MAT6480 - Assignment 2

Due: 14 October 2008, 10:00 AM

1. We proved in class that if
$$e\binom{k}{2}\binom{n-2}{k-2}2^{1-\binom{k}{2}} < 1$$
then the diagonal Ramsey number $R(k,k)$ satisfies $R(k,k) > n$. Use this and Stirling's formula
$$n! = (1+o(1))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$
to prove that
$$R(k,k) \geq (1+o(1))\frac{\sqrt{2}}{e}k \cdot 2^{k/2}.$$

2. Let $\mathcal{F} = \{B_i\}_{i\in I}$ be a k-covering of $\mathbb{R}^3$ with open unit balls, and for $x \in \mathbb{R}^3$ let $E_x = \{B_i \in \mathcal{F} \,:\, x \in B_i\}$. Let H be the hypergraph with vertex set $\mathcal{F}$ and edge set $\{E_x \,:\, x \in \mathbb{R}^3\}$, with each hyperedge appearing only once, and suppose that H has maximum degree at most $d \in \mathbb{N}$ (i.e. so that no vertex (ball B_i) appears in more than d hyperedges), and that the set of centers $\{c_i\}_{i\in I}$ of the balls in $\mathcal{F}$ is nowhere dense.

   Show that there is an increasing sequence of finite subhypergraphs H_i, $i = 1, 2, \ldots$ of H and an increasing sequence of subsets $S_1, S_2, \ldots$ of $\mathbb{R}^3$, so that

   (a) for each i, the hyperedges in H_i induce a k-covering of S_i;
   (b) $\lim_{i\to\infty} S_i = \mathbb{R}^3$; and
   (c) $\lim_{i\to\infty} H_i = H$.

3. For any $m = 1, 2, \ldots$, show that given any directed graph $G = (V, E)$ with minimum outdegree $d > m$ and maximum in+outdegree $k > d$, if
$$e(k+1)^2\frac{d^m}{2^d} < 1$$
then there exists a partition of V into $V_1 \cup V_2$ such that $G[V_1]$ and $G[V_2]$ both have minimum outdegree at least m.

4. For each $n = 3, 4, \ldots$ construct random variables $X_1, \ldots, X_n$ so that $X_1, \ldots, X_n$ are not mutually independent but for each i, X_i is mutually independent of $\{X_j, j \neq i\}$.

5. (Alon + Spencer, 5.8.4, weakened) Let $G = (V, E)$ be a cycle of length $12n$ and let $V = V_1 \cup \ldots \cup V_n$ be a partition of its $12n$ vertices in to n pairwise disjoint subsets, each of cardinality 12. Prove that there must be an independent set of G containing precisely one vertex from each V_i.

6. (Extra credit: Alon + Spencer, 5.8.4): Can we replace 12 by 4 in question 5? Prove or supply a counterexample.

7. (Weighted local lemma): Given events $A_i, i = 1, \ldots, n$, a dependency graph $G = (\{1, \ldots, n\}, E)$ for the A_i, p with $0 \leq p \leq 1/4$ and constants $t_i \geq 1, i = 1, \ldots, n$, show that if

   (a) $\mathbf{P}\{A_i\} \leq p^{t_i}$, for $i = 1, \ldots, n$, and
   (b) $\sum_{ij\in E}(2p)^{t_j} \leq t_i/2$, for $i = 1, \ldots, n$

   then $\mathbf{P}\left\{\bigcap_{i=1}^n \overline{A_i}\right\} > 0$. (Hint: use general local lemma and the fact that $(1-x) \geq e^{-2\ln 2x}$ for $x \leq 1/2$.)