Pólya's conjecture for Euclidean balls

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Abstract

The celebrated Pólya's conjecture (1954) in spectral geometry states that the eigenvalue counting functions of the Dirichlet and Neumann Laplacian on a bounded Euclidean domain can be estimated from above and below, respectively, by the leading term of Weyl's asymptotics. Pólya's conjecture is known to be true for domains which tile Euclidean space, and, in addition, for some special domains in higher dimensions. In this paper, we prove Pólya's conjecture for the disk, making it the first non-tiling planar domain for which the conjecture is verified. We also confirm Pólya's conjecture for arbitrary planar sectors, and, in the Dirichlet case, for balls of any dimension. Along the way, we develop the known links between the spectral problems in the disk and certain lattice counting problems. A key novel ingredient is the observation, made in recent work of the last named author, that the corresponding eigenvalue and lattice counting functions are related not only asymptotically, but in fact satisfy certain uniform bounds. Our proofs are purely analytic, except for a rigorous computer-assisted argument needed to cover the short interval of values of the spectral parameter in the case of the Neumann problem in the disk.

Contents

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§1. Weyl's law and Pólya's conjecture

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Consider the Dirichlet eigenvalue problem for the Laplacian

$$-\Delta := -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

in Ω:

$$\begin{aligned} -\Delta u &= \lambda u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

It is well known that the spectrum of (1.1) is discrete and consists of isolated eigenvalues of finite multiplicity accumulating to $+\infty$,

$$0 < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots \le \lambda_n(\Omega) \le \ldots,$$

which we enumerate with account of multiplicities.

Similarly, assuming additionally that $\partial\Omega$ is Lipschitz, consider the Neumann eigenvalue problem

$$\begin{aligned} -\Delta u &= \mu u & \text{in } \Omega, \\ \partial_n u &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $\partial_n u = \langle \nabla u, n \rangle|_{\partial\Omega}$ denotes the normal derivative of u with respect to the exterior unit normal n on the boundary. The spectrum of (1.2) again consists of isolated eigenvalues of finite multiplicity accumulating to $+\infty$,

$$0 = \mu_1(\Omega) \le \mu_2(\Omega) \le \cdots \le \mu_n(\Omega) \le \ldots,$$

enumerated with account of multiplicities.

Let, for $\lambda \in \mathbb{R}$,

$$\mathcal{N}_\Omega^{\mathrm{D}}(\lambda) := \#\left\{n : \lambda_n(\Omega) \le \lambda^2\right\} \qquad \text{and} \qquad \mathcal{N}_\Omega^{\mathrm{N}}(\lambda) := \#\left\{n : \mu_n(\Omega) \le \lambda^2\right\}$$

denote the *counting functions*[1] of the Dirichlet and Neumann eigenvalue problems on Ω.[2] It follows from the variational principles for (1.1) and (1.2) that

$$\mathcal{N}_\Omega^{\mathrm{D}}(\lambda) \le \mathcal{N}_\Omega^{\mathrm{N}}(\lambda)$$

for any $\lambda \ge 0$.[3]

Under the assumptions stated above, the leading term asymptotics of the counting functions is given by *Weyl's law* [Wey11],

$$\mathcal{N}_\Omega(\lambda) = C_d |\Omega|_d \lambda^d + R(\lambda), \tag{1.3}$$

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[1] Strictly speaking, we are counting the number of eigenvalues less than or equal to a given λ^2, but such normalisation will be convenient to us throughout.

[2] One can also define the counting functions using strict inequalities; this does not affect any of the results below.

[3] This in fact can be improved to $\mathcal{N}_\Omega^{\mathrm{D}}(\lambda) + 1 \le \mathcal{N}_\Omega^{\mathrm{N}}(\lambda)$, see [Fri91] and [Filo4].

where $\mathcal{N}_\Omega(\lambda)$ denotes either $\mathcal{N}_\Omega^{\mathrm{D}}(\lambda)$ or $\mathcal{N}_\Omega^{\mathrm{N}}(\lambda)$, $|\cdot|_d$ denotes the d-dimensional volume, $R(\lambda) = o\left(\lambda^d\right)$ as $\lambda \to +\infty$, and

$$C_d := \frac{1}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2} + 1\right)}$$

is the so-called *Weyl constant*. We refer to [SafVas97] for a historical review, as well as numerous generalisations and improvements.

H. Weyl himself conjectured [Wey12] a sharper version of (1.3) taking into account the boundary conditions: for $\Omega \subset \mathbb{R}^d$ with a piecewise smooth boundary,

$$\mathcal{N}_\Omega(\lambda) = C_d |\Omega|_d \lambda^d \pm C_{\mathrm{b},d} |\partial\Omega|_{d-1} \lambda^{d-1} + o\left(\lambda^{d-1}\right) \qquad \text{as } \lambda \to +\infty, \tag{1.4}$$

where the minus sign is taken for the Dirichlet boundary conditions and the plus sign for the Neumann ones, and

$$C_{\mathrm{b},d} := \frac{1}{2^{d+1} \pi^{\frac{d-1}{2}} \Gamma\left(\frac{d+1}{2}\right)}.$$

We note that for planar domains (1.4) takes the particularly simple form

$$\mathcal{N}_\Omega(\lambda) = \frac{\mathrm{Area}(\Omega)}{4\pi} \lambda^2 \pm \frac{\mathrm{Length}(\partial\Omega)}{4\pi} \lambda + o\left(\lambda\right). \tag{1.5}$$

The two-term Weyl's law (1.4) remains open in full generality. It has been proved by V. Ivrii [Ivr80] under the condition that the set of periodic billiard trajectories in Ω has measure zero. While this condition is conjectured to be satisfied for all Euclidean domains, it has been verified only for a few classes, such as convex analytic domains and polygons, see [SafVas97] and references therein. Specifically for a disk, it was proved by N. Kuznetsov and B. Fedosov in [KuzFed65].

Assuming that the two-term Weyl's asymptotics (1.4) holds for a domain $\Omega \subset \mathbb{R}^d$, we immediately obtain that for λ above some *sufficiently large but unspecified* value Λ_1 we have

$$\mathcal{N}_\Omega^{\mathrm{D}}(\lambda) \le C_d |\Omega|_d \lambda^d \le \mathcal{N}_\Omega^{\mathrm{N}}(\lambda). \tag{1.6}$$

We refer also to [Mel80] for results of the same kind in the Riemannian setting.

In 1954, G. Pólya [Pól54] conjectured that the inequalities (1.6) hold for *all* $\lambda \ge 0$.[4] He later proved this conjecture in [Pól61] for *tiling domains* Ω: that is, domains such that $\mathbb{R}^d$ can be covered, up to a set of measure zero, by a disjoint union of copies of Ω. In fact, in the Neumann case, some additional assumptions were imposed in [Pól61] that have been removed in [Kel66]. It has been also shown that Pólya's conjecture in the Dirichlet case holds for a Cartesian product $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{d_1+d_2}$ if it holds for $\Omega_1 \subset \mathbb{R}^{d_1}$ with $d_1 \ge 2$, and $\Omega_2 \subset \mathbb{R}^{d_2}$ is bounded, see [Lap97, Theorem 2.8]. For general domains, somewhat weakened versions of (1.6) are known to hold as a consequence of the so-called *Berezin–Li–Yau* inequalities: we have

$$\left(\frac{d}{d+2}\right)^{d/2} \mathcal{N}_\Omega^{\mathrm{D}}(\lambda) \le C_d |\Omega|_d \lambda^d \le \frac{d+2}{2} \mathcal{N}_\Omega^{\mathrm{N}}(\lambda)$$

for all $\lambda \ge 0$, see [LiYau83], [Krö92], and [Lap97]. We refer also to [Lin17],[KLS19], [FLP21], and [FreSal22] for some recent results on Pólya's conjecture and further interesting links to other problems in spectral geometry.

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[4]In fact, Pólya's original conjecture was only for planar domains, and in a slightly different form.

Remark 1.1. Pólya's conjecture (1.6) can be equivalently restated as the inequalities for the eigenvalues (instead of the counting functions),

$$\mu_{n+1}(\Omega) \le (C_d|\Omega|_d)^{-\frac{2}{d}} n^{\frac{2}{d}} \le \lambda_n(\Omega) \tag{1.7}$$

for all $n \ge 1$. It is known that inequalities (1.7) hold for any domain in any dimension for $n = 1, 2$. In particular, for $n = 1$ this follows from the celebrated Faber–Krahn and Szegő–Weinberger inequalities, and for $n = 2$ in the Dirichlet case from the Krahn–Szego inequality, see [Hen06]. For $n = 2$ in the Neumann case, we refer to [GNP09], [BucHen19]. These are the only eigenvalues for which it is known in full generality. We refer also to [Fre19] for further results on the validity of the Dirichlet Pólya's conjecture for low eigenvalues in higher dimensions. ◀

Remarkably, since balls do not tile the space, Pólya's conjecture has so far remained open for Euclidean balls, including planar disks.[5] Although all the eigenvalues of the Dirichlet and Neumann Laplacians on the unit disk are explicitly known in terms of zeros of the Bessel functions or their derivatives, see §2 below, in each case the spectrum is given by a *two-parametric* family, and rearranging it into a single monotone sequence appears to be an unfeasible task.

Let $\mathbb{B}^d \subset \mathbb{R}^d$ be the d-dimensional unit ball. Then $|\mathbb{B}^d|_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$. Therefore the leading Weyl's term in (1.3) for $\mathbb{B}^d$ becomes

$$W_d(\lambda) := C_d|\mathbb{B}^d|_d\lambda^d = w_d\lambda^d, \qquad w_d = \frac{1}{2^d\left(\Gamma\left(\frac{d}{2}+1\right)\right)^2}, \tag{1.8}$$

in particular

$$W_2(\lambda) = \frac{\lambda^2}{4} \qquad \text{and} \qquad W_3(\lambda) = \frac{2\lambda^3}{9\pi}.$$

The main results of this paper address the validity of Pólya's conjecture for disks and balls. Namely, we prove the following results.

Theorem 1.2. *The Dirichlet Pólya's conjecture for the unit ball holds in any dimension $d \ge 2$, that is we have*

$$\mathcal{N}_{\mathbb{B}^d}^{\mathrm{D}}(\lambda) < W_d(\lambda)$$

for all $\lambda > 0$.

Our results in the Neumann case are restricted to the case $d = 2$. Higher-dimensional Neumann problems are harder, and we intend to treat them in a subsequent paper.

We first state

Lemma 1.3. *The Neumann Pólya's conjecture for $\mathbb{D} = \mathbb{B}^2$ is valid for all $\lambda \in [0, \Lambda_0]$, where*

$$\Lambda_0 := 2\sqrt{3}.$$

Proof. Immediately follows from the bounds (1.7) with $n = 1$ and $n = 2$. □

We then prove

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[5]As stated in [Lap12, p. 638]: "Remarkably this conjecture still remains open even for such a simple domain as the disc, where the eigenvalues of the Dirichlet Laplacians could be calculated via the roots of Bessel functions." See also [FLW09, p. 1366] and [Lau12, p. 66].

Theorem 1.4. *The Neumann Pólya's conjecture for the unit disk holds for all*

$$\lambda \ge \Lambda_1 := \frac{6\pi}{3\pi - 8}. \tag{1.9}$$

We note that $\Lambda_0 > 3$ and $\Lambda_1 < 14$, so we already have the validity of the Neumann Pólya's conjecture for the disk for all λ outside the interval $(3, 14)$.

Theorem 1.5. *The Neumann Pólya's conjecture for the unit disk holds for all $\lambda \in [3, 14]$.*

The proof of Theorem 1.5 is *rigorous* but *computer-assisted*. More specifically, it is based on a realisation of an algorithm which satisfies two fundamental principles.

Principle 1. The algorithm should complete in a finite number of steps.

Principle 2. The algorithm should operate only with integer or rational numbers, thus avoiding *any* use of floating-point arithmetic and *any* rounding errors.

The combination of Lemma 1.3 and Theorems 1.4 and 1.5 ensure that the Neumann Pólya conjecture for the disk is valid for all $\lambda > 0$, that is we have

Corollary 1.6. $\mathcal{N}_{\mathbb{D}}^{\mathrm{N}}(\lambda) > \frac{\lambda^2}{4}$ *for all $\lambda > 0$.*

Remark 1.7. Since Pólya's conjecture is scale-invariant, its validity for a unit ball immediately implies that it is valid for any ball of the same dimension. ◀

We additionally have the following generalisation of Pólya's result for tiling domains: we show that Pólya's conjecture holds not only for domains which tile Euclidean space, but also for domains which tile another domain for which it is known to be true.

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^d$ be a domain for which either the Dirichlet or the Neumann Pólya's conjecture holds, and let Ω' be a domain which tiles Ω. Then the same Pólya's conjecture also holds for Ω'.*

Proof. Assume that Ω can be tiled by $\ell \ge 2$ congruent copies of Ω', so that $|\Omega|_d = \ell |\Omega'|_d$. We have, by bracketing and since the eigenvalues of all the congruent copies coincide with those of Ω',

$$\ell \mathcal{N}_{\Omega'}^{\mathrm{D}}(\lambda) \le \mathcal{N}_{\Omega}^{\mathrm{D}}(\lambda) < \mathcal{N}_{\Omega}^{\mathrm{N}}(\lambda) \le \ell \mathcal{N}_{\Omega'}^{\mathrm{N}}(\lambda).$$

Assuming now (1.6) for all $\lambda \ge 0$, we get

$$\ell \mathcal{N}_{\Omega'}^{\mathrm{D}}(\lambda) \le C_d |\Omega|_d \lambda^d = C_d \ell |\Omega'|_d \lambda^d \le \ell \mathcal{N}_{\Omega'}^{\mathrm{N}}(\lambda),$$

and the result follows by cancelling ℓ. ☐

Remark 1.9. If the inequalities in Pólya's conjecture (1.6) for Ω are strict, they are also strict for Ω'. ◀

Theorem 1.8 immediately implies the following

Corollary 1.10. *Let $\widetilde{\Omega} \subset \mathbb{S}^{d-1}$ be a spherical domain which tiles $\mathbb{S}^{d-1}$. Then the Dirichlet Pólya's conjecture holds for the spherical cone in $\mathbb{R}^d$ with the base $\widetilde{\Omega}$ and the vertex at the origin.*

In the planar case we get

Corollary 1.11. *Pólya's conjecture holds for any circular sector S_α with an aperture $\alpha = \frac{2\pi}{\ell}$, where $\ell \in \{2, 3, \ldots\}$.*

We refer also to [FreSal22] for an alternative proof of Corollary 1.11 for sufficiently large (but unspecified) ℓ.

We can in fact extend the result of Corollary 1.11 to arbitrary sectors.

Theorem 1.12. *Pólya's conjecture holds for any circular sector S_α with an aperture $\alpha \in (0, 2\pi]$, that is*

$$\mathcal{N}_{S_\alpha}^{\mathrm{D}}(\lambda) < \frac{\alpha \lambda^2}{8\pi} < \mathcal{N}_{S_\alpha}^{\mathrm{N}}(\lambda)$$

for all $\lambda > 0$.

Remark 1.13. The result of Theorem 1.12 in the case $\alpha = 2\pi$ (the disk with the radial slit) follows immediately from Theorems 1.2 ($d = 2$) and 1.4 by Dirichlet–Neumann bracketing.   ◄

Plan of the paper. In the next section we describe two lattice counting problems (2.6) and (2.7), variants of which were originally introduced by N. Kuznetsov and B. Fedosov in [KuzFed65], and which are closely linked to the Dirichlet and Neumann eigenvalue counting problems in the ball. The key novel tool is Theorem 2.3, originally obtained in part in [She22], which gives a *uniform* bound between the eigenvalue and the lattice counting functions, as opposed to asymptotic relations that were previously known. We provide an independent proof of this result in §3. In §4 we state the results on the lattice counting functions which are sufficient for proving Pólya's conjecture for balls. The bulk of the paper, §§5–8, is devoted to the proofs of these results. Theorem 1.12 is proved in §9.

§2. Dirichlet and Neumann eigenvalues of the ball and lattice counting problems

Throughout this paper, with $\nu \geq 0$, and $k \in \mathbb{N}$, let $J_\nu(z)$ be the Bessel functions of order ν, let $j_{\nu,k}$ be the kth positive zero of J_ν, and let $j'_{\nu,k}$ be the kth positive zero of its derivative J'_ν, with the exception of J'_0 for which $j'_{0,1} = 0$.

It is well known that the eigenvalues of the Dirichlet Laplacian in the unit ball are given by the squares of the zeros of the cylindrical Bessel functions. Namely, considering the Dirichlet Laplacian in $\mathbb{B}^d$, we have the simple eigenvalues

$$\lambda_{d,0,k} = \left(j_{d/2-1,k} \right)^2, \qquad k \in \mathbb{N},$$

that is of multiplicity

$$\kappa_{d,0} := 1,$$

and the eigenvalues

$$\lambda_{d,m,k} = \left(j_{m+d/2-1,k} \right)^2, \qquad m, k \in \mathbb{N},$$

of multiplicity

$$\kappa_{d,m} := \binom{m+d-1}{d-1} - \binom{m+d-3}{d-1}.$$

Remark 2.1. We note that the numbers $\kappa_{d,m}$, $d \geq 2$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, coincide with the multiplicity of the eigenvalue $m(m + d - 2)$ of the Laplace–Beltrami operator on the unit sphere $\mathbb{S}^{d-1}$, or alternatively with the dimension of the space of homogeneous harmonic polynomials of degree m in $\mathbb{R}^d$. In the planar case we have

$$\kappa_{2,m} = 2 \qquad \text{for } m \in \mathbb{N}.$$

◀

We therefore have

$$\mathcal{N}_{\mathbb{B}^d}^{\mathrm{D}}(\lambda) = \sum_{m=0}^{\infty} \kappa_{d,m} \# \left\{ k \in \mathbb{N} : j_{m+d/2-1,k} \leq \lambda \right\}. \tag{2.1}$$

Remark 2.2. The sum in (2.1) is in fact finite: we have[6]

$$\mathcal{N}_{\mathbb{B}^d}^{\mathrm{D}}(\lambda) = \sum_{m=0}^{\lfloor \lambda - d/2 + 1 \rfloor} \kappa_{d,m} \# \left\{ k \in \mathbb{N} : j_{m+d/2-1,k} \leq \lambda \right\}, \tag{2.2}$$

This is due to the fact that $j_{\nu,1} > \nu$ [DLMF, Eq. 10.21.3].   ◀

In the planar case the expression (2.1) simplifies to

$$\mathcal{N}_{\mathbb{D}}^{\mathrm{D}}(\lambda) = \# \left\{ k \in \mathbb{N} : j_{0,k} \leq \lambda \right\} + 2 \sum_{m=1}^{\lfloor \lambda - d/2 + 1 \rfloor} \# \left\{ (m,k) \in \mathbb{N}^2 : j_{m,k} \leq \lambda \right\}.$$

Similarly, the eigenvalues of the Neumann Laplacian in the unit disk $\mathbb{D}$ are given by the squares of the zeros of the derivatives of Bessel functions. We have the simple eigenvalues

$$\mu_{0,k} = \left(j'_{0,k} \right)^2, \qquad k \in \mathbb{N},$$

and the double eigenvalues

$$\mu_{m,k} = \left(j'_{m,k} \right)^2, \qquad m, k \in \mathbb{N}.$$

We therefore have

$$\mathcal{N}_{\mathbb{D}}^{\mathrm{N}}(\lambda) = \# \left\{ k \in \mathbb{N} : j'_{0,k} \leq \lambda \right\} + 2 \sum_{m=1}^{\lfloor \lambda - d/2 + 1 \rfloor} \# \left\{ (m,k) \in \mathbb{N}^2 : j'_{m,k} \leq \lambda \right\}, \tag{2.3}$$

where the sum is again finite since $j'_{\nu,1} \geq \nu$ [DLMF, Eq. 10.21.3].

For illustrative purposes only, we show the graphs of the Dirichlet and Neumann eigenvalue counting functions for the disk in Figure 1.

We will be comparing the counting functions $\mathcal{N}_{\mathbb{B}^d}^{\mathrm{D}}(\lambda)$ and $\mathcal{N}_{\mathbb{D}}^{\mathrm{N}}(\lambda)$ with some weighted lattice counting functions. Let

$$h(x) := \frac{1}{\pi} \left(\sqrt{1 - x^2} - x \arccos x \right), \qquad x \in [0,1],$$

and let $\mathbb{P}$ be a planar region under the graph of $h(x)$,

$$\mathbb{P} := \left\{ (x,y) : x \in [0,1], y \in [0, h(x)] \right\}.$$

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[6] Here and further on, $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$ denotes the integer part of $x \in \mathbb{R}$, and $\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$ denotes its ceiling.
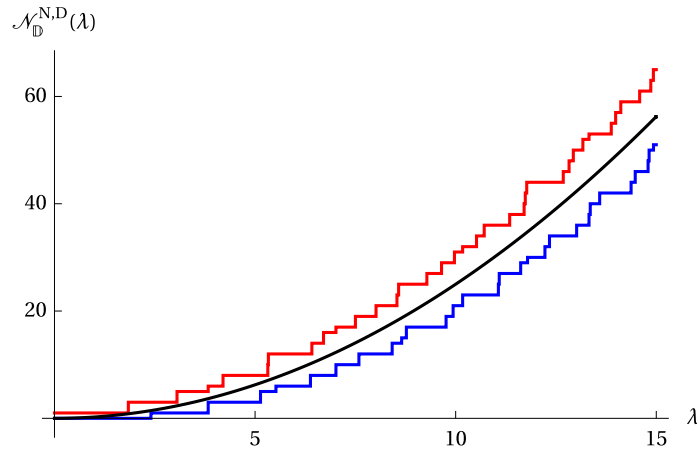
Figure 1: The Dirichlet eigenvalue counting function $\mathcal{N}_{\mathbb{D}}^{\mathrm{D}}(\lambda)$ (blue), the Neumann eigenvalue counting function $\mathcal{N}_{\mathbb{D}}^{\mathrm{N}}(\lambda)$ (red), and the leading Weyl's term $W_d(\lambda) = \frac{\lambda^2}{4}$ (black) in dimension $d = 2$. The plot is produced using the floating-point evaluation of zeros of the Bessel functions and their derivatives. If we were to assume (contrary to the philosophy of this paper) the validity of floating-point arithmetic, this plot would have presented a *numerically assisted* (as opposed to *computer-assisted*) "proof" of Pólya's conjecture for the disk for $\lambda \underset{\approx}{\lesssim} 15$.

Let, for $\lambda > 0$,

$$G_\lambda(z) := \lambda h\left(\frac{z}{\lambda}\right) = \frac{1}{\pi}\left(\sqrt{\lambda^2 - z^2} - z \arccos\frac{z}{\lambda}\right), \qquad z \in [0, \lambda], \tag{2.4}$$

and let $\mathbb{P}_\lambda$ be a dilation of $\mathbb{P}$ with coefficient λ with respect to the origin,

$$\mathbb{P}_\lambda = \left\{(z, y) : 0 \le z \le \lambda, 0 \le y \le G_\lambda(z)\right\}, \tag{2.5}$$

that is, the region under the graph of $G_\lambda(z)$.

Let

$$Q_d^{\mathrm{D}}(\lambda) := \left\{(m, k) + \left(\frac{d}{2} - 1, -\frac{1}{4}\right) \in \mathbb{P}_\lambda : (m, k) \in \mathbb{N}_0 \times \mathbb{N}\right\}$$

and

$$Q_2^{\mathrm{N}}(\lambda) := \left\{(m, k) + \left(0, -\frac{3}{4}\right) \in \mathbb{P}_\lambda : (m, k) \in \mathbb{N}_0 \times \mathbb{N}\right\}$$

be the sets of shifted integer lattice points which lie in $\mathbb{P}_\lambda$, see Figure 2. The definitions of the two sets for $d = 2$ differ by a vertical shift. The reason for choosing this particular notation will become evident later.

We now introduce the *weighted lattice point counting functions*

$$\mathscr{P}_d^{\mathrm{D}}(\lambda) := \sum_{\left(m + \frac{d}{2} - 1, k - \frac{1}{4}\right) \in Q_d^{\mathrm{D}}(\lambda)} \kappa_{d,m} \tag{2.6}$$

and

$$\mathscr{P}_2^{\mathrm{N}}(\lambda) := \sum_{\left(m, k - \frac{3}{4}\right) \in Q_2^{\mathrm{N}}(\lambda)} \kappa_{2,m}. \tag{2.7}$$

It is immediately seen from the definitions (2.5)–(2.7) that with $\aleph \in \{\mathrm{D}, \mathrm{N}\}$ we have

$$\mathscr{P}_d^\aleph(\lambda) = \sum_{m=0}^{\lfloor \lambda - d/2 + 1\rfloor} \kappa_{d,m}\left\lfloor G_\lambda\left(m + \frac{d}{2} - 1\right) + s^\aleph\right\rfloor, \tag{2.8}$$

where

$$s^{\mathrm{D}} := \frac{1}{4}, \qquad s^{\mathrm{N}} := \frac{3}{4}. \tag{2.9}$$
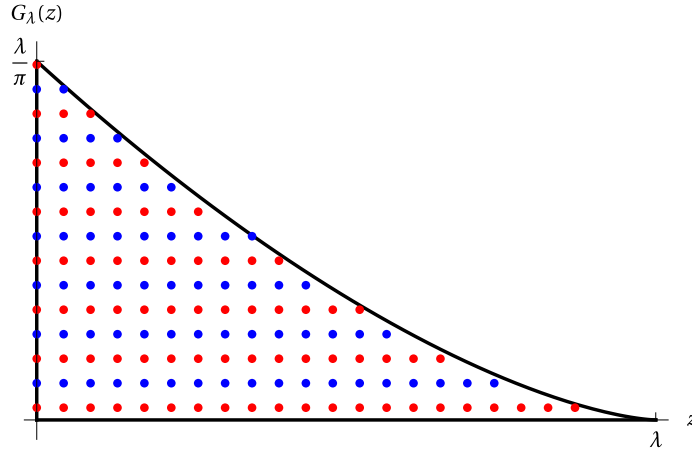
Figure 2: The region $\mathbb{P}_\lambda$, and the sets of shifted lattice points $Q_d^{\mathrm{D}}(\lambda)$ (blue) and $Q_d^{\mathrm{N}}(\lambda)$ (red), shown here for $d = 2$ and $\lambda = 23$.

It is well known that as $\lambda \to +\infty$, the asymptotics of the lattice point counting function $\mathscr{P}_d^{\mathrm{D}}(\lambda)$ is intricately linked to the asymptotics of the eigenvalue counting function $\mathscr{N}_{\mathbb{B}^d}^{\mathrm{D}}(\lambda)$. This was first shown in the planar case in [KuzFed65] and later re-discovered in [CdV10], see also [Gra07]. Namely, in some appropriate sense,

$$\mathscr{N}_{\mathbb{B}^d}^{\mathrm{D}}(\lambda) \sim \mathscr{P}_d^{\mathrm{D}}(\lambda) \qquad \text{as } \lambda \to +\infty.$$

This observation, together with asymptotic bounds on the difference between the two functions, has been used to great effect to estimate the remainder in Weyl's law for the unit ball. In particular, for the Dirichlet problem in the disk the two-term Weyl asymptotics (1.5) holds with an improved remainder estimate

$$O\left(\lambda^{131/208}(\log \lambda)^{18627/8320}\right),$$

see [GMWW21] (the remainder estimate $O\left(\lambda^{2/3}\right)$ was already obtained in [KuzFed65], [CdV10]). Similar improved remainder estimates are also known in the Dirichlet case for higher-dimensional balls [Guo21] and in the planar Neumann case [GWW19].

As has been recently found in [She22] in the Dirichlet case, there is a further simple *non-asymptotic* relation between the lattice point and the eigenvalue counting functions, which lies at the cornerstone of our proofs of Theorems 1.2 and 1.4.

Theorem 2.3. *For any $d \geq 2$ and any $\lambda \geq 0$, we have*

$$\mathscr{N}_{\mathbb{B}^d}^{\mathrm{D}}(\lambda) \leq \mathscr{P}_d^{\mathrm{D}}(\lambda).$$

We also have, for any $\lambda \geq 0$,

$$\mathscr{P}_2^{\mathrm{N}}(\lambda) \leq \mathscr{N}_{\mathbb{D}}^{\mathrm{N}}(\lambda).$$

§3. Proof of Theorem 2.3

We start by introducing some additional notation. Set, for $v \geq 0$, $\lambda \geq 0$, and $\aleph \in \{D, N\}$,

$$A_v^\aleph(\lambda) := \begin{cases} G_\lambda(v) + s^\aleph, & \text{if } \lambda \geq v, \\ s^\aleph, & \text{if } 0 \leq \lambda < v, \end{cases} \tag{3.1}$$

where G_λ is defined by (2.4) and $s^\aleph$ is defined by (2.9). Some typical graphs of the functions $A_v^\aleph(\lambda)$ are shown in Figure 3.

The crucial step in the proof of Theorem 2.3 comes from the following bounds on the number of zeros of Bessel functions and their derivatives below a given number.

Proposition 3.1. *Let $v \geq 0$ and $\lambda \geq 0$. Then*

$$\#\left\{k \in \mathbb{N} : j_{v,k} \leq \lambda\right\} \leq \left\lfloor A_v^D(\lambda) \right\rfloor \tag{3.2}$$

and

$$\#\left\{k \in \mathbb{N} : j'_{v,k} \leq \lambda\right\} \geq \left\lfloor A_v^N(\lambda) \right\rfloor. \tag{3.3}$$

Remark 3.2. For $\lambda \in [0, v]$, the inequalities (3.2) and (3.3) become the trivial identities $0 = \left\lfloor s^\aleph \right\rfloor = 0$. ◀

Remark 3.3. For $v = 0$, the inequality (3.2) is equivalent to $j_{0,k} \geq \pi\left(k - \frac{1}{4}\right)$, which was proved in [Het70]. ◀

Proof of Proposition 3.1. We recall the representations of the Bessel functions of the first and second kind, J_v and Y_v, and their derivatives in terms of the so-called *modulus functions* M_v and N_v and the *phase functions* θ_v and ϕ_v,

$$J_v(x) = M_v(x)\cos\theta_v(x), \quad Y_v(x) = M_v(x)\sin\theta_v(x),$$
$$J'_v(x) = N_v(x)\cos\phi_v(x), \quad Y'_v(x) = N_v(x)\sin\phi_v(x)$$

(see [DLMF, Eqs. 10.18.4–5]). We will be using various properties of the phase functions below; for a review of these properties see [Hor17]. We will be only considering the cases $v \geq 0$ and $x \geq 0$ for which the moduli $M_v(x)$ and $N_v(x)$ are both positive.

Let us concentrate first on (3.2). We have $J_v(x_0) = 0$ if and only if $\cos\theta_v(x_0) = 0$, and so if and only if

$$\frac{1}{\pi}\theta_v(x_0) + \frac{1}{2} \in \mathbb{Z}. \tag{3.4}$$

Note that the phase function $\theta_v(x)$ satisfies $\theta_v(x) \to -\frac{\pi}{2}$ as $x \to +0$ [DLMF, Eq. 10.18.3], and that it is monotone increasing for $x \in (0, +\infty)$ [Hor17, Theorem 1], therefore (3.4) can be replaced by

$$B_v^D(x_0) \in \mathbb{N},$$

where

$$B_v^D(x) := \frac{1}{\pi}\theta_v(x) + \frac{1}{2}, \tag{3.5}$$

and therefore

$$\#\left\{k \in \mathbb{N} : j_{v,k} \leq \lambda\right\} = \left\lfloor B_v^D(\lambda) \right\rfloor. \tag{3.6}$$

We have

$$G_\lambda(v) = \frac{\lambda}{\pi} - \frac{v}{2} + O\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda \to +\infty. \tag{3.7}$$

Using the asymptotics[7] [Hor17, Eq. (21)], [DLMF, Eq. 10.18.18],

$$\theta_v(\lambda) = \lambda - \frac{\pi}{2}\left(v + \frac{1}{2}\right) + O\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda \to +\infty,$$

and (3.5), (3.7), we obtain

$$B_v^{\mathrm{D}}(\lambda) - A_v^{\mathrm{D}}(\lambda) \to 0 \qquad \text{as } \lambda \to +\infty.$$

Further,

$$\frac{\mathrm{d}B_v^{\mathrm{D}}(\lambda)}{\mathrm{d}\lambda} = \frac{1}{\pi}\theta_v'(\lambda) > \frac{\sqrt{\lambda^2 - v^2}}{\pi\lambda} = \frac{\mathrm{d}A_v^{\mathrm{D}}(\lambda)}{\mathrm{d}\lambda}$$

for $\lambda \geq v$ by [Hor17, Eq. (56)]. As we additionally have

$$\frac{\mathrm{d}B_v^{\mathrm{D}}(\lambda)}{\mathrm{d}\lambda} > 0 = \frac{\mathrm{d}A_v^{\mathrm{D}}(\lambda)}{\mathrm{d}\lambda}$$

for $\lambda \in (0, v]$, the function $B_v^{\mathrm{D}}(\lambda) - A_v^{\mathrm{D}}(\lambda)$ is monotone increasing on $(0, +\infty)$ and tends to zero at infinity. Thus, we have proved that

$$B_v^{\mathrm{D}}(\lambda) < A_v^{\mathrm{D}}(\lambda) \qquad \text{for } \lambda \in [0, +\infty). \tag{3.8}$$

Combining (3.6) and (3.8) proves (3.2).

We now prove (3.3). In the same manner we have $J_v'(x_0) = 0$ if and only if

$$\frac{1}{\pi}\phi_v(x_0) + \frac{1}{2} \in \mathbb{Z}. \tag{3.9}$$

We note that the phase function ϕ_v satisfies $\phi_v(x) \to \frac{\pi}{2}$ as $x \to +0$ [DLMF, Eq. 10.18.3]. Also, $\phi_v(x)$ is monotone increasing for $x \in (v, +\infty)$ and monotone decreasing for $x \in (0, v)$ [Hor17, Theorem 1], with $\phi_v(v) > -\frac{\pi}{2}$ [Hor17, formula (60)]. Thus, the condition (3.9) can be replaced by

$$B_v^{\mathrm{N}}(x_0) \in \mathbb{N},$$

where

$$B_v^{\mathrm{N}}(x) := \frac{1}{\pi}\phi_v(x) + \frac{1}{2}, \tag{3.10}$$

and therefore

$$\#\left\{k \in \mathbb{N} : j_{v,k}' \leq \lambda\right\} = \left\lfloor B_v^{\mathrm{N}}(\lambda) \right\rfloor. \tag{3.11}$$

Using the asymptotics [Hor17, Eq. (22)], [DLMF, Eq. 10.18.21],

$$\phi_v(\lambda) = \lambda - \frac{\pi}{2}\left(v - \frac{1}{2}\right) + O\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda \to +\infty,$$

and (3.10), (3.7), we get

$$B_v^{\mathrm{N}}(\lambda) - A_v^{\mathrm{N}}(\lambda) \to 0 \qquad \text{as } \lambda \to +\infty.$$

Also,

$$\frac{\mathrm{d}B_v^{\mathrm{N}}(\lambda)}{\mathrm{d}\lambda} = \frac{1}{\pi}\phi_v'(\lambda) < \frac{\sqrt{\lambda^2 - v^2}}{\pi\lambda} = \frac{\mathrm{d}A_v^{\mathrm{N}}(\lambda)}{\mathrm{d}\lambda}$$

for $\lambda \geq v$ by [Hor17, formula following Eq. (58)]. As we additionally have

$$\frac{\mathrm{d}B_v^{\mathrm{N}}(\lambda)}{\mathrm{d}\lambda} < 0 = \frac{\mathrm{d}A_v^{\mathrm{N}}(\lambda)}{\mathrm{d}\lambda}$$

---

[7] See also [HBRV15] for all the coefficients of the full asymptotic expansion and some useful remarks.

for $\lambda \in (0, \nu]$, the function $B_\nu^N(\lambda) - A_\nu^N(\lambda)$ is monotone decreasing on $(0, +\infty)$ and tends to zero at infinity. Thus, we have proved that

$$B_\nu^N(\lambda) > A_\nu^N(\lambda) \qquad \text{for } \lambda \in [0, +\infty). \tag{3.12}$$

Combining (3.11) and (3.12) proves (3.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

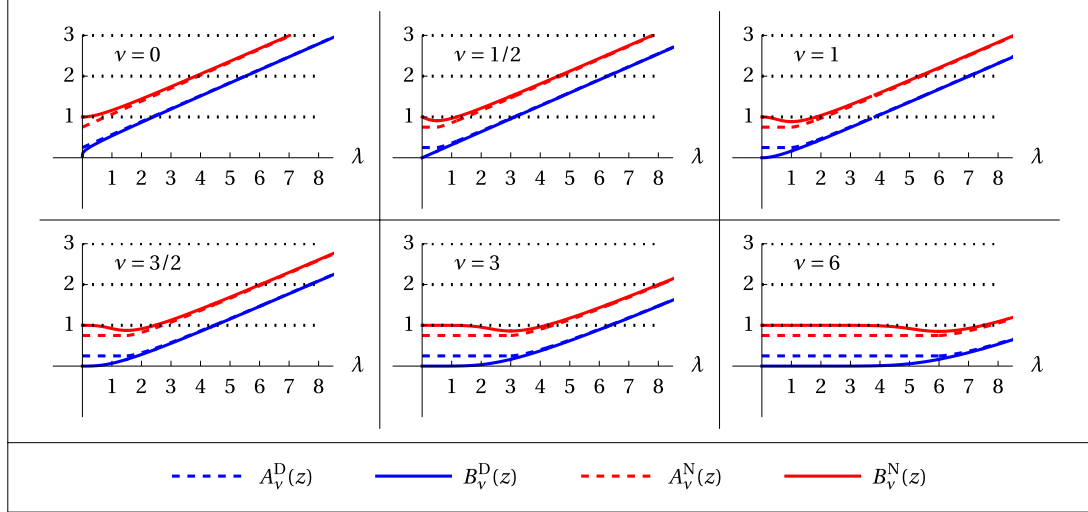We illustrate inequalities (3.8) and (3.12) in Figure 3.

Figure 3: An illustration of inequalities (3.8) and (3.12). The plots of $B_\nu^D(\lambda)$ and $B_\nu^N(\lambda)$ are drawn using the recipe from [Hor17]. We remark that $B_\nu^N(\lambda)$ has a minimum at $\lambda = \nu$.

For methodological purposes we include also an alternative proof of (3.2). This proof does not use any properties of the phase of the Bessel functions but relies instead on the known asymptotics of the Bessel zeros and the Sturm comparison theorem.

Let $\nu \geq 0$, and let us consider the function

$$\sin\left(\pi A_\nu^D(x)\right).$$

Let $a_{\nu,k}$ be its kth zero in $[\nu, +\infty)$, ordered increasingly. Obviously,

$$\#\left\{k \in \mathbb{N} : a_{\nu,k} \leq \lambda\right\} = \left\lfloor A_\nu^D(\lambda) \right\rfloor. \tag{3.13}$$

We will prove

Lemma 3.4. $a_{\nu,k} \leq j_{\nu,k}$ for all $k \in \mathbb{N}$.

Together with (3.13), Lemma 3.4 immediately implies (3.2).

Proof of Lemma 3.4. The function $U_\nu(x) := \sqrt{x} J_\nu(x)$ satisfies the differential equation

$$U_\nu''(x) + \left(1 - \frac{\nu^2 - 1/4}{x^2}\right) U_\nu(x) = 0. \tag{3.14}$$

Consider the function

$$V_\nu(x) := \frac{\sqrt{x}}{(x^2 - \nu^2)^{1/4}} \sin\left(\pi A_\nu^D(x) + b\right)$$

with some $b \in \left[0, \frac{\pi}{4}\right)$. Then

$$V'_\nu(x) = -\frac{\nu^2}{2\sqrt{x}(x^2 - \nu^2)^{5/4}} \sin\left(\pi A^{\mathrm{D}}_\nu(x) + b\right) + \frac{(x^2 - \nu^2)^{1/4}}{\sqrt{x}} \cos\left(\pi A^{\mathrm{D}}_\nu(x) + b\right),$$

$$V''_\nu(x) = \left(\frac{6\nu^2 x^2 - \nu^4}{4x^{3/2}(x^2 - \nu^2)^{9/4}} - \frac{(x^2 - \nu^2)^{3/4}}{x^{3/2}}\right) \sin\left(\pi A^{\mathrm{D}}_\nu(x) + b\right).$$

Therefore,

$$V''_\nu(x) + \left(1 - \frac{\nu^2}{x^2} - \frac{\nu^2(6x^2 - \nu^2)}{4x^2(x^2 - \nu^2)^2}\right) V_\nu(x) = 0 \qquad \text{for } x \in (\nu, +\infty). \tag{3.15}$$

Denote by $\upsilon_{\nu,k} = \upsilon_{\nu,k}(b)$ the kth zero of the function V_ν. By the definitions of V_ν and A^{D}_ν,

$$\sqrt{\upsilon^2_{\nu,k} - \nu^2} - \nu \arccos \frac{\nu}{\upsilon_{\nu,k}} + \frac{\pi}{4} + b = \pi k.$$

As

$$\sqrt{\upsilon^2_{\nu,k} - \nu^2} = \upsilon_{\nu,k} + O\left(\upsilon^{-1}_{\nu,k}\right) \qquad \text{and} \qquad \arccos \frac{\nu}{\upsilon_{\nu,k}} = \frac{\pi}{2} + O\left(\upsilon^{-1}_{\nu,k}\right) \qquad \text{as } k \to \infty,$$

we have

$$\upsilon_{\nu,k}(b) = \pi\left(k + \frac{\nu}{2} - \frac{1}{4}\right) - b + O\left(k^{-1}\right) \qquad \text{as } k \to \infty.$$

On the other hand the asymptotics

$$j_{\nu,k} = \pi\left(k + \frac{\nu}{2} - \frac{1}{4}\right) + O\left(k^{-1}\right) \qquad \text{as } k \to \infty.$$

is well known, see for example [DLMF, Eq. 10.21.19].

Suppose that $b > 0$. Then there exists $K \in \mathbb{N}$ such that

$$j_{\nu,k} > \upsilon_{\nu,k}(b) \qquad \text{for } k \geq K. \tag{3.16}$$

The coefficient in front of U_ν in (3.14) is greater than the coefficient in front of V_ν in (3.15):

$$1 - \frac{\nu^2 - 1/4}{x^2} > 1 - \frac{\nu^2}{x^2} > 1 - \frac{\nu^2}{x^2} - \frac{\nu^2(6x^2 - \nu^2)}{4x^2(x^2 - \nu^2)^2} \qquad \text{for all } x \in [\nu, +\infty).$$

By the Sturm comparison theorem there is a zero of U_ν between $\upsilon_{\nu,k}(b)$ and $\upsilon_{\nu,k+1}(b)$. So, if $j_{\nu,k_0} \leq \upsilon_{\nu,k_0}(b)$ for some number k_0, then $j_{\nu,k_0+1} \leq \upsilon_{\nu,k_0+1}(b)$, and by induction $j_{\nu,k} \leq \upsilon_{\nu,k}(b)$ for all $k \geq k_0$ which contradicts (3.16). Therefore, (3.16) holds for all natural k.

Finally, each function $\upsilon_{\nu,k}(b)$ is continuous in b. Thus,

$$j_{\nu,k} \geq \upsilon_{\nu,k}(0) = a_{\nu,k}.$$

$\square$

Returning now to the proof of Theorem 2.3, we rewrite (2.8) as

$$\mathscr{P}^{\aleph}_d(\lambda) = \sum_{m=0}^{\lfloor \lambda - d/2 + 1 \rfloor} \kappa_{d,m} \left\lfloor A^{\aleph}_{m+d/2-1}(\lambda) \right\rfloor. \tag{3.17}$$

Theorem 2.3 now immediately follows from Proposition 3.1 with account of (3.17), (2.2), and (2.3).

§4. From the weighted lattice point count towards Pólya's conjecture

By Theorem 2.3, the Dirichlet Pólya's conjecture for $\mathbb{B}^d$ would follow immediately if we can prove that

$$\mathscr{P}_d^{\mathrm{D}}(\lambda) < W_d(\lambda) \tag{4.1}$$

for all $\lambda \in (0, +\infty)$, where $W_d(\lambda)$ is defined by (1.8).

Similarly, the Neumann Pólya's conjecture for $\mathbb{D}$ would follow immediately if we can prove that

$$\mathscr{P}_2^{\mathrm{N}}(\lambda) > W_2(\lambda) \tag{4.2}$$

for all $\lambda \in (\Lambda_0, +\infty)$; we note that we have already dealt with $\lambda \le \Lambda_0$ by Lemma 1.3.

We establish (4.1) in the following cases, which will be dealt with separately.

Theorem 4.1. *The inequality*

$$\mathscr{P}_2^{\mathrm{D}}(\lambda) < \frac{\lambda^2}{4} \tag{4.3}$$

holds for all $\lambda > 0$.

Theorem 4.1 will be proved in §5. It directly implies Theorem 1.2 in the planar case.

Theorem 4.2. *The inequalities* (4.1) *hold for all $d \ge 3$ and $\lambda > 0$.*

Theorem 4.2 will be proved in §7. It implies Theorem 1.2 for higher-dimensional balls.

In the Neumann case, the situation is more delicate, as we cannot expect (4.2) to hold for all values of $\lambda \in (0, +\infty)$ since $\mathscr{P}_2^{\mathrm{N}}(\lambda)$ is identically zero for $\lambda < \frac{\pi}{4}$, see Figure 4.
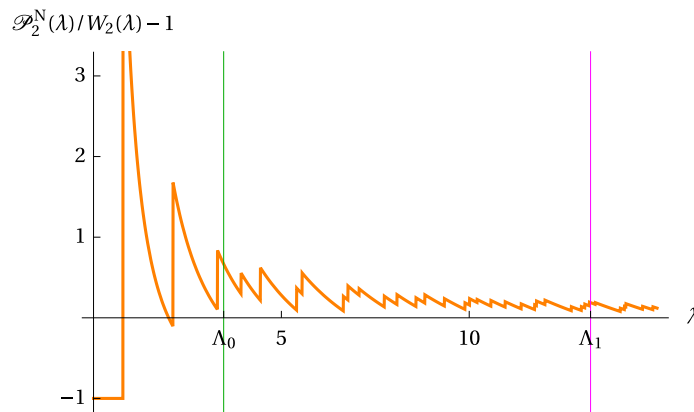

Figure 4: A numerical experiment: the computed $\mathscr{P}_2^{\mathrm{N}}(\lambda)/W_2(\lambda) - 1$ as a function of λ.

We prove the following results.

Theorem 4.3. *The inequality*

$$\mathscr{P}_2^{\mathrm{N}}(\lambda) > W_2(\lambda)$$

holds for all $\lambda \ge \Lambda_1$, where Λ_1 is given by (1.9).

Theorem 4.3 will be proved in §6. It implies Theorem 1.4 in the planar case.

We are further able to eliminate the remaining gap in the Neumann case.

Theorem 4.4. *The inequality* (4.2) *holds for any $\lambda \in (\Lambda_0, \Lambda_1)$.*

The proof of this result, presented in §8, is computer-assisted. Theorem 4.4 implies Theorem 1.5.

In all cases, we deal with estimating a (weighted) count of (shifted) lattice points under the graph of a particular function G_λ. Such problems have been extensively studied in number theory, going back to the Gauss circle problem. Important contributions in the general case can be traced through the works of van der Corput [vdC23] and Krätzel [Krä00] to some very recent results of Laugesen and Liu [Liu17, LauLiu18a]. In particular, [LauLiu18b, Proposition 15] is directly applicable (with account of the fact that Laugesen and Liu do not count the points on the vertical axis and do not double-count the points inside) to our shifted lattice point count $\mathscr{P}_2^{\mathrm{D}}(\lambda)$, yielding the bound

$$\mathscr{P}_2^{\mathrm{D}}(\lambda) \le \frac{\lambda^2}{4} + \left(\frac{2}{3} + \frac{1}{\pi} - \frac{\sqrt{3}}{2\pi}\right)\lambda \approx \frac{\lambda^2}{4} + 0.7093\lambda.$$

Unfortunately, since the coefficient in front of λ in this formula is positive, this bound is weaker than our required bound (4.3). We need therefore to obtain sharper lattice point count bounds than those available generally, and to do so we additionally use some properties of the *derivative* of the function G_λ in addition to the properties of the function itself, see Theorems 5.1 and 6.1, and also Remarks 5.2 and 6.3 for an informal explanation.

For future use, we summarise below some elementary properties of the function G_λ.

The first lemma is checked by a direct calculation.

Lemma 4.5. *The function $G_\lambda : [0, \lambda] \to \left[0, \frac{\lambda}{\pi}\right]$ defined by (2.4) is a strictly monotone decreasing convex C^1 function with*

$$G_\lambda(0) = \frac{\lambda}{\pi}, \qquad G_\lambda(\lambda) = 0,$$
$$G_\lambda'(z) = -\frac{1}{\pi}\arccos\frac{z}{\lambda}, \qquad G_\lambda'(0) = -\frac{1}{2}, \qquad G_\lambda'(\lambda) = 0,$$
$$G_\lambda''(z) = \frac{1}{\pi\sqrt{\lambda^2 - z^2}}.$$

We can therefore define the inverse function $G_\lambda^{-1} : \left[0, \frac{\lambda}{\pi}\right] \to [0, \lambda]$ which is also monotone decreasing and convex. Sometimes, it will be also convenient for us to consider G_λ on the interval $[0, \lceil\lambda\rceil]$ by extending it by zero to $(\lambda, \lceil\lambda\rceil]$: the resulting function, which we for simplicity denote by the same symbol, remains monotone decreasing, convex, and C^1.

Lemma 4.6. *Let $\beta \ge 0$. Then*

$$\int_0^\lambda z^\beta G_\lambda(z)\,\mathrm{d}z = \frac{\Gamma\left(\frac{\beta+1}{2}\right)\lambda^{\beta+2}}{4\sqrt{\pi}(\beta+2)\Gamma\left(\frac{\beta+4}{2}\right)}.$$

In particular,

$$\int_0^\lambda G_\lambda(z)\,\mathrm{d}z = \frac{\lambda^2}{8}. \tag{4.4}$$

Proof. After a change of variables $z = \lambda\cos\tau$, we obtain

$$\int_0^\lambda z^\beta G_\lambda(z)\,\mathrm{d}z = \frac{\lambda^{\beta+2}}{\pi}\int_0^{\pi/2}(\cos\tau)^\beta(\sin\tau - \tau\cos\tau)\sin\tau\,d\tau. \tag{4.5}$$

By [DLMF, Eqs. 5.12.1–2],

$$\int_0^{\pi/2} (\cos\tau)^\rho (\sin\tau)^\sigma d\tau = \frac{\Gamma\left(\frac{\rho+1}{2}\right)\Gamma\left(\frac{\sigma+1}{2}\right)}{2\Gamma\left(\frac{\rho+\sigma+2}{2}\right)}$$

for any $\rho, \sigma \geq 0$. Therefore,

$$\int_0^{\pi/2} (\cos\tau)^\beta (\sin\tau)^2 d\tau = \frac{\sqrt{\pi}\Gamma\left(\frac{\beta+1}{2}\right)}{4\Gamma\left(\frac{\beta+4}{2}\right)},$$

$$\int_0^{\pi/2} \tau(\cos\tau)^{\beta+1}\sin\tau\, d\tau = -\left.\frac{\tau(\cos\tau)^{\beta+2}}{\beta+2}\right|_0^{\pi/2} + \frac{1}{\beta+2}\int_0^{\pi/2}(\cos\tau)^{\beta+2}d\tau = \frac{\sqrt{\pi}\Gamma\left(\frac{\beta+3}{2}\right)}{2(\beta+2)\Gamma\left(\frac{\beta+4}{2}\right)},$$

and so

$$\int_0^{\pi/2} (\cos\tau)^\beta (\sin\tau - \tau\cos\tau)\sin\tau\, d\tau = \frac{\sqrt{\pi}\Gamma\left(\frac{\beta+1}{2}\right)}{4(\beta+2)\Gamma\left(\frac{\beta+4}{2}\right)}.$$

Substituting this into (4.5) we get the result.   □

Corollary 4.7. *Let $d \in \mathbb{N}$, $d \geq 2$. Then*

$$\frac{2}{(d-2)!}\int_0^\lambda z^{d-2}G_\lambda(z)\,dz = W_d(\lambda) = w_d\lambda^d.$$

Proof. Applying Lemma 4.6 with $\beta = d-2$ we get

$$\frac{2}{(d-2)!}\int_0^\lambda z^{d-2}G_\lambda(z)\,dz = \frac{\Gamma\left(\frac{d-1}{2}\right)\lambda^d}{2\,d\,\sqrt{\pi}\,(d-2)!\,\Gamma\left(\frac{d+2}{2}\right)}.$$

The duplication formula [DLMF, Eq. 5.5.5]

$$\sqrt{\pi}(d-2)! = \sqrt{\pi}\Gamma(d-1) = 2^{d-2}\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{d}{2}\right)$$

implies

$$\frac{\Gamma\left(\frac{d-1}{2}\right)\lambda^d}{2d\sqrt{\pi}(d-2)!\Gamma\left(\frac{d+2}{2}\right)} = \frac{\lambda^d}{2^{d-1}\,d\,\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+2}{2}\right)} = \frac{\lambda^d}{2^d\left(\Gamma\left(\frac{d+2}{2}\right)\right)^2} = W_d(\lambda).$$

     □

An important role in our study in the Neumann case will be played by the inverse function value $G_\lambda^{-1}\left(\frac{1}{4}\right)$ (defined for all $\lambda \geq \frac{\pi}{4}$). We will use the following bounds.

Lemma 4.8. *We have*

$$G_\lambda^{-1}\left(\frac{1}{4}\right) < \lambda - 1 \tag{4.6}$$

for all $\lambda \geq 2$. Additionally, for any $\sigma \in \left(0, \frac{\pi}{2}\right]$ we have

$$G_\lambda^{-1}\left(\frac{1}{4}\right) \geq \lambda\cos\sigma \tag{4.7}$$

whenever

$$\lambda \geq r_1(\sigma) := \frac{\pi}{4(\sin\sigma - \sigma\cos\sigma)}. \tag{4.8}$$

Proof. Since G_λ is monotone decreasing, the claim (4.6) is equivalent to

$$G_\lambda(\lambda - 1) < \frac{1}{4}. \tag{4.9}$$

We have $G_2(1) - \frac{1}{4} = \frac{\sqrt{3}}{\pi} - \frac{7}{12} < 0$, and additionally

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(G_\lambda(\lambda - 1)\right) = \frac{1}{\pi}\left(\frac{1}{\lambda}\sqrt{2\lambda - 1} - \arccos\left(1 - \frac{1}{\lambda}\right)\right) < 0$$

as

$$\cos\left(\frac{1}{\lambda}\sqrt{2\lambda - 1}\right) > 1 - \frac{2\lambda - 1}{2\lambda^2} > 1 - \frac{1}{\lambda},$$

thus implying (4.9) for $\lambda \geq 2$.

Similarly, the claim (4.7) is equivalent to

$$G_\lambda(\lambda\cos\sigma) = \frac{\lambda(\sin\sigma - \sigma\cos\sigma)}{\pi} \geq \frac{1}{4},$$

given (4.8). $\qquad\square$

§5. Proof of Theorem 4.1

We first state the following

Theorem 5.1. *Let $b > 0$, and let g be a non-negative decreasing convex function on $[0, b]$ such that $g(b) = 0$ and*

$$\left|g(z) - g(w)\right| \leq \frac{1}{2}|z - w| \tag{5.1}$$

for all $z, w \in [0, b]$. Then

$$\left\lfloor g(0) + \frac{1}{4}\right\rfloor + 2\sum_{m=1}^{\lfloor b\rfloor}\left\lfloor g(m) + \frac{1}{4}\right\rfloor \leq 2\int_0^b g(z)\,\mathrm{d}z. \tag{5.2}$$

The equality is possible only if g is identically zero on $[0, b]$.

Remark 5.2. We explain here, very informally, the ideas behind the proof of Theorem 5.1. The area under the graph of the function g on the interval $[m, m+1]$ is approximately equal to the area under the straight line passing through the points $(m, g(m))$ and $(m+1, g(m+1))$, so

$$\int_m^{m+1} g(z)\mathrm{d}z \approx \frac{1}{2}\left(g(m) + g(m+1)\right).$$

Summing up these equalities over m we obtain

$$2\int_0^b g(z)\mathrm{d}z \approx g(0) + 2\sum_{m=1}^{\lfloor b\rfloor} g(m).$$

If a number x is chosen randomly then $\lfloor x\rfloor \approx x - \frac{1}{2}$ on average. Thus, $\lfloor g(m) + \frac{1}{4}\rfloor \approx g(m) - \frac{1}{4}$, and these extra contributions of $-\frac{1}{4}$ ensure the sign of the inequality in (5.2). In order to prove Theorem 5.1 rigorously we divide the graph by horizontal lines $y = n$, with $n = 0, 1, \ldots, \lfloor g(0)\rfloor$, and we consider what happens in the intervals where $n + 1 \geq g(z) \geq n$. The values of $\lfloor g(m) + 1/4\rfloor$ there are either n or $n+1$. The point m is "bad" if $g(m) \geq n + \frac{3}{4}$ and thus $\lfloor g(m) + 1/4\rfloor = n+1$: these "bad" points contribute more to the sum than we expect "on average". The convexity of the function g and condition (5.1) ensure that the number of such "bad" points is not greater than half of the total number of integer points in an interval, and this yields the required estimate in the interval we are considering. ◀

In order to prove Theorem 5.1 we require the following

Lemma 5.3. *Let $i, j \in \mathbb{Z}$, $i < j$. Let g be a decreasing convex function on $[i, j+1]$ satisfying (5.1) for all $z, w \in [i, j+1]$. Assume additionally that*

$$n + 1 \geq g(i+1) \geq \cdots \geq g(j) \geq n \geq g(j+1) \tag{5.3}$$

for some $n \in \mathbb{Z}$. Then

$$\frac{1}{2} \left\lfloor g(i) + \frac{1}{4} \right\rfloor + \sum_{m=i+1}^{j-1} \left\lfloor g(m) + \frac{1}{4} \right\rfloor + \frac{1}{2} \left\lfloor g(j) + \frac{1}{4} \right\rfloor \leq \int_i^j g(z)\,\mathrm{d}z. \tag{5.4}$$

Proof of Lemma 5.3. The validity of the claim does not change if we add a constant integer number to the function g. So, without loss of generality we can assume $n = 0$, so that (5.3) becomes

$$1 \geq g(i+1) \geq \cdots \geq g(j) \geq 0 \geq g(j+1). \tag{5.5}$$

Additionally, (5.1) implies that $g(i) \leq g(i+1) + \frac{1}{2} \leq \frac{3}{2}$.

Set

$$K = \#\left\{ m \in \{i, \ldots, j\} : \frac{3}{4} \leq g(m) \right\},$$

and consider four cases.

Case $K = 0$. The left-hand side of (5.4) is zero, and the right-hand side is non-negative by (5.5).

Case $K = 1$. Here

$$\frac{5}{4} > g(i) \geq \frac{3}{4} > g(i+1) \geq \cdots \geq g(j) \geq 0,$$

and the left-hand side of (5.4) is equal to $\frac{1}{2}$. The assumption (5.1) yields $g\left(i + \frac{1}{2}\right) \geq \frac{1}{2}$, therefore by non-negativity and convexity of g on $[i, j]$,

$$\int_i^j g(z)\,\mathrm{d}z \geq \int_i^{i+1} g(z)\,\mathrm{d}z \geq g\left(i + \frac{1}{2}\right) \geq \frac{1}{2}.$$

Case $K = 2$. Here

$$\frac{3}{2} \geq g(i) \geq g(i+1) \geq \frac{3}{4} > g(i+2) \geq \cdots \geq g(j) \geq 0 \geq g(j+1),$$

and the left-hand side of (5.4) equals $\frac{3}{2}$. By (5.1) we have $j \geq i + 2$, and therefore by non-negativity and convexity of g,

$$\int_i^j g(z)\,\mathrm{d}z \geq \int_i^{i+2} g(z)\,\mathrm{d}z \geq 2g(i+1) \geq \frac{3}{2}.$$

Case $K \geq 3$. Here

$$\frac{3}{2} \geq g(i) \geq g(i+1) \geq \cdots \geq g(i+K-1) \geq \frac{3}{4} > g(i+K) \geq \cdots \geq g(j) \geq 0 \geq g(j+1).$$

The left-hand side of (5.4) is equal to $K - \frac{1}{2}$. By convexity of g,

$$(K-1)g(i+1) + (K-2)g(i+2K-2) \geq (2K-3)g(i+K-1),$$

and therefore

$$g(i + 2K - 2) \geq \frac{2K - 5}{4(K - 2)} > 0.$$

Thus, $j \geq i + 2K - 2$. Next,

$$\int_i^j g(z)\,dz \geq \int_i^{i+2K-2} g(z)\,dz \geq (2K-2)g(i+K-1) \geq \frac{3K-3}{2} \geq K - \frac{1}{2}$$

as $K \geq 3$.

$\square$

Remark 5.4. One can easily see from the proof that the equality in (5.4) is attained in the following three cases only:

- $g(z) \equiv n$ on $[i, j]$ (if $K = 0$);

- $j = i + 1$ and $g(z) = n + \frac{i-z}{2} + \frac{3}{4}$ on $[i, i+1]$ (if $K = 1$);

- $j = i + 2$ and $g(z) = n + s(i+1-z) + \frac{3}{4}$ on $[i, i+2]$ with $s \in \left[\frac{3}{8}, \frac{1}{2}\right]$ (if $K = 2$).

◀

We can now proceed to the proof of Theorem 5.1 proper.

Proof of Theorem 5.1. Let

$$N = \lfloor g(0) \rfloor.$$

If $N = 0$, then applying Lemma 5.3 with $i = 0$, $j = \lfloor b \rfloor$, and $g(z)$ extended by zero for $z \in (b, \lfloor b \rfloor + 1]$, gives

$$\left\lfloor g(0) + \frac{1}{4} \right\rfloor + 2 \sum_{m=1}^{\lfloor b \rfloor} \left\lfloor g(m) + \frac{1}{4} \right\rfloor \leq 2 \int_0^b g(z)\,dz,$$

and therefore (5.2).

Assume now $N \geq 1$. For $k = 0, 1, \ldots, N$, let

$$L_k := \max\left\{ m \in \{0, \ldots, \lfloor b \rfloor\} : g(m) \geq k \right\}, \tag{5.6}$$

see Figure 5. Therefore, we have

$$0 \leq L_N < L_{N-1} < \cdots < L_0 = \lfloor b \rfloor,$$

where the strict inequalities $L_k < L_{k-1}$, $k = 1, \ldots, N$, follow from (5.1).

We will consider two cases depending on whether $L_N = 0$.

Case $L_N = 0$. We write

$$\left\lfloor g(0) + \frac{1}{4} \right\rfloor + 2 \sum_{m=1}^{\lfloor b \rfloor} \left\lfloor g(m) + \frac{1}{4} \right\rfloor$$
$$= 2 \sum_{k=0}^{N-1} \left(\frac{1}{2} \left\lfloor g(L_{k+1}) + \frac{1}{4} \right\rfloor + \sum_{m=L_{k+1}+1}^{L_k - 1} \left\lfloor g(m) + \frac{1}{4} \right\rfloor + \frac{1}{2} \left\lfloor g(L_k) + \frac{1}{4} \right\rfloor \right). \tag{5.7}$$

For each $k \in \{0, \ldots, N-1\}$ we have

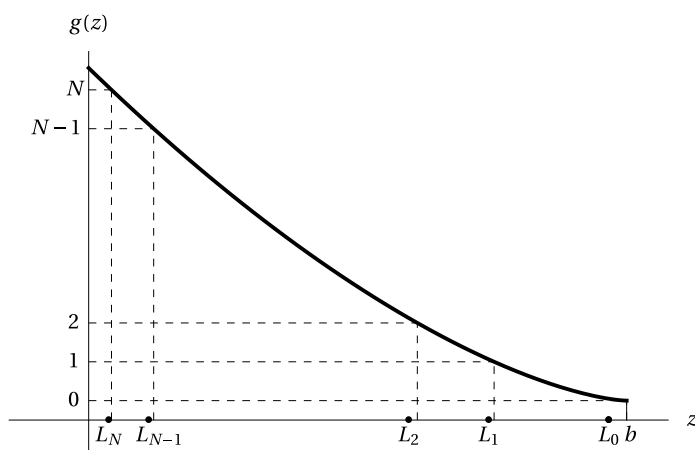$$k + 1 > g(L_{k+1} + 1) \cdots \geq g(L_k) \geq k \geq g(L_k + 1),$$

Figure 5: The numbers L_k, see (5.6) and also Remark 5.5.

and applying Lemma 5.3 with $i = L_{k+1}$, $j = L_k$, and $n = k$ yields

$$\frac{1}{2}\left\lfloor g(L_{k+1}) + \frac{1}{4}\right\rfloor + \sum_{m=L_{k+1}+1}^{L_k-1}\left\lfloor g(m) + \frac{1}{4}\right\rfloor + \frac{1}{2}\left\lfloor g(L_k) + \frac{1}{4}\right\rfloor \le \int_{L_{k+1}}^{L_k} g(z)\,\mathrm{d}z. \qquad (5.8)$$

Substituting (5.8) into (5.7) gives

$$\lfloor g(0)\rfloor + 2\sum_{m=1}^{\lfloor b\rfloor}\left\lfloor g(m) + \frac{1}{4}\right\rfloor \le 2\int_0^{\lfloor b\rfloor} g(z)\,\mathrm{d}z \le 2\int_0^b g(z)\,\mathrm{d}z, \qquad (5.9)$$

as required, where in the last inequality we used non-negativity of g.

Case $L_N > 0$. We write

$$\left\lfloor g(0) + \frac{1}{4}\right\rfloor + 2\sum_{m=1}^{\lfloor b\rfloor}\left\lfloor g(m) + \frac{1}{4}\right\rfloor$$

$$= 2\left(\frac{1}{2}\left\lfloor g(0) + \frac{1}{4}\right\rfloor + \sum_{m=1}^{L_N-1}\left\lfloor g(m) + \frac{1}{4}\right\rfloor + \frac{1}{2}\left\lfloor g(L_N) + \frac{1}{4}\right\rfloor\right) \qquad (5.10)$$

$$+ 2\sum_{k=0}^{N-1}\left(\frac{1}{2}\left\lfloor g(L_{k+1}) + \frac{1}{4}\right\rfloor + \sum_{m=L_{k+1}+1}^{L_k-1}\left\lfloor g(m) + \frac{1}{4}\right\rfloor + \frac{1}{2}\left\lfloor g(L_k) + \frac{1}{4}\right\rfloor\right).$$

We have

$$N + 1 > g(0) \ge \cdots \ge g(L_N - 1) \ge g(L_N) \ge N > g(L_N + 1),$$

therefore applying Lemma 5.3 with $i = 0$, $j = L_N$, and $n = N$, we get

$$2\left(\frac{1}{2}\left\lfloor g(0) + \frac{1}{4}\right\rfloor + \sum_{m=1}^{L_N-1}\left\lfloor g(m) + \frac{1}{4}\right\rfloor + \frac{1}{2}\left\lfloor g(L_N) + \frac{1}{4}\right\rfloor\right) \le 2\int_0^{L_N} g(z)\,\mathrm{d}z. \qquad (5.11)$$

Substituting (5.11) and (5.8) into (5.10) gives (5.9).

Finally, assume that we have the equality in (5.2). Due to Remark 5.4, the function g is linear on each interval $[L_{k+1}, L_k]$, and either $\mathrm{dist}(g(L_k), \mathbb{Z}) \ge \frac{1}{4}$ or $g \equiv 0$ on $[L_{k+1}, L_k]$. The situation when $\mathrm{dist}(g(L_k), \mathbb{Z}) \ge \frac{1}{4}$ and $g \equiv 0$ on $[L_k, L_{k-1}]$ is impossible due to continuity of g. If $\mathrm{dist}(g(L_k), \mathbb{Z}) \ge \frac{1}{4}$ for all k, then in particular $g(L_0) = g(\lfloor B\rfloor) \ge \frac{1}{4}$, and the last inequality in (5.9) is strict. Therefore, the equality in (5.2) requires $g \equiv 0$ on the whole interval $[0, b]$. $\square$

Remark 5.5. If g is strictly monotone on $[0, b]$, then the inverse function g^{-1} is well-defined on $[0, g(0)]$, and the definitions (5.6) may be equivalently rewritten as

$$L_k = \lfloor g^{-1}(k) \rfloor, \qquad k = 0, \dots, \lfloor g(0) \rfloor.$$

◀

We finally use Theorem 5.1 to prove Theorem 4.1.

Proof of Theorem 4.1. We apply (5.2) with $b = \lambda$ and $g = G_\lambda$ (which we can do since Lemma 4.5 ensures that (5.1) holds in this case), and use (4.4), giving the bound (4.3) and therefore confirming the validity of the Dirichlet Pólya's conjecture for the disk. □

§6. Proof of Theorem 4.3

We start by stating

Theorem 6.1. *Let $b > 0$, and let g be a non-negative decreasing convex function on $[0, b]$ such that $g(0) \geq \frac{1}{4}$, $g(b) = 0$, and (5.1) holds for all $z, w \in [0, b]$. Let*

$$M_0 = M_{g,0} := 1 + \max \left\{ m \in \{0, \dots, \lfloor b \rfloor\} : g(m) \geq \frac{1}{4} \right\},$$

and assume that $M_0 \leq b$. Then

$$\sum_{m=0}^{\lfloor b \rfloor} \left\lfloor g(m) + \frac{3}{4} \right\rfloor \geq \int_0^b g(z) \, \mathrm{d}z - \frac{b - 3M_0}{8}. \tag{6.1}$$

Remark 6.2. If g is strictly monotone on $[0, b]$, then the inverse function g^{-1} is well-defined on $[0, g(0)]$, and

$$M_0 = \left\lfloor g^{-1}\left(\frac{1}{4}\right) \right\rfloor + 1, \tag{6.2}$$

cf. Remark 5.5. ◀

Remark 6.3. Once more, we first outline a very informal plan of proving Theorem 6.1. As we have argued in Remark 5.2, we have $\lfloor g(m) + \frac{3}{4} \rfloor \approx g(m) + \frac{1}{4}$, which should in principle ensure the correct inequality sign in (6.1). "Bad" points are now the points with $n \leq g(m) \leq n + \frac{1}{4}$. So, we divide the graph of g by the horizontal lines at $y = n + \frac{1}{4}$, where $n = 0, 1, \dots, \lfloor g(0) + \frac{3}{4} \rfloor$. Again, this guarantees that the number of "bad" points in each resulting interval is less than half the total number of points there. This still leaves an unresolved issue of points m lying under the tail of the graph of g, where $0 \leq g(m) < \frac{1}{4}$. Such points make no contribution to the left-hand side of (6.1), but the tail does contribute to the integral: consider, for example, a toy case of a function $g(z) = \frac{10-z}{80}$ on the interval $[0, 10]$. To account for that, we subtract an additional term in the right-hand side of (6.1). ◀

Before proceeding to the proof of Theorem 6.1, we require

Lemma 6.4. *Let $i, j \in \mathbb{Z}$, $i < j$. Let g be a decreasing convex function on $[i, j]$. Assume that*

$$n + \frac{1}{4} > g(i) \geq \cdots \geq g(j-1) \geq n - \frac{3}{4} > g(j)$$

for some $n \in \mathbb{Z}$. Then

$$\frac{1}{2} \left\lfloor g(i) + \frac{3}{4} \right\rfloor + \sum_{m=i+1}^{j-1} \left\lfloor g(m) + \frac{3}{4} \right\rfloor + \frac{1}{2} \left\lfloor g(j) + \frac{3}{4} \right\rfloor \geq \int_i^j g(z) \, \mathrm{d}z + \frac{j-i}{4} - \frac{1}{2}. \tag{6.3}$$

Proof of Lemma 6.4. The left-hand side of (6.3) is equal to $(j-i)n - \frac{1}{2}$. By convexity of g,

$$\int_i^j g(z)\,\mathrm{d}z \le \frac{j-i}{2}\big(g(j)+g(i)\big) \le (j-i)\left(n-\frac{1}{4}\right).$$

$\square$

Proof of Theorem 6.1. As $g(0) \ge \frac{1}{4}$, we have

$$N := \left\lfloor g(0)+\frac{3}{4}\right\rfloor \ge 1. \tag{6.4}$$

For $k = 1,\dots,N-1$, denote

$$M_k = M_{g,k} := 1 + \max\left\{m \in \{0,\dots,\lfloor b\rfloor\} : g(m) \ge k + \frac{1}{4}\right\}.$$

We also denote $M_N = M_{g,N} := 0$. The assumption (5.1) yields $M_k > M_{k+1}$ for all $k = 0,\dots,N-1$. Therefore, by Lemma 6.4,

$$\frac{1}{2}\left\lfloor g(a)+\frac{3}{4}\right\rfloor + \sum_{m=1}^{\lfloor b\rfloor}\left\lfloor g(m)+\frac{3}{4}\right\rfloor$$

$$= \sum_{n=0}^{N-1}\left(\frac{1}{2}\left\lfloor g(M_{n+1})+\frac{3}{4}\right\rfloor + \sum_{m=M_{n+1}+1}^{M_n-1}\left\lfloor g(m)+\frac{3}{4}\right\rfloor + \frac{1}{2}\left\lfloor g(M_n)+\frac{3}{4}\right\rfloor\right)$$

$$\ge \sum_{n=0}^{N-1}\left(\int_{M_{n+1}}^{M_n} g(z)\,\mathrm{d}z + \frac{M_n-M_{n+1}}{4}-\frac{1}{2}\right) = \int_0^{M_0} g(z)\,\mathrm{d}z + \frac{M_0}{4}-\frac{N}{2}.$$

Due to (6.4) we get

$$\sum_{m=0}^{\lfloor b\rfloor}\left\lfloor g(m)+\frac{3}{4}\right\rfloor \ge \int_0^{M_0} g(z)\,\mathrm{d}z + \frac{M_0}{4}.$$

By convexity of g,

$$\int_{M_0}^b g(z)\,\mathrm{d}z \le \frac{1}{2}g(M_0)(b-M_0) \le \frac{b-M_0}{8},$$

and we get (6.1). $\square$

We can now proceed to the proof of Theorem 4.3. We start with

Proposition 6.5. *Let $\lambda \ge 2$. Then*

$$\mathscr{P}_2^{\mathrm{N}}(\lambda) > \frac{\lambda^2}{4} + \frac{1}{4}R_2(\lambda), \tag{6.5}$$

where

$$R_2(\lambda) := 3G_\lambda^{-1}\left(\frac{1}{4}\right) - \lambda\left(1+\frac{4}{\pi}\right) - 3.$$

Proof. We have, with account of $G_\lambda(0) = \frac{\lambda}{\pi}$,

$$\mathscr{P}_2^{\mathrm{N}}(\lambda) = \left\lfloor G_\lambda(0)+\frac{3}{4}\right\rfloor + 2\sum_{m=1}^{\lfloor \lambda\rfloor}\left\lfloor G_\lambda(m)+\frac{3}{4}\right\rfloor$$

$$= -\left\lfloor \frac{\lambda}{\pi}+\frac{3}{4}\right\rfloor + 2\sum_{m=0}^{\lfloor \lambda\rfloor}\left\lfloor G_\lambda(m)+\frac{3}{4}\right\rfloor.$$

We apply Theorem 6.1 to the sum in the right-hand side, with $g = G_\lambda$, $b = \lambda$, and

$$M_{G_\lambda, 0} = \left\lfloor G_\lambda^{-1} \left(\frac{1}{4} \right) \right\rfloor + 1$$

(see Remark 6.2), taking into account the bound (4.6) (which ensures that $M_{G_\lambda, 0} \leq \lambda$), and the value of the integral from Lemma 4.6, leading to

$$\mathscr{P}_2^{\mathrm{N}}(\lambda) - \frac{\lambda^2}{4} \geq \frac{3 \left\lfloor G_\lambda^{-1} \left(\frac{1}{4} \right) \right\rfloor + 3 - \lambda - 4 \left\lfloor \frac{\lambda}{\pi} + \frac{3}{4} \right\rfloor}{4}.$$

Finally, we use

$$x - 1 < \lfloor x \rfloor \leq x$$

to obtain (6.5). □

We now have

Proposition 6.6. *Let $\lambda \geq \Lambda_1$, where Λ_1 is given by (1.9). Then $R_2(\lambda) \geq 0$.*

Proof of Proposition 6.6. Let $\sigma \in \left(0, \frac{\pi}{2} \right]$ and $\lambda \geq r_1(\sigma)$, which is given by (4.8). Then by Lemma 4.8, the bound (4.7) holds. Therefore

$$R_2(\lambda) \geq \lambda \left(3 \cos \sigma - 1 - \frac{4}{\pi} \right) - 3,$$

and is non-negative whenever

$$\lambda \geq r_2(\sigma) := \frac{3}{3 \cos \sigma - 1 - \frac{4}{\pi}}$$

and $r_2(\sigma) > 0$. Assuming this is satisfied, we have $R_2(\lambda) \geq 0$ for $\Lambda \geq \max\{r_1(\sigma), r_2\sigma)\}$.

Choose now $\sigma = \sigma_\bullet := \arccos \frac{5}{6}$. Then

$$\frac{3\pi/2}{\sqrt{11} - 5 \arccos \frac{5}{6}} = r_1(\sigma_\bullet) < r_2(\sigma_\bullet) = \frac{6\pi}{3\pi - 8} = \Lambda_1$$

(with the inequality easy to prove rigorously using the method of verified rational approximations discussed in §8), and the result follows. □

Theorem 4.3 now immediately follows from Propositions 6.6 and 6.5.

§7. Proof of Theorem 4.2

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We have the following "dimension reduction" formula.

Theorem 7.1. *Let $d \geq 3$. Then*

$$\mathscr{P}_d^{\mathrm{D}}(\lambda) = \sum_{n=0}^{\lfloor \lambda - \frac{d}{2} + 1 \rfloor} \binom{n + d - 3}{d - 3} \tilde{\mathscr{P}}_{n + \frac{d}{2} - 1}^{\mathrm{D}}(\lambda), \tag{7.1}$$

where for $r \in [0, \lambda]$ we denote by

$$\tilde{\mathscr{P}}_r^{\mathrm{D}}(\lambda) := \left\lfloor \tilde{G}_{\lambda, r}(0) + \frac{1}{4} \right\rfloor + 2 \sum_{j=1}^{\lfloor \lambda \rfloor} \left\lfloor \tilde{G}_{\lambda, r}(j) + \frac{1}{4} \right\rfloor = \left\lfloor G_\lambda(r) + \frac{1}{4} \right\rfloor + 2 \sum_{j=1}^{\lfloor \lambda - r \rfloor} \left\lfloor G_\lambda(j + r) + \frac{1}{4} \right\rfloor \tag{7.2}$$

the "standard" two-dimensional weighted shifted lattice point count under the graph of the function $\tilde{G}_{\lambda, r}(t) := G_\lambda(t + r)$, $t \in [0, \lambda - r]$.

We remark that in comparison to our original definition

$$\mathscr{P}_d^{\mathrm{D}}(\lambda) = \sum_{m=0}^{\lfloor \lambda - \frac{d}{2} + 1 \rfloor} \kappa_{d,m} \left\lfloor G_\lambda \left(m + \frac{d}{2} - 1 \right) + \frac{1}{4} \right\rfloor,$$

see (2.8), where the weights $\kappa_{d,m}$ are attached at each individual abscissa m, the formula in the right-hand side of (7.1) attaches weights $\binom{n+d-3}{d-3}$ to the whole counts $\tilde{\mathscr{P}}_{n+\frac{d}{2}-1}^{\mathrm{D}}(\lambda)$, which we will later estimate using the previously proven Theorem 5.1.

Proof of Theorem 7.1. With account of (7.2), the right-hand side of (7.1) reads

$$\sum_{n=0}^{\lfloor \lambda - \frac{d}{2} + 1 \rfloor} \binom{n+d-3}{d-3} \left(\left\lfloor G_\lambda \left(n + \frac{d}{2} - 1 \right) + \frac{1}{4} \right\rfloor + 2 \sum_{j=1}^{\lfloor \lambda - n - \frac{d}{2} + 1 \rfloor} \left\lfloor G_\lambda \left(j + n + \frac{d}{2} - 1 \right) + \frac{1}{4} \right) \right\rfloor \right).$$

For a fixed $m \in \{0\} \cup \mathbb{N}$, the contribution of $\left\lfloor G_\lambda \left(m + \frac{d}{2} - 1 \right) + \frac{1}{4} \right\rfloor$ in this expression appears with the factor

$$\binom{m+d-3}{d-3} + 2 \sum_{j=0}^{m-1} \binom{j+d-3}{d-3} = \sum_{j=0}^{m} \binom{j+d-3}{d-3} + \sum_{j=0}^{m-1} \binom{j+d-3}{d-3}$$

$$= \sum_{i=d-3}^{m+d-3} \binom{i}{d-3} + \sum_{i=d-3}^{m-d-4} \binom{i}{d-3} = \binom{m+d-2}{d-2} + \binom{m+d-3}{d-2}$$

$$= \binom{m+d-1}{d-1} - \binom{m+d-3}{d-1} = \kappa_{d,m},$$

where we have used the standard identity [DLMF, Eq. 26.3.7]

$$\sum_{i=l}^{r} \binom{i}{l} = \binom{r+1}{l+1}, \qquad r \geq l, \tag{7.3}$$

and another identity [DLMF, Eq. 26.3.5],

$$\binom{k}{l} = \binom{k+1}{l+1} - \binom{k}{l+1}, \qquad k \geq l.$$

Thus, the contributions of $\left\lfloor G_\lambda \left(m + \frac{d}{2} - 1 \right) + \frac{1}{4} \right\rfloor$ in both sides of (7.1) coincide. □

Before proceeding to the proof of Theorem 4.2 we will introduce some additional notation and state some auxiliary facts which will be required later. Let, for $x \geq 0$,

$$\Pi_n(x) := \prod_{j=1}^{n} (x + j) = (x+1) \cdots (x+n) \quad \text{for } n \in \mathbb{N}, \qquad \Pi_0(x) := 1.$$

The function $\Pi_n(x)$ is closely related to *Pochhammer's symbol*, or the *rising factorial* $(x)_n := x \cdots (x + n - 1)$ (for which numerous other notation is also used) in the sense that $\Pi_n(x) = (x+1)_n$. We also have

$$\binom{i+j}{i} = \frac{(i+1) \cdots (i+j)}{j!} = \frac{\Pi_j(i)}{j!} = \frac{\Pi_i(j)}{i!}.$$

Let us introduce, for $d \geq 3$, a piecewise-constant function

$$f_d(t) := \begin{cases} 0, & \text{if } t < \frac{d}{2} - 1, \\ \frac{\Pi_{d-2}(m)}{(d-2)!} = \binom{m+d-2}{d-2}, & \text{if } \left\lfloor t - \frac{d}{2} + 1 \right\rfloor = m \geq 0, \end{cases} \tag{7.4}$$

and let

$$F_d(z) := \int_0^z f_d(t) \, dt.$$

In what follows we will require an upper polynomial bound on $F_d(z)$. If $z \leq \frac{d}{2} - 1$, then $F_d(z) = 0$. Let $m = \left\lfloor z - \frac{d}{2} + 1 \right\rfloor \geq 0$. Then

$$\begin{aligned} F_d(z) = \int_0^z f_d(t) \, dt &= \left(\sum_{k=0}^{m-1} \int_{k+d/2-1}^{k+d/2} f_d(t) \, dt \right) + \int_{m+d/2-1}^z f_d(t) \, dt \\ &= \left(\sum_{k=0}^{m-1} \binom{k+d-2}{d-2} \right) + \left(z - m - \frac{d}{2} + 1 \right) \binom{m+d-2}{d-2} \\ &= \binom{m+d-2}{d-1} + \left(z - m - \frac{d}{2} + 1 \right) \binom{m+d-2}{d-2} \\ &= \frac{1}{(d-1)!} \Pi_{d-1}(m-1) + \frac{1}{(d-2)!} \left(z - m - \frac{d}{2} + 1 \right) \Pi_{d-2}(m) \\ &= \frac{\Pi_{d-2}(m)}{(d-1)!} \left(m + (d-1) \left(z - m - \frac{d}{2} + 1 \right) \right) \\ &= \frac{\Pi_{d-2}(m)}{(d-1)!} \left((d-1)z - (d-2)m - \frac{(d-1)(d-2)}{2} \right), \end{aligned} \tag{7.5}$$

where we used (7.3) to evaluate the sum of binomial coefficients.

We will use the following technical

Lemma 7.2. *Let $x \geq 0$, $n \in \mathbb{N}$ and $z \geq 0$. Then*

$$\Pi_n(x) \left((n+1)z - nx - \frac{n(n+1)}{2} \right) \leq z^{n+1}. \tag{7.6}$$

Proof. We apply the AM-GM inequality

$$l \left(\beta_1 \cdots \beta_\ell \right)^{1/l} \leq \beta_1 + \cdots + \beta_l, \qquad l \in \mathbb{N}, \quad \beta_1, \dots, \beta_l \geq 0,$$

with $l = n + 1$ and

$$\beta_j = \begin{cases} (x+j)\Pi_n(x) & \text{if } j \leq n, \\ z^{n+1} & \text{if } j = n+1, \end{cases}$$

yielding

$$(n+1)z\Pi_n(x) \leq \Pi_n(x) \left((x+1) + \cdots + (x+n) \right) + z^{n+1} = \left(nx + \frac{n(n+1)}{2} \right) \Pi_n(x) + z^{n+1},$$

and hence (7.6) after collecting the factors of $\Pi_n(x)$ to one side. $\qquad \square$

We will now establish a bound on $F_d(z)$.

Lemma 7.3. *Let $z \geq 0$. Then*

$$F_d(z) \leq \tilde{F}_d(z) := \frac{z^{d-1}}{(d-1)!}.$$

Proof. If $z < \frac{d}{2} - 1$, then $F_d(z) = 0$, and the bound is trivial. For $z \geq \frac{d}{2} - 1$, we apply (7.6) with $n = d - 2$ and $x = m$ to the right-hand side of (7.5). □

We will require one more auxiliary

Lemma 7.4. *Let f be a non-negative function on $[0, +\infty)$, and let $\underline{F}, \tilde{F} \in C^1[0, +\infty)$ with $\underline{F}(0) = \tilde{F}(0) = 0$ be such that*

$$\underline{F}(z) \leq F(z) := \int_0^z f(t)\,\mathrm{d}t \leq \tilde{F}(z) \qquad \text{for all } z \geq 0.$$

Let $b > 0$, and let $g \in C^1[0, b]$ be a decreasing function such that $g(b) = 0$. Then

$$\int_0^b \underline{F}'(z)g(z)\,\mathrm{d}z \leq \int_0^b f(z)g(z)\,\mathrm{d}z \leq \int_0^b \tilde{F}'(z)g(z)\,\mathrm{d}z.^8 \tag{7.7}$$

Proof. We have

$$\tilde{F}(z) - F(z) = \int_0^z \left(\tilde{F}'(t) - f(t)\right)\mathrm{d}t \geq 0.$$

Integrating by parts, we get

$$\int_0^b \tilde{F}'(z)g(z)\,\mathrm{d}z - \int_0^b f(z)g(z)\,\mathrm{d}z = \int_0^b \left(\tilde{F}'(z) - F'(z)\right)g(z)\,\mathrm{d}z$$

$$= \left.\left(\tilde{F}(z) - F(z)\right)g(z)\right|_0^b - \int_0^b \left(\tilde{F}(z) - F(z)\right)g'(z)\,\mathrm{d}z \geq 0$$

since $\tilde{F}(0) = F(0) = g(b) = 0$ and $\left(\tilde{F}(z) - F(z)\right)g'(z) \leq 0$, thus proving the upper bound in (7.7).

Similarly,

$$F(z) - \underline{F}(z) = \int_0^z \left(f(t) - \underline{F}'(t)\right)\mathrm{d}t \geq 0,$$

and thus

$$\int_0^b f(z)g(z)\,\mathrm{d}z - \int_0^b \underline{F}'(z)g(z)\,\mathrm{d}z = \int_0^b \left(F'(z) - \underline{F}'(z)\right)g(z)\,\mathrm{d}z$$

$$= \left.(F(z) - \underline{F}(z))g(z)\right|_0^b - \int_0^b (F(z) - \underline{F}(z))g'(z)\,\mathrm{d}z \geq 0$$

since we also have $\underline{F}(0) = 0$, thus proving the lower bound in (7.7). □

We now proceed to the proof of Theorem 4.2 proper.

Proof of Theorem 4.2. First of all, we apply Theorem 5.1 with $g = \tilde{G}_{\lambda,r}$, $r = n + \frac{d}{2} - 1$, and $b = \lambda - r = \lambda - n - \frac{d}{2} + 1$ to the right-hand side of (7.2), giving

$$\tilde{\mathscr{P}}^{\mathrm{D}}_{n+\frac{d}{2}-1}(\lambda) < 2\int_0^{\lambda - n - \frac{d}{2} + 1} G_\lambda\left(t + n + \frac{d}{2} - 1\right)\mathrm{d}t = 2\int_{n+\frac{d}{2}-1}^\lambda G_\lambda(z)\,\mathrm{d}z$$

for all $\lambda > 0$. We now substitute the result into (7.1), yielding the bound

$$\mathscr{P}^{\mathrm{D}}_d(\lambda) < 2\sum_{n=0}^{\lfloor \lambda - \frac{d}{2} + 1 \rfloor} \binom{n+d-3}{d-3} \int_{n+\frac{d}{2}-1}^\lambda G_\lambda(z)\,\mathrm{d}z$$

$$= 2\int_0^\lambda \left(\sum_{n=0}^{\lfloor z - \frac{d}{2} + 1 \rfloor} \binom{n+d-3}{d-3}\right) G_\lambda(z)\,\mathrm{d}z \tag{7.8}$$

$$= 2\int_0^\lambda \binom{\lfloor z - \frac{d}{2} + 1 \rfloor + d - 2}{d-2} G_\lambda(z)\,\mathrm{d}z.$$

---

[8] We will only use the upper bound but state and prove both bounds for the sake of completeness.

Using notation (7.4), we rewrite (7.8) as

$$\mathscr{P}_d^{\mathrm{D}}(\lambda) < 2 \int_0^\lambda f_d(z) G_\lambda(z) \, \mathrm{d}z. \tag{7.9}$$

We now apply Lemma 7.4 to the right-hand side of (7.9), taking $f = f_d$, $g = G_\lambda$, and $\tilde{F}(z) = \tilde{F}_d(z) = \frac{z^{d-1}}{(d-1)!}$ by Lemma 7.3, which gives

$$\mathscr{P}_d^{\mathrm{D}}(\lambda) \le 2 \int_0^\lambda \tilde{F}_d'(z) G_\lambda(z) \, \mathrm{d}z = \frac{2}{(d-2)!} \int_0^\lambda z^{d-2} G_\lambda(z) \, \mathrm{d}z.$$

Finally, applying Corollary 4.7 gives

$$\mathscr{P}_d^{\mathrm{D}}(\lambda) \le W_d(\lambda)$$

as required. □

§8. Closing the gap: the proof of Theorem 4.4

We describe the algorithm (based on the two Principles stated in §1) of verifying the statement

$$\mathscr{P}_2^{\mathrm{N}}(\lambda) > \frac{\lambda^2}{4} \qquad \text{for all } \lambda \in [3, 14], \tag{8.1}$$

with the lattice point counts given explicitly by

$$\mathscr{P}_2^{\mathrm{N}}(\lambda) = \sum_{m=0}^{\lfloor \lambda \rfloor} \kappa_{2,m} \left\lfloor G_\lambda(m) + \frac{3}{4} \right\rfloor.$$

The first Principle is easily implemented with the help of the following simple lemma.

Lemma 8.1. *If the inequality* (8.1) *holds for a particular λ_0, that is we have*

$$e(\lambda_0) := \mathscr{P}_2^{\mathrm{N}}(\lambda_0) - \frac{\lambda_0^2}{4} > 0,$$

this inequality also holds for all

$$\lambda \in (\lambda_0, \lambda_0 + \delta(\lambda_0)) = \left(\lambda_0, \sqrt{\lambda_0^2 + 4e(\lambda_0)} \right),$$

where

$$\delta(\lambda_0) := \sqrt{\lambda_0^2 + 4e(\lambda_0)} - \lambda_0.$$

Proof. The result immediately follows from the facts that $\mathscr{P}_2^{\mathrm{N}}(\lambda)$ is non-decreasing in λ and that $\Lambda = \lambda_0 + \delta(\lambda_0)$ is the positive root of the equation

$$\frac{\Lambda^2}{4} = \frac{\lambda_0^2}{4} + e(\lambda_0).$$

□

To implement the second Principle, we work with rational numbers only. Let, for $x \in \mathbb{R}$, $\underline{x} \le x \le \overline{x}$, where $\underline{x}, \overline{x} \in \mathbb{Q}$ are some lower and upper rational approximations of x. The function $G_\lambda(z)$ does not as a rule take rational values even for rational λ and z, so to overcome this we work instead with

$$\underline{\mathscr{P}}_2^{\mathrm{N}}(\lambda) = \sum_{m=0}^{\lfloor \lambda \rfloor} \kappa_{2,m} \left\lfloor \underline{G}_\lambda(m) + \frac{3}{4} \right\rfloor, \qquad \lambda \in \mathbb{Q},$$

where

$$\underline{G}_\lambda(z) = \frac{1}{\pi}\left(\sqrt{\lambda^2 - z^2} - z\,\overline{\arccos}\,\frac{z}{\lambda} \right), \qquad z \in \mathbb{Q} \cap [0, \lambda].$$

Of course, $\underline{G}_\lambda(z) \le G_\lambda(z)$ and therefore $\underline{\mathscr{P}}_2^{\mathrm{N}}(\lambda) \le \mathscr{P}_2^{\mathrm{N}}(\lambda)$, for $\lambda, z \in \mathbb{Q}$. Obviously, taking integer parts of rational numbers, as well as other arithmetic operations on them is exact and does not introduce any numerical errors. We now describe how to construct *verified rational approximations* of the square roots $\underline{\sqrt{\cdot}}$ and arccosines $\overline{\arccos}(\cdot)$. For the former, any guess (say, obtained from numerics) can be directly verified by taking squares and comparing rationals, which is rigorous. To verify our approximations of arccosines (taken at rational points) one may proceed as follows. Define the functions $\underline{\cos}, \overline{\cos} : \mathbb{Q} \to \mathbb{Q}$ as

$$\overline{\cos} x := T_{12}[\cos](x), \qquad \underline{\cos} x := T_{14}[\cos](x),$$

where $T_t[\cos](x)$ is the Taylor polynomial of $\cos x$ at $x = 0$ of degree t, so that $\underline{\cos} x < \cos x < \overline{\cos} x$. Then

$$\overline{\cos}\left(\overline{\beta}\right) < x < \underline{\cos}\left(\underline{\beta}\right) \qquad \text{implies} \qquad \underline{\beta} := \underline{\arccos} x < \beta = \arccos x < \overline{\arccos} x =: \overline{\beta},$$

and the verification is again reduced to elementary operations on rationals. We can also verify the calculated

$$\underline{\pi} = 3\,\underline{\arccos}\,\frac{1}{2}, \qquad \overline{\pi} = 3\,\overline{\arccos}\,\frac{1}{2}$$

in the same manner.

To finish describing our process, we need also to rationalise the square root appearing in the definition of $\delta(\lambda)$: we effectively replace $e(\lambda)$ by a smaller number

$$\underline{e}(\lambda) := \underline{\mathscr{P}}_2^{\mathrm{N}}(\lambda) - \frac{\lambda^2}{4}, \tag{8.2}$$

and also replace $\delta(\lambda)$ by a smaller number

$$\underline{\delta}(\lambda) := \underline{\sqrt{\lambda^2 + 4\underline{e}(\lambda)}} - \lambda, \tag{8.3}$$

where a verification is again by taking squares.

Remark 8.2. In practice, we use the following process to find lower and upper rational approximations of a number $x \in \mathbb{R}$ (which may be a square root, or an arccosine). Throughout, we fix a relatively small number ε (say, $\varepsilon = 10^{-3}$) as an accuracy parameter. We find numerically some approximation x_0 of x (which may be above or below x) with some better accuracy. Then, we define $\underline{x}$ as the rational number in the interval $[x_0 - 3\varepsilon, x_0 - \varepsilon]$ with the smallest possible denominator, and $\overline{x}$ as the rational number in the interval $[x_0 + \varepsilon, x_0 + 3\varepsilon]$ with the smallest possible denominator, using a modification of a fast algorithm for traversing the Stern–Brocot tree [For07]. As we always verify the resulting approximations using the procedures described above, we do not in fact depend on the quality of an original numerical "guess" x_0 as long as $|x_0 - x| < \varepsilon$.   ◄

The algorithm for proving (8.1)

$\lambda \leftarrow \underline{\Lambda_0}$
stepnumber $\leftarrow 0$
while $\lambda \leq \overline{\Lambda_1}$ **do**
    stepnumber $\leftarrow$ stepnumber $+ 1$
    $p \leftarrow \underline{\mathscr{P}_2^{\mathrm{N}}}(\lambda)$
    $e \leftarrow p - \frac{\lambda^2}{4}$
    if $e > 0$ **then**

        $\lambda \leftarrow \underline{\sqrt{\lambda^2 + 4e}}$
    else
        print "Proof failed ☺"; **exit**
    end if
end while
print "Success in ", stepnumber, "steps ☺"

Figure 6: The basic algorithm.

Thus, our main algorithms work as follows. In order to prove (8.1) for $\lambda \in [\Lambda_0, \Lambda_1]$, we move upwards: set $\lambda = \underline{\Lambda_0}$, compute the margin $\underline{e}(\lambda)$ from (8.2), set $\lambda_{\mathrm{new}} = \lambda + \underline{\delta}(\lambda)$ using (8.3), and continue the process. If the margins are positive on each step, the process will stop successfully if after a finite number of steps we reach $\lambda > \overline{\Lambda_1}$, see Figure 6.

The algorithm works extremely fast (when implemented in `Mathematica`, see the footnote on the title page), thus proving Theorem 4.4: in principle, with enough patience the whole implementation can be done by hand. We summarise its outcomes in Table 1.

Step	λ	$\underline{e}(\lambda)$	$\underline{\delta}(\lambda)$
1	$3 < \Lambda_0$	$\frac{3}{4}$	$\frac{6}{13}$
2	$\frac{45}{13}$	$\frac{1355}{676}$	$\frac{223}{221}$
3	$\frac{76}{17}$	$\frac{868}{289}$	$\frac{584}{493}$
4	$\frac{164}{29}$	$\frac{3368}{841}$	$\frac{995}{783}$
5	$\frac{187}{27}$	$\frac{11687}{2916}$	$\frac{29}{27}$
6	8	3	$\frac{43}{60}$
7	$\frac{523}{60}$	$\frac{57671}{14400}$	$\frac{227}{260}$
8	$\frac{374}{39}$	$\frac{6098}{1521}$	$\frac{719}{897}$
9	$\frac{239}{23}$	$\frac{10591}{2116}$	$\frac{339}{368}$
10	$\frac{181}{16}$	$\frac{4103}{1024}$	$\frac{11}{16}$
11	12	6	$\frac{24}{25}$
12	$\frac{324}{25}$	$\frac{2506}{625}$	$\frac{241}{400}$
13	$\frac{217}{16}$	$\frac{7183}{1024}$	$\frac{271}{272}$
14	$\frac{495}{34} > \Lambda_1$		

Table 1: Detailed output of the computer-assisted algorithm.

§9. Proof of Theorem 1.12

Let S_α be a circular sector of aperture $0 < \alpha \le 2\pi$. The eigenvalues of the Dirichlet and Neumann Laplacians on S_α are easily found by separation of variables. They are all simple, and are given by

$$\lambda_{m,k} = \left(j_{\frac{m\pi}{\alpha},k} \right)^2 \qquad m \in \mathbb{N}, \quad k \in \mathbb{N},$$

and

$$\mu_{m,k} = \left(j'_{\frac{m\pi}{\alpha},k} \right)^2, \qquad m \in \mathbb{N} \cup \{0\}, \quad k \in \mathbb{N},$$

respectively. Therefore, the corresponding eigenvalue counting functions are

$$\mathcal{N}_{S_\alpha}^{\mathrm{D}}(\lambda) = \sum_{m=1}^{\lfloor \frac{\alpha\lambda}{\pi} \rfloor} \#\left\{ k \in \mathbb{N} : j_{\frac{m\pi}{\alpha},k} \le \lambda \right\} \qquad \text{and} \qquad \mathcal{N}_{S_\alpha}^{\mathrm{N}}(\lambda) = \sum_{m=0}^{\lfloor \frac{\alpha\lambda}{\pi} \rfloor} \#\left\{ k \in \mathbb{N} : j'_{\frac{m\pi}{\alpha},k} \le \lambda \right\},$$

where the both sums are finite since $j_{\nu,1} > j'_{\nu,1} \ge \nu$.

Assume for the moment that the sector S_α contains a half-disk, that is $\alpha \in [\pi, 2\pi]$. By Proposition 3.1 with account of (3.1), (2.4) and (2.9), we have

$$\mathcal{N}_{S_\alpha}^{\mathrm{D}}(\lambda) \le \sum_{m=1}^{\lfloor \frac{\alpha\lambda}{\pi} \rfloor} \left\lfloor G_\lambda \left(\frac{m\pi}{\alpha} \right) + \frac{1}{4} \right\rfloor. \tag{9.1}$$

Set

$$g_{\lambda,\alpha}(t) := G_\lambda\left(\frac{\pi t}{\alpha}\right), \qquad t \in [0, b],$$

where

$$b := \frac{\alpha\lambda}{\pi}.$$

Then $g_{\lambda,\alpha}$ is a monotone decreasing convex function on $[0, b]$ with $g(b) = 0$; moreover,

$$g'_{\lambda,\alpha}(t) = \frac{\pi}{\alpha}G'_\lambda\left(\frac{\pi t}{\alpha}\right) \in \left[-\frac{\pi}{2\alpha}, 0\right] \subset \left[-\frac{1}{2}, 0\right]$$

due to our assumption $\alpha \geq \pi$. With this notation, the right-hand side of (9.1) becomes

$$\sum_{m=1}^{\lfloor b \rfloor} \left\lfloor g_{\lambda,\alpha}(m) + \frac{1}{4} \right\rfloor,$$

and we can estimate it from above directly by Theorem 5.1 with $g = g_{\lambda,\alpha}$, giving

$$\sum_{m=1}^{\lfloor b \rfloor} \left\lfloor g_{\lambda,\alpha}(m) + \frac{1}{4} \right\rfloor < \int_0^{\alpha\lambda/\pi} g_{\lambda,\alpha}(t)\,\mathrm{d}t = \frac{\alpha}{\pi}\int_0^\lambda G_\lambda(z)\,\mathrm{d}z = \frac{\alpha\lambda}{8\pi}.$$

Substituting this into (9.1) proves the Dirichlet Pólya's conjecture for sectors containing a half-disk.

We now turn to the Neumann problem in S_α, still assuming that $\alpha \in [\pi, 2\pi]$. Following the same argument as in Lemma 1.3 we conclude that the Neumann Pólya's conjecture for S_α holds for $\lambda \leq 2\sqrt{\frac{6\pi}{\alpha}}$. We therefore may assume that $\lambda > 2\sqrt{\frac{6\pi}{\alpha}} \geq 2\sqrt{3}$ from now on.

Using once more Proposition 3.1 yields

$$\mathscr{N}_{S_\alpha}^{\mathrm{N}}(\lambda) \geq \sum_{m=0}^{\lfloor \frac{\alpha\lambda}{\pi} \rfloor} \left\lfloor G_\lambda\left(\frac{m\pi}{\alpha}\right) + \frac{3}{4} \right\rfloor = \sum_{m=0}^{\lfloor b \rfloor} \left\lfloor g_{\lambda,\alpha}(m) + \frac{3}{4} \right\rfloor, \tag{9.2}$$

with the same function $g_{\lambda,\alpha}$ and parameter b as above.

We are now going to use Theorem 6.1 with $g = g_{\lambda,\alpha}$ to estimate the right-hand side of (9.2) from below. In this case by (6.2)

$$M_0 = M_{g_{\lambda,\alpha},0} = \left\lfloor g_{\lambda,\alpha}^{-1}\left(\frac{1}{4}\right) \right\rfloor + 1 = \left\lfloor \frac{\alpha}{\pi}G_\lambda^{-1}\left(\frac{1}{4}\right) \right\rfloor + 1,$$

and in order to apply Theorem 6.1 we need to ensure that $M_0 \leq b = \frac{\alpha\lambda}{\pi}$. But this is true since, with account of $\alpha \geq \pi$,

$$M_0 - b \leq \frac{\alpha}{\pi}G_\lambda^{-1}\left(\frac{1}{4}\right) + 1 - \frac{\alpha\lambda}{\pi} \leq \frac{\alpha}{\pi}\left(G_\lambda^{-1}\left(\frac{1}{4}\right) + 1 - \lambda\right) < 0$$

by (4.6). Then, (6.1) implies

$$\sum_{m=0}^{\lfloor b \rfloor} \left\lfloor g_{\lambda,\alpha}(m) + \frac{3}{4} \right\rfloor \geq \frac{\alpha\lambda}{8\pi} - \frac{b - 3M_0}{8}. \tag{9.3}$$

We now reason as in the proof of Theorem 4.3: if we can show that $b - 3M_0 < 0$ for all $\lambda \geq 2\sqrt{3}$, this would prove, via the combination of (9.2) and (9.3), that $\mathscr{N}_{S_\alpha}^{\mathrm{N}}(\lambda) > \frac{\alpha\lambda}{8\pi}$. We have

$$b - 3M_0 = \frac{\alpha\lambda}{\pi} - 3\left\lfloor \frac{\alpha}{\pi}G_\lambda^{-1}\left(\frac{1}{4}\right) \right\rfloor - 3 \leq \frac{\alpha}{\pi}\left(\lambda - 3G_\lambda^{-1}\left(\frac{1}{4}\right)\right) < \frac{\alpha}{\pi}\left(\lambda - 2G_\lambda^{-1}\left(\frac{1}{4}\right)\right).$$

We now apply the second statement of Lemma 4.8 with $\sigma = \frac{\pi}{3}$ which guarantees that $b - 3M_0 < 0$ for

$$\lambda > r_1\left(\frac{\pi}{3}\right) = \frac{3\pi}{6\sqrt{3} - 2\pi}.$$

As $\frac{3\pi}{6\sqrt{3} - 2\pi} < 2\sqrt{3}$, this finishes the proof of Theorem 1.12 for sectors S_α of aperture $\alpha \in [\pi, 2\pi]$.

To complete the proof of Theorem 1.12 we now need to consider the case $\alpha \in (0, \pi)$. Set

$$\ell := \left\lfloor \frac{2\pi}{\alpha} \right\rfloor \geq 2.$$

Then $\tilde{\alpha} = \ell\alpha \in [\pi, 2\pi]$, Pólya's conjecture holds for $S_{\tilde{\alpha}}$, and S_α tiles $S_{\tilde{\alpha}}$. By Theorem 1.8, Pólya's conjecture holds for S_α.

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