PERSISTENT TRANSCENDENTAL BÉZOUT THEOREMS

LEV BUHOVSKY[1], IOSIF POLTEROVICH[2], LEONID POLTEROVICH[3], EGOR SHELUKHIN[4] AND VUKAŠIN STOJISAVLJEVIĆ[5]

ABSTRACT. An example of Cornalba and Shiffman from 1972 disproves in dimension two or higher a classical prediction that the count of zeros of holomorphic self-mappings of the complex linear space should be controlled by the maximum modulus function. We prove that such a bound holds for a modified coarse count based on the theory of persistence modules originating in topological data analysis.

1. INTRODUCTION AND MAIN RESULTS

1.1. **The transcendental Bézout problem.** The classical Bézout theorem states that the number of common zeros of n polynomials in n variables is generically bounded by the product of their degrees. The *transcendental Bézout problem* is concerned with the count of zeros of entire maps $\mathbb{C}^n \to \mathbb{C}^n$. It is motivated by a number of influential mathematical ideas. The starting point is Serre's famous G.A.G.A. [25], by now understood as a meta-mathematical principle stating that complex *projective* analytic geometry reduces to algebraic geometry. A prototypical result is a theorem of Chow [9], by which every closed complex submanifold of $\mathbb{C}P^n$ is necessarily algebraic, i.e., is given as the set of solutions of a system of polynomial equations. However, as the following simple example shows, Chow's theorem fails in the affine setting.

Example 1.1. Consider an analytic function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = e^z + 1 = (e^x \cos y + 1) + ie^x \sin y,\ z = x + iy\ .$$

Zeros of f form an infinite discrete set $\{(2k+1)\pi i,\ k \in \mathbb{Z}\}$. It is not biholomorphically equivalent to any algebraic (and hence finite) proper subset of $\mathbb{C}$.

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In order to revive at least some parts of G.A.G.A. in the affine framework one needs a substitute of the notion of the degree of a polynomial for entire mappings $f : \mathbb{C}^n \to \mathbb{C}^n$. As it is put in [14], *"A transcendental entire function that can be expanded into an infinite power series can be viewed as a "polynomial of infinite degree", and the fact that the degree is infinite brings no additional information to the statement that an entire function is not a polynomial."* To this end, one introduces the *maximum modulus*

$$\mu(f,r) = \max_{z \in B_r} |f(z)| ,$$

where B_r stands for the closed ball of radius r. This quantity has at least two degree-like features. First, assume that

$$\limsup_{r\to\infty} \frac{\log \mu(f,r)}{\log r} < k+1.$$

Then, remarkably, f is a polynomial of the total degree $\leq k$. This is a minor generalization of Liouville's classical theorem. Thus one can distinguish polynomials in terms of the maximum modulus.

The second feature of the maximum modulus is given by a statement which readily follows from Jensen's formula. Let $\zeta(f,r)$ be the number of zeros of an entire function $f : \mathbb{C} \to \mathbb{C}$ inside the ball B_r. Then, provided $f(0) \neq 0$, one has for every $a > 1$

$$\zeta(f,r) \leq C \log \mu(f,ar) \ \ \forall r > 0 , \tag{1}$$

where C is a positive constant depending on a and $f(0)$. For instance, in Example 1.1 both ζ and $\log \mu$ grow linearly in r.

These two features might have given a hope that $\log \mu(r)$ is an appropriate substitute of the degree for an entire map $f : \mathbb{C}^n \to \mathbb{C}^n$ (this was known as the transcendental Bézout problem). However, this analogy was overturned by Cornalba and Shiffman [11] who famously constructed, for $n = 2$, an entire map f with $\log \mu(f,r) \leq C_\epsilon r^\epsilon$ for every $\epsilon > 0$ (and hence of growth *order* zero), with $\zeta(f,r)$ growing arbitrarily fast. As Griffiths wrote in [15] *"This is the first instance known to this author when the analogue of a general result in algebraic geometry fails to hold in analytic geometry."*

1.2. **Coarse zero count.** One of the motivations for the present paper is to further explore the Cornalba–Shiffman example using the notion of *coarse zero count* introduced in [7], which is based on topological persistence. The idea, roughly speaking, is to discard the zeros corresponding to small oscillations of the map. It turns out that with such a count we are able to get a Jensen-type estimate (1), albeit with a somewhat worse power of $\log \mu(r)$ in the right-hand side, see (2) below.

Given an analytic map $f : \mathbb{C}^n \to \mathbb{C}^n$ and positive numbers $\delta, r > 0$, we define the counting function $\zeta(f, r, \delta)$ of δ-*coarse zeros* of f inside a ball B_r as the number of connected components of the set $f^{-1}(B_\delta) \cap B_r$ which contain zeros of f, see Figure 1.

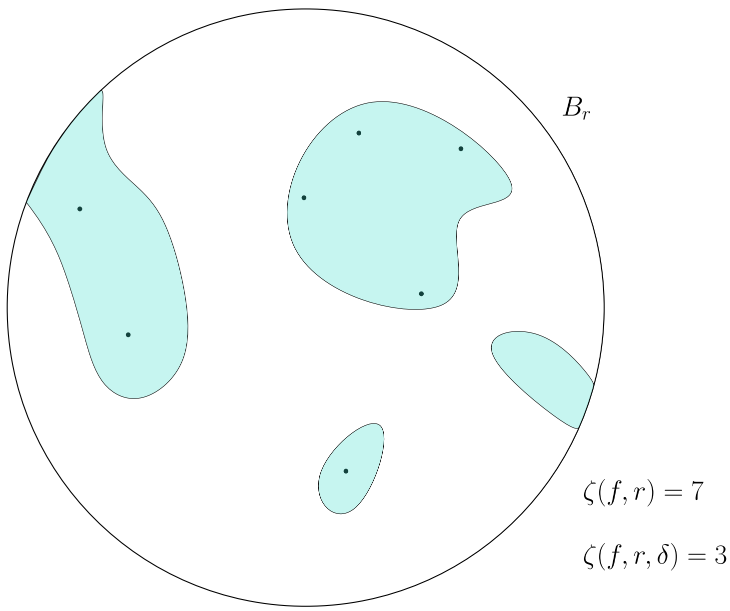

FIGURE 1. Dots represent zeros of f, while shaded regions depict the set $f^{-1}(B_\delta)$

Theorem 1.2. *For any analytic map $f : \mathbb{C}^n \to \mathbb{C}^n$ and any $a > 1$, $r > 0$, and $\delta \in (0, \frac{\mu(f,ar)}{e})$, we have*

$$\zeta(f, r, \delta) \leq C \left(\log \left(\frac{\mu(f, ar)}{\delta} \right) \right)^{2n-1}, \tag{2}$$

where the constant C depends only on a and n.

Note that by Liouville's theorem, unless f is constant, $\mu(f, ar)$ is unbounded. Therefore, for any given $\delta > 0$, the condition $\delta \in (0, \mu(f, ar)/e)$ holds for all r large enough.

Remark 1.3. Consider a higher-dimensional generalization of Example 1.1: take an analytic map $f : \mathbb{C}^n \to \mathbb{C}^n$ given by

$$f(z_1, \dots, z_n) = (e^{z_1} + 1, \dots, e^{z_n} + 1).$$

It is easy to see that $\log \mu(f, r)$ grows linearly in r and $\zeta(f, r, \delta)$ grows as r^n when $r \to \infty$, for δ sufficiently small. It would be interesting to understand whether the power of the logarithm in (2) is sharp or it can be improved, possibly, to n.

It follows from Theorem 1.2 that for the Cornalba-Shiffman example the coarse count of zeros grows slower than any positive power of r, see Theorem 1.5 below for precise asymptotics.

Remark 1.4. Consider a function f of growth order $\leq \rho$, that is, for all $\epsilon > 0$, there exist positive constants A_ϵ, B, such that

$$|f(z)| \leq A_\epsilon e^{B|z|^{\rho+\epsilon}}$$

everywhere. Then by (2), $\zeta(f,r,\delta)$ grows slower than $r^{(2n-1)\rho+\epsilon}$ for every $\epsilon > 0$. At the same time, it was shown in [8, equation (1.9)] that for any $\alpha > 0$, $\zeta(f+c,r)$ grows slower than $r^{(2n-1)\rho+1+\alpha}$ for almost all $c > 0$ small enough. While the growth rate in (2) is slightly sharper, it is interesting to note that the power $(2n-1)\rho$ appears in both bounds.

1.3. **Cornalba–Shiffman example: a coarse perspective.** Let us remind the Cornalba–Shiffman construction. Let $g : \mathbb{C} \to \mathbb{C}$ be given by

$$g(z) = \prod_{i=1}^{\infty} \left(1 - \frac{z}{2^i}\right).$$

For $k \geq 1$ an integer, let

$$g_k(z) = \frac{g(z)}{1 - \frac{z}{2^k}}$$

be the function defined by the same product with k-th term excluded. All the infinite products converge uniformly on compact subsets of $\mathbb{C}$ and hence g and g_k are holomorphic by Weierstrass' theorem. For a positive integer c we define a polynomial $P_c : \mathbb{C} \to \mathbb{C}$ as

$$P_c(w) = \prod_{j=1}^{c} \left(w - \frac{1}{j}\right).$$

Given a strictly increasing sequence of positive integers $\mathfrak{c} = \{c_i\}$, $c_1 < c_2 < \ldots$ define $f : \mathbb{C}^2 \to \mathbb{C}$ as

$$f(z,w) = \sum_{i=1}^{\infty} 2^{-c_i^2} g_i(z) P_{c_i}(w).$$

f converges uniformly on compact sets and is hence holomorphic by Weierstrass' theorem in several variables. Finally, we define a map $F : \mathbb{C}^2 \to \mathbb{C}^2$, $F(z,w) = (g(z), f(z,w))$. As shown in [11], for all $\mathfrak{c}$, F is of order zero. However, the zero set of F is given by

$$F^{-1}(0) = \left\{ \left(2^i, \frac{1}{j}\right) \mid i = 1,2,\ldots; j = 1,\ldots,c_i \right\},$$

as depicted in Figure 2. The dots represent zeros of F and the number of zeros $\zeta(F,r) = \zeta(F,r,0)$ equals the number of dots inside the circle.

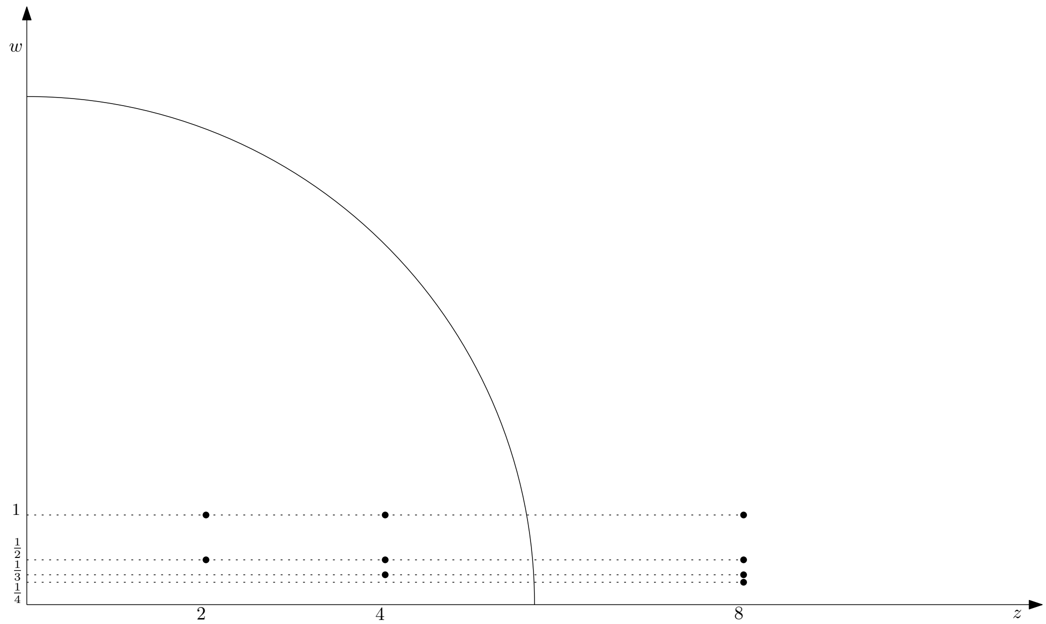

FIGURE 2. Classical count of zeros

We now see that by taking $\mathfrak{c}$ which increases sufficiently fast $\zeta(F,r)$ can grow arbitrarily fast which disproves the two-dimensional transcendental Bézout problem. More precisely, in [11] Cornalba and Shiffman made a remark that $c_i = 2^{2^i}$ would suffice. Indeed, it is not difficult to check that for $\lambda > 0$, if $c_i = \lfloor 2^{\lambda i} \rfloor$ then $\zeta(F,r) = \Theta(r^\lambda)$ i.e. the order of growth of the number of zeros is λ, while for $c_i = 2^{2^i}$, $\log_2 \zeta(F,r) = \Theta(r)$ and the order of growth of $\zeta(F,r)$ is infinite. Here and further on we write $a(r) = \Theta(b(r))$ if $a(r) = O(b(r))$ and $b(r) = O(a(r))$ as $r \to \infty$; we will also write $a(r) \sim b(r)$ if $\lim_{r\to\infty} a(r)/b(r) = 1$.

Let us re-examine the same class of examples from the coarse point of view.

Theorem 1.5. *Let $\mathfrak{c}$ be an arbitrary increasing sequence of positive integers. When $r \to +\infty$ it holds*

$$\log \mu(F,r) = \Theta((\log_2 r)^2),$$

and for a fixed $\delta > 0$

$$\zeta(F,r,\delta) \sim \log_2 r.$$

Let us explain the geometric picture behind Theorem 1.5, while referring the reader to Section 5 for detailed proofs. For a fixed δ, we show

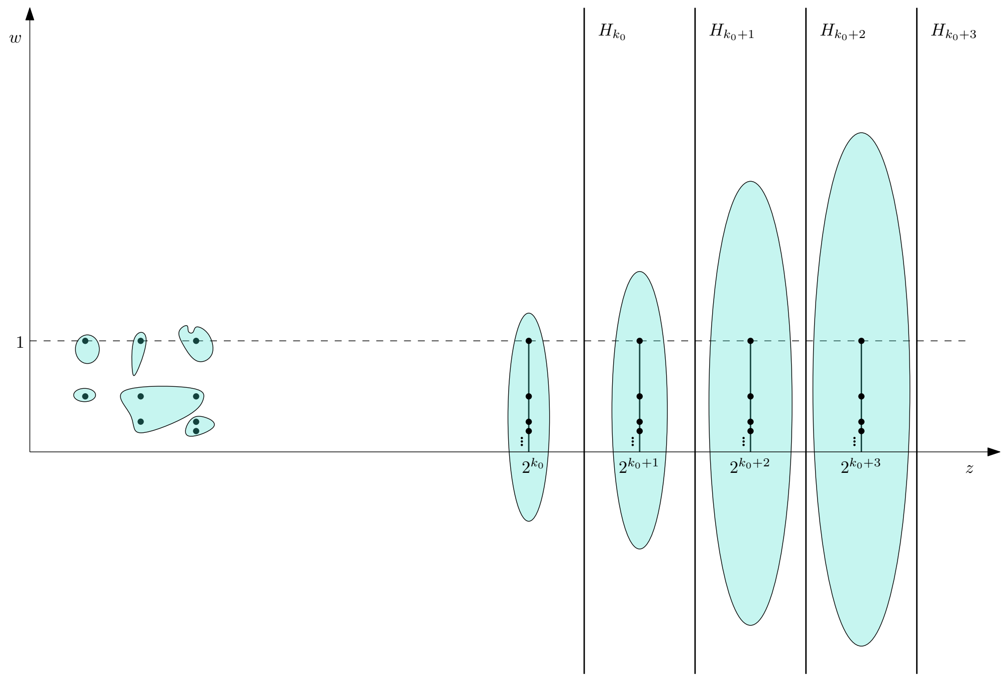

FIGURE 3. Coarse count of zeros

that the set $\{|F| \leq \delta\}$, while possibly being complicated for small radius r, stabilizes for large radii and can be described rather accurately. More precisely, we show that there exists k_0, which depends only on δ, such that $\{|F| \leq \delta\}$ contains intervals $\{2^k\} \times [0,1]$ for all $k \geq k_0$. Thus, for $k \geq k_0$ the zeros on each of the intervals $\{2^k\} \times [0,1]$ are counted coarsely as one zero and the coarse count increases at the rate $\log r$. This implies $\zeta(F,r,\delta) = O(\log r)$. Furthermore, for $k \geq k_0$, $\{|F| \leq \delta\}$ will never intersect hyperplanes $H_k = \{(w,z) \mid Re(z) = 2^k + 2^{k-1}\}$. In other words, $\{|F| \leq \delta\}$ consists of parts contained between those hyperplanes and which contain intervals $\{2^k\} \times [0,1]$, as shown on Figure 3 (shaded regions represent the set $\{|F| \leq \delta\}$). This implies that $\zeta(F,r,\delta) = \Theta(\log r)$, which we can improve to $\zeta(F,r,\delta) \sim \log r$ as claimed by Theorem 1.5.

Putting together Theorems 1.2 and 1.5, we come to the following conclusion. It follows from Theorem 1.2 for $n = 2$ that

$$\zeta(F,r,\delta) \leq C_a(\log \mu(F,ar) - \log \delta)^3.$$

On the logarithmic scale, this inequality tells us that for fixed a and δ,

$$\log \zeta(F,r,\delta) = O\left(\log\log \mu(F,ar)\right), \tag{3}$$

when $r \to +\infty$. Theorem 1.5 implies that (3) is asymptotically sharp as

$$\log \zeta(F, r, \delta) = \Theta(\log\log \mu(F, ar)). \tag{4}$$

In other words, Cornalba–Shiffman examples exhibit highly oscillatory behaviour on small scales, which increases the count of zeros in an uncontrollable way and contradicts the transcendental Bézout problem. However, equality (4) shows that if we discard small oscillations, the same examples behave essentially as predicted by the coarse version of the transcendental Bézout problem.

1.4. **Islands vs peninsulas.** A connected component of $f^{-1}(B_\delta) \cap B_r$ is called an *island* if it is disjoint from $S_r = \partial B_r$ and a *peninsula* otherwise. It is not hard to see that every island contains at least one zero of f, see Remark 3.2. Let $\zeta^0(f, r, \delta)$ denote the number of islands, and let $\tau(f, r, \delta)$ denote the total number of zeros of f with multiplicities *contained in islands* in B_r. Since an island can contain more than one zero, clearly

$$\zeta^0(f, r, \delta) \le \tau(f, r, \delta).$$

The following result is a consequence of Rouché's theorem for analytic mappings.

Theorem 1.6. *For all $a > 1$, $r > 0$, and $\delta \in (0, \mu(f, ar)/e)$*

$$\tau(f, r, \delta) \le C_1\big(\log(\mu(f, ar)/\delta)\big)^n, \tag{5}$$

where C_1 depends only on a and n.

Note that in view of Remark 1.3, estimate (5) is sharp.

Remark 1.7. Estimates analogous to (5) for the usual count of zeros have been proven under positive lower bounds on the Jacobian of f in [18, 19, 20]. Upper bounds in Theorems 1.2 and 1.6 apply to all holomorphic mappings $f : \mathbb{C}^n \to \mathbb{C}^n$. A detailed comparison of our results with those in [18, 19, 20] are carried out in Section 7. In particular, we give a different proof of a result from [20] using Theorem 1.6.

The proof of Theorem 1.6 does not use persistence techniques and does not extend to an estimate on $\zeta(f, r, \delta)$, as it does not control peninsulas. In general, $\zeta(f, r, \delta)$ and $\zeta^0(f, r, \delta)$ can behave rather differently. Indeed, this is the case for the Cornalba–Shiffman example. as we discuss below.

Let F be a Cornalba-Shiffman map defined previously. It is natural to ask what is the possible growth of the coarse count of islands $\zeta^0(F, r, \delta)$. We show that, as opposed to $\zeta(F, r, \delta)$, $\zeta^0(F, r, \delta)$ can grow arbitrarily slow, with an upper bound depending on $\mathfrak{c}$. More precisely, we prove the following theorem.

Theorem 1.8. *Let $\lambda, \delta > 0$, $l \geq 1$ an integer and denote by $\exp_2(x) = 2^x$. If $c_i = \lfloor \underbrace{\exp_2 \ldots \exp_2}_{l \text{ times}}(\lambda i) \rfloor$ then there exists a constant $m_{l,\lambda,\delta}$ such for all $r \geq \underbrace{\exp_2 \ldots \exp_2}_{l+1 \text{ times}}(1)$ it holds*

$$\zeta^0(F, r, \delta) \leq \frac{1}{\lambda} \underbrace{\log_2 \ldots \log_2}_{l+1 \text{ times}} r + m_{l,\lambda,\delta}.$$

In particular, for $c_i = 2^{2^i}$ as in [11], we have that $\zeta^0(F, r, \delta) = O(\log_2 \log_2 \log_2 r)$.

From the geometric perspective, the slow growth of $\zeta^0(F, r, \delta)$ is due to elongation of $\{|F| \leq \delta\}$ in the w-direction. Namely, as r increases, new groups of zeros of roughly the same modulus r appear, while the components of $\{|F| \leq \delta\}$ which contain these zeros grow in the w-direction faster than r (the diameter of their w-projection grows faster than r). Hence, it takes larger r for a component of $\{|F| \leq \delta\}$ to be fully contained in B_r i.e. to contribute to $\zeta^0(F, r, \delta)$.

1.5. **Discussion.** Below we discuss some extensions of our results as well as directions for further research.

1.5.1. *Analytic mappings from $\mathbb{C}^n$ to $\mathbb{C}^k$.* We expect that a bound analogous to Theorem 1.2 should hold for entire mappings $f : \mathbb{C}^n \to \mathbb{C}^k$. However, to have geometric meaning in this case, the definition of $\zeta(f, r, \delta)$ should be generalized. There are two essentially dual directions of doing so. First, we can look at coarse homology groups of the zero set: for $0 \leq d \leq 2n-1$, set

$$\zeta_d(f, r, \delta) = \dim \operatorname{Im}\Big(H_d(\{f = 0\} \cap B_r) \to H_d(\{|f| \leq \delta\} \cap B_r)\Big).$$

Considering generic algebraic maps f, we expect only $0 \leq d \leq n-k$ to have geometric significance. Of particular interest is $d = n-k$, since this is the dimension where vanishing cycles appear. We expect the upper bound

$$\zeta_d(f, r, \delta) \leq C\Big(\log(\mu_{ar}(f)/\delta)\Big)^{2n}.$$

Second, we may set

$$m_d(f, r, \delta) = \dim \operatorname{Im}\Big(H_d(\{|f| > \delta\} \cap B_r) \to H_d(\{|f| > 0\} \cap B_r)\Big).$$

We expect the upper bound

$$m_d(f, r, \delta) \leq C\Big(\log(\mu_{ar}(f)/\delta)\Big)^{2n}$$

for $0 \leq d \leq 2n-1$.

1.5.2. *Affine varieties.* It would be interesting to generalize our main results to more general affine algebraic varieties $Y \subset \mathbb{C}^N$. The starting case would be varieties which compactify to smooth projective varieties $X \subset \mathbb{C}P^N$ by a normal crossings divisor $D = X \setminus Y$. We expect that the methods of [10] combined with those of [7], in particular the subadditivity theorem for persistence barcodes, should be useful for this purpose.

1.5.3. *Harmonic mappings.* A result analogous to Theorem 1.2 but in the context of harmonic maps can be proven in a similar way. Namely, let $h = (h_1, \ldots, h_d) : \mathbb{R}^d \to \mathbb{R}^d$ be a harmonic map in the sense that $h_j : \mathbb{R}^d \to \mathbb{R}$ is harmonic for all j. In this case, the coarse counts $\zeta(h,r,\delta), \zeta^0(h,r,\delta)$ being defined analogously to the above, should satisfy the upper bound

$$\zeta^0(h,r,\delta) \leq \zeta(h,r,\delta) \leq C_2\Big(\log(\mu_{ar}(h)/\delta)\Big)^d , \tag{6}$$

for all $a > C_1$, $r > 0$, and $\delta \in (0, \mu_{ar}(h)/e)$, where C_1 depends on d only, and C_2 depends on a and d, but not on r or δ. (By Liouville's theorem the condition on δ holds for all r large enough, if our mapping is not constant.) This bound is sharp asymptotically in r for all fixed $\delta > 0$, as can be easily seen from the example $h = (h_1, \ldots, h_d)$ where $h_i(x_1, \ldots, x_d) = (e^{x_{i+1}} \sin(x_i))$ for $1 \leq i \leq d-1$ and $h_d(x_1, \ldots, x_d) = (e^{x_1} \sin(x_d))$. Note that $\log(\mu_{ar}(h))$ is closely related to the notion of the doubling index of the harmonic function (see for example [21, Equation (12)]).

SInce our proof of inequality (6) is based on Cauchy estimates, as in Proposition 4.1, and bounds on barcodes of polynomials as in Proposition 3.1 (see also [7, Proposition 4.12]), it is not hard to extend (6) to a certain quasi-analytic class of functions.

1.5.4. *Near-holomorphic mappings.* Let us also note that Theorem 1.2 implies the following result about the count of islands for continuous functions that are close to holomorphic ones.

Corollary 1.9. *Fix $b < 1$ and $\delta > 0$, and let $h : \mathbb{C}^n \to \mathbb{C}^n$ be a continuous function such that there exists a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}^n$ with $d_{C^0}(h,f) < \frac{b}{2}\delta$. Then*

$$\zeta(h,r,(1+b)\delta) \leq C\Big(\log(\mu(h,ar)/\delta)\Big)^{2n-1} ,$$

where C depends on a, b, n only.

1.5.5. *A dynamical interlude.* A dynamical counterpart of the transcendental Bézout problem is the count of periodic orbits of entire maps $f : \mathbb{C}^n \to \mathbb{C}^n$. Here by a k-periodic orbit we mean a fixed point of the iteration $f^{\circ k} = f \circ \cdots \circ f$ (k times). There exists a vast literature on the orbit growth of algebraic maps f (see e.g. [2]). For instance, it follows from the Bézout theorem that if the components of f are generic polynomials of degree $\le d$, the number of k-periodic orbits does not exceed d^{kn}. Can one expect a bound on the number of k-periodic orbits in the ball of radius r in terms of the maximum modulus function $\mu(f,r)$? The naive answer is "no" due to the the Cornalba-Shiffman example. Nevertheless, Theorem 1.2 above readily yields such a bound on the *coarse count* $\zeta(f_k, r, \delta)$, where

$$f_k(z) := f^{\circ k}(z) - z \ .$$

One can check that the maximum modulus function behaves nicely under the composition and the sum:

$$\mu(f \circ g, r) \le \mu(f, \mu(g,r)), \quad \mu(f+g, r) \le \mu(f,r) + \mu(g,r) \ .$$

Fix $a > 1$ and $\delta > 0$, set $\tilde{\mu}(r) := \mu(f, ar)$, and put

$$\mu_k(r) = \tilde{\mu}^{\circ k}(r) + r \ .$$

By Theorem 1.2 we have the desired estimate

$$\zeta(f_k, r, \delta) \le C \max\left(\log\left(\frac{\mu_k(r)}{\delta}\right)^{2n-1}, 1\right) \ . \tag{7}$$

A few questions are in order.

Question 1.10. *Can one find a transcendental entire map f for which estimate* (7) *is sharp?*

A natural playground for testing this question are transcendental Hénon maps whose entropy as restricted to a family of concentric discs grows arbitrarily fast [1].

Further, recall that a k-periodic orbit of an entire map $f : \mathbb{C}^n \to \mathbb{C}^n$ is called *primitive* if it is not m-periodic with $m < k$. Denote by $\nu_k(f,r)$ the number of primitive periodic orbits lying in the ball of radius r.

Question 1.11. *Does there exist a transcendental entire map f of order* 0 *(i.e., the modulus $\mu(f,r)$ grows slower than e^{r^ϵ} for every $\epsilon > 0$) such that $\nu_k(f,r)$ grows arbitrarily fast in k and r?*

For instance, taking $f(z) = h(z) + z$, where h is the Cornalba-Shiffman map, we see that $\nu_1(f,r)$ can grow arbitrarily fast. Can one generalize this construction to $k \ge 2$?

Finally, let us mention that the failure of the transcendental Bézout theorem appears as one of the substantial difficulties in the work [16] dealing with a dynamical problem of a completely different nature, namely with embeddings of $\mathbb{Z}^k$-actions into the shift action on the infinite dimensional cube (see (2) on p. 1450 in [16]). In particular, the authors analyze the structure of zeroes of so-called *tiling-like band-limited maps* (see p.1477 and Lemma 5.9). It would be interesting to perform our coarse count of zeroes (i.e., to calculate ζ) for this class of examples.

1.5.6. *Multiparameter persistence.* An entire map $f : \mathbb{C}^n \to \mathbb{C}^n$ naturally gives rise to a persistence module $H_*(\{|f| \leq t\} \cap B_r)$ in two parameters r and t. In this paper we considered it as an r-parametrized family of persistence modules with one parameter t. It would be interesting to study this persistence module from the viewpoint of multiparameter persistence, see [4] and references therein, for example by using the recently introduced language of signed barcodes, see [5, 6, 23].

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2. Persistence modules and barcodes

Recall that for a Morse function $f : M \to \mathbb{R}$ on a compact manifold and a coefficient field $\mathbb{K}$, its *barcode* in degree $q \in \mathbb{Z}$ is a finite multi-set $\mathscr{B}_q(f;\mathbb{K})$ of intervals with multiplicities (I_j, m_j), where $m_j \in \mathbb{N}$ and I_j is finite, that is of the form $[a_j, b_j)$ or infinite, that is of the form $[c_j, \infty)$. The number of infinite bars is equal to the Betti number $b_q(M;\mathbb{K}) = \dim H_q(M;\mathbb{K})$.

This barcode is obtained algebraically from the *persistence module* $V_q(f;\mathbb{K})$ consisting of vector spaces $V_q(f;\mathbb{K})^t = H_q(\{f \leq t\};\mathbb{K})$ parametrized by $t \in \mathbb{R}$ and connecting maps $\pi^{s,t} : V_q(f;\mathbb{K})^s \to V_q(f;\mathbb{K})^t$ induced by the inclusions $\{f \leq s\} \hookrightarrow \{f \leq t\}$ for $s \leq t$. These maps satisfy the structure relations of a persistence module: $\pi^{s,s} = \mathrm{id}_{V_q(f;\mathbb{K})^s}$ for all s and $\pi^{s_2,s_3} \circ \pi^{s_1,s_2} = \pi^{s_1,s_3}$ for all $s_1 \leq s_2 \leq s_3$. The total barcode of f is set to be

$$\mathscr{B}(f;\mathbb{K}) = \sqcup_{q\in\mathbb{Z}} \mathscr{B}_q(f;\mathbb{K})$$

where $\sqcup$ stands for the sum operation on multisets. This is the barcode of the persistence module

$$V(f;\mathbb{K}) = \oplus_{q\in\mathbb{Z}} V_q(f;\mathbb{K}).$$

On a compact manifold M with boundary ∂M and a Morse function $f : M \to \mathbb{R}$ in the sense of manifolds with boundary, we may define the persistence module and barcode of f as above.

One simple property of the barcode of this persistence module is that for $x \in \mathbb{R}$ the number of bars starting at x is $\dim \operatorname{coker}(\pi^{x-\epsilon,x+\epsilon})$ for $\epsilon > 0$ sufficiently small. Moreover, the number of bars in the barcode clearly coincides with the number of their starting points. Another property is that the number of bars containing a given interval $[a, b]$ is $\dim \operatorname{Im}(\pi^{a,b})$.

Recall that the length of a finite bar $[a,b)$ is $b-a$ and the length of an infinite bar $[c,\infty)$ is $+\infty$. We require the following notion: for $\delta \geq 0$, let $\mathcal{N}_\delta(f;\mathbb{K})$ denote the number of bars of length $>\delta$ in the barcode $\mathcal{B}(f;\mathbb{K})$. Similarly $\mathcal{N}_{q,\delta}(f;\mathbb{K})$ is the number of bars of length $>\delta$ in the barcode $\mathcal{B}_q(f;\mathbb{K})$ and $\mathcal{N}_\delta(f;\mathbb{K}) = \sum_q \mathcal{N}_{\delta,q}(f;\mathbb{K})$.

We refer to [24] for a systematic introduction to persistence modules with a view towards applications in topology and analysis. The only result which we require here is the following direct consequence of the algebraic isometry theorem [3] (see [24, Theorem 2.2.8, Equation (6.4)]).

Theorem 2.1. *Let $f,g : M \to \mathbb{R}$ two functions on a compact manifold M with boundary such that $d_{C^0}(f,g) \leq c - \epsilon/2$ for $c > \epsilon/2 > 0$. Then for all $q \in \mathbb{Z}$, $\mathcal{N}_{q,2c}(f) \leq \mathcal{N}_{q,\epsilon}(g)$.*

3. Barcode for a complex polynomial mapping on a ball

In this section we prove the following result which provides necessary bounds on the number of degree zero bars in the barcode of the norm of an equidimensional polynomial mapping over a ball.

Proposition 3.1. *Let $k \geq 1$. Let $p_1,\dots,p_n : \mathbb{C}^n \to \mathbb{C}$ be complex polynomials of degree at most k. Set $p : \mathbb{C}^n \to \mathbb{C}^n$, $p = (p_1,\dots,p_n)$ for the induced polynomial mapping. Let B be a ball in $\mathbb{C}^n$. Then for all $\delta > 0$, the number $\mathcal{N}_{0,\delta}(|p|)$ of degree 0 bars of length at least δ in the barcode of $|p|$ on the ball B satisfies*

$$\mathcal{N}_{0,\delta}(|p|) \leq C_n k^{2n-1}$$

for a constant C_n depending on n only. Moreover, the degree 0 barcode of $|p|$ is finite with $\mathcal{N}_{0,0}(|p|) \leq C_n k^{2n-1}$ finite bars.

Proof. We start the proof by perturbing p to another polynomial mapping $g = (g_1,\dots,g_n)$ with $d_{C^0(Q)}(p,g) < \delta/2$ and $\deg(g_j) \leq k_1 = Ak + B$ for constants $A \geq 1$, $B \geq 0$ depending only on n, for all j. Moreover, we choose g such that $h = |g|^2$ is a Morse function on B in the sense of manifolds with boundary: in particular, the critical points of h lie in the interior of B, where h is Morse, and $h|_{\partial B}$ is Morse (see [17] for a slightly more general notion). In addition we choose $g : \mathbb{C}^n \to \mathbb{C}^n$ to be a proper mapping. It is sufficient to prove that the number of degree 0 bars in the barcode $\mathcal{B}(h;\mathbb{K})$ of h is at most $C_n k_1^{2n-1}$. In order to do this we shall estimate the number of boundary critical points first and subsequently the number of interior critical points. That this produces the required upper bound is classical Morse theory for interior critical points together with [17, Theorem 8] for critical points on the boundary.

First, by assumption, there exists a point $N \in \partial B$ which is regular for $h|_{\partial B}$. Consider complex linear coordinates $(z_1,\dots,z_n)$, $z_j = x_{2j-1} + i x_{2j}$, in which B is the unit ball, so ∂B is the unit sphere, and N is the base vector $(0,\dots,0,i)$. Note that h is a polynomial of degree at most $2k_1$ in these coordinates. Following an idea of Nonez, we consider inverse stereographic projection $\theta : \mathbb{R}^{2n-1} \to \partial B \setminus \{N\}$, $\theta(u_1,\dots,u_{2n-1}) = (x_1,\dots,x_{2n})$, $x_j = \frac{2u_j}{|u|^2+1}$ for $1 \le j \le 2n-1$, $x_{2n} = \frac{|u|^2-1}{|u|^2+1}$. Then $\theta^* h = \frac{q}{(|u|^2+1)^{2k_1}}$ for a polynomial q in $u_1,\dots,u_{2n-1}$ of degree at most $4k_1$. The critical points of $\theta^* h$ are in bijection with those of $h|_{\partial B}$, and are given by $2n-1$ polynomial equations $\partial_{u_j} q(u)(|u|^2+1) - 4k_1 u_j q(u) = 0$, $1 \le j \le 2n-1$, each one of degree at most $4k_1 + 1 \le 5k_1$. In total, by an estimate of Milnor [22] we obtain that there are at most $(5k_1)^{2n-1} = C_n k^{2n-1}$ such critical points.

It remains to estimate the number of degree 0 critical points in the interior of B. Such critical points are local minima of $|g|^2$ and hence of $|g|$. Since g is proper, it is finite, and as it is equidimensional it is therefore an open mapping [12, Proposition 3, Section 2.1.3]. Therefore the interior local minima of $|g|$ are necessarily zeros of g. Hence their number is bounded by the number of the connected components of the zero variety $Z_g = g^{-1}(0)$. In turn, this number is bounded by k_1^n in view of Bézout's theorem (see [13, Example 8.4.6] for example). This finishes the proof of the main part of the theorem.

The moreover part follows immediately, since the estimate on $\mathcal{N}_{0,\delta}(|p|)$ is uniform in δ.

□

Remark 3.2. In fact, we did not need to arrange g to be proper. Indeed, an interior Morse minimum p of $h = |g|^2$ must be a zero of g. If $g(p) \neq 0$, then $d_p h = 0$ implies that $d_p g : \mathbb{C}^n \to \mathbb{C}^n$ has non-trivial kernel K. This implies that the Hessian of h restricted to K is the real part of a complex quadratic form, and hence has signature zero. Therefore it cannot be positive definite. Note that the same argument can be applied to show that for a holomorphic function, every island contains at least one zero.

4. Proof of Theorem 1.2

We first recall a version of the classical Cauchy estimates for complex analytic mappings.

Proposition 4.1. *Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be a complex analytic mapping, $a > 1$, and $R_k = f - p_k$ be the Taylor remainder for the approximation of f by the Taylor polynomial mapping p_k at 0 of degree $< k$. Then for all $r > 0, k \ge 0$,*

$$\mu_r(R_k) \le C_a a^{-k} \mu_{ar}(f)$$

for the constant $C_a = \frac{a}{a-1}$ depending only on a.

We give a proof for clarity.

Proof. Suppose first that $m = 1$. Let $v \in S^{2n-1} \subset \mathbb{C}^n$ and $u \in B_r(\mathbb{C}) \subset \mathbb{C}$. Write a point z in $B_r = B_r(\mathbb{C}^n)$ as $z = uv$. Then $R_k(z) = f(uv) - p_k(uv)$. Now take $v' \in S^{2n-1}$ and set $g(u) = \langle f(uv), v' \rangle$, $q_k(u) = \langle p_k(uv), v' \rangle$. Then q_k is the Taylor polynomial of g of degree $< k$ and $r_k = g - q_k$ is the corresponding Taylor remainder. It is enough to bound $r_k(u)$ uniformly in v, v' for $u \in B_r(\mathbb{C})$.

Let $0 < r < \rho$. The Cauchy integral formula yields for $|w| = \rho$, $|u| = r$ that

$$r_k(u) = \frac{1}{2\pi i} \int_{S^1_\rho} g(w) \Big(\frac{u}{w}\Big)^k \frac{1}{w-u} dw.$$

Therefore

$$|r_k(u)| \leq \Big(\frac{r}{\rho}\Big)^k \frac{\rho}{\rho - r} \mu_\rho(g)$$

and picking $\rho = ar$, we get

$$|r_k(u)| \leq C_a a^{-k} \mu_{ar}(g) \leq C_a a^{-k} \mu_{ar}(f).$$

So taking maxima over u, v and v', we obtain

$$\mu_r(R_k) \leq C_a a^{-k} \mu_{ar}(f).$$

□

Proof of Theorem 1.2. Let us now suppose that $f = (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a complex analytic self-mapping. Let $h = |f|$. Proposition 4.1 implies that on the ball B_r of radius r, there exists a polynomial mapping $p_k : \mathbb{C}^n \to \mathbb{C}^n$ of degree $< k$ such that

$$d_{C^0}(f, p_k) < C_a a^{-k} \mu_{ar}(f).$$

In particular for $k = \lceil \log_a(\mu_{ar}(f)/\delta) \rceil = \lceil \log(\mu_{ar}(f)/\delta)/\log(a) \rceil$, which makes sense as a positive integer since $a > 1$,

$$d_{C^0}(f, p_k) < C_a \delta,$$

and therefore

$$d_{C^0}(h, |p_k|) < C_a \delta.$$

Hence by Theorem 2.1 and Proposition 3.1 from Section 3

$$\mathcal{N}_{0,C'_a\delta}(h) \leq \mathcal{N}_{0,\epsilon}(|p_k|) \leq C_n k^{2n-1} \leq C_{n,a}\Big(\log(\mu_{ar}(f)/\delta)\Big)^{2n-1},$$

for $C'_a = 2C_a$, $C_{n,a} = C_n((\log(a))^{-1} + 1)^{2n-1}$ and suitable small $\epsilon > 0$. In turn by a change of variable

$$\mathcal{N}_{0,\delta}(h) \leq C'_{n,a}\Big(\log(\mu_{ar}(f)/\delta)\Big)^{2n-1},$$

for $C'_{n,a} = (\log(C'_a)+1)^{2n-1}C_{n,a}$. We used the condition $\mu_{ar}(f)/\delta > e$ in order to absorb additive terms into suitable multiplicative terms in both inequalities.

Now, $\zeta(r,f,\delta) \le \mathcal{N}_{0,\delta}(h)$ by definition of the persistence module of h. $\square$

5. Coarse analysis of the Cornalba-Shiffman example

The goal of this section is to prove Theorem 1.5. We will break down its proof into Propositions 5.1, 5.4 and 5.6. Since there are no zeros of F when $r < 2$ we always assume $r \ge 2$. Firstly, we estimate $\mu_r(F)$ as needed for the first part of Theorem 1.5.

Proposition 5.1. *For all $\mathfrak{c}$ and all $r \ge 2$ it holds*

$$\frac{1}{2}(\log_2 r)^2 - \frac{3}{2}\log_2 r + 1 \le \log_2 \mu_r(F) \le \frac{3}{2}(\log_2 r)^2 + \frac{7}{2}\log_2 r + C,$$

where $C = 4 + \log_2\left(\prod_{i=1}^{\infty}(1+2^{-i})\right)$.

Proof. Let $k \ge 1$ be an integer such that $2^k \le r < 2^{k+1}$. To prove the first inequality it is enough to take $(z,w) = (-r,0)$. Now

$$|F(-r,0)| \ge |g(-r)| \ge |g(-2^k)| = \prod_{i=1}^{\infty}(1+2^{k-i}),$$

and

$$\prod_{i=1}^{\infty}(1+2^{k-i}) > \prod_{i=1}^{k}(1+2^{k-i}) = \prod_{j=0}^{k-1}(1+2^j) > 2^{\frac{k(k-1)}{2}} > \left(\frac{r}{2}\right)^{\frac{k-1}{2}}.$$

Since $\log_2 r < k+1$ we have that $\frac{k-1}{2} > \frac{\log_2 r - 2}{2}$ and hence $|F(-r,0)| \ge \left(\frac{r}{2}\right)^{\frac{\log_2 r}{2}-1}$. Applying logarithm to both sides proves the first inequality.

To prove the second inequality we firstly notice that if $|z| \le r$ then

$$|g(z)| \le \prod_{i=1}^{\infty}(1+2^{-i}|z|) < \prod_{i=1}^{\infty}(1+2^{k+1-i}) = \prod_{j=0}^{k}(1+2^j)\cdot\prod_{i=1}^{\infty}(1+2^{-i}).$$

Denoting $C_1 = \prod_{i=1}^{\infty}(1+2^{-i})$ and estimating

$$\prod_{j=0}^{k}(1+2^j) \le \prod_{j=1}^{k+1}2^j = 2^{\frac{(k+1)(k+2)}{2}} \le (2r)^{\frac{k+2}{2}} \le (2r)^{\frac{\log_2 r}{2}+1},$$

yields

$$|g(z)| < C_1(2r)^{\frac{\log_2 r}{2}+1}. \tag{8}$$

Similarly,

$$|g_i(z)| < C_1(2r)^{\frac{\log_2 r}{2}+1},$$

for all $i \geq 1$. Using this inequality we further estimate that if $|(z,w)| \leq r$ then

$$|f(z,w)| \leq \sum_{i=1}^{\infty} \frac{|g_i(z)| \cdot |P_{c_i}(w)|}{2^{c_i^2}} \leq C_1(2r)^{\frac{\log_2 r}{2}+1} \sum_{i=1}^{\infty} \frac{|P_{c_i}(w)|}{2^{c_i^2}}.$$

On the other hand

$$\sum_{i=1}^{\infty} \frac{|P_{c_i}(w)|}{2^{c_i^2}} = \sum_{i=1}^{\infty} \frac{\prod_{j=1}^{c_i} |(w-1/j)|}{2^{c_i^2}} < \sum_{i=1}^{\infty} \frac{(r+1)^{c_i}}{2^{c_i^2}} \leq \sum_{i=1}^{\infty} \frac{(r+1)^i}{2^{i^2}}.$$

To bound the last term we proceed as follows

$$\sum_{i=1}^{\infty} \frac{(r+1)^i}{2^{i^2}} = \sum_{1 \leq i \leq \log_2(r+1)} \left(\frac{r+1}{2^i}\right)^i + \sum_{i > \log_2(r+1)} \left(\frac{r+1}{2^i}\right)^i <$$

$$< \sum_{1 \leq i \leq \log_2(r+1)} (r+1)^i + \sum_{j=0}^{\infty} \frac{1}{2^j} < (r+2)^{\log_2(r+1)} + 2 < (2r)^{\log_2 2r} + 2.$$

Putting all the inequalities together, we obtain

$$|f(z,w)| < C_1(2r)^{\frac{\log_2 r}{2}+1}((2r)^{\log_2 2r} + 2). \tag{9}$$

Since $|F(z,w)| = \sqrt{|g(z)|^2 + |f(z,w)|^2}$, combining (8) and (9) proves the desired inequality. $\square$

We will now estimate $\zeta(F,r,\delta)$ from above. Before we carry out the relevant computations, let us explain the geometric intuition behind the estimate.

Zeros of F belong to intervals $\{2^k\} \times [0,1]$, $k \geq 1$. For a fixed δ, we wish to prove that there exists k_0 such that for all $k \geq k_0$, each of the intervals $\{2^k\} \times [0,1]$ is fully contained in $\{|F| \leq \delta\}$. This is the content of Corollary 5.3. Now, on each of these intervals all zeros belong to the same connected component of $\{|F| \leq \delta\}$ and are thus counted at most once in the coarse count $\zeta(F,r,\delta)$, see Figure 4. In other words, each of the intervals $\{2^k\} \times [0,1]$, $k \geq k_0$ contributes at most one to $\zeta(F,r,\delta)$ and since they appear at rate $\log_2 r$ we have that

$$\zeta(F,r,\delta) \leq \log_2 r + \text{the error term}.$$

The error term comes from zeros on intervals $\{2^k\} \times [0,1]$ for $k < k_0$ where we can not guarantee merging of zeros in $\{|F| \leq \delta\}$, i.e. we observe no coarse effects. Moreover, since k_0 depends only on δ, the error terms only depends on $\mathfrak{c}$ and δ. These considerations are formally proven in Proposition 5.4.

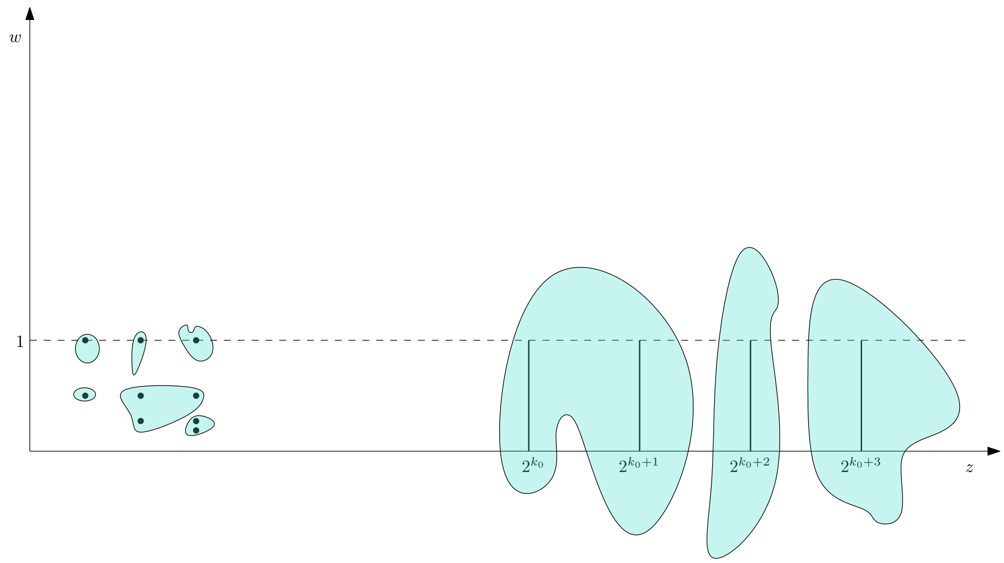

FIGURE 4. Merging of zeros starting from 2^{k_0}

Lemma 5.2. *For each $\delta \geq 2^{\frac{(i-1)i}{2}-c_i^2}$ the whole interval $\{2^i\}\times[0,2^{c_i-\frac{(i-1)i}{2c_i}}\delta^{\frac{1}{c_i}}]$ is contained in $\{|F|\leq\delta\}$.*

Proof. Firstly we notice that

$$|g_i(2^i)| = \prod_{j=1}^{i-1}(2^{i-j}-1)\cdot\prod_{j=i+1}^{\infty}(1-2^{i-j}) < \prod_{j=1}^{i-1}2^{i-j} = 2^{\frac{(i-1)i}{2}}.$$

Secondly

$$|F(2^i,w)| = |f(2^i,w)| = 2^{-c_i^2}|g_i(2^i)||P_{c_i}(w)| < 2^{\frac{(i-1)i}{2}-c_i^2}|P_{c_i}(w)|.$$

Now, if $w\in[0,1]$, $|P_{c_i}(w)|<1$ and the claim follows by the assumption on δ. If $w\in(1,2^{c_i-\frac{(i-1)i}{2c_i}}\delta^{\frac{1}{c_i}}]$, $|P_{c_i}(w)|<w^{c_i}$ and thus $|F(2^i,w)|<2^{\frac{(i-1)i}{2}-c_i^2}w^{c_i}\leq\delta$ and the claim follows. □

Corollary 5.3. *If $\delta\geq 2^{\frac{-i(i+1)}{2}}$ then the whole interval $\{2^i\}\times[0,1]$ is contained in $\{|F|\leq\delta\}$.*

Proof. Since $c_i^2\geq i^2$, $\delta\geq 2^{\frac{-i(i+1)}{2}}$ implies that $\delta\geq 2^{\frac{(i-1)i}{2}-c_i^2}$ and thus

$$\{2^i\}\times[0,1]\subset\{2^i\}\times[0,2^{c_i-\frac{(i-1)i}{2c_i}}\delta^{\frac{1}{c_i}}]\subset\{|F|\leq\delta\}$$

by Lemma 5.2. □

Proposition 5.4. *The following estimates hold for $r > 2$:*

$$\zeta(F,r,\delta) \leq \begin{cases} \sum\limits_{1\leq i\leq \log_2 r} c_i, \text{ if } 0<\delta<2^{\frac{-\log_2 r(\log_2 r+1)}{2}} \\ \log_2 r + 2 - \sqrt{-2\log_2\delta} + \sum\limits_{1\leq i<\sqrt{-2\log_2\delta}} c_i, \text{ if } 2^{\frac{-\log_2 r(\log_2 r+1)}{2}} \leq \delta < \frac{1}{2} \\ \log_2 r, \text{ if } \delta \geq \frac{1}{2} \end{cases}.$$

Proof. Firstly, we notice that for all $r \geq 2$

$$\dim H_0(F^{-1}(0)\cap B_r) = \text{ number of zeros in } B_r \leq \sum_{1\leq i\leq \log_2 r} c_i,$$

and hence $\zeta(F,r,\delta) \leq \sum\limits_{1\leq i\leq \log_2 r} c_i$ which proves the first case of the proposition.

Now, we treat the third case, i.e. $\delta \geq \frac{1}{2}$. In this case, by Corollary 5.3 all intervals $\{2^i\}\times[0,1], i \geq 1$ are contained in $\{|F| \leq \delta\}$. Thus $\dim H_0(\{|F|\leq\delta\}\cap B_r)$ equals the number of intervals $\{2^i\}\times[0,1], i \geq 1$ which intersect B_r. This number is not greater than $\log_2 r$ and thus $\zeta(F,r,\delta) \leq \log_2 r$.

Finally, we treat the second case, i.e. $2^{\frac{-\log_2 r(\log_2 r+1)}{2}} \leq \delta < \frac{1}{2}$. Denote by $r_0 > 2$ the unique real number such that $\delta = 2^{\frac{-\log_2 r_0(\log_2 r_0+1)}{2}}$. By the assumption, $r_0 \leq r$. Let $k \geq 1$ be an integer such that $2^k < r_0 \leq 2^{k+1}$. Now $2^{\frac{-(k+1)(k+2)}{2}} \leq \delta$ and hence by Corollary 5.3, $\{|F| \leq \delta\}$ contains all interval $\{2^i\}\times[0,1], i \geq k+1$. Thus

$$\dim(H_0(\{|F|\leq\delta\}\cap B_r) \leq \mathrm{I} + \mathrm{II}, \tag{10}$$

where

$$\mathrm{I} = \text{ the number of zeros in } B_r \cap \cup_{i=1}^k \{2^i\}\times[0,1],$$

and

$$\mathrm{II} = \text{ the number of intervals } \{2^i\}\times[0,1], i\geq k+1 \text{ which intersect } B_r.$$

From (10) it follows that

$$\zeta(F,r,\delta) \leq \mathrm{I} + \mathrm{II}.$$

Since $k < \log_2 r_0 < \sqrt{-2\log_2\delta}$ we have that

$$\mathrm{I} \leq \sum_{1\leq i<\sqrt{-2\log_2\delta}} c_i.$$

On the other hand, $r \geq r_0$ and thus $\log_2 r \geq \log_2 r_0 > k$ as well as

$$\mathrm{II} \leq \log_2 r - k.$$

Lastly, we use $\sqrt{-2\log_2 \delta} < \log_2 r_0 + 1 \leq k + 2$ to obtain the desired inequality. $\square$

We will now provide a lower bound for $\zeta(F, r, \delta)$. As before, we start by explaining the geometric intuition.

As r increases,new zeros of F appear on intervals $\{2^k\} \times [0, 1]$, i.e. at a rate $\log_2 r$. We wish to prove that zeros on different intervals will not be counted as one zero in the coarse count $\zeta(F, r, \delta)$. Precisely, in Lemma 5.5, we prove that for a fixed δ, there exists k_0 which depends only on δ, such that the set $\{|F| \leq \delta\}$ does not intersect any of the hyperplanes $H_k = \{Re(z) = 2^k + 2^{k-1}\}$ for $k \geq k_0$. Since H_k separates intervals $\{2^k\} \times [0, 1]$ and $\{2^{k+1}\} \times [0, 1]$, $\{|F| \leq \delta\}$ can not contain zeros from different intervals for $k \geq k_0$, see Figure 5. Similarly to the case of the upper bound, this implies that

$$\zeta(F, r, \delta) \geq \log_2 r + \text{the error term},$$

where the error term comes from zeros on intervals $\{2^k\} \times [0, 1]$ for $k < k_0$ where we can not guarantee separation of components of $\{|F| \leq \delta\}$. These considerations are formally proven in Proposition 5.6.

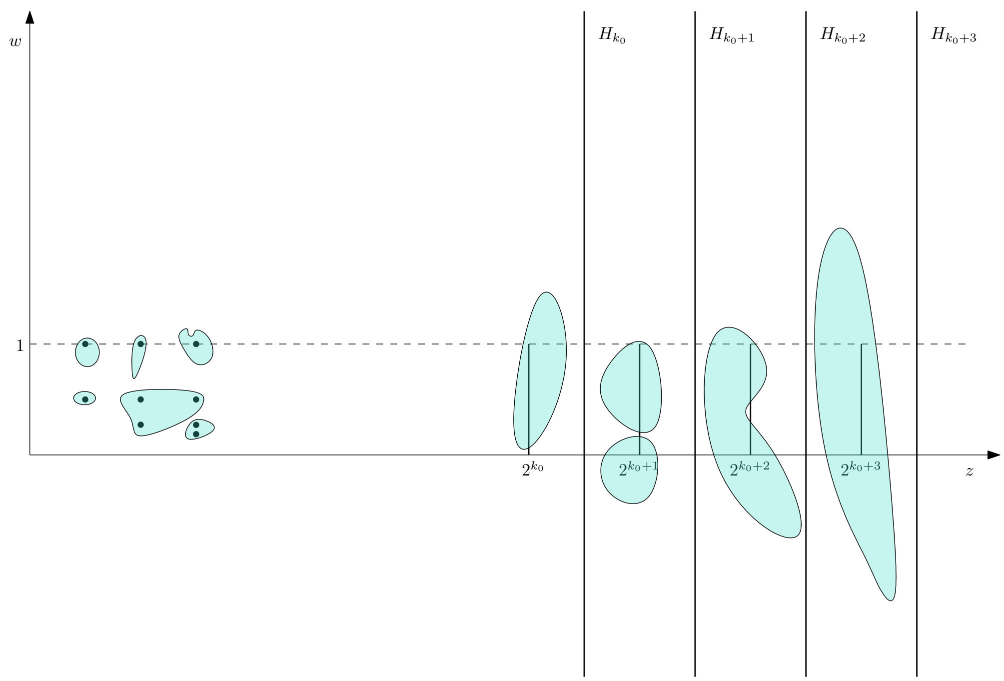

FIGURE 5. Separation of zeros starting from 2^{k_0}

In the lemma and the proposition that follow, we denote by $C_0 = \frac{1}{2}\prod_{i=1}^{\infty}\left(1 - \frac{3}{2^{i+1}}\right)$.

Lemma 5.5. *Let $z \in \mathbb{C}$ be such that $Re(z) = 2^k + 2^{k-1}$ for some integer $k \geq 1$. Then for all $w \in \mathbb{C}$ it holds*

$$|F(z,w)| > C_0 2^{\frac{(k-1)k}{2}}.$$

Proof. We estimate

$$|F(z,w)| \geq |g(z)| = \prod_{i=1}^{\infty} |1 - 2^{-i} z| \geq \prod_{i=1}^{\infty} |1 - 2^{-i} Re(z)| =$$

$$= \prod_{i=1}^{k-1} \left(\frac{2^k + 2^{k-1}}{2^i} - 1 \right) \cdot \frac{1}{2} \cdot \prod_{i=k+1}^{\infty} \left(1 - \frac{2^k + 2^{k-1}}{2^i} \right) > C_0 2^{\frac{(k-1)k}{2}}.$$

□

Proposition 5.6. *For all $r \geq 2$ it holds*

$$\zeta(F, r, \delta) \geq \begin{cases} \lfloor \log_2 r \rfloor - 1, \ \textit{if } \delta \leq C_0 \\ \lfloor \log_2 r \rfloor - \sqrt{2 \log_2 \delta - 2 \log_2 C_0} - 2, \ \textit{if } C_0 < \delta \leq C_0 r^{\frac{\log_2 r - 1}{2}} \end{cases}.$$

Proof. We first prove the case $\delta \leq C_0$. By Lemma 5.5, we see that on each hyperplane $\{Re(z) = 2^i + 2^{i-1}\}, i \geq 1$ it holds $|F(z,w)| > C_0 \geq \delta$. Hence $\{|F| \leq \delta\}$ does not intersect any of these hyperplanes and in particular zeros $(2^i, 1), i \geq 1$ all belong to different connected components of $\{|F| \leq \delta\}$. In other words

$$\zeta(F, r, \delta) \geq \text{the number of points } (2^i, 1), i \geq 1 \text{ in } B_r \geq \lfloor \log_2 r \rfloor - 1,$$

which finishes the proof of the first case.

To prove the second case, we firstly denote by $r_0 > 2$ the unique real number such that $\delta = C_0 r_0^{\frac{\log_2 r_0 - 1}{2}}$. By assumption $r_0 \leq r$ and we denote by $k \geq 1$ an integer such that $2^k \leq r_0 < 2^{k+1}$. Now $\delta < C_0 2^{\frac{k(k+1)}{2}}$ and Lemma 5.5 implies that $\{|F| \leq \delta\}$ does not intersect hyperplanes $\{Re(z) = 2^i + 2^{i-1}\}, i \geq k+1$. As in the first case, zeros $(2^i, 1), i \geq k+1$ belong to different connected components of $\{|F| \leq \delta\}$ and thus

$$\zeta(F, r, \delta) \geq \text{the number of points } (2^i, 1), i \geq k+1 \text{ in } B_r \geq \lfloor \log_2 r \rfloor - 1 - k.$$

Since $\log_2 \delta = \log_2 C_0 + \frac{1}{2} \log_2 r_0 (\log_2 r_0 - 1) > \log_2 C_0 + \frac{1}{2}(k-1)^2$, we have that $k < \sqrt{2 \log_2 \delta - 2 \log_2 C_0} + 1$ and the claim follows. □

6. Counting islands

Let us start with the proof of Theorem 1.6 which, as was mentioned in the introduction, is a an easy corollary of Rouché's theorem.

Proof of Theorem 1.6. By Rouché's theorem for analytic mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$ (see e.g. [12, Section 2.1.3]) if $g : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial mapping of degree at most k such that $d_{C^0}(f|_{B_r}, g|_{B_r}) < \delta/2$, then

$$\tau(f, r, \delta) \leq \tau(g, r, \delta/2).$$

By Bézout's theorem, however,

$$\tau(g, r, \delta/2) \leq k^n.$$

By Proposition 4.1, it is enough to take k such that $C_a a^{-k}\mu_{ar}(f) < \delta/2$. It is easy to see that the optimal such k satisfies

$$k \leq C_a' \log(\mu_{ar}(f)/\delta).$$

Combining the three displayed inequalities finishes the proof. $\square$

Before we give a formal proof of Theorem 1.8, let us briefly explain the geometric intuition as we did for the proof of Theorem 1.5. As we already explained, for a fixed δ, the sublevel set $\{|F| \leq \delta\}$ stabilizes starting from $r = 2^{k_0}$ into components which contain intervals $\{2^k\} \times [0,1]$, $k \geq k_0$, but which are separated by hyperplanes $H_k = \{Re(z) = 2^k + 2^{k-1}\}$. However, we will show that these components in fact contain intervals $\{2^k\} \times [0, L(k)]$ where $L(k)$ can grow arbitrarily fast, the lower bound on growth depending on $\mathfrak{c}$. This follows from Lemma 5.2. In other words, components of $\{|F| \leq \delta\}$ get elongated in w-direction and thus they partly remain outside of B_r for very large r, as shown on Figure 6. Due to this elongation, most of the components of $\{|F| \leq \delta\}$ only contribute to $\zeta(F, r, \delta)$ and not to $\zeta^0(F, r, \delta)$, which leads to the upper bound given by Theorem 1.8.

Proof of Theorem 1.8. Since $\zeta^0(F, r, \delta)$ is decreasing in δ, it is enough to prove the statement for $\delta \leq 1$. Denote by $b_i = 2^{c_i - \frac{(i-1)i}{2c_i}} \delta^{\frac{1}{c_i}}$. Let $i_0(l, \lambda, \delta)$ be the smallest index such that

1) For all $i \geq i_0$, $\delta \geq \max(2^{-c_i}, 2^{\frac{(i-1)i}{2} - c_i^2})$;
2) $\log_2 b_{i_0} \geq i_0$;
3) For all $i \geq i_0$, b_i is increasing.

Now,

$$\zeta^0(F, r, \delta) \leq \sum_{1 \leq i \leq \log_2 b_{i_0}} c_i + \mathrm{I}, \tag{11}$$

where

$$\mathrm{I} = \text{ the number of islands with zeros from } \{2^i\} \times [0,1], i > \log_2 b_{i_0}.$$

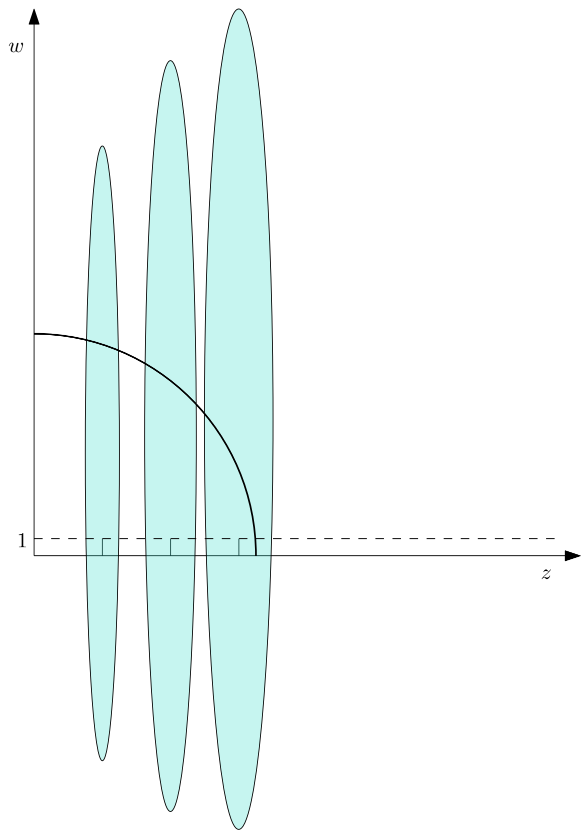

FIGURE 6. Components count for $\zeta(F,r,\delta)$, but not for $\zeta^0(F,r,\delta)$

Since the first term of the right-hand side of this inequality depends only on δ and $\mathfrak{c}$, we wish to estimate I. If $r \le b_{i_0}$ then $2^i > r$ and $\mathrm{I} = 0$. Thus, we assume that $r > b_{i_0}$. Firstly, from 2) it follows that

$$\mathrm{I} \le \text{ the number of islands with zeros from } \{2^i\} \times [0,1], i > i_0. \tag{12}$$

Now, 3) implies that there exists a unique integer $k \ge i_0$ such that $b_k \le r < b_{k+1}$. Due to 1) we may apply Lemma 5.2 to conclude that $\{|F| \le \delta\}$ contains intervals $\{2^i\} \times [0, b_i]$ for all $i \ge i_0$. This fact, combined with (12) implies

$$\mathrm{I} \le \text{ the number of intervals } \{2^i\} \times [0, b_i], i > i_0 \text{ contained in } B_r \le k - i_0.$$

Going back to (11) we have that

$$\zeta^0(F,r,\delta) \le \sum_{1 \le i \le \log_2 b_{i_0}} c_i - i_0 + k. \tag{13}$$

Finally, 1) gives us that $\delta^{\frac{1}{c_k}} \geq \frac{1}{2}$ and hence $2^{c_k - \frac{(k-1)k}{2c_k} - 1} \leq b_k \leq r$. Taking logarithms we obtain that

$$c_k \leq \log_2 r + \frac{(k-1)k}{2c_k} + 1.$$

Since $\frac{(k-1)k}{2c_k} + 1$ has an upper bound which only depends on λ, taking logarithms l times gives us

$$\underbrace{\log_2 \ldots \log_2}_{l \text{ times}} c_k \leq \underbrace{\log_2 \ldots \log_2}_{l+1 \text{ times}} r + a_\lambda,$$

where a_λ depends only on λ. Substituting the desired value of c_k in this inequality together with (13) finishes the proof. □

7. Comparison with other results

The goal of this section is to compare the results of this paper to the results of [20]. More precisely, we will deduce Theorem 5.1 in [20] from Theorem 1.6, as well as show that Theorem 1.2 does not follow from Theorem 5.1 in [20]. We start by recaling this result.

For an entire map $f : \mathbb{C}^n \to \mathbb{C}^n$ let J_f denote the complex Jacobian matrix. Given a sequence of zeros $\xi = \{\xi_i\} \subset f^{-1}(0)$ we define $\zeta_\xi(f, r)$ to be the number of elements of ξ inside a ball B_r.

Theorem 7.1 (Theorem 5.1 in [20]). *Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be an entire map and $\xi = \{\xi_i\}$ a sequence of zeros of f. If there exist real numbers $c > 0$ and b such that*

$$(\forall i)\ |\det J_f(\xi_i)| \geq c(\mu(f, |\xi_i|))^{-b},$$

then for any $a > 1$ it holds

$$\zeta_\xi(f, r) = O((\log \mu(f, ar))^n),$$

when $r \to \infty$.

Firstly, we give a proof of Theorem 7.1 using Theorem 1.6. The strategy of the proof follows [18]. Namely, the main results of [18], Theorems 1.1 and 1.2, are proven using a lemma which was referred to as "Weak Bézout estimate", see [18, Lemma 3.1]. This lemma establishes an inequality

$$\tau(f, r, \delta) \leq C_n \left((r+1) \frac{\mu(f, r+1)}{\delta} \right)^{2n}, \tag{14}$$

with C_n which depends only on n. The proof of (14) relies on a global version of the Chern-Levine-Nirenberg inequality, see [18, Theorem 2.1] and references therein. Substituting (14) with Theorem 1.6 and using the same general arguments as in [18] proves Theorem 7.1. To implement this strategy, we will need the following lemma.

Lemma 7.2. *Let $f:\mathbb{C}^n \to \mathbb{C}^n$ be an entire map and ξ a zero of f such that $J_f(\xi) \neq 0$. Then, for all $z \in \mathbb{C}^n$ such that*

$$|z| \leq \frac{1}{2\left(n!\frac{(\mu(f,|\xi|+1))^n}{|\det J_f(\xi)|}+1\right)}$$

it holds

$$|f(\xi+z)| \geq \frac{|\det J_f(\xi)|\cdot|z|}{2n!(\mu(f,|\xi|+1))^{n-1}}.$$

The proof of Lemma 7.2 can be extracted from the proof of Theorem 1.1 in [18]. We present it here for the sake of completeness. We will use the following auxiliary statement, which is a direct consequence of Schwarz lemma, see [18, Lemma 3.4] for details.

Lemma 7.3. *Let $f:\mathbb{C}^n \to \mathbb{C}^n$ be an entire map such that $f(0)=0, J_f(0)=0$. Then for all z such that $|z| \leq \frac{1}{2\mu(f,1)}$ it holds $|f(z)| \leq \frac{1}{2}|z|$.*

Proof of Lemma 7.2. We start by proving the following auxiliary inequality

$$\|J_f(\xi)^{-1}\|_{op} \leq n!\frac{(\mu(f,|\xi|+1))^{n-1}}{|\det J_f(\xi)|}, \tag{15}$$

where $\|\cdot\|_{op}$ denotes the operator norm. Indeed,

$$\|J_f(\xi)^{-1}\|_{op} = \frac{1}{|\det J_f(\xi)|}\|\operatorname{adj}(J_f(\xi))\|_{op},$$

and thus we need to prove that

$$\|\operatorname{adj}(J_f(\xi))\|_{op} \leq n!(\mu(f,|\xi|+1))^{n-1}.$$

By Cauchy-Schwarz inequality

$$\|\operatorname{adj}(J_f(\xi))\|_{op} \leq n\cdot\max_{i,j}|\operatorname{adj}(J_f(\xi))_{i,j}|,$$

and we are left to prove that

$$\max_{i,j}|\operatorname{adj}(J_f(\xi))_{i,j}| \leq (n-1)!(\mu(f,|\xi|+1))^{n-1}.$$

For each i and j, $\operatorname{adj}(J_f(\xi))_{i,j}$ is a sum of $(n-1)!$ terms, each of which is a product of $n-1$ partial derivatives of f. Thus

$$\max_{i,j}|\operatorname{adj}(J_f(\xi))_{i,j}| \leq (n-1)!\cdot(\max_i|\partial_i f(\xi))|)^{n-1}.$$

Finally, Cauchy's inequality yields that $\max_i|\partial_i f(\xi))| \leq \mu(f,|\xi|+1)$ which completes the proof of (15).

Now, let $g : \mathbb{C}^n \to \mathbb{C}^n$ be an entire map given by $g(z) = (J_f(\xi))^{-1} f(\xi + z)$. Since $g(0) = 0$ and $J_g(0) = \mathrm{id}_{\mathbb{C}^n}$ we may apply Lemma 7.3 to the map $g(z) - z$, which gives us

$$|g(z) - z| \leq \frac{1}{2}|z|, \tag{16}$$

for all z, such that $|z| \leq \frac{1}{2\mu(g(z)-z,1)}$. Moreover, $\mu(g(z)-z, 1) \leq \mu(g, 1) + 1$ implies that (16) holds for all z with $|z| \leq \frac{1}{2(\mu(g,1)+1)}$ and triangle inequality further implies that

$$|g(z)| \geq \frac{1}{2}|z| \tag{17}$$

as long as $|z| \leq \frac{1}{2(\mu(g,1)+1)}$. From the definition of g and (17) it follows that

$$|f(\xi + z)| \cdot \|(J_f(\xi))^{-1}\|_{op} \geq \frac{1}{2}|z|, \tag{18}$$

for all z such that $|z| \leq \frac{1}{2(\mu(g,1)+1)}$. Applying (15) to (18) yields

$$|f(\xi + z)| \cdot n! \frac{(\mu(f, |\xi| + 1))^{n-1}}{|\det J_f(\xi)|} \geq \frac{1}{2}|z|, \tag{19}$$

for all z such that $|z| \leq \frac{1}{2(\mu(g,1)+1)}$. Using (15) again gives us

$$|g(z)| \leq |f(\xi + z)| \cdot \|(J_f(\xi))^{-1}\|_{op} \leq |f(\xi + z)| \cdot n! \frac{(\mu(f, |\xi| + 1))^{n-1}}{|\det J_f(\xi)|},$$

and thus

$$\mu(g, 1) \leq n! \frac{(\mu(f, |\xi| + 1))^{n}}{|\det J_f(\xi)|}.$$

Combining the last inequality and (19) finishes the proof. □

Proof of Theorem 7.1. For simplicity we assume that $b \geq 0$ and $f(0) \neq 0$. The general case readily reduces to this one.

Let $r \geq 0$ and $\xi_i \in B_r$. By the assumption

$$|\det J_f(\xi_i)| \geq c(\mu(f, |\xi_i|))^{-b} \geq c(\mu(f, r))^{-b} \geq c(\mu(f, r + 1))^{-b},$$

and since $\mu(f, |\xi_i| + 1) \leq \mu(f, r + 1)$, Lemma 7.2 gives us that

$$|f(\xi_i + z)| \geq \delta := \frac{c|z|}{2n!(\mu(f, r + 1))^{n-1+b}}, \tag{20}$$

for all z such that

$$|z| \leq \varepsilon := \frac{1}{2(c^{-1} \cdot n!(\mu(f, r + 1))^{n+b} + 1)}.$$

Since $\varepsilon < \frac{1}{2} < 1$ a connected component of $\{|f| \leq \delta\}$ which contains ξ_i is itself fully contained in B_{r+1}. The same holds for each $\xi_j \in B_r$ and thus

$$\zeta_\xi(f,r) \leq \tau(f,r+1,\delta).$$

Finally, by (20), $\delta/e \leq \mu(f,a(r+1))/e$ for all $a>1$ and we may apply Theorem 1.6 which yields

$$\zeta_\xi(f,r) \leq \tau(f,r+1,\delta) \leq C_{n,a}\left(\log\frac{e\cdot\mu(f,a(r+1))}{\delta}\right)^n.$$

Substituting the expression for δ into this inequality and using $\mu(f,r+1) \leq \mu(f,a(r+1))$ gives us

$$\zeta_\xi(f,r) \leq C_{n,a}\left(\log\frac{2en!\cdot(\mu(f,a(r+1)))^{n+b}}{c|z|}\right)^n$$

for all z such that $|z| \leq \varepsilon$. Taking $|z| = \varepsilon$ yields

$$\zeta_\xi(f,r) \leq C_{n,a}\left(A_{n,b}\log\mu(f,a(r+1)) + B_{n,c}\right)^n,$$

where A,B,C depend on n,a,b,c as indicated. This completes the proof. $\square$

So far we proved that Theorem 7.1 follows from Theorem 1.6. Now, we wish to show that Theorem 1.2 does not follow from Theorem 7.1. Namely, one may imagine the following scenario. Let $f:\mathbb{C}^n \to \mathbb{C}^n$ be an entire map. While the classical count of zeros of f might not satisfy the transcendental Bézout bound, it may still happen that for a fixed $\delta > 0$, we can choose a sequence of zeros $\{\xi_i\}$ from $f^{-1}(B_\delta)$ such that Theorem 7.1 applies to this sequence and $\zeta(f,r,\delta) = O(\zeta_\xi(f,r))$. In this case, Theorem 7.1 would imply Theorem 1.2 (at least up to a constant which depends on δ). However, our next result rules out this possibility.

Proposition 7.4. *Let $\mathfrak{c} = \{c_i\}$ be an increasing sequence of positive integers such that $\lim_{i\to+\infty}\frac{i}{c_i} = 0$ and F the corresponding Cornalba-Shiffman map. For all real numbers $c>0$ and b the inequality*

$$|\det J_f(\xi)| \geq c(\mu(f,|\xi|))^{-b},$$

holds for at most finitely many zeros $\xi \in F^{-1}(0)$.

Proof. Since the zeros of Cornalba-Shiffman maps are isolated and $\mu(F,r) \to +\infty$ as $r \to +\infty$, it is enough to prove the proposition for $b>0$. Since $F(z,w) = (g(z), f(z,w))$, we have that

$$\det J_F = \partial_z g\,\partial_w f - \partial_w g\,\partial_z f = \partial_z g\,\partial_w f.$$

All relevant infinite sums and products converge uniformly on compact sets and thus we may compute

$$\partial_z g = \sum_{l=1}^{\infty} -2^{-l} g_l,$$

as well as

$$\partial_w f = \sum_{l=1}^{\infty} 2^{-c_l^2} g_l \partial_w P_{c_l}.$$

We wish to evaluate J_F at zeros of F. To this end, let us denote the zeros of F by $\xi_{i,j} = (2^i, 1/j)$ for $i \geq 1$ and $1 \leq j \leq c_i$. We calculate

$$\partial_z g(2^i) = -2^{-i} g_i(2^i),$$

as well as

$$\partial_w f(\xi_{i,j}) = 2^{-c_i^2} g_i(2^i) \partial_w P_{c_i}(1/j) = 2^{-c_i^2} g_i(2^i) \prod_{1 \leq l \leq c_i, l \neq j} \left(\frac{1}{j} - \frac{1}{l} \right).$$

Since $\left| \prod_{1 \leq l \leq c_i, l \neq j} \left(\frac{1}{j} - \frac{1}{l} \right) \right| \leq 1$, we get that for all i and j

$$|\det J_F(\xi_{i,j})| \leq 2^{-c_i^2 - i} (g_i(2^i))^2. \tag{21}$$

Moreover,

$$g(2^i) = \prod_{l=1}^{i-1} (1 - 2^{i-l}) \cdot \prod_{l=i+1}^{\infty} (1 - 2^{i-l}) = \prod_{l=1}^{i-1} (1 - 2^l) \cdot \prod_{l=1}^{\infty} (1 - 2^{-l}),$$

and thus we have that

$$|g_i(2^i)| < \prod_{l=1}^{i-1} (2^l - 1) < \prod_{l=1}^{i-1} 2^l = 2^{\frac{i(i-1)}{2}}.$$

Combining this inequality with (21) yields

$$|\det J_F(\xi_{i,j})| < 2^{-c_i^2 + i^2 - 2i}, \tag{22}$$

for all i and j. By the assumption on $\mathfrak{c}$, we have that for any $b > 0$, $c_i^2 \geq 5b(i+1)^2 + i^2$ for all but finitely many i. Thus, for all but finitely many i, it holds

$$|\det J_F(\xi_{i,j})| < 2^{-5b(i+1)^2 - 2i} \tag{23}$$

for all j. Since $|\xi_{i,j}| = |(2^i, 1/j)| < 2^{i+1}$, we have that

$$2^{-5b(i+1)^2} < 2^{-5b(\log_2 |\xi_{i,j}|)^2} < \left(2^{\frac{3}{2}(\log_2 |\xi_{i,j}|)^2 + \frac{7}{2} \log_2 |\xi_{i,j}|} \right)^{-b}. \tag{24}$$

Now, by Proposition 5.1

$$\left(2^{\frac{3}{2}(\log_2 |\xi_{i,j}|)^2 + \frac{7}{2} \log_2 |\xi_{i,j}|} \right)^{-b} \leq 2^{bC} (\mu(F, |\xi_{i,j}|))^{-b}, \tag{25}$$

where $C > 0$ is an absolute constant. Putting (23), (24) and (25) together gives us

$$|\det J_F(\xi_{i,j})| < 2^{-2i+bC}(\mu(F, |\xi_{i,j}|))^{-b}$$

for all but finitely many indices i, j. Since for every $a > 0, 2^{-2i+bC} < a$ for all but finitely many i, the proof follows. $\square$

Remark 7.5. Different results in the spirit of Theorem 7.1 were obtained in [18] and [19]. In one way or another, these results rely on lower bounds on $\det J_f$ at zeros of f. Namely, in [18], $|\det J_f(\xi_i)|$ is assumed to be bounded from below by a constant, while in [19] the upper bound for $\zeta_\xi(f, r)$ involves terms of the form $\log \frac{1}{\det J_f(\xi_i)}$. From (22), it follows that by taking $\{c_i\}$ which increases sufficiently fast, we can make $|\det J_F(\xi_{i,j})|$ of Cornalba-Shiffman maps decrease arbitrarily fast. Since, by Theorem 1.5, $\zeta(F, r, \delta)$ increases as $\log_2 r$ independently of $\{c_i\}$, we conclude that Theorem 1.2 can not be deduced from the results of [18] and [19] using the above-described strategy.

Remark 7.6. It is interesting to notice that Proposition 7.4 does not rule out a possibility that Theorem 1.6, or at least the same bound for ζ^0, can be deduced from Theorem 7.1. Indeed, we have not proven any lower bound on the count of islands of Cornalba-Shiffman maps. As a matter of fact, it is not even clear if for each $\delta > 0$ and each sequence $\{c_i\}$, $\zeta^0(F, r, \delta) \to +\infty$ as $r \to +\infty$. Namely, it may happen that starting from certain finite r_0, all connected components of $\{|F| \le \delta\}$, which contain zeros of F, elongate all the way to infinity in w-direction and thus never become islands, but rather remain peninsulas for all $r > r_0$.

Question 7.7. *Is it true that for each $\delta > 0$ and all sequences $\mathfrak{c}$, $\zeta^0(F, r, \delta) \to +\infty$ as $r \to +\infty$? If so, what is the possible growth rate of $\zeta^0(F, r, \delta)$ depending on parameters δ and $\mathfrak{c}$?*

Remark 7.8. The main technical ingredient in [20] and [19] is Theorem 3.6 from [20] (slightly modified in [19]). While the proof of this result has certain similarities with the proof of Theorem 1.6, it seems that the two approaches are fundamentally different. Namely, approximation of a holomorphic map by a polynomial is the key idea in the proof of Theorem 1.6, while it is not directly used in the proof of Theorem 3.6 in [20]. It would be interesting to explore how the methods of [20] and [19] relate to the coarse counts of zeros. In the opposite direction, it would be interesting to deduce the results of [19] using the same strategy as above, by proving a suitable analogue of Theorem 1.6.

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