (in particular, we do not use the algebra structure on the homology of N) and we will work under the bubbling threshold.

REMARK 3.7. Another technique to detect Floer trajectories through a point appeared in [**9**] but the module multiplication approach is simpler to implement here.

Let $f : W \to \mathbb{R}$ be a Morse function constructed as follows. Recall that $W = \mathbb{R} \times \{1\} \times L_\epsilon$. Let $f_0 : L_\epsilon \to \mathbb{R}$ be a fixed Morse function with a single maximum and a single minimum and put $f(s, l) = -s^2 + f_0(l)$, $s \in \mathbb{R}$, $l \in L_\epsilon$. Fix also a Riemannian metric g_0 on L_ϵ so that the pair (f_0, g_0) is Morse–Smale. Extend the metric g_0 to the metric $g = ds^2 \oplus g_0$ on W and denote by γ_t the negative gradient flow of f with respect to g.

Denote by $CM(f, W)$ the Morse complex associated to (f, g). Note that the obvious map $CM(f_0, L_\epsilon) \to CM(f, W)$ is an isomorphism (ignoring grading). Consider also the complex $C = CF^a_b(W, V'; -H, \widetilde{J})$. Notice that there is a module multiplication map:

$$\mu_C : CM(f, W) \otimes C \to C \ , \ \mu_C(a, \tilde{x}) = \sum_y \#_2 \mathcal{M}(a; \tilde{x}, \tilde{y}; -H, \widetilde{J})\tilde{y},$$

where $a \in Crit(f)$, $\tilde{x} \in \widetilde{\Gamma}_{-H}$ and $\mathcal{M}(a; \tilde{x}, \tilde{y}; -H, \widetilde{J})$ is the moduli space of paths $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{P}_0}$ that verify (8) and that, additionally, satisfy the relation $\lim_{t \to -\infty} p(\gamma_t(\tilde{u}(0, 0))) = a$. By the same methods that were discussed in the last section it follows without difficulty that the map μ_C is a chain map (in particular, working under the bubbling threshold is important here).

Similarly, we have corresponding chain maps μ_{C_i}, $i = 1, 2$ associated to the complexes C_i that are defined in similar ways as above. These maps are related in the obvious sense through the comparison maps ψ_{G_i}. To be more explicit, we claim that $\mu_C(id_{CM} \otimes \psi_{G_2}) \simeq \psi_{G_2}(\mu_{C_2})$ and similar identities up to chain homotopy for the other comparison map (see [**10**, **12**] for related analysis).

Using these module multiplications we can reduce our claim (11) to an algebraic identity, as follows. We first pick the Morse function $f_0 : L_\epsilon \to \mathbb{R}$ above so that its minimum point is $R \in L_\epsilon$ that appears in (11). Notice that, by the definition of the function $f : W \to \mathbb{R}$ and due to the definition of the metric g on W, the unstable manifold of the critical point $R' = \{0\} \times \{1\} \times \{R\} \in W$ of f is precisely the line $\mathbb{R} \times \{1\} \times \{R\}$. In view of these choices, it is immediate to see that (11) is implied by the following non-vanishing of the module multiplication

$$R' * HF^a_b(W, V'; -H, \widetilde{J}) \neq 0. \tag{17}$$

Further, by using the comparison maps ψ_{G_i} and their compatibility with the multiplications $\mu_C, \mu_{C'_i}$ we see that (17) is implied by:

$$R' * \mathrm{image}(\psi_{2,1}) \neq 0, \tag{18}$$

where we recall that $\psi_{2,1} : C_2 \to C_1$ is the monotone comparison map described in Lemma 3.5, and we use the same notation for the map it induces on homology.

3.9. Reduction to an identity in the fibre over P. The purpose of this subsection is to rewrite (18) in terms of a morphism of complexes of the type $CF(L_\epsilon, L; -, -)$.

We start by making more precise the choice of the function $\kappa : L \to \mathbb{R}$ from §3.4. Recall that this function has the property that the graph of $d\kappa$ is the Lagrangian L_ϵ. Recall also that the point $\tilde{m}_0$ is the base point for the actions we use here and that m_0 is the minimum point of κ and w_0 is the maximum point of κ. We assume that κ is small enough, and that

$$\kappa(m_0) = 0 \text{ and } \kappa(w_0) < \zeta - \delta''. \tag{19}$$

Notice that picking κ in this way is possible because the choice of the constants ζ, δ'' is independent of the choice of L_ϵ.

We will consider truncated Floer complexes of the form

$$CF_b^a(L_\epsilon, L; \eta, \widetilde{J}_+).$$

Here η is a constant $\eta \in \mathbb{R}$ and in our argument it will only take two values: $\eta_1 = -\operatorname{osc}(H) - \delta''$ and $\eta_2 = 0$ (to fix ideas, recall that the Hamiltonians on $\mathbb{C} \times M$ are both constant too, $H_2 = 0$, $H_1 = -\operatorname{osc}(H) - \delta''$). The construction of this truncated Floer complex follows the same procedure as in §3.5 but the choice of path space in use requires some special attention.

Recall that $P = \frac{3}{2} + i = \pi(W) \cap \pi(V')$. Consider the component $\mathcal{P}_0(L_\epsilon, L)$ of m_0 in the space $\mathcal{P}(L_\epsilon, L)$ of paths (in M) from L_ϵ to L, and the inclusion $j : \mathcal{P}_0(L_\epsilon, L) \to \mathcal{P}_0$ induced by $\{P\} \times M \hookrightarrow \mathbb{C} \times M$. Let $\widetilde{\mathcal{P}}_0(L, L_\epsilon)$ be the pull-back of the covering space $\widetilde{\mathcal{P}}_0 \to \mathcal{P}_0$ (recall that $\mathcal{P}_0$ is the component of m_0 of the of space of paths joining W to V' and $\widetilde{\mathcal{P}}_0$ is the covering of $\mathcal{P}_0$ associated to the morphism induced by integration of Ω). We use the path space $\widetilde{\mathcal{P}}_0(L, L_\epsilon)$ to construct $CF_b^a(L_\epsilon, L; \eta, \widetilde{J}_+)$, the point $\tilde{m}_0$, which belongs to $\widetilde{\mathcal{P}}_0(L, L_\epsilon)$, is taken as the base point for the relevant action. We emphasize that our construction implies that $\mathcal{A}_\eta(\tilde{m}_0) = \kappa(m_0) + \eta = \eta$.

Lemma 3.8. *With the notation above*

$$CF_b^a(L_\epsilon, L; \eta, \widetilde{J}_+) = CF_b^a(W, V'; \eta, \widetilde{J}),$$

where the complex $CF_b^a(W, V'; \eta, \widetilde{J})$ is the complex constructed in §3.5 for G the constant Hamiltonian $G \equiv \eta$ defined on $\mathbb{C} \times M$.

Proof. First, the generators of both complexes are appropriate lifts of the intersection points $W \cap V' = L_\epsilon \cap L$. The choice of path space used in the construction of $CF_b^a(L_\epsilon, L; -, -)$ shows that these lifts coincide and, thus, the generators of the two complexes are the same. Concerning the Floer differential recall that over $[1, \infty) \times \mathbb{R}$ the almost complex structure $\widetilde{J}$ is of the form $i \times \widetilde{J}_+$. In view of this, the result will follow by a simple application of the open mapping theorem as in [**12**]. Indeed, for any solution u of the Floer equation associated to the data $(\eta, \widetilde{J})$ with boundary conditions W, V' let $v = \pi \circ u$. Then v is holomorphic in the region $[1, \infty) \times \mathbb{R}$. Moreover, P is both a $-\infty$ asymptotic limit for v as well as a $+\infty$ asymptotic limit. But this means (for orientation reasons) that, if v is non-constant, the image of v intersects one of the unbounded regions of $([0,1) \times \mathbb{R}) \backslash (\pi(V') \cup \pi(W))$. By the open mapping theorem this contradicts the fact that v is of finite energy. Thus, v is constant. Therefore, the differentials of the two complexes coincide too. q.e.d.

It is easy to see that this identification is again compatible with the multiplications involved so that (18) becomes:

$$R * \mathrm{image}(\psi_{\eta_2, \eta_1}) \neq 0, \tag{20}$$

where

$$\psi_{\eta_2, \eta_1} : CF_b^a(L_\epsilon, L; \eta_2, \widetilde{J}_+) \to CF_b^a(L_\epsilon, L; \eta_1, \widetilde{J}_+)$$

is the comparison morphism associated to a monotone homotopy relating the two constant Hamiltonians $\eta_2 = 0$ and $\eta_1 = -\operatorname{osc}(H) - \delta''$ defined on M (we use the same notation for the map on the homology level). This morphism is easiest to understand if we use an interpolating monotone homotopy S^λ as in §3.7, Remark 3.6.

We can further simplify this equation by taking into account that L and L_ϵ are Hamiltonian isotopic. Indeed, recall from §3.4, §3.9 that L_ϵ is the graph of the form $d\kappa$. In particular, there is a Hamiltonian $\bar{\kappa} : M \to \mathbb{R}$ of oscillation equal to the oscillation of the function κ so that $\phi_1^{\bar{\kappa}}(L) = L_\epsilon$. Moreover, on a Weinstein neighbourhood of L, $\bar{\kappa}$ has the form $\bar{\kappa} = \kappa \circ p_L$ where p_L is the projection on the base on that neighbourhood. We shall use this Hamiltonian and a naturality type transformation to transform equation (20) into an equation only involving the Floer theory of the pair (L_ϵ, L_ϵ). The only subtelty is that we also need to transform appropriately the covering $p' : \widetilde{\mathcal{P}}_0(L_\epsilon, L) \to \mathcal{P}_0(L_\epsilon, L)$ into a covering

$$p'' : \widetilde{\mathcal{P}}_0(L_\epsilon, L_\epsilon) \to \mathcal{P}_0(L_\epsilon, L_\epsilon).$$

For this purpose consider the Hamiltonian $\overline{K}$ on $\mathbb{C} \times M$ given as $0 \oplus \overline{\kappa}$ cut off to 0 away from a neighbourhood of $\{P\} \times M$. Let Ψ' be the transformation

$$\Psi' : \mathcal{P}_0 \to \mathcal{P}_0(W, V''') \ , \ \Psi' : \gamma(t) \to \phi_t^{\overline{K}}(\gamma(t)).$$

Put $a_0 = \Psi'(m_0)$ and notice that this is actually a constant path (because $m_0 \in Crit(\kappa)$). Denote by $V''' = \phi_1^{\overline{K}}(V')$ and let $\mathcal{P}_0(W, V''')$ be the component of the space of path from W to V''' that contains a_0. Let, as usual, $\tilde{p} : \widetilde{\mathcal{P}}_0(W, V''') \to \mathcal{P}_0(W, V''')$ be the covering space associated to the kernel of the morphism $\pi_1(\mathcal{P}_0(W, V''')) \to \mathbb{R}$ given by integrating Ω. Finally, let $\mathcal{P}_0(L_\epsilon, L_\epsilon)$ be the component of the space of paths from L_ϵ to itself (in M) that contains a_0. The covering p'' is the pullback of the covering $\tilde{p}$ by the inclusion $\mathcal{P}_0(L_\epsilon, L_\epsilon) \to \mathcal{P}_0(W, V''')$ induced by $\{P\} \times M \subset \mathbb{C} \times M$. The transformation Ψ' defines a homeomorphism that relates the two coverings p' and p''. Denote by $\tilde{a}_0$ the image of $\tilde{m}_0$.

Now define the Floer complex $CF_b^a(L_\epsilon, L_\epsilon; \bar{h}, \bar{J})$ by the same procedure as in §3.5 but by using as base-point $\tilde{a}_0$ and the path space given by the covering p''. Here $\bar{h}$ is a Hamiltonian on M.

Lemma 3.9. *The map Ψ' induces an identification:*

$$\Psi' : CF_b^a(L_\epsilon, L; \eta, \widetilde{J}_+) \to CF_b^a(L_\epsilon, L_\epsilon; \eta + \bar{\kappa}, \bar{J}), \tag{21}$$

where $\bar{J} = (\phi_t^{\bar{\kappa}})_*(\widetilde{J}_+)$ and $\eta \in \mathbb{R}$. The morphism is compatible with the action of $CM(f_0, L_\epsilon)$ and (20) becomes:

$$R * \mathrm{image}(\bar{\psi}_{2,1}) \neq 0, \tag{22}$$

where

$$\bar{\psi}_{2,1} : HF_b^a(L_\epsilon, L_\epsilon; \eta_2 + \bar{\kappa}, \bar{J}) \to HF_b^a(L_\epsilon, L_\epsilon; \eta_1 + \bar{\kappa}, \bar{J})$$

is the natural comparison map.

Proof. A standard calculation – such as in §3.2.3 [**9**] – shows that Ψ' takes the intersection points $L_\epsilon \cap L$ to orbits of $\phi^{\bar{k}}$ going from L_ϵ to L_ϵ and that Ψ' is action preserving. Further, Ψ' also transforms the Floer equation written for the data $(\eta, \widetilde{J}_+)$ to the Floer equation associated to the data $(\eta + \bar{k}, \bar{J})$. This shows that Ψ' is an isomorphism. Concerning the action of $CM(f_0, L_\epsilon)$ notice from §3.8 that the incidence condition used in the definition of the moduli spaces giving this module action only involves the $\mathbb{R} \times \{0\}$ boundary of the strip $\mathbb{R} \times [0, 1]$. As a consequence, Ψ' also identifies the moduli spaces giving the module action in the domain and target of Ψ'. Finally, it is immediate to see that, in homology, Ψ' also intertwines the two comparison maps. q.e.d.

We have reduced our argument to showing (22) which will be done in the next subsection.

Remark 3.10. It is tempting to directly argue that the complex $CF_b^a(L_\epsilon, L_\epsilon; \eta + \bar{\kappa}, \bar{J})$ can be identified with a Morse type complex by reasoning like in Floer's work [**20**]. However, here we are not in an exact setting and all such identifications are very sensitive to the action window. In the next subsection we will compare the complex $CF^a_b(L_\epsilon, L_\epsilon)$ with an appropriate Morse complex but the argument is more involved.

3.10. A PSS type comparison argument and the proof of (22). We start with the remark that for any Morse function $\sigma : L_\epsilon \to \mathbb{R}$ so that the pair (σ, g_0) is Morse–Smale, and the pairs (σ, g_0) and (f_0, g_0) are suitably in general position with respect to one another, there is a multiplication $CM(f_0, L_\epsilon) \otimes CM(\sigma, L_\epsilon) \to CM(\sigma, L_\epsilon)$ so that $R * HM(\sigma, L_\epsilon)$ is non-trivial. This is because its induced product in homology is identified with the singular intersection product which has a unit (given by the fundamental class). The next step is to compare $CM(\sigma, L_\epsilon)$ and $CF^a_b(L_\epsilon, L_\epsilon; \eta + \overline{\kappa}, \bar{J})$ by means of the so-called PSS [40] maps. It is necessary to work with a certain extension of the complex $CM(\sigma, L_\epsilon)$ that takes into account the path space used to define the Floer complex $CF(L_\epsilon, L_\epsilon)$. To define this extension, let $p''' : \widetilde{L}_\epsilon \to L_\epsilon$ be the pull-back covering induced from p'' by the map $j_{L_\epsilon} : L_\epsilon \to \mathcal{P}_0(L_\epsilon, L_\epsilon)$ which sends each point in L_ϵ to the constant path. Denote by $CM(\sigma, \widetilde{L}_\epsilon)$ the obvious lift of the Morse complex of σ. In other words, this is the Morse complex of $\sigma \circ p'''$: the generators are the lifts of the critical points of σ and the connecting trajectories are paths in $\widetilde{L}_\epsilon$ that project to negative gradient trajectories of σ. It is useful to understand the complex $CM(\sigma, \widetilde{L}_\epsilon)$ better. First of all notice that $\tilde{a}_0 \in \widetilde{L_\epsilon}$. Each point in $\widetilde{L_\epsilon}$ is identified with a pair formed by a point z in L_ϵ together with a "weight" $\Omega(z)$ given by the integral of Ω over a path in $\mathcal{P}_0(W, V''')$ that starts at $j_{L_\epsilon}(a_0)$ and ends at $j_{L_\epsilon}(z)$. Obviously, the integral of Ω along any path in $\mathcal{P}_0(W, V''')$ that is completely in L_ϵ vanishes. Therefore, the Morse differential does not modify this weight. As a consequence, if we only look at the generators of $CM(\sigma, \widetilde{L}_\epsilon)$ that are of weight 0, they form a subcomplex $CM(\sigma, \widetilde{L}_\epsilon; 0)$ which is actually a factor of $CM(\sigma, \widetilde{L}_\epsilon)$, and is obviously isomorphic to $CM(\sigma, L_\epsilon)$. The product $R * HM(\sigma, \widetilde{L}_\epsilon; 0)$ obviously continues to be non-trivial.

In summary, to show our claim (22), it is enough to show:

Lemma 3.11. *For any $\eta \in [\eta_1, \eta_2]$, there exist two chain maps*

$$\phi_\eta : CM(\sigma, L_\epsilon) = CM(\sigma, \widetilde{L}_\epsilon, 0) \to CF^a_b(L_\epsilon, L_\epsilon; \eta + \bar{\kappa}, \bar{J})$$

and

$$\phi'_\eta : CF^a_b(L_\epsilon, L_\epsilon; \eta + \bar{\kappa}, \bar{J}) \to CM(\sigma, \widetilde{L}_\epsilon; 0),$$

so that: both maps are compatible with the multiplication with $CM(f_0, L_\epsilon)$, $\phi'_\eta \circ \phi_\eta$ is chain homotopic to the identity, and, moreover, $\bar{\psi}_{2,1} \circ \phi_{\eta_2}$ is chain homotopic to ϕ_{η_1}.

Proof. The two maps ϕ_η and ϕ'_η are particular examples of the so-called PSS maps introduced in [40]. These maps (with the description that we review below) have been studied by a variety of authors such as [3], [10], [11], [30], [39]. It is useful to keep in mind that all the argument takes place in the fiber over the point P, in other words in the symplectic manifold (M, ω) (so that no compactness issues related to cobordisms are present) and we work at all times under the bubbling threshold so that bubbling off issues are not present either. Thus, these previous works apply without difficulty to our situation as soon as we provide the appropriate action estimates. This is the main task in the proof.

To simplify notation we shall denote the Hamiltonian $\eta + \bar{\kappa}$ by F and we assume $\eta \in [\eta_1, \eta_2]$. We also denote the projections in the respective covering spaces by p. The construction of $\phi = \phi_\eta$ is based on counting trajectories (u, γ) where $u : \mathbb{R} \to \widetilde{\mathcal{P}}_0(L_\epsilon, L_\epsilon)$, $\gamma : (-\infty, 0] \to \widetilde{L}_\epsilon$ and if we put $u' = p(u)$, $\gamma' = p(\gamma)$, then we have:

$$u'(\mathbb{R} \times \{0,1\}) \subset L\ ,\ \partial_s(u') + \bar{J}(u')\partial_t(u') + \theta(s)\nabla F(u') = 0\ , u(+\infty) = y$$

and

$$\frac{d\gamma'}{dt} = -\nabla\sigma(\gamma')\ ,\ \gamma(-\infty) = x\ ,\ \gamma(0) = u(-\infty),$$

where x is a lift of a critical point of σ, with $\Omega(x) = 0$, and y is a generator of $CF^a_b(L_\epsilon, L_\epsilon; \eta + \bar{\kappa}, \bar{J})$; θ is a smooth cut-off function which is increasing and vanishes for $s \leq 1/2$ and equals 1 for $s \geq 1$.

The energy of such an element (u, γ) is defined in the obvious way by $E(u, \gamma) = \int ||\partial_s u'||^2 dsdt$ and it is easy to see that:

$$E(u, \gamma) = I(u) + \int_{\mathbb{R}\times[0,1]} (u')^*\omega - \int_0^1 F(y(t))dt,$$

where $I(u) = \int_{\mathbb{R}\times[0,1]} \beta'(s)F(u'(s))dsdt$. The energy verifies

$$E(u, \gamma) = I(u) - \mathcal{A}_F(y) \leq \sup(F) - \mathcal{A}_F(y).$$

Recall now from (10) that $a = \zeta$ and $b = -\operatorname{osc}(H) - \zeta$ and we also have from (19):

$$\sup(F) = \eta + \sup(\bar{\kappa}) < \eta + \zeta - \delta'' \leq \zeta = a.$$

Therefore, $\mathcal{A}_F(y) < a$ so that ϕ is well defined. We also need to notice that our definition keeps the energy under the bubbling threshold so that this map is a chain map. As $CF^a_b = CF^a/CF^b$ it follows that the only orbits of interest have action in between $[-\operatorname{osc}(H) - \zeta, \zeta]$, therefore, by (9) we have $E(u, \gamma) \leq \operatorname{osc}(H) + 2\zeta < \delta'(V, J)$.

The construction of the map $\phi' = \phi'_\eta$ is similar. We consider orbits that join lifts of Hamiltonian orbits to lifts of critical points of σ, again of weight 0, except that the pairs (u, γ) considered here, start as semi-tubes and end as flow lines of σ. The equation verified by u is similar to the one before but instead of the cut-off function θ we use the cut-off function $1 - \theta$. The energy estimate in this case gives

$$E(u, \gamma) \leq \mathcal{A}_F(y) - \inf(F).$$

Thus, $E(u,\gamma) \leq \mathcal{A}_F(y) - \eta \leq \mathrm{osc}(H) + \zeta + \zeta < \delta'(V,J)$ so that the bubbling threshold is again respected. This implies that both ϕ_η and ϕ'_η are well-defined chain morphisms.

The next step is to show that the composition of the two chain morphisms $\phi'_\eta \circ \phi_\eta$ is chain homotopic to the identity as long as $\eta \in [\eta_1, \eta_2]$. The usual PSS technique applies to prove this statement. Again the only point worth making explicit concerns the energy estimates. To discuss this, recall from [**3**] that the construction of the chain homotopy between the identity and $\phi'_\eta \circ \phi_\eta$ appeals to a new type of configuration that we denote by (r, γ, u, γ_1). Here $u : \mathbb{R} \to \widetilde{\mathcal{P}}_0(L_\epsilon, L_\epsilon)$, $\gamma : (-\infty, 0] \to \widetilde{L}_\epsilon$, $\gamma_1 : [0, \infty) \to \widetilde{L}_\epsilon$ and with the notation $u' = p(u)$, $\gamma' = p(\gamma)$, $\gamma'_1 = p(\gamma_1)$ we have:

$$u'(\mathbb{R} \times \{0,1\}) \subset L \ , \ \partial_s(u') + \bar{J}(u')\partial_t(u') + \theta_r(s)\nabla F(u') = 0,$$
$$\frac{d\gamma'}{dt} = -\nabla\sigma(\gamma') \ , \ \gamma(-\infty) = x \ , \ \gamma(0) = u(-\infty),$$
$$\frac{d\gamma'_1}{dt} = -\nabla\sigma(\gamma'_1) \ , \ \gamma_1(+\infty) = x' \ , \ \gamma_1(0) = u(+\infty).$$

where x and x' are generators of $CW(\sigma, \widetilde{L}_\epsilon; 0)$. The family of functions $\theta_r : \mathbb{R} \to [0,1]$ is chosen so that when $r \to 0$ the family goes uniformly to 0 and for sufficiently large r it has support inside $[-r-1, r+1]$ and it is constant equal to 1 in $[-r, r]$ and is increasing in the interval $[-r-1, -r]$ and decreasing in the interval $[r, r+1]$. It is again easy to estimate the energy of such configurations using the same formula as before. The conclusion in this case is that because x, x' are of weight 0, we obtain $E(r, \gamma, u, \gamma_1) \leq \mathrm{osc}(F) \leq \mathrm{osc}(H) + 2\zeta < \delta'(V,J)$ so that we can deduce that $\phi' \circ \phi$ are chain homotopic to the identity by the usual PSS reasoning.

Finally, we need to notice that $\bar{\psi}_{2,1} \circ \phi_{\eta_2}$ is chain homotopic to ϕ_{η_1}. For this notice that the two Hamiltonians involved here are $F_1 = \eta_1 + \bar{\kappa} = -\mathrm{osc}(H) - \delta'' + \bar{\kappa}$ and $F_2 = \eta_2 + \bar{\kappa} = \bar{\kappa}$. Thus, F_1 and F_2 only differ by a constant. In particular, F_1 and F_2 have the same Hamiltonian flows. It follows that the difference between ϕ_{η_1} and ϕ_{η_2} only consists in the way the truncation is applied. In other words the actual underlying moduli spaces are the same but when the respective chain morphisms are defined the truncations take into account the difference between F_1 and F_2. But this is precisely the effect of $\bar{\psi}_{2,1}$ (see Remark 3.6). q.e.d.

This concludes the proof of Proposition 3.2 in the case of simple cobordisms and under the simplifying geometric assumption in §3.3.

3.11. Dropping the special assumptions. The statement of the proposition has been proved in the preceding sections under two assumptions:

a. the cobordism V is simple.

b. the projection of V in the plane is as described at the beginning of §3.3 (as in Figure 2) and the constant δ at point iii. in §3.3 can be assumed as small as desired.

We start by explaining how to drop condition *b.* while assuming for the moment that V is simple.

To start this argument, first notice that in the proof discussed in the previous sections we can take without any difficulty the points $Q, P \in R + i$ so that $Re(Q) < -a$, $Re(P) > a$ for a constant $a > 0$ as large as desired instead of assuming $Re(Q) \in (-2, -1)$, $Re(P) = \frac{3}{2}$.

Fix $\delta > 0$. The finite part of the outline (recall Definition 1.1), $ou'_V = ou_V \cap \{-a/2 \leq Re(z) \leq a/2\} \subset \mathbb{C}$, is measurable. Therefore, it can be approximated by a union U of closed planar rectangles so that $ou'_V \subset U$ and $\text{Area}(U) \leq \mathcal{S}(V) + \frac{\delta}{2}$. The boundary of U is piecewise linear and, thus, can be approximated by a union of two smooth, embedded planar curves C_1 and C_2 so that $C_1 \cap C_2$ consists of two points x and y that satisfy $\{x, y\} \subset C_j \cap \mathbb{R} + i$, $x < y$, C_j is tangent to $\mathbb{R} + i$ at both x and y, $j = 1, 2$ and the bounded region enclosed by $C_1 \cup C_2$ is of area at most $\mathcal{S}(V) + \frac{2\delta}{3}$. To fix ideas, we assume that C_1 is above C_2 in the plane. It is a simple exercise to show that there exists a constant a and a symplectic diffeomorphism $\psi : \mathbb{C} \to \mathbb{C}$ with support in $[-a, a] \times \mathbb{R}$ so that if we put $\bar{\psi} = \psi \times id : \mathbb{C} \times M \to \mathbb{C} \times M$, then $\tilde{V} = \bar{\psi}(V)$ has the projection $\pi(\tilde{V})$ as in Figure 2. This symplectic diffeomorphism takes the curve C_1 to a subset of $\mathbb{R} + i$ and the curve C_2 to the graph of a function β. To ensure that the function β is smooth an additional rounding of the corners might be needed at the points $\psi(x)$ and $\psi(y)$ but, in all cases, we obtain $\int_{-\infty}^{+\infty}(1 - \beta(x))dx \leq \mathcal{S}(V) + \delta$.

We can now construct a Hamiltonian $\tilde{H}$, and $\tilde{V}'$, $\tilde{V}''$, $\tilde{W}$ as described in §3.3 and in §3.4 (where the notation skips the ˜) but now starting from the cobordism $\tilde{V}$ and relative to two points Q and P so that $Re(Q) < -a$, $Re(P) > a$. Next define $H = \tilde{H} \circ \bar{\psi}$, $V' = \bar{\psi}^{-1}(\tilde{V}')$, $V'' = \bar{\psi}^{-1}(\tilde{V}'')$, $W = \bar{\psi}^{-1}(\tilde{W})$. Clearly, $\mathcal{S}(\tilde{V}) = \mathcal{S}(V)$ so that the oscillation of H continues to be controlled by the shadow of V. From this point on we continue the proof exactly as in Sections §3.4–§3.10 but for H, V', V'', W as just defined. Notice, in particular, that because the points Q and P are away from the support of ψ the arguments involving the behaviour of holomorphic curves near Q and P do not need any adjustment. This ends the proof of the proposition in the case of a simple cobordism.

Dropping the simplicity assumption (*a.* above) is immediate using the simple case because the condition $L \cap L_k = \emptyset \ \forall\ 1 \leq k \leq r$ and $L \cap L'_m = \emptyset \ \forall\ 1 \leq m \leq s$ ensures that the ends of the cobordism V that are different from both L and L' do not interfere in any way in the proof. This concludes the proof of Proposition 3.2.

4. Proof of Theorem 1.3

Each point of the Theorem is the subject of one of the subsections below.

4.1. The metrics d^* for $* \geq wm$. We first remark that formula (1) defines a pseudo-metric. To start, notice that cobordisms can be glued. More precisely, if $V : L \rightsquigarrow L'$, $V' : L' \rightsquigarrow L''$ are two cobordisms, assumed both to be cylindrical outside $[0,1] \times \mathbb{R}$, then $V'' : L \rightsquigarrow L''$ is obtained as the union $\bar{V} \cup \tilde{V'}$ where $\bar{V} = V \setminus (-\infty, -2] \times \mathbb{R} \times M$ and $\tilde{V'} = \{(x-3, y, m) \mid (x, y, m) \in V' \setminus [2, +\infty) \times \mathbb{R} \times M\}$. It is easy to see that $\mathcal{S}(V'') = \mathcal{S}(V) + \mathcal{S}(V')$. As a consequence d^* verifies the triangle inequality. Further, for each cobordism $V : L \rightsquigarrow L'$ we can use the planar transformation $z \to -z$ followed by an appropriate translation in $\mathbb{C}$ to construct a cobordism $\hat{V} : L' \rightsquigarrow L$ so that $\mathcal{S}(V') = \mathcal{S}(V)$. As a consequence d^* is symmetric and, thus, a pseudo-metric.

For point i. we now need to see that when $* \geq wm$ this pseudo-metric is non-degenerate. Consider two distinct Lagrangians $L, L' \in \mathcal{L}^*(M)$ and a cobordism $V : L \rightsquigarrow L'$, $V \in \mathcal{L}^*_{cob}(\mathbb{C} \times M)$. By Proposition 3.2, for arbitrarily small ϵ_0, there exists an almost complex structure J such that $d^*(L, L') \geq \min\{w(L, L') - \epsilon_0, \delta(V; J)\}$. Given that $L \neq L'$, $w(L, L') > 0$. For $* = wm$, if u is a J-holomorphic disk or sphere (as in (3)), then $\omega(u) = \rho\mu(u)$ and as $\mu(u) \in \mathbb{Z}$ it follows that $\omega(u) \geq |\rho|$. Thus, $\delta(V; J)$ is bounded from below by $|\rho|$ (and is equal to $+\infty$ when $\rho = 0$). This argument also applies for $* = m$ as well as for $* = e$. Thus, if $* = wm, m, e$ we obtain that the pseudo-metric d^* is non-degenerate. The Hamiltonian orbit case reduces to a statement that is well-known but we will sketch a direct proof for completeness in the next section.

4.2. Relation to the Hofer norm. To prove point ii. we now consider the (Lagrangian) Hofer distance on the space of Lagrangians L Hamiltonian isotopic to a fixed Lagrangian L_0 as recalled in Remark 1.4 d. To show that d_H is non-degenerate one can proceed just as in our argument for the first point of Proposition 3.2. For completeness, we sketch the proof below, omitting most technical details.

We want first to show that if $L = \phi_1^G(L')$, then for $\epsilon_0 > 0$ there exists an almost complex structure J on M such that

$$\int_0^1 (\max_{x \in M} G(t, x) - \min_{x \in M} G(t, x)\ )dt \geq \min\{\delta(L, J), w(L, L') - \epsilon_0\}.$$

We first pick J as J_- was chosen in §3.4. We assume that the quantity

$$\operatorname{osc}(G) = \int_0^1 (\max_{x \in M} G(t, x) - \min_{x \in M} G(t, x))dt$$

is smaller than the bubbling threshold $\delta(L, J)$. By a naturality transformation such as Ψ' in (21) together with the module action from §3.8, the proof reduces to show that

$$R * HF_b^a(L, L; G, \bar{J}) \neq 0, \tag{23}$$

where $(\phi_t^G)_* \bar{J} = J$. This non-vanishing is in perfect analogy to formula (22). Here $a = \int_0^1 (\max_{x \in M} G(t,x))dt + \zeta$ and $b = \int_0^1 (\min_{x \in M} G(t,x)\ )dt - \zeta$ where ζ is an arbitrarily small constant. To show (23) we apply the construction of the PSS maps as described in §3.10. The energy estimates in this case are exactly what is required for the argument to work. Indeed, in the argument in §3.10 we dealt with a time independent Hamiltonian F but if we apply the same energy calculations to a time dependent Hamiltonian G, then the oscillation of F is replaced by $\mathrm{osc}(G)$, $\sup(F)$ by $\int_0^1 (\max_{x \in M} G(t,x))dt$ and $\inf(F)$ by $\int_0^1 (\min_{x \in M} G(t,x)\ )dt$. The final step is to see that max and min in these formulas can be taken as in the definition of d_H, that is over $\phi_t^G(L)$. But this is easy to do by truncating $G(t,z)$ for each t outside a neighbourhood of $\phi_t^G(L)$.

The shadow of cobordisms and the Hofer norm are naturally related through the Lagrangian suspension construction.

Fix a connected, closed Lagrangian $L \subset M$ and let its Lagrangian suspension along G be L^G as in §2.5. It is immediate to see from Definition 1.1 that:

$$\mathcal{S}(L^G) = \int_0^1 [\ \max_{z \in \phi_t^G(L)} G(t,z) - \min_{z \in \phi_t^G(L)} G(t,z)\ ]dt. \tag{24}$$

Thus, for a Lagrangian L_0 in M, the metric d^{L_0} defined on the orbit $\mathcal{L}^{L_0}(M)$ of L_0 under the action of the Hamiltonian group, as provided by Theorem 1.3, satisfies:

$$d^{L_0}(L, L') = d_H(L, L'),$$

for any $L, L' \in \mathcal{L}^{L_0}(M)$.

Remark 4.1. For any $\phi \in \mathrm{Ham}(M, \omega)$ that is not equal to the identity, there is some Lagrangian L in M – possibly taken in a sufficiently small Darboux chart – so that $\phi(L) \neq L'$. Thus, the non-degeneracy of the Lagrangian metric $d_H(-,-)$ implies that the Hofer norm on $\mathrm{Ham}(M, \omega)$ itself is non-degenerate. This method to show the non-degeneracy of the Hofer norm is due to Polterovich [**42**] and Chekanov [**16**].

4.3. Surgery and non-isotopic Lagrangians at finite d^{wm} distance. In this subsection we prove point iii. An example of two non-isotopic Lagrangians and a weakly monotone cobordism relating them was constructed in [**12**]. We give here an outline of the construction because we want to also discuss the shadow of this cobordism.

We start by recalling the Lagrangian surgery construction, [**32**], [**41**]. This is based on a simple local construction. Fix the following two Lagrangians: $L_1 = \mathbb{R}^n \subset \mathbb{C}^n$ and $L_2 = i\mathbb{R}^n \subset \mathbb{C}^n$.

Consider a curve $\chi \subset \mathbb{C}$, $\chi(t) = a(t) + ib(t)$, $t \in \mathbb{R}$, so that (see also Figure 5): χ is smooth; $(a(t), b(t)) = (t, 0)$ for $t \in (-\infty, -1]$; $(a(t), b(t)) = (0, t)$ for $t \in [1, +\infty)$; $a'(t), b'(t) > 0$ for $t \in (-1, 1)$. Let

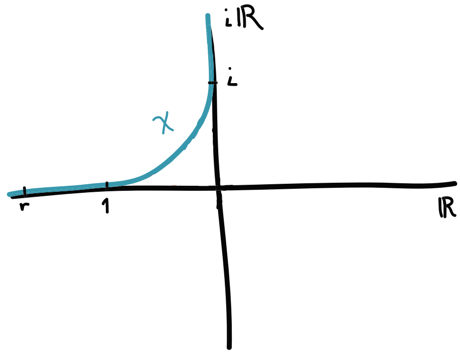

Figure 5. The curve $\chi \subset \mathbb{C}$.

$$L = \Big\{ \big((a(t) + ib(t))x_1, \ldots, (a(t) + ib(t))x_n \big) \mid t \in \mathbb{R}, \sum x_j^2 = 1 \Big\} \subset \mathbb{C}^n.$$

It is easy to see that L is Lagrangian. By an abuse of notation because we omit the handle χ from the notation and we will denote $L = L_1 \# L_2$. Notice that different choices of handles χ produce Hamiltonian isotopic Lagrangians L (for $n > 1$). By choosing the handle small enough, we can have the result of the surgery be contained in an arbitrarily small neighbourhood of $L_1 \cup L_2$.

There is a Lagrangian cobordism $L \rightsquigarrow (L_1, L_2)$ constructed as follows. Define

$$\widehat{\chi} = \Big\{ \big((a(t)+ib(t))x_1, \ldots, (a(t)+ib(t))x_{n+1} \big) \mid t \in \mathbb{R}, \sum x_j^2 = 1 \Big\} \subset \mathbb{C}^{n+1}$$

and notice that $\widehat{\chi}$ is also Lagrangian. Consider the projection $\pi : \mathbb{C}^{n+1} \to \mathbb{C}$, $\pi(z_1, \ldots z_{n+1}) = z_1$ and denote by $\widehat{\pi}$ its restriction to $\widehat{\chi}$.

Let $S_+ = \{(x, y) \in \mathbb{R}^2 \mid y \geq x\}$ (as usual, we identify $\mathbb{R}^2$ with $\mathbb{C}$ under $(x, y) \to x + iy$). Consider $W_0 = \widehat{\pi}^{-1}(S_+) \cap \pi^{-1}([-2, 0] \times [0, 2])$ (see Figure 6). It is not difficult to see that W_0 is a manifold with boundary and that $\partial W_0 = \{(-2, 0)\} \times L_1 \cup \{(0, 2)\} \times L_2 \cup \{0, 0\} \times L$. To finish the construction of the cobordism we adjust W_0 (as described explicitly in [**12**]) so as to continue the L-boundary component to be cylindrical. The resulting Lagrangian W' provides the cobordism desired between L and (L_1, L_2) – see also Figure 7.

Going from the local argument above to a global one is easy. Suppose that we have two Lagrangians L' and L'' that intersect transversely, possibly in more than a single point. At each intersection point we fix symplectic coordinates mapping (locally) L' to $\mathbb{R}^n \subset \mathbb{C}^n$ and mapping

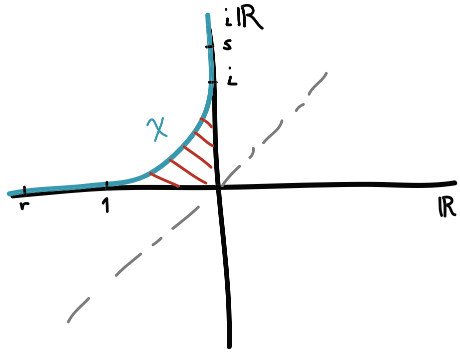

Figure 6. The projection of W_0 is the red region together with the two semi-axes $(-\infty, 0] \subset \mathbb{R}$ and $i[0, +\infty) \subset i\mathbb{R}$ and the curve χ.

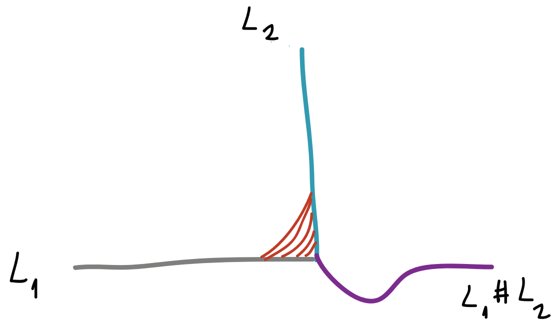

Figure 7. The trace of the surgery after projection on the plane.

(again locally) L'' to $i\mathbb{R}^n \subset \mathbb{C}^n$. We then apply the construction above at each of these intersection points. This produces a new Lagrangian submanifold $L'\#L''$ as well as a cobordism $W'' : L'\#L'' \rightsquigarrow (L', L'')$ (caveat: $L'\#L''$ is topologically not a connected sum if there are several intersection points).

We are interested in the shadows of the Lagrangians resulting from this construction. There are two useful remarks in this direction. First, for the cobordism W'' above, we see easily that for any $\epsilon > 0$ we can find a sufficiently small handle χ so that $\mathcal{S}(W'') \leq \epsilon$. However, different handles lead to different outputs of the surgery as the resulting $L'\#L''$ are Hamiltonian isotopic (for $n > 1$) but not identical. Secondly, if in the place of L' and L'' we take cobordisms $V' = \gamma' \times N'$, $V'' = \gamma'' \times N''$ where γ' and γ'' are appropriate curves in $\mathbb{C}$ that intersect transversely at a single point, then for any ϵ we may construct the surgered Lagrangian $V'\#V''$ so that its shadow is smaller than ϵ by again using in the construction a sufficiently small handle H.

Our example is based on the construction described above. We will start our construction in the ambient manifold $M' = \mathbb{C}\backslash\{P_1, P_2, P_3\}$ where P_1, P_2, P_3 are three points $P_i \in \mathbb{C}$. One may worry that excising points creates concave ends in our symplectic manifold, but this is easily overcome by a positivity of intersections argument. Alternatively one may glue in small handles at the three points P_1, P_2, P_3. Finally, one can work inside a large ball, and compactify it to a sphere, adding a handle at infinity. This gives us an example in the closed surface of genus 4. The following arguments apply uniformly in all cases.

We consider two circles $A = \{z \in \mathbb{C} \ : \ |z + 1/2| = 1\}$ and $B = \{x \in \mathbb{C} \ : \ |z - 1/2| = 1\}$ and denote by $D(A)$ and $D(B)$ the two disks bounded by A and B, respectively. We assume that the positions of the points P_i relative to the circles A, B are such that $P_1 \in D(A)\backslash D(B)$, $P_2 \in D(A) \cap D(B)$ and $P_3 \in D(B)\backslash D(A)$ as at the middle of Figure 9. We consider two smooth curves in the plane $\mathbb{C}$, $\gamma_1 : [-1, 1] \to \mathbb{C}$ and $\gamma_2 : [-1, 1] \to \mathbb{C}$ so that – see Figure 8:

i. $\gamma_1(t) = t$ for $t \in [-1, -1/2]$,
ii. $\gamma_1(t) = 1 + (1 - t)i$ for $t \in [1/2, 1]$,
iii. $Re(\gamma_1(t))$ is strictly increasing for $t \in (-1/2, 1/2 - \epsilon)$. $Im(\gamma_1(t))$ is strictly increasing for $t \in (-1/2, 1/2 - \epsilon)$ and strictly decreasing for $t \in (1/2 - \epsilon, 1/2)$,
iv. $\gamma_2(t) = -\gamma_1(t)$ for all $t \in [-1, 1]$.

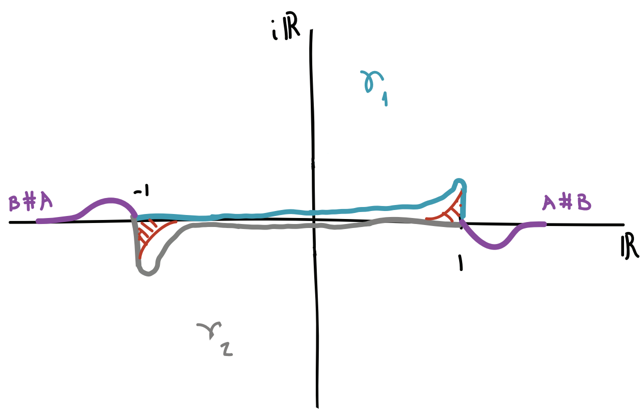

Figure 8. The projection of V on $\mathbb{C}$; the surgery regions; and the curves γ_1, γ_2.

We now consider the Lagrangians $A' = \gamma_2 \times A \subset \mathbb{C} \times M$ and $B' = \gamma_1 \times B \subset \mathbb{C} \times M$. By performing surgery at both intersection points $A \cap B$ we can extend the union of the two Lagrangians $A' \cup B'$ towards the positive end as well as towards the negative end as in Figure 8, thus, obtaining a cobordism $V : A\#B \rightsquigarrow B\#A$.

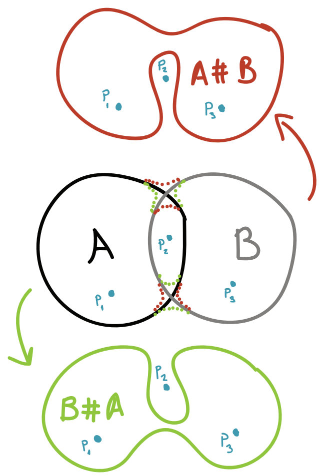

Figure 9. The two circles A and B as well as $A\#B$ and $B\#A$. The three puncture points are indicated as well.

Put $L = A\#B$ and $L' = B\#A$. It is easy to see that L and L' look as in Figure 9 and are exact. We notice that L and L' are not smoothly isotopic in M'. However, it is shown in [**12**] that, by choosing the handles associated with the surgeries in the two intersection points of A and B appropriately, V can be made monotone with minimal Maslov class 1. It is also clear that the shadow of V can be made as close as desired to the sum of two areas, one corresponding to the shadow of one handle used at the $A\#B$ end of the surgery and the other corresponding to the shadow of the handle used at the $B\#A$ end – as suggested by Figure 8. Notice, however, that diminishing the size of these handles also modifies the ends $L = A\#B$ and $L' = B\#A$. Indeed, Proposition 3.2 shows that if L and L' are fixed the shadow of the cobordism V cannot be arbitrarily small.

It is also noticed in [**12**] that L and L' are not cobordant via a monotone cobordism V with $N_V \geq 2$. In short, this is shown by observing that the Floer homologies $HF(S, L)$ and $HF(S, L')$ where S is the vertical semiaxis in $\mathbb{C}$, pointing up and starting at P_2 are non-isomorphic. But by the results in [**12**, **13**] if a monotone simple cobordism would relate L and L' then for any other Lagrangian N we would have $HF(N, L) \cong HF(N, L')$ (indeed, L and L' would even be isomorphic in the appropriate Fukaya category).

In summary, the two exact Lagrangians $L, L' \in \mathcal{L}^e(M')$ constructed before are not smoothly isotopic and

$$0 < d^{wm}(L, L') < \infty \ , \ d^m(L, L') = \infty.$$

We will see in §4.4 that we also have $d^g(L, L') = 0$.

4.4. The pseudo-metric d^g is degenerate. We, finally, consider point iv. We consider here the pseudo-metric d^g given as in the formula in Theorem 1.3 but for $* = g$. Recall that g stands for general. In other words, there is no constraint imposed on either the Lagrangians or the cobordisms involved.

The following construction was suggested to us by Emmy Murphy.

Consider two Lagrangians L and L' and a cobordism $V : L \rightsquigarrow L'$. We shall construct other cobordisms $L \rightsquigarrow L'$ whose outlines have arbitrarily small areas (recall Definition 1.1).

For $\epsilon > 0$ define a map $\tilde{\epsilon} : \mathbb{C} \times M \to \mathbb{C} \times M$ given by rescaling in the imaginary direction in the plane $\tilde{\epsilon}(x + iy, z) = (x + i\epsilon y, z)$. Put $V^\epsilon = \tilde{\epsilon}(V)$. Denote by $e : V \to \mathbb{C} \times M$ the natural embedding, and define a smooth embedding $e_\epsilon : V \to \mathbb{C} \times M$ by $e_\epsilon = \tilde{\epsilon} \circ e$.

Let R be a compact rectangle in $\mathbb{C}$ outside whose interior V is cylindrical: for example, one can take $R = [\beta_-, \beta_+] \times [Y_-, Y_+]$, where $Y_- < \inf \mathrm{Im}(\pi(V) \cap [\beta_-, \beta_+] \times \mathbb{R})$, and $Y_+ > \sup \mathrm{Im}(\pi(V) \cap [\beta_-, \beta_+] \times \mathbb{R})$. Clearly, $V^1 = V$ and

$$\mathrm{Area}(ou_{V^\epsilon}) = \epsilon \ \mathrm{Area}(ou_V) < \epsilon \mathrm{Area}(R).$$

For a point $v \in \mathbb{C} \times M$, consider the symplectomorphism $l_{\epsilon,v} : \mathbb{C} \times M \to \mathbb{C} \times M$ given by

$$l_{\epsilon,v}(x + iy, z) = (x + iy - i(1 - \epsilon)\mathrm{Im}(\pi_{\mathbb{C}}(v)), z).$$

Viewed as a family of maps, $l_{\epsilon,v}$ depends smoothly on both parameters. Note that $l_{\epsilon,v}(v) = \tilde{\epsilon}(v)$. Hence, $dl_{\epsilon,v} : T_v(\mathbb{C} \times M) \to T_{\tilde{\epsilon}(v)}(\mathbb{C} \times M)$ is an isomorphism of symplectic vector spaces.

Define a smooth bundle map $\phi^\epsilon : TV \to e_\epsilon^* T(\mathbb{C} \times M)$ by

$$\phi^\epsilon(\xi) = dl_{\epsilon,e(a)} \circ de(\xi),$$

for each $\xi \in T_a V$. The image of this bundle map is a Lagrangian subbundle and, because this bundle map covers the smooth embedding e_ϵ, we may apply the Gromov–Lees h-principle [**26**, **33**] to e_ϵ. Properly speaking, we need here a relative version of Theorem 1 in [**33**] as we want to modify e_ϵ only away from the ends of the cobordism V. However, Lees' proof adjusts trivially to this relative case (because his proof goes by induction over handle attachments, see page 221 in [**33**]). As a result we obtain that e_ϵ can be approximated arbitrarily well in C^0 norm by Lagrangian immersions. In particular, for any $\delta > 0$ we may find a Lagrangian immersion $e_{\epsilon,\delta} : V \to \mathbb{C} \times M$ with image $V^{\epsilon,\delta}$ so that $\mathrm{Area}(ou_{V^{\epsilon,\delta}}) \leq \epsilon \mathrm{Area}(R) + \delta$.

We now modify $V^{\epsilon,\delta}$ twice: first we perturb the immersion to a new immersion with only transverse double points and, secondly, surger all the self-intersection points by using very small Lagrangian handles so as to get an embedded Lagrangian $V^{\epsilon,\delta,\delta'}$ so that $\mathrm{Area}(ou_{V^{\epsilon,\delta,\delta'}}) \leq \epsilon \mathrm{Area}(R) + \delta + \delta'$.

By taking the (generic) perturbation of the immersion $e_{\epsilon,\delta}$ small enough and by taking the surgery handles to be also small enough, we may assume that $\delta' \leq \epsilon$ and $\delta \leq \epsilon$. Hence, we get a cobordism $V^{\epsilon,\delta,\delta'} : L \rightsquigarrow L'$ with

$$\mathrm{Area}(ou_{V^{\epsilon,\delta,\delta'}}) \leq \epsilon\ (\mathrm{Area}(R) + 2).$$

Therefore, $d^g(L, L') = 0$ and, thus, the pseudo-metric d^g is degenerate.

This concludes the proof of Theorem 1.3.

Remark 4.2. An alternative argument that does not involve the h-principle was suggested to us by Lev Buhovsky.

5. Additional comments

5.1. Relation to spectral distance. The argument for the proof of Proposition 3.2 suggests that in the setting where L and L' are Hamiltonian isotopic and exact (the same would hold in the weakly exact case $\omega|_{\pi_2(M,L)} = 0$, at least under the additional assumption that the Maslov class $\mu|_{\pi_2(M,L)} = 0$), assuming that the cobordism is monotone, one can replace $w(L, L')$ in the statement of the Proposition by $d_S(L, L')$, the spectral distance between L and L' (introduced in [**49**], see also [**29**] for additional references). For a fixed Lagrangian L recall from the work of Milinkovic [**35**] that, if L' is sufficiently C^1-close to L, then $d_S(L, L') = d_H(L, L')$. Therefore, we expect that, at least under this additional proximity assumption, $d^*(L, L') = d_H(L, L')$ for all $* \geq m$.

5.2. Lower bound for the shadow in the monotone case. We believe that an adaptation of the proof of Proposition 3.2 shows that, under the assumptions of the Proposition, and if, additionally, $L, L' \in \mathcal{L}^m(M)$, $V \in \mathcal{L}^m_{cob}(\mathbb{C} \times M)$, then we have:

$$\mathcal{S}(V) \geq w(L, L').$$

This inequality fits with the leitmotiv of the paper: more rigid topological constraints lead to sharper inequalities. Indeed, in the setting of the proposition, this expected inequality shows that monotonicity is sufficient to eliminate $\delta(V, J)$ from the general inequality given by Proposition 3.2. Further, as seen in §4.2, if we assume that $V : L \rightsquigarrow L'$ is a Lagrangian suspension, then the inequality becomes even stronger as we can replace $w(L, L')$ by the Hofer distance $d_H(L, L')$ between L and L'.

5.3. Categorical view-point. A somewhat more conceptual perspective on the construction of the metrics d^* from Theorem 1.3 is as follows. Using the notion of cobordism, one can define – as in [**12**] – various categories that have as objects Lagrangians in M and have morphisms given by Lagrangian cobordisms. As before, the specific Lagrangians and cobordisms involved are subject to the constraints encoded in the superscript $-^*$, where $*$ can be any of the conditions listed in §2. The simplest such category, $\mathcal{L}ag^*_s(M)$, has as objects the Lagrangians in $\mathcal{L}^*(M)$ and as morphisms the horizontal isotopy classes of simple cobordisms $V : L \rightsquigarrow L'$ so that $V \in \mathcal{L}^*_{cob}(C \to M)$.

Given a small category $\mathcal{C}$ assume that the morphisms of $\mathcal{C}$ are endowed with a valuation $\nu : \mathrm{Mor}_{\mathcal{C}} \to [0, \infty)$ in the sense that $\nu(f \circ g) \leq \nu(f) + \nu(g)$ for all composable morphisms f and g, and, for each morphism $f \in \mathrm{Mor}(A, B)$, there is a morphism $\bar{f} \in \mathrm{Mor}(B, A)$ with $\nu(f) = \nu(\bar{f})$. Such a valuation induces a pseudo-metric d_ν on $\mathcal{O}b(\mathcal{C})$ that is given by:

$$d_\nu(A, B) = \inf_{\varphi \in \mathrm{Mor}(A,b)} \nu(\varphi).$$

This number is taken to be infinite in case there are no morphisms from A to B. In case the valuation is non-degenerate in the sense that $\nu(f) = 0$ iff $f = id_X$ for some object X, then the pseudo-metric is a true metric (with this definition the metric is finite only for objects that are related by some morphism).

The shadow of cobordisms, as given in Definition 1.1, provides a valuation on the category $\mathcal{L}ag^*_s(M)$ by putting for each morphism $[V]$ represented by a cobordism V:

$$\nu([V]) = \inf_{V'} \{\mathcal{S}(V') : V' \text{ horizontally isotopic to } V\}. \tag{25}$$

Obviously, Theorem 1.3 shows that the resulting pseudo-metrics $d^* = d^*_\nu$ are non-degenerate for $* \geq wm$ and degenerate for $* = g$.

5.4. Immersed Lagrangian cobordism. Following the work of Akaho [**1**] as well as Akaho–Joyce [**2**] (see also [**4**]), Floer theory is also defined for a class of immersed Lagrangians so-called unobstructed, following [**22**, **23**]. The cobordism machinery can also be adapted without any trouble to this setting and we expect that there are variants of both Proposition 3.2 and Theorem 1.3 in this context.

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Department of Mathematics and Statistics
University of Montreal
Montreal, QC H3C 3J7
Canada
Institute for Advanced Study
1 Einstein Drive
Princeton NJ 08540
USA
E-mail address: cornea@dms.umontreal.ca

Department of Mathematics and Statistics
University of Montreal
Montreal, QC H3C 3J7
Canada
Institute for Advanced Study
1 Einstein Drive
Princeton NJ 08540
USA
E-mail address: shelukhin@dms.umontreal.ca, egorshel@ias.edu