MAT6480 - Assignment 1

Due: 15 September 2008, 10:00 AM

1. Do all the exercises given in class. (No credit, do not hand in)

2. If X is a non-negative random variable, establish the identity

$$\mathbf{E}[X] = \int_0^\infty \mathbf{P}\{X > t\}\, dt$$

and more generally for any $0 < r < \infty$

$$\mathbf{E}[X^r] = r\int_0^\infty t^{r-1}\mathbf{P}\{X > t\}\, dt.$$

(Tao Vu 1.1.1)

3. Let A, B be non-empty subsets of a finite commutative group $(Z, +)$. Show that there exists an $x \in Z$ such that

$$1 - \frac{|A \cap (B+x)|}{|Z|} \leq \left(1 - \frac{|A|}{|Z|}\right)\left(1 - \frac{|B|}{|Z|}\right),$$

and a $y \in Z$ such that

$$1 - \frac{|A \cap (B+y)|}{|Z|} \geq \left(1 - \frac{|A|}{|Z|}\right)\left(1 - \frac{|B|}{|Z|}\right).$$

(Tao Vu 1.1.5)

4. Consider a set A as above. Show that there exists a subset $\{v_1, \ldots, v_d\}$ of Z with $d = O(\log(|Z|/|A|))$ such that

$$|A + [0,1]^d \cdot (v_1, \ldots, v_d)| \geq |Z|/2.$$

(Tao Vu 1.1.6)

5. (Extra credit.) Consider a set A as above. Show that there exists a subset $\{v_1, \ldots, v_d\}$ of Z with $d = O(\log(|Z|/|A|) + \log\log(10 + |Z|))$ such that

$$|A + [0,1]^d \cdot (v_1, \ldots, v_d)| = |Z|.$$

6. Suppose $n \geq 4$ and let H be an n-uniform hypergraph (all hyperedges have n elements) with at most $4^{n-1}/3^n$ edges. Prove that there is a colouring of the vertices of H by four colours so that in every edge all four colours are represented. (Alon Spencer 1.6.2)

7. Prove that for every two independent, identically distributed real random variables X and Y,

$$\mathbf{P}\{|X - Y| \leq 2\} \leq 3\mathbf{P}\{|X - Y| \leq 1\}.$$

(Alon Spencer 1.6.3)

8. When does equality hold in Chebyshev's inequality? (Tao Vu 1.2.1)

9. If X and Y are two random variables, verify the *Cauchy-Schwartz inequality*

$$|\mathbf{E}\left[(X-\mathbf{E}X)(Y-\mathbf{E}Y)\right]| \leq \sqrt{\mathbf{Var}\left[X\right]\mathbf{Var}\left[Y\right]}$$

and the *triangle inequality*

$$\sqrt{\mathbf{Var}\left[X+Y\right]} \leq \sqrt{\mathbf{Var}\left[X\right]} + \sqrt{\mathbf{Var}\left[Y\right]}.$$

When does equality occur? (Tao Vu 1.2.2)

10. By obtaining an upper bound on the fourth moment, prove that

$$\frac{|\{x \in \{1,\ldots,n\} \ : \ |\nu(x) - \log\log N| > K\sqrt{\log\log n}\}|}{N} = O(K^{-4}).$$

(Tao Vu 1.2.6) (Extra credit: what can you say if 4 is replaced by another integer m?)

11. Show that there is a positive constant c such that the following holds. For any n reals $a_1,\ldots,a_n$ satisfying $\sum_{i=1}^n a_i^2 = 1$, if $(x_1,\ldots,x_n)$ is a $\{-1,1\}$ random vector obtained by chooxing each x_i independently with uniform distribution to be either 1 or -1, then

$$\mathbf{P}\left\{\left|\sum_{i=1}^n x_i a_i\right| \leq 1\right\} > c.$$

(Alon Spencer 4.8.2)

12. (Extra credit) Show that there is a positive constant c such that the following holds. For any n vectors $a_1,\ldots,a_n \in \mathbb{R}^2$ satisfying $\sum_{i=1}^n \|a_i\|^2 = 1$ and $\|a_i\| \leq 1/10$, where $\|\cdot\|$ denotes the Euclidian norm, if $(x_1,\ldots,x_n)$ is a $\{-1,1\}$ random vector obtained by chooxing each x_i independently with uniform distribution to be either 1 or -1, then

$$\mathbf{P}\left\{\left\|\sum_{i=1}^n x_i a_i\right\| \leq 1\right\} > c.$$

(Alon Spencer 4.8.3)

13. Let $X_1,\ldots,X_n$ be jointly independent random variables, taking finitely many values, with $a_i \leq X_i \leq b_i$ for all i and some real numbers $a_i < b_i$. Let $S_n = \sum_{i=1}^n X_i$. Using the exponential moment method, show that for all $\lambda > 0$,

$$\mathbf{P}\left\{|X - \mathbf{E}X| > \lambda\left(\sum_{i=1}^n |b_i - a_i|^2\right)^{1/2}\right\} \leq 2e^{-2\lambda^2}.$$

(Tao Vu 1.3.5)