MAT6480 - Assignment 3

Due: 30 October 2008, 10:00 AM

1. (a) Prove that if $|A+A| = O(|A|)$ then $|A \cdot A| = \Omega(|A|^2/\log|A|)$. You may use any results proved in class.

   (b) Let n be a large integer. Show that all but at most $o(n^2)$ elements of $\{1,\ldots,n\}\cdot\{1,\ldots,n\}$ have $(2+o(1))\log\log n$ prime divisors. (Note that the convergence of the sum $\sum_{m=1}^{\infty}$ shows that one can neglect those elements which have a large square factor.) Conclude that $\{1,\ldots,n\}\cdot\{1,\ldots,n\} = o(n^2)$. (Hint: you may want to use a theorem proved earlier in the term, on the number of prime divisors of a "typical" (random) element of $\{1,\ldots,n\}$.)

2. Prove that for any set $A \subset \mathbb{R}$ with $2 \leq |A| < \infty$, $|A+A|^4|A\cdot A| = \Omega(|A|^6/\log|A|)$. You may use the fact proved in class that the claim holds if all elements of A are positive.

3. Suppose G is a multigraph in which each edge has multiplicity exactly m, and let G^* be the simple graph underlying G. Prove that $\text{cross}(G) = m^2\text{cross}(G^*)$.

4. Prove that if $P \subset \mathbb{R}^2$, $|P| < \infty$, and $k \geq 2$ then

$$|\{(p,\ell) : p \in P, \ell \text{ a line}, p \in \ell, |P \cap \ell| \geq k\}| = O\left(\frac{|P|^2}{k^2} + |P|\log|P|\right).$$

(Hint: apply rich lines result dyadically (to intervals of the form $(2^i, 2^{i+1})$.)