**Spectral asymptotics and dynamics on Riemannian manifolds**

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According to Bohr's correspondence principle in quantum mechanics, the asymptotic properties of the eigenvalues and the eigenfunctions of the Laplacian on a Riemannian manifold are linked to the behavior of the geodesic flow. In particular, the growth of various spectral quantities (such as the spectral function, the error term in Weyl's law, eigenfunctions) depends on the underlying dynamics.

Let $M$ be a compact Riemannian manifold of dimension $n$ without boundary. Consider the Laplacian $\Delta$ on $M$ with the eigenvalues $0 < \lambda_1^2 \le \lambda_2^2 \le \dots$, and the corresponding orthonormal basis $\{\phi_i\}$ of eigenfunctions: $\Delta\phi_i = \lambda_i^2\phi_i$. Let $N_{x,y}(\lambda) = \sum_{\lambda_i<\lambda}\phi_i(x)\phi_i(y)$, where $x, y \in M$, be the spectral function of the Laplacian. As $\lambda \to \infty$, it satisfies

$$N_{x,y}(\lambda) = O(\lambda^{n-1}), \qquad x \neq y. \tag{1}$$

The asymptotics of the spectral function on the diagonal is given by the pointwise Weyl's law:

$$N_{x,x}(\lambda) = \frac{\lambda^n}{(4\pi)^{\frac{n}{2}}\,\Gamma(\frac{n}{2}+1)} + R_x(\lambda), \;\; R_x(\lambda) = O(\lambda^{n-1}). \tag{2}$$

Integrating (2) over $M$ one gets the well-known Weyl's law for the eigenvalue counting function $N(\lambda) = \#\{\lambda_i < \lambda\}$:

$$N(\lambda) = \int_M N_{x,x}(\lambda) = \frac{\mathrm{Vol(M)}\,\lambda^n}{(4\pi)^{\frac{n}{2}}\,\Gamma(\frac{n}{2}+1)} + R(\lambda), \;\; R(\lambda) = O(\lambda^{n-1}). \tag{3}$$

The upper bounds on the spectral function and the error terms (due to Avakumovič and Levitan) are sharp and attained on a round sphere. At the same time, the upper bound on $R(\lambda)$ (respectively, $R_x(\lambda)$ and $N_{x,y}(\lambda)$) can be improved to $o(\lambda^{n-1})$ under the condition that the measure of directions corresponding to closed geodesics (respectively, geodesic loops at $x$ and geodesic segments from $x$ to $y$) is zero (see [DG], [Saf]).

Further improvements of estimates (1-3) are expected on manifolds whose geodesic flow is either completely integrable or ergodic (cf. [St, Conjecture I]). For the sake of simplicity, we focus here on manifolds of dimension two. Consider the integrable case first.

**Conjecture 1.** *Let $M$ be a sphere with a generic (cf.* [CdV, section 6]*) metric of revolution. Then $R(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$. Moreover, if $x$ is not a pole, then $R_x(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$. Also, if at least one of the points $x \neq y \in M$ is not a pole, then $N_{x,y}(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$.*

It was shown in [CdV] that on a generic surface of revolution $R(\lambda) = O(\lambda^{2/3})$. The genericity condition is important, since one has to eliminate such " degenerate" cases as a round sphere or a Zoll surface, for which the remainder estimate (3) is attained. Note also that at poles of a surface of revolution the pointwise estimate (2) is attained (see [Saf]), and, most likely, (1) is attained if $x, y$ are both poles. A dynamical explanation of this phenomenon is that all geodesics starting from one pole pass through another pole and return back.

**Conjecture 2.** *Let $M$ be a torus with a Liouville metric. Then $R(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$, $R_x(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$, $N_{x,y}(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$, for any $\varepsilon > 0$ and for all $x \neq y \in M$.*

It is shown in [La] that for a large class of Liouville tori $R(\lambda) = O(\lambda^{2/3})$. As in [CdV], the main idea is to approximate the eigenvalue count by a lattice count (in the integrable case this is possible, as follows from the EBK quantization conditions, see [St] for a discussion). Then one can apply the van der Corput method of exponential sums and obtain a $O(\lambda^{2/3})$ remainder estimate.

For the flat square torus, counting eigenvalues is equivalent to counting integer points inside a circle (the latter is known as the Gauss's circle problem). In this case, a slightly better estimate on $R(\lambda)$ was proved by Huxley. The celebrated Hardy's conjecture about the error term in the Gauss' circle problem is an important motivation for Conjectures 1 and 2 (in fact, it is a special case of Conjecture 2 when $M$ is a flat square torus).

The estimates in Conjectures 1 and 2 can not be significantly improved. On a flat torus, the Hardy-Landau lower bound yields $R(\lambda) \equiv R_x(\lambda) \neq O(\sqrt{\lambda})$, and using the methods of [JP] one can also show that in this case $N_{x,y}(\lambda) \neq O(\sqrt{\lambda})$. It is not known whether $\varepsilon$ can be removed from any of the estimates in Conjecture 1, but it seems unlikely. It was proved in [Sar] that $R(\lambda) \neq o(\sqrt{\lambda})$ on any surface with integrable geodesic flow. For $R_x(\lambda)$ and $N_{x,y}(\lambda)$ such a lower bound was proved in a much higher generality in [JP, LPS].

Consider now the negatively curved case. As was proved by Anosov, the geodesic flow on a surface of negative curvature is ergodic.

**Conjecture 3.** *Let $M$ be a generic negatively curved surface. Then $R(\lambda) = O(\lambda^{\varepsilon})$, $R_x(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$, $N_{x,y}(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$ and for all $x \neq y \in M$.*

Note that we expect $R(\lambda)$ to grow much slower on a generic negatively curved surface than in the integrable case (cf. [St, Conjecture I]). Moreover, lower bounds on $R_x(\lambda)$ proved in [JP] indicate that substantial cancelations occur when the pointwise remainder is integrated over a negatively curved surface. In particular, results of [JP] imply that $N_{x,y}(\lambda) \neq O(\sqrt{\lambda})$ and $R_x(\lambda) \neq O(\sqrt{\lambda})$ for all points on a negatively curved surface.

The genericity assumption in Conjecture 3 can not be removed: indeed, as shown by Hejhal and Selberg [Hej], on arithmetic surfaces of constant negative curvature $R(\lambda) \neq o\left(\frac{\sqrt{\lambda}}{\log \lambda}\right)$. Note that arithmetic surfaces have exceptionally high multiplicities in the length spectrum. As mentioned in [Sar], this gives a dynamical explanation for the faster growth of the remainder. Randol (see [Ran]) conjectured that for arithmetic surfaces (and, in general, for all surfaces of constant negative curvature) $R(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$ — similarly to the integrable case.

It is shown in [JPT] that on any negatively curved surface $R(\lambda) \neq o\left((\log \lambda)^{\alpha}\right)$ for some positive constant $\alpha$ that can be expressed in terms of certain dynamical characteristics of the geodesic flow. This lower bound is consistent with Conjecture 3. For surfaces of constant negative curvature (in this case one can take any $\alpha < 1/2$) it was proved by Randol and Hejhal using the Selberg zeta function techniques.

Conjecture 3 is very far from being proved. The best result up-to-date is $R_x(\lambda) = O\left(\frac{\lambda}{\log \lambda}\right)$ and $R(\lambda) = O\left(\frac{\lambda}{\log \lambda}\right)$ obtained in [Ber] (see also [Vol]).

Let us conclude by a recent result showing that the off-diagonal bounds on $N_{x,y}(\lambda)$ in Conjectures 1–3 hold *on average* in a much higher generality.

**Theorem 1.** [LPS] *Let $M$ be an arbitrary compact $n$-dimensional Riemannian manifold. For any finite measure $\nu$ on $\mathbb{R}$ and any $x \in M$ there exists a subset $M_{x,\nu} \subset M$ of full measure such that*

$$\frac{N_{x,y}(\lambda)}{1+\lambda^{\frac{n-1}{2}}} \in L^2(\mathbb{R}, \nu), \quad \forall y \in M_{x,\nu}. \tag{4}$$

Theorem 1 is inspired by [Ran] where similar averaging was considered. We prove it using purely analytic methods. It would be interesting to find a dynamical interpretation of this result — in particular, to check whether (4) is true if $x$ is not conjugate to $y$ along any geodesic.

## References


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